MINISTRY OF EDUCATION AND SCIENCE OF THE REPUBLIC OF KAZAKHSTAN

SULEYMAN DEMIREL UNIVERSITY

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SULEYMAN DEMIREL UNIVERSITY

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Some properties of ordered algebraic structures

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Kaskelen, 2019

Аңдатпа

Джон Гудрик реттелген дп-минимальді құрылымдарын зерттеумен айналысты. Ол, атап айтқанда, Кантор жиыны предикат P ретінде берілген $(\mathbb{R}, <, P)$ құрылымының қарапайым T теориясы дп-минимальді болатынын дәлелдеді. Дәлелдеу ретінде кванторларды жою әдісін осы T теориясына қолданады. Бұл жұмыста біз осы теорияға байланысты басқа сигнатурамен берілген кванторларды жою әдісі арқылы басқа дәлел ұсынамыз.

Кантор жиыны предикат P ретінде берілген $(\mathbb{R}, <, P, r, l, 0, 1)$ реттелген сандар жиыны мен екі r және l функцияларын (төменде анықталған) қарастырайық.

Егер x саны $[0, 1]$ кесіндісінде жатпайтын болса, онда $l(x) = r(x) = x$.

Егер x саны $[0, 1]$ кесіндісіне жататын болса, онда $l(x)$ - x санынан кіші болатын Кантор жиынының ең үлкен мәні, болмаса $l(x)) = x$.

r функциясы тура осылай анықталады. Егер x саны $[0, 1]$ кесіндісінде жатса, онда $r(x)$ - x санынан үлкен болатын Кантор жиынының ең аз мәні, болмаса $r(x) = x$.

$(\mathbb{R}, <, P, r, l, 0, 1)$ құрылымның осы қарапайым теориясы кванторларды жоюды рұқсат ететіні дәлелденеді. .

Сонымен қатар, реттелген тұрақты топтардағы формулалар жиынтығының қасиеттері зерттеледі.

Аннотация

В [5] Джон Гудрик начал изучать дп-минимальные упорядоченные структуры, где, в частности, он доказал, что элементарная теория T структуры $(R, <, P)$, где P выделяет канторово множество, является дп-минимальной. Для этого он доказал, что эта теория T допускает элиминвцию кванторов. В этой работе мы представляем еще одно доказательство элиминации кванторов для данной теории относительно другой сигнатуры.

Рассмотрим упорядоченное множество вещественных чисел с выделенным канторовом множеством $(\mathbb{R}, <, P, r, l, 0, 1)$, в котором предикат P пронитерпретировано как канторово множество и две унарные функции r и l определены следующим образом.

Если число x не лежит в отрезке $[0, 1]$, то $l(x) = r(x) = x$.

Если число x лежит в отрезке $[0, 1]$, то $l(x)$ - это максимальное число из канторово множества, строго меньшее, чем x, если оно существует, в противном случае $l(x)) = x$.

Функция r определяется аналогичным образом. Если число x лежит в отрезке $[0, 1]$, то $r(x)$ - это минимальное число из множества Кантора, строго превышающее x, если оно существует, в противном случае $r(x) = x$.

Доказывается, что эта элементарная теория структуры $(\mathbb{R}, <, P, r, l, 0, 1)$ допускает элиминацию кванторов.

Кроме того, исследуются свойства подмножеств формул в упорядоченно устойчивых упорядоченных группах.

Acknowledgements

This work would not have been possible without the financial support and state grant.

I am grateful to my supervisor with whom I have had the pleasure to work during this and other related works. Each of the members of my Dissertation Committee has provided me extensive personal and professional guidance and taught me a great deal about both scientific research and life in general. I would especially like to thanks Dr. Verbovskiy V. V. and Dr. Baizhanov B. S.. As my teachers and mentors, they have taught me more than I could ever give them credit for here. They have shown me, by his example, what a good scientist and person should be. Nobody has been more important to me in the pursuit of this project than the members of my family. I would like to thank my parents and husband, whose love and guidance are with me in whatever I pursue. They are the ultimate role models.

Contents

Nomenclature

$\exists$	Exists
$\forall$	For all
$\mathcal{L}$	First order language
$\mathfrak{M}$	Model of T
$\models$	Model
$\neg$	Negation
$\prec$	Elementary extension
$\rightarrow$	Implication
$\subset$	Subset
$\upharpoonright$	Bounded
$\vee$	Disjunction
$\wedge$	Conjunction
T	Theory

1. Introduction

1.1 Motivation

Quantifier elimination is one of the most important tools in model theory. Indeed, if a theory allows quantifier elimination, then this theory is complete, and the description of all definable subsets can be reduced to describing only those subsets that are defined by a quantifier-free formula. One of the most important mathematical structures is the linearly ordered set of real numbers. On it, you can set an ordered group and field. It is known that the elementary theory of these structures admits quantifier elimination, and since these theories are computably axiomatizable, quantifier elimination implies their solvability.

1.2 Aims and Objectives

In [5], John Goodrick began to study dp-minimal ordered structures, where, in particular, he proved that the elementary theory T structures $(\mathbb{R}, <, P)$, where P distinguishes Cantor's one third set, is dp-minimal. In order to prove this, J.Goodrick proved that this theory T admits quantifier elimination. In this work, we present another proof of quantifier elimination for the given theory with respect to another signature.

We work this the set of real numbers $(\mathbb{R}, <, P, r, l, 0, 1)$, in which Cantor's set and two unary functions r and l are defined as follows.

If the number x does not lie in the interval $[0, 1]$, then $l(x) = r(x) = x$.

If the number x lies in the interval $[0, 1]$, then $l(x)$ is the maximum number from Cantor's set that is strictly less than x, if such exists, otherwise $l(x) = x$.

The function r is defined similarly. If the number x lies in the interval $[0, 1]$, then $r(x)$ is the minimum number from Cantor's set that is strictly greater than x, if such exists, otherwise $r(x) = x$.

It is proved that this elementary theory of structure $(\mathbb{R}, <, P, r, l, 0, 1)$ admits the elimination of quantifiers.

In addition, the properties of formula subsets in orderly stable ordered groups are investigated.

1.3 Thesis Outline

This chapter 3 is written with the help of the book of J. Barwise [2] and it is concerned with the fundamental relationship between mathematical statements (axioms), on the one hand, and mathematical structures (models) which satisfy them, on the other. The emphasis is on the model theory of first-order statements. This chapter, written for those with no prior knowledge of first-order logic, explains the most basic notions. This material is really pre-model theory and is needed for most of the chapters.

In this chapter 4 we introduce the notions of o-stable(see) and superstable theories (see 4.2). We investigate the properties of formula subsets in orderly stable ordered groups.

In the penultimate chapter 5, it is proved that an ordered set of real numbers with a distinguished Cantor set $(\mathbb{R}, <, P, r, l, 0, 1)$, in which the predicate P is perfected as a Cantor set, admits elimination of quantifiers (see 5.2).

The final chapter 6 presents the conclusion.

2. Preliminaries

Definition 2.1. A *basic type* over T is a set of basic formulas $\Phi(x)$ with one variable which is maximal consistent with T. Given an element m of a model $\mathfrak{M}$ of T, the set of all basic formulas (quantifier-free formula) satisfied by m is a basic type, called the *basic type* of m.

Note that if every set of basic formulas consistent with T then we can extend it to a basic type over T.

Definition 2.2. A formula $\phi(x)$ is said to be *complete* in T if the set

$$\{\theta(x) : T \models \phi(x) \to \theta(x)\}$$

is a type over T. A type which is generated by a complete formula is called an *isolated type.*

3. FUNDAMENTALS OF MODEL THEORY

3.1 An introduction to first-order logic

Let Σ be a given collection of function symbols, relation symbols and constant symbols. We make no restriction on the size of the collection Σ, though usually Σ is finite or countably infinite, but also it can be uncountable. Each function symbol $f \in \Sigma$ has a positive integer $\#(f)$ assigned to it; if $n = \#(f)$, then f is called an n-ary function symbol. Similarly, each relation symbol $R \in \Sigma$ comes with a positive integer $\#(R)$; if $n = \#(R)$ then R is said to be an n-ary relation symbol.

Example 3.1. *For the signature $\Sigma = \{+, 0\}$ appropriate to group theory there are no relation symbols and $\#(f) = 2$. For the signature $\Sigma = \{\in\}$ of collection theory, there are no functions or constant symbols and $\#(\in) = 2$.*

Given a signature Σ we have a natural notion of a *structure* or a *model* for Σ. A structure $\mathfrak{M}$ assigns a nonempty collection M of objects over which the quantifiers range, and $\mathfrak{M}$ also assigns appropriate interpretations of the basic primitive relation, function and constant symbols of Σ.

Definition 3.1. A (collection-theoretic) *structure* for Σ is a pair $\mathfrak{M} = \langle M, F\rangle$ where M is a nonempty collection and F is an operation with domain Σ such that, writing $x^{\mathfrak{M}}$ for $F(x)$,

(i) if $R \in \Sigma$ is an n-ary relation symbol, then $R^{\mathfrak{M}} \subseteq M^n$;

(ii) if $f \in \Sigma$ is an n-ary function symbol, then $f^{\mathfrak{M}} : M^n \to M$;

(iii) if $c \in \Sigma$, where c some constant symbol then $c^{\mathfrak{M}} \in M$.

Example 3.2. *If $\Sigma = \{+, 0\}$ is the signature appropriate to group theory, then a structure for Σ has the form $\mathfrak{N} = \langle N, +^N, 0^N\rangle$ where N is a nonempty collection,*

$+^N : N \times N \to N$ *and* $0^N \in N$. *We usually use* G *rather than* $\mathfrak{N}$ *and drop the superscripts.*

We now turn to syntactic notions of first-order logic. Recall the basic building blocks $\vee, \wedge, \neg, \to, =, \forall, \exists, x, y, z, \ldots), ($. Let Σ be a fixed signature. Any finite sequence, each element of which is one of these basic symbols or an element of Σ, is called an *expression*. From the collection of expressions we want to single out the ones to which we can assign a meaning.

Definition 3.2. The *terms* of Σ form the smallest collection of expressions containing the variables $x, y, z, \ldots$, all constant symbols of Σ (if any) and closed under the formation rule: if $t_1, \ldots, t_n$ are terms of Σ and if $f \in \Sigma$ is an n-ary function symbol from Σ, then the expression $f(t_1 \ldots t_n)$ is a term of Σ. A *closed term* is a term in which no variable appears.

If there are no function symbols in Σ then the formation rule is vacuous so the only terms are variables and the constants of Σ.

Example 3.3. *If* $\Sigma = \{+, 0\}$ *then, strictly speaking, the terms are expressions like*

$$+(xy), \quad +(0 + (x0)).$$

We naturally agree to abbreviate these by the more natural

$$x + y, \quad 0 + (x + 0),$$

respectively, thus moving the symbol $+$ *inside and leaving off the outer parentheses if no confusion arises. We use* nx *as an abbreviation of* $(\ldots((x+x)+x)+\cdots+x)$, n *times, for* $n \geq 1$. *For this language the only closed terms are the expressions built up from* 0 *and* $+$, *none of which are very interesting from a group theoretic point of view.*

Definition 3.3. An *atomic formula* of Σ is an expression of either of the two forms:

$$(t_l = t_2), \quad R(t_1, \ldots, t_n)$$

where, in the first case, t_1 and t_2 are terms of Σ. In the second case $R \in \Sigma$ is any n-ary relation symbol and $t_1, \ldots, t_n$ are terms of Σ.

Example 3.4. *In the language* $\Sigma = \{+, 0\}$ *of group theory there are no any relation symbols, so the only atomic formulas are statements of equalities between terms, expressions like*

$$(x + y = z), \quad (x + y = y + x), \quad (x + y) + z = x + (y + z).$$

Definition 3.4. The first-order *formulas* of Σ form the smallest collection of expressions containing the atomic formulas and closed under the following formation rules:

(i) If ϕ, θ are formulas so are the expressions

$$\neg\phi, \quad (\phi \wedge \theta), \quad (\phi \vee \theta), \quad (\phi \rightarrow \theta);$$

(ii) if ϕ is a formula and v is a free variable, then $(\exists\, v\, \phi)$ and $(\forall\, v\, \phi)$ are formulas.

We associate parentheses to the right in strings where the same symbol is repreated. Thus $\phi \wedge \psi \wedge \theta$ is $(\phi \wedge (\psi \wedge \theta))$ and $\phi \rightarrow \psi \rightarrow \theta$ is $(\phi \rightarrow (\psi \rightarrow \theta))$.

Given a signature Σ we construct the collection of all formulas, which we call a language and usually denote by the symbol $\mathcal{L}$. Note, that $\mathcal{L}$ is uniquely defined by Σ and from $\mathcal{L}$ we can recover the signature Σ. So, sometimes we do not distinguish a language and a signature.

Example 3.5. *Let* $\mathcal{L} = \{+, 0\}$. *The following are formulas:*

$$(x + y = 0),$$

$$\exists y(x + y = 0)),$$

$$\forall x(\exists y(x + y = 0))).$$

Note that in the first formula both x and y are sort of "floating free", in the second formula y is "bound up" by $\exists$ and in the last formula both x and y are "bound". Only the last formula makes any intuitive sense as an axiom. This is similar to the situation in elementary calculus where

$$x^2 + 2x + 1$$

is an expression which has a variable in it, but the expression

$$\int_0^1 (x^2 + 2x + 1)dx$$

has x "bound"; it has a meaning independent of x. The definite integral is performing roughly the same syntactic role that the quantifiers $\exists$ and $\forall$ play in logic. The next definition makes the notion of "free variable" precise. One can think of it as defined by induction on the length of the formula ϕ.

Definition 3.5. The collection $S\ (\phi)$ of *free variables* of a formula ϕ is defined following:

(i) If ϕ is an atomic formula, then $S\ (\phi)$ is just the collection of variables appearing in the expression ϕ,

(ii) $S(\neg\phi) = S(\phi)$,

(iii) $S(\phi \wedge \theta) = S(\phi \vee \theta) = S(\phi \rightarrow \theta) = S(\phi) \cup S(\theta)$,

(iv) $S(\exists\ v\ \phi) = S(\forall\ v\ \phi) = S(\phi) - \{v\}$.

Definition 3.6. A *(first-order) expression* of $\mathcal{L}$ is a formula without any free variables.

So far the terms, formulas and expressions of $\mathcal{L}$ are finite strings. We need to check that our logical expression is true. This is done by the *satisfaction relation* $\mathfrak{M} \models \phi$ between structures on the one hand (the left one) and expressions on the other.

Definition 3.7. Let $\mathfrak{M} = \langle M, \ldots \rangle$ be given. For t a term of $\mathcal{L}$ define $t^{\mathfrak{M}}$ as follows:

(i) all variables and constants are terms, that is, $t^{\mathfrak{M}}(s) = c^{\mathfrak{M}}$ and $t^{\mathfrak{M}}(s) = s(v)$ for all s;

(ii) if t is the term of function then, for all s, define

$$t^{\mathfrak{M}}(s) = f^{\mathfrak{M}}(t_1^{\mathfrak{M}}(s), \ldots, t_n^{\mathfrak{M}}(s)).$$

In (ii), since each of $t_1, \ldots, t_n$ is simpler than t we can assume by induction on (the complexity of) terms that $t_1^{\mathfrak{M}}, \ldots, t_n^{\mathfrak{M}}$ are already defined. $f^{\mathfrak{M}}$ is defined since $\mathfrak{M}$ is a structure for $\mathcal{L}$ and $f \in \mathcal{L}$. If $s_1(v) = s_2(v)$, then $t^{\mathfrak{M}}(s_1) = t^{\mathfrak{M}}(s_2)$. Thus $t^{\mathfrak{M}}$, as a function, depends on a finite number of s.

Example 3.6. *Let $\mathfrak{L}$ be the language of rings and let t be the term, or polynomial,*

$$x^2 + 2x + 1$$

Then $t^{\mathfrak{M}}$, for any ring $\mathfrak{R}$, is the corresponding *polynomial function* from $\mathfrak{R}$ into $\mathfrak{R}$. If $s(x) = a$, then $t^{\mathfrak{M}}(s) = a^2 + 2a + 1$, the operations of $+$ and $\cdot$ being those of the ring $\mathfrak{R}$.

In the following definition we use $s\binom{a}{v}$ for the assignment s' which agrees with s except that $s'(v) = a$.

Definition 3.8. Let $\mathfrak{M}$ be an $\mathcal{L}$-structure. For all s and all formulas ϕ, we define a relation

$$\mathfrak{M} \models \phi[s],$$

as follows.

(i) $\mathfrak{M} \models (y_1 = y_2)[s] \Leftrightarrow y_1^{\mathfrak{M}}(s) = y_2^{\mathfrak{M}}(s)$,

(ii) $\mathfrak{M} \models R(y_1 \ldots y_n)[s] \Leftrightarrow (y_1^{\mathfrak{M}}(s), \ldots, y_n^{\mathfrak{M}}(s)) \in R^{\mathfrak{M}}$,

(iii) $\mathfrak{M} \models \neg\phi[s] \Leftrightarrow$ not $\mathfrak{M} \models \phi[s]$,

(iv) $\mathfrak{M} \models (\phi \wedge \theta)[s] \Leftrightarrow \mathfrak{M} \models \phi[s]$ and $\mathfrak{M} \models \theta[s]$,

(v) $\mathfrak{M} \models (\phi \vee \theta)[s] \Leftrightarrow \mathfrak{M} \models \phi[s]$ or $\mathfrak{M} \models \theta[s]$ or both,

(vi) $\mathfrak{M} \models (\phi \rightarrow \theta)[s] \Leftrightarrow$ either not $\mathfrak{M} \models \phi[s]$ or else $\mathfrak{M} \models \theta[s]$,

(vii) $\mathfrak{M} \models (\exists\, v\, \phi)[s] \Leftrightarrow a \in M$ such that $\mathfrak{M} \models \phi[s\binom{a}{v}]$,

(viii) $\mathfrak{M} \models (\forall\, v\, \phi)[s] \Leftrightarrow a \in M$, $\mathfrak{M} \models \phi[s\binom{a}{v}]$.

We note that the falsity or truth of $\mathfrak{M} \models \phi[s]$ depends only on the values of $s(v)$ for variables v. That is, if $s_1(v) = s_2(v)$ for all v free in ϕ, then $\mathfrak{M} \models \phi[s_1]$ iff $\mathfrak{M} \models \phi[s_2]$. Thus, if ϕ is $\phi(v_1 \ldots v_n)$ and $a_1 = s(v_1), \ldots, a_n = s(v_n)$, then we may write $\mathfrak{M} \models \phi[a_1, \ldots, a_n]$ for $\mathfrak{M} \models \phi[s]$ without confusion.

A structure $\mathfrak{M}$ is a *model of a collection* Φ of expressions if $\mathfrak{M} \models \phi$ for all $\phi \in \Phi$. Given two structure $\mathfrak{M}$, $\mathfrak{N}$ for $\mathcal{L}$, we say that $\mathfrak{M}$ and $\mathfrak{N}$ are *elementarily equivalent*, and write $\mathfrak{M} \equiv \mathfrak{N}$, iff for all expressions ϕ of $\mathcal{L}$, $\mathfrak{M} \models \phi$ iff $\mathfrak{N} \models \phi$. If $\mathfrak{M} \cong \mathfrak{N}$ (i.e. $\mathfrak{M}$ is isomorphic to $\mathfrak{N}$, in the obvious sense) then $\mathfrak{M} \equiv \mathfrak{N}$. Finally, let $\mathfrak{K}$ be a class of structures for a language $\mathcal{L}$. $\mathfrak{K}$ is (*finitely*) *axiomatizable* if there is a (finite) collection Φ of first-order expressions of $\mathcal{L}$ such that, for all structures $\mathfrak{M}$, $\mathfrak{N} \in \mathfrak{K}$ iff $\mathfrak{M}$ is a model of Φ.

3.2 The Completeness Theorem

Surely the most important discovery for mathematics by the ancient Greeks was of the notion of *proof*, turning mathematics into the science of deduction. Note that the theorem ϕ must have a proof or *axioms*. The proof can demonstrate that the conclusion ϕ follows from the axioms in T by the laws of logic alone. The mathematician implicitly assumes that he understands the notion of proof and that, in particular, he will be able to check in a rigorous manner whether a purported complete proof does indeed establish the conclusion from the stated assumptions. The natural question is: Can the notions "laws of logic" and "proof" be made mathematically precise?

There is a seeming obstacle to our program. How can we hope to prove such a result without knowing in advance what it means to follow by the laws of logic alone? Luckily, we do not need to know. All we need is to agree that, whatever it means, it at least *implies* that ϕ will hold in all collection theoretic structures which are models of T; i.e., it implies $T \models \phi$. Thus, to realize our goal, it more than suffices to provide valid rules of inference and show that ϕ is provable from T if and only if $T \models \phi$. This is the content of Godel's Completeness Theorem.

Propositional logic is expeditious to break a study of first-order logic up into two parts, the trivial part having to do with the propositional connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$ and then the part having to do with equality and the quantifiers $\forall$ and $\exists$.

Let P be a collection of objects called *prime formulas*. They might be expressions of some natural language or letters $\phi, \psi, \theta, \ldots$ of the alphabet, for example. In our application, they will be those first-order formulas and this formulas are not propositional combinations of simpler formulas, that is, atomic formulas and formulas beginning with a quantifier. The collection of *propositional formulas* of

P form the smallest collection-of statements which are consists the members of P and closed by the following rules: if ϕ, θ are propositional formulas then so are $\neg\phi$, $(\phi \wedge \theta)$, $(\phi \vee \theta)$ and $(\phi \to \theta)$. The *prime constituents* of a propositional formula ϕ are just the prime formulas out of which ϕ is built.

Example 3.7. *Suppose $P = \{\phi, \psi, \theta\}$. The following are propositional formulas P:*

$$\phi,\ \theta,\ (\phi \vee \theta),\ (\theta \vee \phi),\ ((\phi \vee \theta) \to (\theta \vee \phi)).$$

We want to show exactly how the truth or falsity of a propositional formula depends on the truth or falsity of its prime constituents. Then, going a step further, we show how to decide which propositional formulas are always true, regardless of the truth or falsity of their prime constituents,formulas like $(\phi \vee \neg\theta)$, $(((\phi \to \theta) \wedge \neg\theta) \to \neg\phi)$ etc. Such formulas are called propositional *tautologies*, since they are true by virtue of their syntactic form alone. These tautologies provide a small first step in isolating the laws of logic.

Suppose that t and f are new symbols and values of them can be chosen as "true" and "false". A *truth assignment* for a collection P of prime formulas is, by definition, a function $u : P \to \{t, f\}$. For each truth assignment u we define its extension $\bar{u}$ to the collection of all propositional formulas of P by induction on length of formulas as follows:

$$\begin{aligned}
\bar{u}(\phi) &= u(\phi) \text{ if } \phi \text{ is prime;}\\
\bar{u}(\neg\phi) &= f \text{ if } \bar{u}(\phi) = t,\\
\bar{u}(\neg\phi) &= t \text{ if } \bar{u}(\phi) = f;\\
\bar{u}(\phi \wedge \theta) &= t \text{ if } \bar{u}(\phi) = \bar{u}(\theta) = t,\\
\bar{u}(\phi \wedge \theta) &= f \text{ if } \bar{u}(\phi) = \bar{u}(\theta) = f,\\
\bar{u}(\phi \vee \theta) &= t \text{ if } \bar{u}(\phi) = t \text{ or } \bar{u}(\theta) = t \text{ or both,}\\
\bar{u}(\phi \vee \theta) &= f \text{ if } \bar{u}(\phi) = f \text{ or } \bar{u}(\theta) = f \text{ or both,}\\
\bar{u}(\phi \to \theta) &= f \text{ if } \bar{u}(\phi) = t \text{ and } \bar{u}(\theta) = f,\\
\bar{u}(\phi \to \theta) &= t \text{ if } \bar{u}(\phi) = f \text{ and } \bar{u}(\theta) = t.
\end{aligned}$$

Finally we can write by *truth table* given definition:

ϕ	θ	$\neg\phi$	$(\phi \wedge \theta)$	$(\phi \vee \theta)$	$(\phi \to \theta)$
t	t	f	t	t	t
t	f	f	f	t	f
f	t	t	f	t	t
t	f	t	f	f	t

By constructing such truth tables we can completely analyze how the truth or falsity of a propositional formula depends on the truth or falsity of its prime constituents. We illustrate the method for the formula

$$((\neg(\phi \wedge \neg\theta) \wedge \theta) \to \phi).$$

We simplify the table by leaving out some of the **t**'s.

ϕ	θ	$\neg\theta$	$(\phi \wedge \neg\theta)$	$\neg(\phi \wedge \neg\theta)$	$\neg(\phi \wedge \neg\theta) \wedge \theta$	$(\neg(\phi \wedge \neg\theta) \wedge \theta) \to \phi$
t	t	f	f			
t	f			f	f	
f	t	f	f			f
f	f		f		f	

Thus, the only circumstances under which our final formula is false is when ϕ is false and θ is true.

Definition 3.9. A propositional formula ϕ of P is a *tautology* if $\bar{u}(A) = t$ for all truth assignments $u : P \to \{t, f\}$. A is *consistent* if $\bar{u}(A) = t$ for some $u : P \to \{t, f\}$.

The method of truth tables makes it a trivial matter to see whether a propositional formula is a tautology or not, or whether it is consistent or not. If we write $\phi \leftrightarrow \theta$ for $(\phi \to \theta) \wedge (\theta \to \phi)$, then we see that the following are tautologies:

$$(\phi \vee \neg\phi) \quad \text{(law of the excluded middle)},$$

$$\neg(\phi \wedge \neg\phi) \quad \text{(law of contradiction)},$$

$$\left.\begin{array}{l} \neg(\phi \wedge \theta) \leftrightarrow (\neg\phi \vee \neg\theta) \\ \neg(\phi \vee \theta) \leftrightarrow (\neg\phi \wedge \neg\theta) \end{array}\right\} \quad \text{(de Morgan's laws)},$$

$$\neg\neg\phi \leftrightarrow \phi \quad \text{(law of double negation)}.$$

Just to make sure the method of truth tables is perfectly clear, we present an example with three prime constituents ϕ, ψ, θ:

$$\underbrace{[((\phi \wedge \psi) \rightarrow \theta)}_{\text{E}} \ \wedge \ \underbrace{(\neg\theta \rightarrow \psi)]}_{\text{F}} \ \rightarrow \ \underbrace{(\phi \rightarrow \theta)}_{\text{G}}$$

ϕ	ψ	θ	$\phi \wedge \psi$	E	$\neg\theta$	F	$E \wedge F$	G	$[E \wedge F] \rightarrow G$
t	t	t			f				
t	t	f		f			f	f	
t	f	t	f		f				
t	f	f	f			f	f	f	
f	t	t	f		f				
f	t	f	f						
f	f	t	f		f				
f	f	f	f			f	f		

Thus, since no falses turn up in the last column, the formula is indeed a tautology.

In practise, there is a much shorter method to check to see whether a formula is or is not a tautology. One works backwards, trying to find a consistent assignment which makes the formula false. Applied to the above, to make $[E \wedge F] \rightarrow G$ false, we need to have $\bar{u}(E) = \bar{u}(F) = t$ but $\bar{u}(G) = t$. To make $\bar{u}(G) = t$, we must make $\bar{u}(\phi) = t$, $\bar{\theta} = f$. To have $\bar{u}(F) = t$ we must have $\bar{u}(\psi) = t$, since $\bar{u}(\theta) = t$. But now we have $\bar{u}(\phi) = \bar{u}(\psi) = t$ and $\bar{u}(\theta) = f$ which gives $\bar{u}(E) = f$, a contradiction. Thus the above formula is a tautology.

A collection T of propositional formulas is said to be *consistent* if there is a truth assignment u such that $\bar{u}(E) = t$ for all $E \in T$.

Theorem 3.10. *A collection T of propositional formulas is consistent $\Leftrightarrow$ if every finite subset of T is consistent.*

3.3 Model theory

Model theory may be described as the union of logic and universal algebra. The leading characters are the expressions ϕ and the structures $\mathfrak{M}$ for a language $\mathcal{L}$. Classical examples in algebra have led to many notions in model theory. The use of expressions as mathematical objects is a powerful tool which has led to unexpected applications outside of model theory.

For example, 300 years ago Leibniz conjectured that calculus could be developed rigorously by extending the real number system $\mathbb{R}$ to a structure $^*\mathbb{R}$ such that $^*\mathbb{R}$ contains infinitesimals but every statement true of $\mathbb{R}$ is true of $^*\mathbb{R}$. There was no chance of solving this problem until model theory developed to the point where the statements could be defined precisely and treated as mathematical objects. Abraham Robinson solved the problem in 1960, taking the statements to be the formulas of first order logic. In the simplest formulation one starts with the structure $\mathfrak{R}$ which has the collection of reals $\mathbb{R}$ for its universe, and a symbol for each real relation and function. The first step is to form a proper elementary extension $^*\mathfrak{R}$ of $\mathfrak{R}$, that is, a proper extension which satisfies the same first order formulas as $\mathfrak{R}$. This elementary extension is obtained easily using general results in model theory, either the compactness theorem or the Zos ultraproduct theorem. The surprise was the next step; Robinson showed that the early intuitive arguments with infinitesimals could be rigorously carried out in $^*\mathbb{R}$, and new results quickly followed.

In North America these are often called western and eastern model theory, because Tarski has lived on the west coast since the 1940's, and Robinson was on the east coast from 1967 to his premature death in 1975. The distinction no longer has anything to do with geography, but it is still mathematically useful.

Western model theory is in the tradition of Skolem and Tarski. It is largely motivated by problems in number theory, analysis, and collection theory. It emphasizes the collection of all formulas of first order logic.

Eastern model theory is in the tradition of Malcev and Robinson. It is motivated by problems in abstract algebra, where theories usually have at most two blocks of quantifiers. It emphasizes the collection of quantifier-free formulas and the collection of existential formulas.

Many model theorists move back and forth between western and eastern model theory. In fact, Tarski and Robinson made major conributions to both areas (Tarski's work on real closed fields and on equational classes are eastern model theory, while Robinson's consistency theorem and his infinitesimal analysis are western model theory).

The deeper proofs in model theory usually depend on the construction of a model with certain properties. The constructions almost always use one or more methods from following list:

- Elementary chains,
- diagrams and other expansions of the language,
- compactness theorem,
- downward Lowenheim-Skolem theorem,
- omitting types theorem,
- forcing,
- ultraproducts,
- homogeneous collections.

3.4 Theories

In this section we present some of the fundamental notions of model theory. In particular we shall discuss various ways of classifying theories in first order logic.

Let study a first order language $\mathcal{L}$. By a *theory* T in $\mathcal{L}$ we define a collection of expressions of $\mathcal{L}$. The notion $\mathfrak{M} \models T$, read $\mathfrak{M}$ is a *model* of T, means that $\mathfrak{M}$ is a structure for $\mathcal{L}$ such that every $\phi \in T$ holds in $\mathfrak{M}$. Let given theories T_1 and T_2 and this theories are *equivalent* if models of this theories are the same. T is *consistent*, or *satisfiable*, if it has at least one model. There are two ways in which theories commonly arise.

First, consider a class K of structures for $\mathcal{L}$. The *theory of* K a collection $\mathrm{Th}(K)$ of all expressions ϕ of $\mathcal{L}$ such that ϕ holds in every $\mathfrak{M} \in K$. For example, if K is the class of finite groups then $\mathrm{Th}(K)$, the theory of finite groups, a collection of all expressions such that it is true in all finite groups.

K is said to be an *elementary class*, or an EC_Δ class, if K is the class of all models of some theory T (hence $\mathrm{Th}(K)$). The class of all groups is an elementary class. However the class of finite groups is not an elementary class. That is, there are infinite groups $\mathfrak{M}$ which satisfy every expression true of all finite groups. To characterize the notion of a finite group one must go beyond first order logic.

Given a structure $\mathfrak{M}$, the *theory of* $\mathfrak{M}$, $\mathrm{Th}(\{\mathfrak{M}\})$ or $\mathrm{Th}(\mathfrak{M})$, is the collection of all expressions true in $\mathfrak{M}$.

Definition 3.11. A theory T of a language $\mathcal{L}$ is *complete* if T is equivalent to $\mathrm{Th}(\mathfrak{M})$ for some structure $\mathfrak{M}$, or, that is the same, either $T \vdash \phi$, or $T \vdash \neg\phi$ for each expression ϕ of the language $\mathcal{L}$.

Two structure $\mathfrak{M}$ and $\mathfrak{N}$ are *elementarily equivalent*, $\mathfrak{M} \equiv \mathfrak{N}$, if $\mathrm{Th}(\mathfrak{M}) = \mathrm{Th}(\mathfrak{N})$. Thus $\mathfrak{M} = \mathfrak{N}$ means that $\mathfrak{M}$ and $\mathfrak{N}$ satisfy exactly the same expressions.

The second way in which theories commonly arise is as simple collections of expressions. A collection of expressions which is equivalent to $\mathrm{Th}(K)$ is called a *collection of axioms* for K. Examples are the theories of groups, abelian groups, divisible groups, and torsion-free groups. The first two examples are finitely axiomatizable. A language $\mathcal{L}$ is said to be *recursive* if $\mathcal{L}$ is finite or countable with a recursive collection of symbols.

Definition 3.12. A theory T is *finitely axiomatizable* if it is equivalent to a finite collection of expressions. A theory T is *recursively axiomatizable* if it is equivalent to a recursive collection of expressions in a recursive language.

The compactness theorem can often be used to show that a theory is not finitely axiomatizable. Of course, finite axiomatizability implies recursive axiomatizability. We list some additional examples.

Example 3.8. *The following theories are finitely axiomatizable:*

- *Groups,*
- *abelian groups,*
- *rings,*
- *integral domains,*
- *fields,*
- *fields of characteristics p ($p \neq 0$, fixed),*
- *ordered fields,*
- *linear order,*
- *lattices,*

- *Boolean algebras,*
- *Bernays-Godel collection theory,*
- *the inconsistent theory.*

Example 3.9. *The following theories are recursively but not finitely axiomatizable:*

- *Divisible groups,*
- *finite fields,*
- *Zermelo-Fraenkel collection theory,*
- *Peano arithmetic.*

Here is a conventient characterization of recursively axiomatizable theories. A expression ϕ is a *consequence* of T, $T \models \phi$, if every model of T is a model of ϕ.

Theorem 3.13. *A theory T in a recursive language is recursively axiomatizable if and only if the collection of all consequences of T is recursively enumerable.*

Usually the collection of consequences of a theory T is not recursive. Church's Theorem shows that if $\mathcal{L}$ has at least one binary relation or function symbol, the collection of consequences of the empty theory (i.e. the valid expressions) is not recursive.

Definition 3.14. A theory T in a recursive language is *decidable* if the collection of all consequences of T is recursive.

Intuitively this means that there is an algorithm for deciding whether an arbitrary ϕ is a consequence of T.

Theorem 3.15. *Every complete recursively axiomatizable theory is decidable.*

Model theory has developed some powerful techniques for showing that a theory is complete; when the theory in question is a recursive collection of expressions, we can conclude that the theory is decidable. For example, in a classical paper, Tarski and McKinsey showed that the theory of real closed fields is complete and hence decidable. The problem can also be viewed as follows. Given a structure $\mathfrak{M}$ (such as the field of reals), determine whether $\mathrm{Th}(\mathfrak{M})$ is decidable, and if so, find a recursive collection of axioms.

Definition 3.16. A *universal formula* is a formula of the form

$$\forall x_1 \ldots \forall x_n \phi$$

where ϕ has no quantifiers. A *universal theory* is a theory which is equivalent to a collection of universal expressions.

$\mathfrak{N}$ is a substructure of $\mathfrak{M}$, $\mathfrak{N} \subseteq \mathfrak{M}$, if the universe of $\mathfrak{N}$ is a subset of the universe of $\mathfrak{M}$ and the interpretation of each relation, function, or constant symbol in $\mathfrak{N}$ is the restriction of the corresponding interpretation in $\mathfrak{M}$. Equivalently, for every atomic formula ϕ and assignment s in $\mathfrak{N}$,

$$\mathfrak{N} \models \phi[s] \text{ iff } \mathfrak{M} \models \phi[s].$$

Given a nonempty subset $X \subseteq M$, there is a least substructure $\mathfrak{N} \subseteq \mathfrak{M}$ containing X, called the *substructure generated by* X. A *finitely generated substructure* of $\mathfrak{M}$ is a substructure generated by a finite $X \subseteq M$.

Proposition 3.17. *If T is a universal theory, then every substructure of a model of T is a model of T.*

We shall prove the converse of this fact.

For example, the theory of groups formulated in the language $\{+, 0, -\}$ is universal, and every substructure of a group is a group. However, when group theory is formulated in the language $\{+, 0\}$, it is no longer a universal theory because an existential quantifier is needed to assert the existence of an inverse,

$$\forall a \exists b (a + b = 0 \wedge b + a = 0).$$

When formulated in the language $\{+, 0\}$, a substructure of a group is a semigroup with unit but not necessarily a group.

Some other examples of universal theories are the theories of abelian groups, rings, integral domains, lattices, linear order, Boolean algebras, and torison-free groups.

The western analogue of a substructure is an elementary substructure (Tarski and Vaught). $\mathfrak{N}$ is an *elementary substructure* of $\mathfrak{M}$, $\mathfrak{N} \prec \mathfrak{M}$, if $\mathfrak{N} \subseteq \mathfrak{M}$ and for every formula ϕ and assignment s in $\mathfrak{N}$,

$$\mathfrak{N} \models \phi[s] \quad \text{iff} \quad \mathfrak{M} \models \phi[s].$$

We also call $\mathcal{M}$ an *elementary extension* of $\mathfrak{N}$ and write $\mathfrak{M} \succ \mathfrak{N}$. Obviously,

$$\mathfrak{N} \prec \mathfrak{M} \quad \text{implies} \quad \mathfrak{N} \equiv \mathfrak{M}.$$

It is often more convenient to work with embeddings instead of substructures. An *isomorphic embedding* $f : \mathfrak{N} \to \mathfrak{M}$ is a mapping of N into M such that for every atomic formula ϕ and assignment s in $\mathfrak{N}$,

$$\mathfrak{N} \models \phi[s] \quad \text{iff} \quad \mathfrak{M} \models \phi[s \circ f].$$

f must be one-to-one because $x = y$ is atomic. Similarly an *elementary embedding* $f : \mathfrak{N} \to \mathfrak{M}$ is a mapping such that for every formula ϕ and assignment s in $\mathfrak{N}$,

$$\mathfrak{N} \models \phi[s] \quad \text{iff} \quad \mathfrak{M} \models \phi[s].$$

An isomorphic embedding $f : \mathfrak{N} \to \mathfrak{M}$ is called an *isomorphism* if M is the range of f, in symbols $f : \mathfrak{N} \cong \mathfrak{M}$. This is the usual notion in algebra, and

$$f : \mathfrak{N} \cong \mathfrak{M} \Rightarrow f : \mathfrak{N} \to_{\prec} \mathfrak{M} \Rightarrow \mathfrak{N} \equiv \mathfrak{M}.$$

Although the notion of an elementary extension is very strong, there is an abundance of examples in model theory. The following criterion is useful.

Theorem 3.18. *$\mathfrak{N} \prec \mathfrak{M}$ if and only if $N \subseteq M$ and for every formula $\psi(x, y_1, \ldots, y_n)$ and $a \in M$, $b_1, \ldots, b_n \in N$, if $\mathfrak{M} \models \psi[a, b_1, \ldots, b_n]$, then there exists $b \in N$ such that $\mathfrak{M} \models \psi[b, b_1, \ldots, b_n]$.*

Example 3.10. *Let F be a field and let X, Y be infinite collections of variables with $X \subseteq Y$. Then $F[X] \prec F[X]$ where $F[X]$ is the ring of polynomials in X over F, and $F(X) \prec F(Y)$ where $F(X)$ is the pure transcendental extension of F by X. If $G(X)$ is the group freely generated by X, then $G(X) \prec G(Y)$.*

Definition 3.19. A theory T is said to be *model-complete* if whenever $\mathfrak{M}, \mathfrak{N}$ are models of T and $\mathfrak{M} \subseteq \mathfrak{N}$, $\mathfrak{M}$ is an elementary substructure of $\mathfrak{N}$. Equivalently, every isomorphic embedding between models of T is an elementary embedding.

Example 3.11. *The following theories are model-complete. Algebraically closed fields, real closed ordered fields, atomless Boolean algebras, dense linear order. Thus, for example, the ordered field of real numbers is an elementary extension of the ordered field of real algebraic numbers.*

Each of the above examples has the following property which is even stronger than model-completeness. Given a formula $\phi(a_1, \ldots, a_n)$, the notation

$$T \models \phi(a_1, \ldots, a_n)$$

means that the expression $\forall a_1 \ldots \forall a_n \phi$ is a consequence of T.

Definition 3.20. A theory T *admits elimination of quantifiers* if every formula $\phi(a_1, \ldots, a_n)$ of L is T-equivalent to a quantifier-free formula $\psi(a_1, ..., a_n)$ of $\mathcal{L}$, that is,

$$T \models \phi(a_1, \ldots, a_n) \leftrightarrow \psi(a_1, \ldots, a_n).$$

It is easy to see that every T which admits elimination of quantifiers is model-complete. In fact, for T to be model-complete it is sufficient that every formula of $\mathcal{L}$ is T-equivalent to a universal formula of $\mathcal{L}$. We shall see in the next section that this condition is also necessary. An example of a theory which is model-complete but does not admit elimination of quantifiers is the theory of real closed fields. The order relation can be defined from the field operations but a quantifier is needed.

Eastern model theory concentrates on inductive theories, defined below.

Definition 3.21. A theory T said to be *inductive* if T is equivalent to a collection of $\forall\exists$ expressions, that is, expressions of the from

$$\forall a_1 \ldots \forall a_n \exists b_1 \ldots \exists b_n \psi$$

where ψ is quantifier-free.

Inductive theories are also called $\forall\exists$ theories. Most theories arising in algebra are inductive. All the theories listed in 3.8 and 3.9 are inductive except finite fields, Bernays-Godel collection theory, Zermelo-Fraenkel collection theory, and Peano arithmetic. The next result is the reason for the use of the name "inductive".

Proposition 3.22. *If T is inductive, then the union of any increasing chain*

$$\mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \mathfrak{M}_2 \subseteq \dots$$

of models of T is a model of T.

The western analogue of a chain of structures is an *elementary chain*

$$\mathfrak{M}_0 \prec \mathfrak{M}_1 \prec \mathfrak{M}_2 \prec \dots$$

Elementary chains are an extremely important construction in model theory. Their importance is based on the following fundamental result of Tarski and Vaught.

Theorem 3.23. *(Elementary Chain Theorem). The union $\mathfrak{M}$ of an elementary chain*

$$\mathfrak{M}_0 \prec \mathfrak{M}_1 \prec \mathfrak{M}_2 \prec \dots$$

is an elementary extension of each structure $\mathcal{M}_n$ in the chain.

Proof. Show by induction on the length of formulas ϕ that for each n and each assignment s in $\mathfrak{M}_n$

$$\mathfrak{M}_n \models \phi[s] \quad \text{iff} \quad \mathfrak{M} \models \phi[s].$$

□

It follows that for any theory T, the union of an elementary chain of models of T is a model of T. 3.22 and 3.23 hold not only for countable chains of structures, but also for uncountable chains and even for arbitrary upward directed systems of structures.

4. GENERAL PROPERTIES OF ORDERED STABLE GROUPS

4.1 Definition of an o-stable theory

The thought of an o-minimal structure was presented in [10]. Afterward, different speculations were proposed (for instance, the idea of a weakly o-stable structure [3, 4]). As is appeared in [8, 7], each o-minimal theory is quasi-o-minimal. It is anything but difficult to see that, for every cut in an o-minimal structure, there exists a unique complete type over the model extending this cut, cf. [10]. A similar assertion for weakly o-minimal structures is proven in [6]. Let $\mathfrak{M}$ be a totally ordered structure. Then $\mathfrak{M}$ is weakly o-minimal if and only if the following conditions hold:

(1) for every cut $\langle C, D\rangle$ in $\mathfrak{M}$, there exist at most two complete 1-types over M extending $\langle C, D\rangle$;

(2) if there exist two complete 1-types over M extending $\langle C, D\rangle$ then, for each of these types, the set of realizations is convex in every elementary extension of $\mathfrak{M}$.

It is immediate from the definition that, for every cut in a quasi-o-minimal model $\mathfrak{M}$, there exists at most continuum many complete types over M extending this cut.

Thus, for all these versions of o-minimality, every cut has a few extensions. Recall that stable theories have a few types. It is natural to combine these observations. We obtain the notion of an o-stable theory: For every cut in every model of the theory, there exist a few complete types extending this cut.

Throughout the work, $\mathfrak{M} = (M, <, \dots)$ means a structure with definable (without parameters) linear order, which we will call a *linearly ordered structure*, and its elementary theory a *linearly ordered theory*.

After such identifying, we obtain that on $\overline{M}$ we can naturally define a linear order extending the linear order by $(M, <)$: $\langle C_1, D_1\rangle \leq \langle C_2, D_2\rangle$, if and only if $C_1 \subseteq C_2$. Obviously, $(M, <)$ is an everywhere dense subset $(\overline{M}, <)$.

A subset A of a linearly ordered set M is called *convex* if, for any $a, b \in A$, the segment $[a, b]$ lies in A. We define the *length* of a convex set A as $\sup\{a - b : a, b \in A\}$. The *convex component* of the set A is the maximal convex subset of A and we define the *convex hull* A^c of the set A as follows in the following way:

$$A^c = \{b \in M : \exists a_1, a_2 \in A(a_1 \leq b \leq a_2)\},$$

that is, it is the smallest convex set containing A. An ordered structure is said to be *weakly o-minimal* if any definable subset consists of a finite number of convex components [3, 4].

Let P be some condition. We say that the condition P is *holds eventually* on the set A if there exists an element $a \in M$ such that $a < \sup A$ and the condition P holds at the intersection $(a, \infty) \cap A$. If $A = M$, then we simply write that the condition P holds eventually. If sets B and C are eventually equal on A, then we denote it's like $B \overset{\infty}{=}_A C$. Let $B \subseteq A \subseteq M$. The set B is called *dense* in A if, for any $a_1, a_2 \in A$ of A, there exists $b \in B$ such that $a_1 < b < a_2$. If $A = M$, then we omit it. The *dense component* of B in A is the maximum subset B_0 of B, which is dense in $A \cap (\inf B_0, \sup B_0)$.

Let T be a theory of the language $\mathcal{L}$, and $\phi(\bar{x}, \bar{y})$ be some formula.

Let T be a theory of a language $\mathcal{L}$. A formula $\phi(\bar{x}, \bar{y})$ possesses the *independence property* (relative to T) if, for every $n < \omega$, there exists a model $\mathfrak{M} \models T$ and two sequences $(\bar{a}_i : i < n)$ and $(\bar{b}_J : J \subseteq n)$ in M such that $\mathfrak{M} \models \phi(\bar{a}_i, \bar{b}_J)$ if and only if $i \in J$. A theory T possesses the independence property if some formula possesses the independence property relative to T.

Notation 4.1. Let s be a partial n-type and let A be a set. Put

$$S^n_s(A) \triangleq \{p \in S^n(A) : p \cup s \text{ is consistent}\}.$$

Notice that s need not to be a partial type over A.

Definition 4.2. 1. An ordered structure $\mathfrak{M}$ is *o-stable in* λ (or *o-λ-stable*) if, for every $A \subseteq M$ with $|A| \leq \lambda$ and every cut $\langle C, D\rangle$ in $\mathcal{M}$, at most λ

1-types over A are consistent with the cut $\langle C, D\rangle$, i.e.

$$\left|S^1_{\langle C,D\rangle}(A)\right| \leq \lambda.$$

2. A theory T is *o-stable in* λ (or *o-λ-stable*) if every model of T is.

3. A theory T is *o-stable* if there exists an infinite cardinal λ such that T is o-stable in λ.

4. A theory T is *o-superstable* if there exists a cardinal λ such that T is o-stable in every μ with $\mu \geq \lambda$.

5. A theory T is *strongly o-stable* if T is o-stable and every definable cut in every model $\mathfrak{M}$ of T is definable in the language of pure order (or, equivalently, $\sup A \in M$ for every definable subset A of M).

Lemma 4.3. *[12] (the strict order property in a cut). Each o-stable theory lacks the strict order property in a cut.*

The following property of o-stable theories in proven in [1].

Fact 4.4. [12] Each o-stable theory lacks the independence property.

4.3 and 4.4 entail the following criterion of o-stability.

Theorem 4.5. *[12] Let the language $\mathcal{L}$ contain the symbol $<$, and the theory T of the language $\mathcal{L}$ contain the axioms of linear order for the predicate $x < y$. A theory T is o-stable if and only if it lacks both the independence property and the strict order property in a cut.*

Let T be a theory of a language $\mathcal{L}$ and let $\mathfrak{M}$ and $\mathfrak{N}$ be models of T such that $\mathfrak{M} \prec \mathfrak{N}$ and $\mathfrak{N}$ is $|\mathfrak{M}|^+$-saturated. For every $\phi(\bar{x}, \bar{\alpha})$ with parameters $\bar{\alpha}$ in N, we introduce a new relation symbol $P_{\phi(\bar{x},\bar{\alpha})}(\bar{x})$ with $P_{\phi(\bar{x},\bar{\alpha})}(M) = \phi(N, \alpha) \cap M^k$. Let $\mathcal{L}^*$ denote the expanded language.

4.2 General properties of o-stability

In this section, by G we mean an ordered group with the unit element e, whose elementary theory is o-stable.

Let H be a convex subgroup of a group G (possibly, not necessarily definable). Below we will work in cut $\sup H$. As in the theory of stable groups, for each formula $\phi(x, \bar{y})$ there exists a natural number n such that each chain $L_1 \cap H \subset L_2 \cap H \subset \cdots \subset L_m \cap H$ has a length not more than n, provided that L_i is defined by the formula $\phi(x, \bar{a}_i)$ for some a_i and each subgroup L_i is unbounded in a convex subgroup H. We call this the *trivial chain condition* for H. Here, in a sense, we used only the fact that L_i is a subset of the group G, in other words, the triviality condition of the chain follows from the absence of the strict order property in a cut. For two subsets E and F of the group G, we put

$$E \vec{\sqcap} F \triangleq \begin{cases} E \cap F & \text{if } E \cap F \quad \text{is unbounded in } F, \\ \{e\} & \text{otherwise.} \end{cases}$$

Note that here $E \vec{\sqcap} F$ is not necessarily equal to $F \vec{\sqcap} E$. We use this notation to rewrite the trivial chain condition as follows: for any formula $\phi(x, \bar{y})$ there exists a natural number n such that the length of any chain $L_1 \vec{\sqcap} H \subset L_2 \vec{\sqcap} H \subset \cdots \subset L_m \vec{\sqcap} H$ is at most n provided that $L_i = \phi(G, \bar{a}_i)$.

Lemma 4.6. *If H and L are definable subgroups of G then $(H \cap L)^c = H^c \cap L^c$, i.e., if $L^c \leq H^c$ then $H \cap L$ is unbounded in L.*

This statement is proved in [12].

Lemma 4.7. *The set of definable unbounded subgroups of G possesses the least element.*

This statement is proved in [12].

Lemma 4.8. *Let G be an ordered group. Then for any element $g \in G$ there exists a definable convex subgroup H, containing g, such that for any $a \in [g; +\infty) \cap H$ the centralizer $C(a)$ intersected with H is not bounded in H.*

Proof. First we construct a convex subgroup L, which contains g and satisfies the condition that $C(a) \cap L$ is not bounded in L for any $a \in [g; +\infty) \bigcap L$.

It is easy to see, that L considered as the convex hull of $\langle g \rangle$ satisfies our requirements, where $\langle g \rangle$ stands for the subgroup generated by g, which is equal to $\{g^n : n \in \mathbb{Z}\}$. Indeed, $\langle g \rangle$ is a subgroup of $C(a)$, because

$$g \cdot g^n = g \cdot (g \cdot \ldots \cdot g) = (g \cdot \ldots \cdot g) \cdot g = g^n \cdot g$$

Let $a \in L$ be bigger than g. By definition of the convex hull there exists some positive n such that $g^n \leq a \leq g^{n+1}$. Then for each m there is k such that $g^m \leq a^k$, it means that $\langle a \rangle$ is not bounded in L. Since $\langle a \rangle$ is a subgroup of $C(a) \cap L$ and is not bounded in K, so $C(a) \cap L$ is not bounded in L.

But this L is not necessary definable. We know that for each $a \in L \cap [g; +\infty)$ and for each $b \in L \cap (a; +\infty)$ the following holds:

$$(a^{-1}b, ab) \cap C(a) \neq \varnothing$$

We use this property in order to define H. Let

$$\begin{aligned} \phi(x, a) \ &:= \ (xa = ax) \wedge x > a \wedge \\ &\wedge \forall y(a < y < x \rightarrow \exists z(a^{-1}y < z < ay \wedge za = az)) \end{aligned}$$

This formula says that $x \in C(a)$ and any interval (b, c) of length a^2 with $a < b < c < x$ contains an element from $C(a)$.

Let

$$\psi(x, a) = \exists y(\phi(y, a) \wedge |x| \leq y),$$

where

$$|x| \leq y \Leftrightarrow (x \geq e \wedge x \leq y) \vee (x < e \wedge e \leq xy).$$

Note that $\langle a \rangle$ is a subgroup of $\psi(G, a)$, which is a convex subgroup of G.

Let

$$\theta(x, z, a) := \forall y(a \leq y \leq z \rightarrow \psi(x, y)).$$

If $x = z$, then

$$\{g \in G : G \models \theta(g, g, a)\} = \psi(G, a).$$

If $z = a^n$, then

$$\theta(G, a^n, a) \geq K,$$

so $a^n < \sup \theta(G, a^n, a)$.

If $a < b_1 < b_2$ then

$$\sup \theta(G, b_1, a) \geq \sup \theta(G, b_2, a).$$

Let $\mu(x, a) = \theta(x, x, a)$. Since $\theta(G, b, a) \geq L$ for each $b \in L$, we obtain that

$\mu(G, a) \geq L$.

Note that $\mu(G, a)$ is a convex subgroup as an intersection of convex subgroups. We denote $\mu(G, a)$ by H. We observe that

1) $L \leq H$, as we have seen above;

2) $\forall b \in [a; +\infty) \cap H$ it holds that $C(b) \cap H$ is not bounded in H.

Let $b \in H$ be bigger that a. Then $G \models \mu(b, a)$, that is $G \models \theta(b, b, a)$. So

$$G \models \forall y(a \leq y \leq b \rightarrow \psi(b, y))$$

$$\theta(G, b, a) = \bigcap_{a \leq c \leq b} \psi(G, c)$$

$$H = \mu(G, a) = \bigcap \left\{ \psi(G, c) : a \leq c \text{ and } c \in \bigcap_{a \leq c} \psi(G, c) \right\}$$

Let (C, D) be the cut, such that

1) $a \in C$;

2) for any $c \in C \cap (a; +\infty)$, $\sup \bigcap_{a \leq c_1 \leq c} \psi(G, c_1) \geq \sup C$;

3) if $\sup \bigcap_{a \leq c_1 \leq c} \psi(G, c_1) > c$, then $c \in C$.

Let $b \in H \cap [a; +\infty)$. Let $g \in H \cap [b; +\infty)$. Then $(b^{-1}g, bg) \cap C(b) \neq \varnothing$, then this intersection contains some c, then $bc \in C(b)$ and $bc > g$. It means that $C(b)$ is not bounded in H. □

Let B be a convex subset of G. The boundaries of the convex set B are determined with the help of convex subgroups H_B^+ and H_B^-:

$$\begin{aligned} H_B^+ &= \{g \in G | \forall b \in B(b + |g| \in B)\}, \\ H_B^- &= \{g \in G | \forall b \in B(b - |g| \in B)\}. \end{aligned}$$

G is non-valuational if for any convex definable bounded set B

$$H_B^- = H_B^+ = 0$$

Fact 4.9. G has no definable non-trivial convex subgroup if only if G is non-valuational.

If $L \leq G$ and L is convex, then $H_L^- = H_L^+ \subseteq L$.

Theorem 4.10. *Let G be a non-valuational o-stable ordered group. We consider only densely ordered groups and let f be a definable continuous function. Then f is piecewise monotone, that is there exists a definable partition $X, A_1, \dots A_n$ of $dom f$, where X is finite and A_i is convex for each i, and $f \restriction A$ is monotone.*

Continuity of f is necessary, because $(\mathbb{R}, <, +, f)$, where f is Dirichlet's function

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

is o-stable [12].

This f is not piecewise monotone in our sense. But if we consider $f \restriction \mathbb{Q}$ and $f \restriction (\mathbb{R} \setminus \mathbb{Q})$, then each function is a constant, that is f is piecewise monotone, provided that pieces need not to be convex.

Lemma 4.11. *Let f be continuous and $A \subseteq dom f$ be open. Then for any $g \in G$ both $\{a \in A | f(a) > g\}$ and $\{a \in A | f(a) < g\}$ are open.*

Proof. Let $f(a) = h > g$. Then

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x (|x - a| < \delta \to |f(x) - f(a)| < \varepsilon = h - g).$$

Then $f(x) < g$ for any $x \in V_\delta(a)$. Then $V_\delta(a) \subseteq \{a \in A | f(a) > g\}$, so this set is open. □

Lemma 4.12. *Let A be a definable open subset of a non-valuational o-stable oredered group G. Then A is a finite union of convex sets.*

Proof. Assume the contrary, that A has infinitely many convex components A_i, for $i \in I$. Since A is open, each convex component of A is infinite, not a singleton.

Let $A_{i_1} < A_{i_2} < \dots A_{i_n} < \dots$ be an increasing (or decreasing) sequence of convex components of A. Since we can assume that G is χ_0^+-saturated, that is any type over a countable subset is realized, we may assume that there is some $\delta \in G$ such that $\inf A_{i_n} + k\delta < \sup A_{i_n}$ for each $k \in \mathbb{N}$.

Note that $H_B^- = \{0\}$, for any convex definable set B, then $\inf B + \delta > \inf B$

$$\inf\{b+\delta | b \in B\} > \inf\{b | b \in B\}.$$

Let $C = \sup \bigcup_{n=1}^{\infty} A_{i_n}$, that is,

$$C = \{g \in G : g < \sup \bigcup_{n=1}^{\infty} A_{i_n}\}$$

and $D = G \setminus C$. Then $\langle C, D \rangle$ is a cut in G.

Let

$$f(x, \delta) = x \in \inf A_{i_n} + \delta),$$

for every $c \in C$, we obtain $(c, \sup C) \cap f(x, \delta) \subset (c, \sup C) \cap f(x, 2\delta)$

Let

$$\begin{aligned} \varepsilon(x, y) \Leftrightarrow x \in A \wedge y \in A \ &\wedge \ (x \leq y \rightarrow \forall z(x \leq z \leq y \rightarrow z \in A)) \wedge \\ &\wedge \ (y \leq x \rightarrow \forall z(y \leq z \leq x \rightarrow z \in A)). \end{aligned}$$

Classes of ε are exactly convex components of A. We denote ε-classes of by $[x]$:

$$f(x, \delta) = x \in (\inf[x], \inf[x] + \delta)$$

which can be written as $\varepsilon(x, x + \delta)$.

Clearly, f has the strict order property in $\langle C, D \rangle$, that contradicts o-stability of G. $\square$

5. SOME PROPERTIES OF ORDERED ALGEBRAIC STRUCTURES

5.1 On quantifier elimination for the ordered set of real numbers with named Cantor's set

Several years ago Shelah worked on non-independence theories [11]. In order to generalize the theory of stability in the context of the theories with the NIP, a lot of work has been done. As a result, a simplest class appeared called *dp-minimal theories*, which were introduced by Onshuus and Usvyatsov in [9].

Definition 5.1. 1. (Shelah) A sequence of pairs $\langle(\phi^\alpha(x;\bar{y}^\alpha),\kappa^\alpha) : \alpha<\kappa\rangle$ of formulas and natural numbers such that there is an array $(\bar{b}^\alpha_i : \alpha<\kappa)$ of *witnesses* is called an *independent partition pattern of length* κ, or *inp-pattern*, that is, for all $\alpha<\kappa$, the "row" $\left\{\phi^\alpha(x;\bar{b}^\alpha_i) : i<\omega\right\}$ is k^α–inconsistent, but every "path" $\left\{\phi^\alpha(x;\bar{b}^\alpha_{\eta(\alpha)}) : \alpha<\kappa\right\}$ (for $\eta\in\omega^\kappa$) is consistent.

2. A theory T does not inp-pattern of length 2 is called an *inp-minimal.*

3. (Onshuus and Usvyatsov) A theory T which is minimal and does not have the Independence Property (i.e. NIP) will be dp-minimal.

In [5] he began to study dp-minimal ordered structures, where he proved that the elementary theory T structures $(\mathbb{R},<,P)$, where P defines the Cantor set, is dp-minimal. For this, he proved that this theory T admits the elimination of quantifiers. However, his proof is not entirely clear; therefore, we present another proof of the elimination of quantifiers for this theory.

The Cantor middle third set (see 5.1) is constructed by removing the interval of the middle third of the segment and then repeating this process with the remaining shorter segment, i.e. from the unit segment $C_0=[0,1]$ remove the middle third, that is, the interval $(1/3,2/3)$. The remaining point set is denoted by C_1. The set

$C_1 = [0, 1/3] \cup [2/3, 1]$ consists of two segments; we now delete the middle third of each segment and denote the remaining set by C_2. Repeating this procedure again, removing the middle thirds of all four segments, we get C_3. Further in the same way we obtain the sequence of closed sets $C_0 \supset C_1 \supset C_2 \supset \ldots$. The intersection

$$C = \bigcap_{i=0}^{\infty} C_i$$

is called a *Cantor middle third set.*

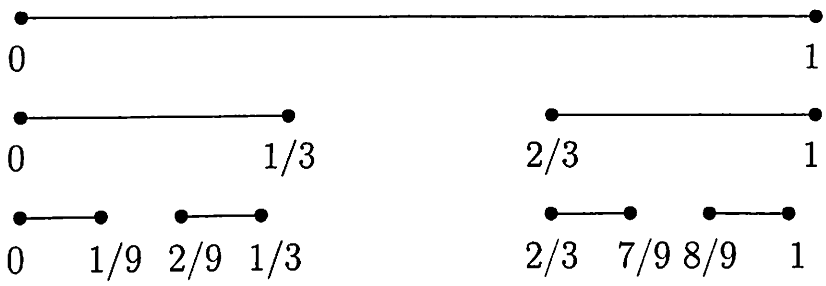

Figure 5.1: Cantor middle third set

Consider $(\mathbb{R}, <, P)$, where the predicate P is interpreted as a Cantor set. If while constructing the Cantor set we dropped the interval (a, b) and $x \in [a, b]$, then $l(x) = a$ and $r(x) = b$, but if $x \notin [0, 1]$, we understand that $l(x) = x$ and $r(x) = x$. We can see that this two functions are equivalent:

$$\begin{aligned}
y = r(x) \;&\Leftrightarrow\; (x = y \wedge [x \leqslant 0 \vee x \geqslant 1]) \wedge \\
&\wedge\; (0 < x < 1 \wedge P(y) \wedge x \leqslant y \wedge \forall z(x < z < y \rightarrow \neg P(z))) \\
y = l(x) \;&\Leftrightarrow\; (x = y \wedge [x \leqslant 0 \vee x \geqslant 1]) \wedge \\
&\wedge\; (0 < x < 1 \wedge P(y) \wedge y \leqslant x \wedge \forall z(y < z < x \rightarrow \neg P(z))).
\end{aligned}$$

Theorem 5.2. *The elementary theory of structure* $(\mathbb{R}, <, P, r, l, 0, 1)$ *admits the elimination of quantifiers.*

Proof. By Tarskii criterion, in order to prove the theorem, it suffices to eliminate the existential quantifier in a formula of the form $\exists x \bigwedge_{i=1}^{n} \phi_i(x, \bar{y})$, where $\phi_i(x, \bar{y})$ of one of the following forms:

(1). $t(x) = t(y)$;

(2). $t(x) \neq t(y)$;

(3). $t(x) < t(y)$;

(4). $t(x) \not< t(y)$;

(5). $t(x) > t(y)$;

(6). $t(x) \not> t(y)$;

(7). $P(t(x))$;

(8). $\neg P(t(x))$.

Note that we can assume that $\exists x \bigwedge\limits_{i=1}^{n} \phi_i(x, \bar{y})$ does not contain formulas like (2), (4) and (6). Indeed takes place

$$t(x) \neq t(y) \Leftrightarrow t(x) < t(y) \vee t(y) < t(x)$$

then, if $\phi_i(x, \bar{y})$ of the form (2), then

$$\begin{aligned} \exists x(t(x) \neq t(y) \wedge \theta(x, z)) &\Leftrightarrow \exists x((t(x) < t(y) \vee t(y) < t(x)) \wedge \theta(x, z)) \Leftrightarrow \\ &\Leftrightarrow \exists x(t(x) < t(y) \wedge \theta) \vee (t(y) < t(x) \wedge \theta) \Leftrightarrow \\ &\Leftrightarrow \exists x(t(x) < t(y) \wedge \theta) \vee \exists x(t(y) < t(x) \wedge \theta). \end{aligned}$$

Thus, we can get rid of all formulas of the form (2). In addition, there are equivalences

$$\begin{aligned} t(x) \not< t(y) &\Leftrightarrow t(y) < t(x) \vee t(x) = t(y), \\ t(x) \not> t(y) &\Leftrightarrow t(x) < t(y) \vee t(x) = t(y). \end{aligned}$$

So, similarly, we can get rid of formulas like (4) and (6).

Consider the following types of atomic formulas obtained from formulas of the form (1), (3), (5), (7), (8), in which we defined the content of the term $t(x)$:

(1). $x = t(y)$;

(2). $x < t(y)$;

(3). $x > t(y)$;

(4). $l(x) < t(y)$;

(5). $l(x) = t(y)$;

(6). $l(x) > t(y)$;

(7). $r(x) < t(y)$;

(8). $r(x) = t(y)$;

(9). $r(x) > t(y)$;

(10). $P(x)$;

(11). $\neg P(x)$.

Here, only $P(x)$ can be considered, not $P(l(x))$, $P(r(x))$, $\neg P(l(x))$ and $\neg P(r(x))$, since

$$\begin{aligned} P(l(x)) &\Leftrightarrow 0 \leqslant x \leqslant 1, \\ P(r(x)) &\Leftrightarrow 0 \leqslant x \leqslant 1, \\ \neg P(l(x)) &\Leftrightarrow x < 0 \vee x > 1, \\ \neg P(r(x)) &\Leftrightarrow x < 0 \vee x > 1. \end{aligned}$$

If one of $\phi_i(x, \bar{y})$ of the form $x = t(y)$, then $\exists x \bigwedge_{i=1}^{n} \phi_i(x, \bar{y})$ is equivalent to the formula $\bigwedge_{i=1}^{n} \phi_i(t(y), \bar{y})$, more precisely, let

$$\phi_j(x, \bar{y}) := x = t(y),$$

then

$$\exists x \bigwedge_{i=1}^{n} \phi_i(x, \bar{y}) \Leftrightarrow \bigwedge_{i \neq j} \phi_i(t(y), \bar{y}).$$

Thus, we can assume that $\exists x \bigwedge_{i=1}^{n} \phi_i(x, \bar{y})$ does not contain formulas of the form (1).

From the definition of the functions l and r, it is easy to deduce the following equivalences; note that

$$l(l(x)) = l(x), \quad r(r(x)) = r(x), \quad l(r(x)) = l(x), \quad r(l(x)) = l(x).$$

The left boundary of the variable x is less than or greater than t, then $x < t$ or $x > t$, respectively, and if x does not fall into the Cantor set, then respectively $x < t$, or its left boundary coincides with the left boundary of t or the variable x is greater than the right boundary of t. Consider formulas of the form (4), (6):

$$\begin{aligned} l(x) < t \;&\Leftrightarrow\; [(x \leqslant 0 \vee x \geqslant 1 \vee P(x)) \wedge x < t] \vee \\ &\vee\; [0 < x < 1 \wedge \neg P(x) \wedge (x < t \vee (l(x) = l(t) \wedge l(t) < t))]; \\ l(x) > t \;&\Leftrightarrow\; [(x \leqslant 0 \vee x \geqslant 1 \vee P(x)) \wedge x > t] \vee \\ &\vee\; [0 < x < 1 \wedge \neg P(x) \wedge x > r(t)]. \end{aligned}$$

The right boundary of the variable x is less than or greater than t, then $x < t$ or $x > t$, and if x does not fall into the Cantor set, then the variable x is less than the left border t or $x > t$, or its right boundary coincides with the right boundary of t. Consider formulas of the form (7), (9):

$$\begin{aligned} r(x) < t \;&\Leftrightarrow\; [(x \leqslant 0 \vee x \geqslant 1 \vee P(x)) \wedge x < t] \vee \\ &\vee\; 0 < x < 1 \wedge \neg P(x) \wedge x < l(t)]; \\ r(x) > t \;&\Leftrightarrow\; [(x \leqslant 0 \vee x \geqslant 1 \vee P(x)) \wedge x > t] \vee \\ &\vee\; [0 < x < 1 \wedge \neg P(x) \wedge (x > t \vee (r(x) = r(t) \wedge t < r(t)))]. \end{aligned}$$

Therefore, we can assume that there are no formulas of the form (4), (6), (7) and (9). The following cases remain: (2), (3), (5), (8), (10), (11).

The formula $\phi_i(x, \bar{y})$ can be considered a formula in disjunctive normal form. Recall that this means that there is a concept of conjunctions and each concept of several literals.

$$\exists x \left(\bigwedge_{i=1}^{n} \phi_i(x, \bar{y}) \right) \Leftrightarrow \exists x \left((\bigwedge_{i=1}^{n} \phi_i(x, \bar{y})) \wedge (x \in [0, 1] \vee x \notin [0, 1]) \right) \Leftrightarrow$$

$$\Leftrightarrow \exists x \left[\left(\bigwedge_{i=1}^{n} \phi_i(x, \bar{y}) \wedge x \in [0,1] \right) \vee \left(\bigwedge_{i=1}^{n} \phi_i(x, \bar{y}) \wedge x \notin [0,1] \right) \right] \Leftrightarrow$$

$$\Leftrightarrow \exists x \left(\bigwedge_{i=1}^{n} \phi_i(x, \bar{y}) \wedge x \in [0,1] \right) \vee \exists x \left(\bigwedge_{i=1}^{n} \phi_i(x, \bar{y}) \wedge x \notin [0,1] \right).$$

In the formula $\exists x(\bigwedge_{i=1}^{n} \phi_i(x, \bar{y}) \wedge x \notin [0,1])$ we have $l(x) = x$, $r(x) = x$, and the formula P is false. Therefore, it suffices to consider ϕ_i of the form $x < t$, $x > t$, $x = t$. We eliminate the quantifier of existence as for the theory of dense linear order.

Consider the formula

$$\exists x \left(\bigwedge_{i=1}^{n} \phi_i(x, \bar{y}) \wedge 0 \leqslant x \leqslant 1 \right),$$

if there is a formula $x < t_2$ and $x > t_1$, then

$$\exists x \left(t_1 < x < t_2 \wedge 0 \leqslant x \leqslant 1 \wedge \bigwedge_{i=1}^{n} \phi_i(x, \bar{y}) \right).$$

Let one of ϕ_i be $P(x)$. Then $l(x) = x$ and $r(x) = x$ and consider ϕ_i of the form $x < t$, $x > t$ and $x = t$, then the variable x occurs only in inequalities. In other words, we are asked if there is a value of x that is greater than some variables and less than some other.

$$\begin{aligned}
\exists x(t_1 < x < t_2 \wedge x = t_3 \wedge P(x)) &\Leftrightarrow t_1 < t_3 < t_2 \wedge P(t_3), \\
\exists x(t_1 < x < t_2 \wedge P(x)) &\Leftrightarrow t_1 < t_2 \wedge [t_1 < l(t_2) \vee r(t_1) < t_2], \\
\exists x(t_1 < x \wedge P(x)) &\Leftrightarrow t_1 < 1, \\
\exists x(x < t_2 \wedge P(x)) &\Leftrightarrow t_2 > 0.
\end{aligned}$$

Consider only a formula of the form $P(x)$, then

$$\exists x P(x) \Leftrightarrow 0 = 0.$$

Let one of ϕ_i be $\neg P(x)$. Consider a formula where there are also $x < t_2$,

$x > t_1$ formulas, that is,

$$\exists x \left(t_1 < x < t_2 \wedge 0 \leqslant x \leqslant 1 \wedge \neg P(x) \wedge \bigwedge_{i=1}^{n} \phi_i(x, \bar{y}) \right).$$

As before, we can assume that there is exactly one inequality of the form $t < x$ and exactly one inequality of the form $x < t$. Therefore, we can say that x does not fall into the Cantor set.

The case when the formula states that the element x lies in the interval but does not lie in the Cantor set, the point from the Cantor set nearest to it coincides with t_3, and the point from the Cantor set nearest to it coincides with t_4 as follows:

$$\exists x(t_1 < x < t_2 \wedge \neg P(x) \wedge l(x) = t_3 \wedge r(x) = t_4) \Leftrightarrow$$

$$\Leftrightarrow t_3 < t_4 \wedge l(t_4) = t_3 \wedge t_1 < t_4 \wedge t_2 > t_3 \wedge t_1 < t_2,$$

The case when the formula states that the element x lies in the interval but does not lie in the Cantor set, and the point from the Cantor set nearest to its left coincides with t_3 as follows:

$$\exists x(t_1 < x < t_2 \wedge \neg P(x) \wedge l(x) = t_3) \Leftrightarrow$$

$$\Leftrightarrow \exists x(t_1 < x < t_2 \wedge \neg P(x) \wedge l(x) = t_3 \wedge r(x) = r(t_3)) \Leftrightarrow$$

$$\Leftrightarrow t_3 < r(t_3) \wedge l(r(t_3)) = t_3 \wedge t_1 < r(t_3) \wedge t_2 > t_3 \wedge t_1 < t_2.$$

The case when the formula states that the element x lies in the interval but does not lie in the Cantor set, and the point from the Cantor set nearest to its right coincides with t_4 as follows:

$$\exists x(t_1 < x < t_2 \wedge \neg P(x) \wedge r(x) = t_4) \Leftrightarrow$$

$$\Leftrightarrow \exists x(t_1 < x < t_2 \wedge \neg P(x) \wedge r(x) = t_4 \wedge l(x) = l(t_4)) \Leftrightarrow$$

$$\Leftrightarrow l(t_4) < t_4 \wedge r(l(t_4)) = t_4 \wedge t_1 < t_4 \wedge t_2 > l(t_4) \wedge t_1 < t_2.$$

It remains to consider the case

$$\exists x(t_1 < x < t_2 \wedge \neg P(x)) \Leftrightarrow t_1 < t_2.$$

In all cases, we have reduced the quantifier of the existence of the element x. The theorem is proved. □

The fact that it is a dp-minimal structure is proved in the paper itself [5].

Note that all Cantor sets (for example, Smith-Volterra-Cantor set, etc.) with an order relation, equality, and bounds admit the elimination of quantifiers. Terms are treated in the same way as in the middle -thirds Cantor set.

Thus, the elimination of quantifiers establishes the solvability of the elementary theory of the structures of Cantor sets with equality, bounds, and order relation.

6. Conclusion

In conclusion, the elimination of quantifiers establishes the solvability of the elementary theory of the structures of Cantor sets with an equality, bounds, and order relation.

Quantifier elimination is one of the most important tools in model theory. Indeed, if a theory allows quantifier elimination, then this theory is complete, and the description of all definable subsets can be reduced to describing only those subsets that are defined by a quantifier-free formula. One of the most important mathematical structures is the linearly ordered set of real numbers. On it, you can set an ordered group and field. It is known that the elementary theory of these structures admits quantifier elimination, and since these theories are computably axiomatizable, quantifier elimination implies their solvability. In this work, we added not operations to the order, but select a Cantor subset on it and proved the elimination of quantifiers for the resulting structure.

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