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FACTORIZATIONS AND HARDY TYPE INEQUALITIES

THESIS

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Declaration

I confirm that this is my original work and that all references to additional sources have been appropriately and completely acknowledged.

Kuralay Apseit

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Dedication

This thesis is dedicated to:

- To express my sincere gratitude to my supervisor, Prof. Nurgissa Yessirkegenov, a man who devoted his life to the development of science in Kazakhstan and constantly strives to make a significant contribution to its prosperity, for the invaluable support, advice and guidance at all stages of writing this thesis. Your deep knowledge, patience and willingness to help have been a huge support to me in the research process. Without your guidance and inspiration, this work could not have been completed.

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Abstract

In this thesis, we derive the Hardy and critical Hardy inequalities using any homogeneous quasi-norm in a unified way. Specifically, we establish a sharp remainder formula for these inequalities. Our identity extend to Hardy and critical Hardy inequalities for the radial derivative operator with any homogeneous quasi-norm, offering enhanced versions of classical results. Our approach is based on the factorization method of differential operators introduced by Gesztesy and Littlejohn. As an application, we show Caffarelli-Kohn-Nirenberg type inequalities with more general weight. Because of the freedom in the choice of any homogeneous quasi-norm, our results give new insights already in both anisotropic $\mathbb{R}^n$ and isotropic $\mathbb{R}^n$. Our results not only generalize existing inequalities but also uncover new perspectives in the study of partial differential equations and functional inequalities. These advances could have significant implications for theoretical research and applications in which anisotropic and isotropic properties are relevant.

Аңдатпа

Бұл диссертациялық жұмыста біз кез келген біртекті квазинорманы біртұтас әдіспен пайдаланып Харди және критикалық Харди теңсіздіктерін шығарамыз. Атап айтқанда, біз осы теңсіздіктер үшін нақты қалдық формуласын белгіледік. Біздің сәйкестендіру классикалық нәтижелердің кеңейтілген нұсқаларын ұсына отырып, кез келген біртекті квазинормасы бар радиалды туынды оператор үшін Харди теңсіздіктері мен критикалық Харди теңсіздіктеріне таралады. Біздің көзқарасымыз Гештеси мен Литтлжон ұсынған дифференциалды операторды факторизациялау әдісіне негізделген. Қолданысы ретінде біз жалпы салмақтары бар Каффарелли-Кон-Ниренберг типті теңсіздіктерді көрсетеміз. Кез келген біртекті квазинорманы таңдау еркіндігінің арқасында біздің нәтижелеріміз анизотропты $\mathbb{R}^n$ және изотропты $\mathbb{R}^n$. бойынша да жаңа түсінік береміз. Біздің нәтижелеріміз бар теңсіздіктерді жалпылама, сонымен қатар дербес дифференциалдық теңдеулер мен функционалдық теңсіздіктерді зерттеуде жаңа перспективалар ашады. Бұл жетістіктер теориялық зерттеулер мен анизотропты және изотропты қасиеттер маңызды болып табылатын қолданбаларға маңызды әсер етуі мүмкін.

Аннотация

В этой диссертационной работе мы выводим Харди и критические неравенства Харди, используя любую однородную квазинорму единым способом. В частности, мы установили точную формулу остатка для этих неравенств. Наше тождество распространяется на неравенства Харди и критические неравенства Харди для оператора радиальной производной с любой однородной квазинормой, предлагая расширенные версии классических результатов. Наш подход основан на методе факторизации дифференциальных операторов, предложенном Гестези и Литтлджоном. В качестве приложения мы показываем неравенства типа Каффарелли-Кона-Ниренберга с более общим весом. Благодаря свободе выбора любой однородной квазинормы наши результаты дают новое понимание уже как в анизотропном $\mathbb{R}^n$, так и в изотропном $\mathbb{R}^n$. Наши результаты не только обобщают существующие неравенства, но и открыть новые перспективы в изучении уравнений в частных производных и функциональных неравенств. Эти достижения могут иметь важное значение для теоретических исследований и приложений, где важны анизотропные и изотропные свойства.

Contents

1. Background and motivations

1.1 Introduction

For certainly one hundred years, from 1918 to the present, mathematicians have been continuously researching the fascinating topic of Hardy inequalities. In [1], the original Hardy inequality was published by G. H. Hardy.

The classical Hardy inequalities, which are one of the most well-known and widely applied inequalitaies in mathematics. They are significant in many branches of mathematical physics and analysis. The Hardy-type inequalities, along with their variations and generalizations, are the subject of several theories and a large literature. We refer the interested reader to the historical monographs [2], [3], [4], [5], and [6] which are some of the standard references on the topic.

The principal objective of this note is to use the method of factorizations of differential operators to study the improved Hardy-type inequalities. For a comprehensive review of the history and properties of the factorization method, the interested reader is referred to [7].

The main aim of this paper is to derive unified Hardy inequalities with any homogeneous quasi-norm via the method of factorization. Gesztesy introduced the factorization method to obtain the classical Hardy inequality in [8] then in [9] the radial and logarithmic refinements. More recently, in the noncommutative setting, the authors in [10] successfully applied the factorization method to obtain improved Hardy and Hardy-Rellich type inequalities in general stratified groups and weighted Hardy inequalities on general homogeneous Lie groups.

1.2 A brief analysis of Hardy inequalities

Recall that the Hardy inequality for all functions $f \in \mathbb{C}_0^\infty(\mathbb{R}^n)$ can be formulated as

$$\int_{\mathbb{R}^n} |(\nabla f)(x)|^2 dx \geqslant \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2} dx, \quad n \geqslant 3 \tag{1.2.1}$$

where ∇ is the standard gradient in $\mathbb{R}^n$, $|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$ is the Euclidean norm on $\mathbb{R}^n$ and the constant $\left(\frac{n-2}{2}\right)^2$ is known to be sharp. If the right-hand side of this inequality is finite, equality holds if and only if f(x) = 0 almost everywhere.

The Hardy inequality has many applications in the analysis of linear and non-linear PDEs. For example, the criteria on existence (or nonexistence) of solutions for the following second order partial differential equations

$$\left.\begin{matrix} 0 \\ u_t \\ u_{tt} \end{matrix}\right\} - \Delta u = \lambda \frac{|u|^s}{|x|_E^2} \tag{1.2.2}$$

depends on a relation between the constant λ and the sharp constant $\frac{2}{n-2}$ from the Hardy inequality. In the equation, instead of the Hardy potential $\frac{1}{|x|_E^2}$ one may have a more general function which motivates the study of weighted Hardy inequalities in [11] and [12].

Today, as I mentioned in (1.1), there are many works on various versions of Hardy's inequality. Let's mention again only classic monographs such as [2], [3], [4], [5], and [6] and the recent open access book [13] on this subject and for its applications.

This note aims to establish more general Hardy inequality, unifying both sub-critical and critical cases, by employing the factorization method for differential expressions.

In this direction, recently Gesztesy and Littlejohn in [7] gave a simple proof of (1.2.1) by using the non-negativity of $T_\alpha^+ T_\alpha$ on $C_0^\infty(\mathbb{R}^n \setminus 0)$ and selecting special value for the parameter α, where

$$T_\alpha := \nabla + \alpha |x|^{-2} x,$$

and T_α^+ its formal adjoint

$$T_\alpha^+ = -div(\cdot) + \alpha |x|^{-2} x \cdot, \ x \in \mathbb{R}^n \setminus 0,$$

for $\alpha \in \mathbb{R}$, $\quad x \in \mathbb{R}^n \setminus 0$ and $n \geq 3$.

There, the authors also applied the factorization method to obtain Rellich inequalities with remainder terms. For this method, we also refer to [14], [8], [9] in the Euclidean setting as well as [10] on stratified Lie groups. In this note, we plan to continue the research from [10] by applying the method to unified Hardy inequalities with any homogeneous quasi-norm in a Euclidean setting and generalizing all results to homogeneous Lie groups in [15]. Actually, in this non-commutative setting, we show Hardy identities that imply improved subcritical and critical Hardy inequalities.

Theorem 1.2.1 (Gesztesy, Littlejohn, Michael, and Pang [9]). *Let $\Omega \subseteq \mathbb{R}^n$ open, $n \in \mathbb{N}, n \geqslant 2$.*

- *Then, for all $f \in C_0^\infty(\Omega)$,*

$$\int_\Omega |(\nabla f)(x)|^2 d^n x \geqslant \int_\Omega |(\partial_r f)(x)|^2 d^n x \geqslant \frac{(n-2)^2}{4} \int_\Omega |x|^{-2} |f(x)|^2 d^n x.$$

- *Let $m \in \mathbb{N}$, and suppose in addition that $\Omega \subset \mathbb{R}^n$ is bounded with $x_0 \in \Omega$. Assume $\gamma > 0$ is such that $0 < \operatorname{diam}(\Omega) < \gamma / e_m$, where e_m is given as in (1.4), and let $\ln_k(\gamma / |x - x_0|), k \in \mathbb{N}$, be as in (1.3), (1.4). Then, for all $f \in C_0^\infty(\Omega)$,*

$$\begin{aligned} \int_\Omega |(\nabla f)(x)|^2 d^n x &\geqslant \int_\Omega |(\partial_r f)(x)|^2 d^n x \\ &\geqslant \int_\Omega |x - x_0|^{-2} |f(x)|^2 \left\{ \frac{(n-2)^2}{4} + \frac{1}{4} \sum_{j=1}^m \prod_{k=1}^j [\ln_k(\gamma / |x - x_0|)]^{-2} \right\} d^n x. \end{aligned}$$

Here, as you can see, this author improves the (1.2.1) by replacing the gradient with the radial derivative ∂_r.

2. Preliminaries

In this section we briefly review the necessary definitions and notations from [9], [13]. A few other details that are necessary for our analysis will also be covered.

2.1 Logarithmic improvement

The Hardy inequality extended involving the logarithmic refinements, which were derived in [7]. In [7] this extension is called remainder terms (the latter often associated with logarithmic refinements or with boundary terms).

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with $x_0 \in \Omega$, $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We use the following notations:

$$\begin{aligned} \ln_1(\gamma/|x-x_0|) &:= \ln(\gamma/|x-x_0|), \quad 0 < |x-x_0| < \gamma, \\ \ln_{k+1}(\gamma/|x-x_0|) &:= \ln(\ln_k(\gamma/|x-x_0|)), \quad 0 < |x-x_0| < \gamma/e_{k+1}, \quad k \in \mathbb{N}, \end{aligned} \tag{2.1.1}$$

for $\gamma > 0$, $x \in \mathbb{R}^n \backslash \{x_0\}$ with $0 < |x-x_0| < \operatorname{diam}(\Omega) < \gamma/e_m$, where,

$$e_1 := 1, \quad e_{k+1} := e^{e_k}, \quad k \in \mathbb{N}. \tag{2.1.2}$$

We denote $\sum_{j=1}^{0}(\cdot) := 0$ and $\prod_{k=1}^{0}(\cdot) := 1$.

2.2 Homogeneous quasi-norm

Definition 2.2.1. A homogeneous quasi-norm refers to a continuous non-negative function

$$\mathbb{R}^n \ni x \mapsto |x| \in [0, \infty) \tag{2.2.1}$$

satisfying the following properties:

- $|-x| = |x|$ for all $x \in \mathbb{R}^n$,
- $|\lambda x| = \lambda |x|$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$,
- $|x| = 0$ if and only if $x = 0$.

The following polar decomposition in $\mathbb{R}^n$ will be very useful for our analysis: there is a (unique) positive Borel measure σ on the unit sphere

$$\mathfrak{S} := \{x \in \mathbb{R}^n : |x| = 1\} \tag{2.2.2}$$

such that for all $f \in L^1(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_{\mathfrak{S}} f(ry)\, r^{n-1}\, d\sigma(y)\, dr. \tag{2.2.3}$$

We use the notation

$$\partial_{|x|} f := \frac{df(x)}{dr}. \tag{2.2.4}$$

3. Weighted Hardy identities with remainder terms

In this section, using differential expression factorization, we obtain Hardy-type identities with remainder terms. The obtained results are already new in the standard setting of $\mathbb{R}^n$ in particular, including $\mathbb{R}^n$ with both isotropic and anisotropic structures. Moreover, all the results are generalized to general homogeneous groups in [15], without weight in [16].

More precisely, by applying the factorization of suitable differential operators, it is shown that:

3.1 Weighted Hardy identities with remainder terms with any homogeneous quasi-norm

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded with a smooth section boundary $\partial\Omega$, $n \geqslant 2$, $x_0 \in \Omega$, $n \in \mathbb{N}$, $m \in \mathbb{N}$. Let $|\cdot|$ be any homogeneous quasi-norm. Assume that $\gamma > 0$, is such that $0 < diam(\Omega) < \frac{\gamma}{e_m}$, where e_m is as in* (2.1.2) *and let $\ln_k(\gamma/|x-x_0|)$, $k \in \mathbb{N}$, $\alpha \in \mathbb{R}$ for all $f \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega \frac{|(\partial_{|x|}f(x)|^2}{|x-x_0|^{2\alpha}}dx = \left[\alpha(\alpha-n+2)+\frac{(n-2)^2}{4}\right]\int_\Omega \frac{|f(x)|^2}{|x-x_0|^{2\alpha}}dx$$
$$+\frac{1}{4}\int_\Omega \frac{\sum_{j=1}^m \prod_{k=1}^j \left(\ln_k \frac{\gamma}{|x-x_0|}\right)^{-2}}{|x-x_0|^{2\alpha}}|f(x)|^2dx$$
$$+\int_\Omega \left|\frac{\partial_{|x|}}{|x-x_0|^\alpha}f(x)+\frac{n-2\alpha-2}{2|x-x_0|^{\alpha+1}}f(x)+\frac{\sum_{j=1}^m \prod_{k=1}^j \left[\ln_k\left(\frac{\gamma}{|x-x_0|}\right)\right]^{-1}}{2|x-x_0|^{\alpha+1}}f(x)\right|^2 dx, \tag{3.1.1}$$

where

$$\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}=\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2}+\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2}\left[\ln\left(\ln\frac{\gamma}{|x|}\right)\right]^{-2}+\cdots$$
$$\cdots+\underbrace{\left[\left(\ln\frac{\gamma}{|x|}\right)\right]^{-2}\left[\ln\left(\ln\frac{\gamma}{|x|}\right)\right]^{-2}\cdots\left[\ln\cdots\left(\ln\left(\frac{\gamma}{|x|}\right)\right)\right]^{-2}}_{\text{m times}}.$$

Next one using non-negativity of $(T_{\alpha,m}T^*_{\alpha,m}f)(x)$, we have following Theorem (3.1.2) :

Theorem 3.1.2. *Let $\Omega\subset\mathbb{R}^n$ be an open and bounded with a smooth section boundary $\partial\Omega$, $n\geqslant 2$, $x_0\in\Omega$, $n\in\mathbb{N}$, $m\in\mathbb{N}$. Let $|\cdot|$ be any homogeneous quasi-norm. Assume that $\gamma>0$, is such that $0<diam(\Omega)<\frac{\gamma}{e_m}$, where e_m is as in (2.1.2) and let $\ln_k(\gamma/|x-x_0|)$, $k\in\mathbb{N}$, $\alpha\in\mathbb{R}$ for all $f\in C_0^\infty(\Omega)$ we have*

$$\begin{aligned}\int_\Omega\frac{|(\partial_{|x|}f(x))|^2}{|x-x_0|^{2\alpha}}dx&=\frac{n(n-4\alpha-4)}{4}\int_\Omega\frac{|f(x)|^2}{|x-x_0|^{2\alpha+2}}dx\\&-\frac{3}{4}\sum_{j=1}^{m}\int_\Omega\frac{|f(x)|^2}{|x-x_0|^{2\alpha+2}}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x-x_0|}\right)\right]^{-2}dx\\&+\sum_{j=1}^{m}\int_\Omega\frac{|f(x)|^2}{|x-x_0|^{2\alpha+2}}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x-x_0|}\right)\right]^{-1}dx\\&-\int_\Omega\frac{|f(x)|^2}{|x-x_0|^{2\alpha+2}}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x-x_0|}\right)\right]^{-1}\sum_{j=1}^{m-1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x-x_0|}\right)\right]^{-1}dx\\&+\int_\Omega\left|\frac{-\partial_{|x|}}{|x-x_0|^{\alpha}}f(x)-\frac{n}{2|x-x_0|^{\alpha+1}}f(x)+\frac{\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x-x_0|}\right)\right]^{-1}}{2|x-x_0|^{\alpha+1}}f(x)\right|^2dx.\end{aligned}\tag{3.1.2}$$

Remark 3.1.3. From Theorem (3.1.1) by dropping the last term in (3.1.1) we have following weighted Hardy inequality:

$$\begin{aligned}\int_\Omega\frac{|(\partial_{|x|}f(x)|^2}{|x-x_0|^{2\alpha}}dx&\geqslant\left[\alpha(\alpha-n+2)+\frac{(n-2)^2}{4}\right]\int_\Omega\frac{|f(x)|^2}{|x-x_0|^{2\alpha}}dx\\&+\frac{1}{4}\int_\Omega\frac{\sum_{j=1}^{m}\prod_{k=1}^{j}\left(\ln_k\frac{\gamma}{|x-x_0|}\right)^{-2}}{|x-x_0|^{2\alpha}}|f(x)|^2dx.\end{aligned}\tag{3.1.3}$$

Remark 3.1.4. From Theorem (3.1.2) by dropping the last term in (3.1.2) we have following weighted Hardy inequality:

$$\begin{aligned}\int_\Omega \frac{|(\partial_{|x|} f(x))|^2}{|x-x_0|^{2\alpha}} dx \geqslant \frac{n(n-4\alpha-4)}{4} \int_\Omega \frac{|f(x)|^2}{|x-x_0|^{2\alpha+2}} dx \\ - \frac{3}{4} \sum_{j=1}^{m} \int_\Omega \frac{|f(x)|^2}{|x-x_0|^{2\alpha+2}} \prod_{k=1}^{j} \left[\ln_k\left(\frac{\gamma}{|x-x_0|}\right)\right]^{-2} dx \\ + \sum_{j=1}^{m} \int_\Omega \frac{|f(x)|^2}{|x-x_0|^{2\alpha+2}} \prod_{k=1}^{j} \left[\ln_k\left(\frac{\gamma}{|x-x_0|}\right)\right]^{-1} dx \\ - \int_\Omega \frac{|f(x)|^2}{|x-x_0|^{2\alpha+2}} \sum_{j=1}^{m} \prod_{k=1}^{j} \left[\ln_k\left(\frac{\gamma}{|x-x_0|}\right)\right]^{-1} \sum_{j=1}^{m-1} \prod_{k=1}^{j} \left[\ln_k\left(\frac{\gamma}{|x-x_0|}\right)\right]^{-1} dx.\end{aligned} \tag{3.1.4}$$

Remark 3.1.5. Let $|\cdot|$ represent the standard Euclidean distance. Inequality (3.1.3) with $m = 0$ and $x_0 = 0$, offers an improved weighted form of the classical Hardy inequality (1.2.1). This improvement is achieved by substituting the full gradient with the radial derivative in a bounded domain.

$$\int_\Omega \frac{|(\partial_{|x|} f(x)|^2}{|x|^{2\alpha}} dx \geqslant \left[\alpha(\alpha - n + 2) + \frac{(n-2)^2}{4}\right] \int_\Omega \frac{|f(x)|^2}{|x|^{2\alpha}} dx. \tag{3.1.5}$$

Remark 3.1.6. Let $\alpha = 0$ and $|\cdot|$ is the usual Euclidean distance, and with the extension $|\nabla f(x)| \geqslant |\partial_{|x|} f(x)|$, we refer to [17].

Remark 3.1.7. Let $\alpha = 0$ and $|\cdot|$ is the usual Euclidean distance with radial derivative , we refer to [9]. Moreover, it was shown in [18] that the constants $(n-2)^2/4$ and $1/4$ are optimal.

Proof of Theorem (3.1.3). Introducing the differential operators T_0 on $C_0^\infty(\Omega)$, respectively, $\Omega \subseteq \mathbb{R}^n$ open and $\Omega = B(0;\rho)$ on $C_0^\infty(B(0;\rho)\backslash\{0\})$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $n \geqslant 2$ as follows,
for $m = 0$:

$$T_{\alpha,0} := |x|^{-\alpha}\partial_{|x|} + \frac{n-2\alpha-2}{2}|x|^{-\alpha-1}, \tag{3.1.6}$$

and $m \in \mathbb{N}$:

$$T_{\alpha,m} := |x|^{-\alpha}\partial_{|x|} + \frac{n-2\alpha-2}{2}|x|^{-\alpha-1} + \frac{1}{2}|x|^{-\alpha-1} \sum_{j=1}^{m} \prod_{k=1}^{j} \left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}. \tag{3.1.7}$$

Let us find the formal adjoint of a differential operator, denoted by $T_{\alpha,m}^{+}$, which is defined such that it satisfies certain inner product relationships with respect to a given function space. The inner product on the space of square-integrable functions $L^2(\Omega)\colon = L^2(\Omega; dx)$ is defined as

$$\langle T_{\alpha,m} f, g\rangle_{L^2} = \langle f, T^+_{\alpha,m} g\rangle_{L^2}. \tag{3.1.8}$$

To find the formal adjoint $T^+_{\alpha,m}$ in $C_0^\infty(\Omega)$, where $\mathfrak{S}$ is the quasi-sphere as in (2.2.2), and using (2.2.3), one calculates:

$$\begin{aligned}
\langle T_{\alpha,m} f, g\rangle_{L^2} &= \int_{L^2(\Omega)} T_{\alpha,m} f(x)\overline{g(x)}dx \\
&= \int_\Omega |x|^{-\alpha}\partial_{|x|} f(x)\overline{g(x)}dx + \int_\Omega \frac{n-2\alpha-2}{2}|x|^{-\alpha-1} f(x)\overline{g(x)}dx \\
&+ \int_\Omega \frac{1}{2}|x|^{-\alpha-1}\sum_{j=1}^m\prod_{k=1}^j\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x)\overline{g(x)}dx \\
&= \int_0^\infty\int_{\mathfrak{S}} r^{-\alpha}\frac{d}{dr}(f(ry))\overline{g(ry)}r^{n-1}d\sigma(y)dr \\
&+ \int_\Omega \frac{n-2\alpha-2}{2}|x|^{-\alpha-1} f(x)\overline{g(x)}dx \\
&+ \int_\Omega \frac{1}{2}|x|^{-\alpha-1}\sum_{j=1}^m\prod_{k=1}^j\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x)\overline{g(x)}dx \\
&= -\int_0^\infty\int_{\mathfrak{S}} r^{-\alpha+n-1} f(ry)\frac{d}{dr}(g(ry))d\sigma(y)dr \\
&+ (\alpha-n+1)\int_0^\infty\int_{\mathfrak{S}} r^{-\alpha+n-2} f(ry)\overline{g(ry)}d\sigma(y)dr \\
&+ \int_\Omega \frac{n-2\alpha-2}{2}|x|^{-\alpha-1} f(x)\overline{g(x)}dx \\
&+ \int_\Omega \frac{1}{2}|x|^{-\alpha-1}\sum_{j=1}^m\prod_{k=1}^j\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x)\overline{g(x)}dx \\
&= \int_\Omega |x|^{-\alpha} f(x)\overline{(-\partial_{|x|} g(x))}dx - \frac{n}{2}\int_\Omega |x|^{-\alpha-1} f(x)\overline{g(x)}dx \\
&+ \int_\Omega \frac{1}{2}|x|^{-\alpha-1}\sum_{j=1}^m\prod_{k=1}^j\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x)\overline{g(x)}dx.
\end{aligned}$$

Thus, the formal adjoint operator of $T_{\alpha,m}$ has the following form:

$$T^*_{\alpha,m} := -|x|^{-\alpha}\partial_{|x|} - \frac{n}{2}|x|^{-\alpha-1} + \frac{1}{2}|x|^{-\alpha-1}\sum_{j=1}^m\prod_{k=1}^j\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}, \quad m\in\mathbb{N}. \tag{3.1.9}$$

When $m=0$, we have a following the adjoint of $T_{\alpha,0}$ is $T^*_{\alpha,0}$:

$$T^*_{\alpha,0} := -|x|^{-\alpha}\partial_{|x|} - \frac{n}{2}|x|^{-\alpha-1}. \tag{3.1.10}$$

Remark 3.1.8. Given differential expressions $S_j, j=1,2$, their product S_1S_2 is used in the usual sense, that is,

$$(S_1S_2f)(x) = (S_1(S_2f))(x),$$

for $f(x)$ in the underlying function space, and analogously for products of three and more differential expressions.

We write

$$0 \le \int_\Omega |(T_{\alpha,m}f)(x)|^2\,dx = \int_\Omega \overline{f(x)}\left(T^*_{\alpha,m}T_{\alpha,m}f\right)(x)dx. \tag{3.1.11}$$

Now, show that

$$\begin{aligned}(T^*_{\alpha,m}T_{\alpha,m}f)(x) = &-|x|^{-2\alpha}\partial^2_{|x|}f(x) - (n-\alpha-1)|x|^{-2\alpha-1}\partial_{|x|}f(x)\\ &-\left[\alpha(\alpha-n+2)+\left(\frac{n-2}{2}\right)^2\right]|x|^{-2\alpha-2}f(x)\\ &-\frac{1}{4}|x|^{-2\alpha-2}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}f(x).\end{aligned} \tag{3.1.12}$$

Using the mathematical induction method on $m\in\mathbb{N}$. For $m=0$, one observes

$$\begin{aligned}(T^*_{\alpha,0}T_{\alpha,0}f)(x) &= \left(-|x|^{-\alpha}\partial_{|x|} - \frac{n}{2}|x|^{-\alpha-1}\right)\left(|x|^{-\alpha}\partial_{|x|} + \frac{n-2\alpha-2}{2}|x|^{-\alpha-1}\right)f(x)\\ &= -|x|^{-2\alpha}\partial^2_{|x|}f(x) - |x|^{-\alpha}\partial_{|x|}\left(\frac{n-2\alpha-2}{2}|x|^{-\alpha-1}f(x)\right)\\ &-\frac{n}{2}|x|^{-2\alpha-1}\partial_{|x|}f(x) - \frac{n}{2}\left(\frac{n-2\alpha-2}{2}\right)|x|^{-2\alpha-2}f(x)\\ &= -|x|^{-2\alpha}\partial^2_{|x|}f(x) - \frac{n-2\alpha-2}{2}|x|^{-2\alpha-1}\partial_{|x|}f(x)\\ &+(\alpha+1)\left(\frac{n-2\alpha-2}{2}\right)|x|^{-2\alpha-2}f(x) - \frac{n}{2}|x|^{-2\alpha-1}\partial_{|x|}f(x)\\ &-\frac{n}{2}\left(\frac{n-2\alpha-2}{2}\right)|x|^{-2\alpha-2}f(x)\\ &= -|x|^{-2\alpha}\partial^2_{|x|}f(x) - (n-\alpha-1)|x|^{-2\alpha-1}\partial_{|x|}f(x)-\\ &-\left[\alpha(\alpha-n+2)+\left(\frac{n-2}{2}\right)^2\right]|x|^{-2\alpha-2}f(x).\end{aligned}$$

Finally $(T^*_{\alpha,0}T_{\alpha,0}f)(x)$, we have

$$
\begin{aligned}
(T^*_{\alpha,0}T_{\alpha,0}f)(x) &= -|x|^{-2\alpha}\partial^2_{|x|}f(x) - (n-\alpha-1)|x|^{-2\alpha-1}\partial_{|x|}f(x) - \\
&- \left[\alpha(\alpha-n+2) + \left(\frac{n-2}{2}\right)^2\right]|x|^{-2\alpha-2}f(x).
\end{aligned}
\tag{3.1.13}
$$

For $m=1$, a direct computation, employing (3.1.13), yields,

$$
\begin{aligned}
&(T^*_{\alpha,1}T_{\alpha,1}f)(x) \\
&= \left(T^*_{\alpha,0} + \frac{1}{2}|x|^{-\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)\left(T_{\alpha,0} + \frac{1}{2}|x|^{-\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)f(x) = \\
&= (T^*_{\alpha,0}T_{\alpha,0}f)(x) + T^*_{\alpha,0}\left(\frac{1}{2}|x|^{-\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\right) \\
&+ \frac{1}{2}|x|^{-\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}T_{\alpha,0}f(x) + \frac{1}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2}f(x) \\
&= (T^*_{\alpha,0}T_{\alpha,0}f)(x) + I_1 + I_2 + \frac{1}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2}f(x).
\end{aligned}
\tag{3.1.14}
$$

Let is direct calculate I_1:

$$
\begin{aligned}
I_1 =& T^*_{\alpha,0}\left(\frac{1}{2}|x|^{-\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f\right)(x) \\
&= \left(-|x|^{-\alpha}\partial_{|x|} - \frac{n}{2}|x|^{-\alpha-1}\right)\left(\frac{1}{2}|x|^{-\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f\right)(x) \\
&= -\frac{1}{2}|x|^{-\alpha}\partial_{|x|}\left(|x|^{-\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\right) - \frac{n}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x) \\
&= -\frac{1}{2}|x|^{-\alpha}\left((-\alpha-1)|x|^{-\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1} + |x|^{-\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2}\right)f(x) \\
&- \frac{1}{2}|x|^{-2\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}f(x) - \frac{n}{4}|x|^{-2\alpha-2}\left[\ln(\frac{\gamma}{|x|})\right]^{-1}f(x) \\
&= \frac{\alpha+1}{2}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x) - \frac{1}{2}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2}f(x) \\
&- \frac{1}{2}|x|^{-2\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}f(x) - \frac{n}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)
\end{aligned}
$$

$$= -\frac{n-2\alpha-2}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x) - \frac{1}{2}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2} f(x)$$
$$- \frac{1}{2}|x|^{-2\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1} \partial_{|x|} f(x). \tag{3.1.15}$$

Now, calculate I_2:

$$\begin{aligned}
I_2 =& \frac{1}{2}|x|^{-\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1} T_{\alpha,0} f(x)\\
=& \left(\frac{1}{2}|x|^{-\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)\left(|x|^{-\alpha}\partial_{|x|} + \frac{n-2\alpha-2}{2}|x|^{-\alpha-1}\right) f(x)\\
=& \left(\frac{1}{2}|x|^{-2\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|} + \frac{n-2\alpha-2}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right) f(x).
\end{aligned} \tag{3.1.16}$$

Putting together (3.1.13), (3.1.15)- (3.1.16) in (3.1.14):

$$\begin{aligned}
&(T^*_{\alpha,1} T_{\alpha,1} f)(x)\\
&= -|x|^{-2\alpha}\partial^2_{|x|} f(x) - (n-\alpha-1)|x|^{-2\alpha-1}\partial_{|x|} f(x) -\\
&- \left(\left[\alpha(\alpha-n+2) + \left(\frac{n-2}{2}\right)^2\right] - \frac{n-2\alpha-2}{4}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)|x|^{-2\alpha-2} f(x)\\
&- \frac{1}{2}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2} f(x) - \frac{1}{2}|x|^{-2\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|} f(x)\\
&+ \left(\frac{1}{2}|x|^{-2\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|} + \frac{n-2\alpha-2}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right) f(x)\\
&+ \frac{1}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2} f(x) = -|x|^{-2\alpha}\partial^2_{|x|} f(x) - (n-\alpha-1)|x|^{-2\alpha-1}\partial_{|x|} f(x)\\
&- \left[\alpha(\alpha-n+2) + \left(\frac{n-2}{2}\right)^2\right]|x|^{-2\alpha-2} f(x) - \frac{1}{4}|x|^{-2\alpha-2}\left[\ln(\frac{\gamma}{|x|})\right]^{-2} f(x).
\end{aligned}$$

Finally $(T^*_{\alpha,1} T_{\alpha,1} f)(x)$, we have:

$$
\begin{aligned}
(T^*_{\alpha,1}T_{\alpha,1}f)(x) &= -|x|^{-2\alpha}\partial^2_{|x|}f(x) - (n-\alpha-1)|x|^{-2\alpha-1}\partial_{|x|}f(x)\\
&\quad - \left[\alpha(\alpha-n+2) + \left(\frac{n-2}{2}\right)^2\right]|x|^{-2\alpha-2}f(x)\\
&\quad - \frac{1}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2} f(x).
\end{aligned}
\tag{3.1.17}
$$

Next, we evaluate the following identity for $m := \mathbb{N}\cup\{0\}$:

$$
\begin{aligned}
&\partial_{|x|}|x|^{\alpha-1}\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x) - |x|^{-\alpha-1}\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}f(x)\\
&= -(\alpha+1)|x|^{-\alpha-2}\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x) + |x|^{-\alpha-1}\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}f(x)\\
&+ |x|^{-\alpha-1}\partial_{|x|}\left(\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right) f(x) - |x|^{-\alpha-1}\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}f(x)\\
&= -(\alpha+1)|x|^{-\alpha-2}\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x) + |x|^{-\alpha-2}\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2} f(x)\\
&+ |x|^{-\alpha-2}\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m-1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x).
\end{aligned}
\tag{3.1.18}
$$

where again note that $\sum_{j=1}^{0}(\cdot) := 0, \quad \prod_{k=1}^{0}(\cdot) = 1.$

Using (3.1.18), one can prove the following identity, which will be useful for establishing Theorem (3.1.1)

$$
\begin{aligned}
&T^*_{\alpha,m}|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x) + |x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1} T_{\alpha,m}f(x) =\\
&= A_1 + A_2 = -|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2} f(x), n\in\mathbb{N}, n\geqslant 2, \text{and } m\in\mathbb{N}_0.
\end{aligned}
\tag{3.1.19}
$$

First, one notes,

$$\begin{aligned}A_1 =&T_{m,\alpha}^{+}|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x) =\\&-|x|^{-\alpha}\partial_{|x|}\left(|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\right)-\\&-\frac{n}{2}|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\\&+\frac{1}{2}|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x),\end{aligned}\tag{3.1.20}$$

and,

$$\begin{aligned}A_2 =&|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}T_{m,\alpha}f(x) =\\&=|x|^{-\alpha}|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}f(x)\\&+\frac{n-2\alpha-2}{2}|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\\&+\frac{1}{2}|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x).\end{aligned}\tag{3.1.21}$$

Thus, applying (3.1.19) yields

$$\begin{aligned}&A_1+A_2=\\&T_{m,\alpha}^{+}|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)+|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}T_{m,\alpha}f(x)\\&=-|x|^{-\alpha}\left(\partial_{|x|}\left(|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)\right.\\&\left.+|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}\right)f(x)-(\alpha+1)|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\\&+|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\end{aligned}$$

$$= -|x|^{-\alpha}\left(-(\alpha+1)|x|^{\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right.$$

$$\left.+|x|^{-\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}\right)f(x)-(\alpha+1)|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\quad +|x|^{-}$$

$$= -|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}f(x),$$

which is (3.1.19).

Now, we return back to the induction, that is, using (3.1.19) and assuming that (3.1.12) is true for $m\in\mathbb{N}_0$, we prove it for $m+1$ as follows:

$$\begin{aligned}
(T^*_{\alpha,m+1}T_{\alpha,m+1}f)(x) &= \left(T^+_{\alpha,m}+\frac{1}{2}|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)\\
&\times\left(T_{\alpha,m}+\frac{1}{2}|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)f(x)\\
&= (T^+_{\alpha,m}T_{\alpha,m}f)(x)+\frac{1}{2}\left(T^+_m|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}+\right.\\
&\left.+|x|^{-\alpha-1}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}T_m\right)f(x)\\
&+\frac{1}{4}|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}f(x)\\
&= (T^+_{\alpha,m}T_{\alpha,m}f)(x)-\frac{1}{2}\left(|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}\right)f(x)\\
&+\frac{1}{4}|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}f(x)\\
&= (T^+_{\alpha,m}T_{\alpha,m}f)(x)-\frac{1}{4}|x|^{-2\alpha-2}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}f(x)\\
&= -|x|^{-2\alpha}\partial^2_{|x|}f(x)-(n-\alpha-1)|x|^{-2\alpha-1}\partial_{|x|}f(x)
\end{aligned}$$

$$
\begin{aligned}
&- \left[\alpha(\alpha-n+2)+\left(\frac{n-2}{2}\right)^{2}\right]|x|^{-2\alpha-2}f(x) \\
&- \frac{1}{4}|x|^{-2\alpha-2}\sum_{j=1}^{m+1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2} f(x).
\end{aligned} \tag{3.1.22}
$$

Thus, we have proved (3.1.12). By the non-negativity of $\left(T^*_{\alpha,m}T_{\alpha,m}f\right)(x)$, one calculates,

$$
0 \leq \int_\Omega \left|\left(T_{\alpha,m}f\right)(x)\right|^2 dx = \int_\Omega \overline{f(x)}\left(T^*_{\alpha,m}T_{\alpha,m}f\right)(x)dx, \tag{3.1.23}
$$

where $T_{\alpha,m}$ and $T^*_{\alpha,m}$as in (3.1.7), (3.1.9), respectively. For simplicity we will work with $f \in C_0^\infty\left(B(0;\rho)\backslash\{0\}\right)$ for $m \in \mathbb{N}$.

The (3.1.23) takes the form:

$$
\begin{aligned}
0 &\leq \int_\Omega \overline{f(x)}\left(T^*_{\alpha,m}T_{\alpha,m}f\right)(x)dx \\
&= \int_\Omega \overline{f(x)}\Bigg(-|x|^{-2\alpha}\partial^2_{|x|}f(x) - (n-\alpha-1)|x|^{-2\alpha-1}\partial_{|x|}f(x) \\
&- \left[\alpha(\alpha-n+2)+\left(\frac{n-2}{2}\right)^{2}\right]|x|^{-2\alpha-2}f(x) \\
&-\frac{1}{4}|x|^{-2\alpha-2}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2} f(x)\Bigg)dx \\
&= -\int_\Omega \overline{f(x)}|x|^{-2\alpha}\partial^2_{|x|}f(x)dx - (n-\alpha-1)\int_\Omega \overline{f(x)}|x|^{-2\alpha-1}\partial_{|x|}f(x)dx \\
&- \left[\alpha(\alpha-n+2)+\left(\frac{n-2}{2}\right)^{2}\right]\int_\Omega \overline{f(x)}|x|^{-2\alpha-2}f(x)dx \\
&- \frac{1}{4}\int_\Omega \overline{f(x)}|x|^{-2\alpha-2}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2} f(x)dx.
\end{aligned} \tag{3.1.24}
$$

Considering the identity,

$$
\begin{aligned}
&\int_\Omega \overline{f(x)}|x|^{-2\alpha}\left(\partial^2_{|x|}f\right)(x)dx = -\int_\Omega |x|^{-2\alpha}\left|\left(\partial_{|x|}f\right)(x)\right|^2 dx \\
&\quad - (n-\alpha-1)\int_\Omega |x|^{-2\alpha-1}\overline{f(x)}\left(\partial_{|x|}f\right)(x)dx, \quad f \in C_0^\infty\left(B_n(0;\rho)\right),
\end{aligned} \tag{3.1.25}
$$

(3.1.24) has a form,

$$
\begin{aligned}
0 \leqslant & \int_\Omega \overline{f(x)} \left(T^*_{\alpha,m} T_{\alpha,m} f\right)(x) dx \\
& = \int_\Omega |x|^{-2\alpha} \left|\left(\partial_{|x|} f\right)(x)\right|^2 dx + (n-\alpha-1) \int_\Omega |x|^{-2\alpha-1} \overline{f(x)} \left(\partial_{|x|} f\right)(x) dx \\
& - (n-\alpha-1) \int_\Omega \overline{f(x)} |x|^{-2\alpha-1} \partial_{|x|} f(x) dx \\
& - \left[\alpha(\alpha-n+2) + \left(\frac{n-2}{2}\right)^2\right] \int_\Omega \overline{f(x)} |x|^{-2\alpha-2} f(x) dx \\
& - \frac{1}{4} \int_\Omega \overline{f(x)} |x|^{-2\alpha-2} \sum_{j=1}^m \prod_{k=1}^j \left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2} f(x) dx \\
& = \int_\Omega |x|^{-2\alpha} \left|\left(\partial_{|x|} f\right)(x)\right|^2 dx \\
& - \left[\alpha(\alpha-n+2) + \frac{(n-2)^2}{4}\right] \int_\Omega |x|^{-2\alpha-2} |f(x)|^2 dx \\
& - (1/4) \sum_{j=1}^m \int_\Omega |x|^{-2\alpha-2} |f(x)|^2 \prod_{k=1}^j \left[\ln_k(\gamma/|x|)\right]^{-2} dx,
\end{aligned}
\tag{3.1.26}
$$

implying,

$$
\begin{aligned}
\int_\Omega |x|^{-2\alpha} |(\partial_{|x|} f)(x)|^2 dx \geqslant & \left[\alpha(\alpha-n+2) + \frac{(n-2)^2}{4}\right] \int_\Omega |x|^{-2\alpha-2} |f(x)|^2 dx \\
& + \frac{1}{4} \sum_{j=1}^m \int_\Omega \prod_{k=1}^j \left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2} |x|^{-2\alpha-2} |f(x)|^2 dx.
\end{aligned}
\tag{3.1.27}
$$

□

Next, we prove Theorem (3.1.4) using non-negativity $(T_{\alpha,m} T^*_{\alpha,m} f)(x)$.

Proof of Theorem (3.1.4).

$$
0 \leq \int_\Omega \left|\left(T^*_{\alpha,m} f\right)(x)\right|^2 dx = \int_\Omega \overline{f(x)} \left(T_{\alpha,m} T^*_{\alpha,m} f\right)(x) dx. \tag{3.1.28}
$$

Let us now show that

$$(T_{\alpha,m}T^*_{\alpha,m}f)(x) = -|x|^{-2\alpha}(\partial^2_{|x|}f)(x) - (n-\alpha-1)|x|^{2\alpha+1}(\partial_{|x|}f)(x)$$
$$\left(-\frac{n(n-4\alpha-4)}{4}|x|^{-2\alpha-2} + \frac{3}{4}|x|^{-2\alpha-2}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2}\right) f(x)$$
$$-(\alpha+1)|x|^{2\alpha+2}\sum_{j=1}^{m}\prod_{k=1}^{j}\ln\left[\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x)$$
$$+|x|^{2\alpha+2}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m-1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x). \tag{3.1.29}$$

For this we use induction on $m \in \mathbb{N}_0$. For $m = 0$ we have,

$$(T_0T_0^*f)(x) = \left(|x|^{-\alpha}\partial_{|x|} + \frac{n-2\alpha-2}{2}|x|^{-\alpha-1}\right)\left(-|x|^{-\alpha}(\partial_{|x|}f) - \frac{n}{2}|x|^{-\alpha-1}f\right)(x)$$
$$= -|x|^{-2\alpha}(\partial^2_{|x|}f)(x) - \frac{n}{2}|x|^{-\alpha}\partial_{|x|}\left(|x|^{-\alpha-1}f(x)\right) - \frac{n-2\alpha-2}{2}|x|^{-2\alpha-1}\partial_{|x|}f(x)$$
$$-\frac{n(n-4\alpha-4)}{4}|x|^{-2\alpha-2}f(x) = -|x|^{-2\alpha}(\partial^2_{|x|}f)(x) - \frac{n}{2}|x|^{-\alpha}\left((-\alpha-1)|x|^{-\alpha-2}f\right.$$
$$\left.+|x|^{-\alpha-1}\partial_{|x|}f\right)(x) - \frac{n-2\alpha-2}{2}|x|^{-2\alpha-1}\partial_{|x|}f(x) - \frac{n(n-4\alpha-4)}{4}|x|^{-2\alpha-2}f(x)$$
$$= \left(-|x|^{-2\alpha}(\partial^2_{|x|}f) - (n-\alpha-1)|x|^{-2\alpha-1}(\partial_{|x|}f) - \frac{n(n-4\alpha-4)}{4}|x|^{-2\alpha-2}f\right)(x).$$

Finally for $(T_0T_0^*f)(x)$:

$$(T_0T_0^*f)(x)$$
$$= \left(-|x|^{-2\alpha}(\partial^2_{|x|}f) - (n-\alpha-1)|x|^{-2\alpha-1}(\partial_{|x|}f) - \frac{n(n-4\alpha-4)}{4}|x|^{-2\alpha-2}f\right)(x). \tag{3.1.30}$$

For $m = 1$, by a direct computation we get

$$(T_{\alpha,1}T^*_{\alpha,1}f)(x)$$
$$= \left(T_{\alpha,0} + \frac{1}{2|x|^{\alpha+1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]}\right)\left((T^*_{\alpha,0}f)(x) + \frac{1}{2}\frac{f(x)}{|x|^{\alpha+1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]}\right)$$
$$= (T_{\alpha,0}T^*_{\alpha,0}f)(x) + T_{\alpha,0}\left(\frac{1}{2}\frac{f(x)}{|x|^{\alpha+1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]}\right)$$

$$
\begin{aligned}
&+\left(\frac{1}{2|x|^{\alpha+1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]}\right)(T^*_{\alpha,0}f)(x)+\frac{1}{4}\frac{f(x)}{|x|^{2\alpha+2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^2}\\
&=(T_{\alpha,0}T^*_{\alpha,0}f)(x)+J_1+J_2+\frac{1}{4}\frac{f(x)}{|x|^{2\alpha+2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^2}.
\end{aligned}
\tag{3.1.31}
$$

Let is direct calculate J_1:

$$
\begin{aligned}
J_1 &= T_{\alpha,0}\left(\frac{1}{2}\frac{f(x)}{|x|^{\alpha+1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]}\right)\\
&=\left(|x|^{-\alpha}\partial_{|x|}+\frac{n-2\alpha-2}{2}|x|^{-\alpha-1}\right)\left(\frac{1}{2}\frac{f(x)}{|x|^{\alpha+1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]}\right)\\
&=|x|^{-\alpha}\partial_{|x|}\left(\frac{1}{2}\frac{f(x)}{|x|^{\alpha+1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]}\right)+\frac{n-2\alpha-2}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\\
&=\frac{1}{2}|x|^{-\alpha}\left((-\alpha-1)|x|^{-\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}+|x|^{-\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2}\right.\\
&\left.+|x|^{-\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}\right)f(x)+\frac{n-2\alpha-2}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\\
&=-\frac{\alpha+1}{2}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)+\frac{1}{2}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2}f(x)\\
&+\frac{1}{2}|x|^{-2\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}f(x)+\frac{n-2\alpha-2}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\\
&=\frac{n-4\alpha-4}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)+\frac{1}{2}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2}f(x)\\
&+\frac{1}{2}|x|^{-2\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}f(x).
\end{aligned}
\tag{3.1.32}
$$

Now, J_2:

$$
\begin{aligned}
J_2 &= \left(\frac{1}{2|x|^{\alpha+1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]}\right)(T^*_{\alpha,0}f)(x) \\
&= \left(\frac{1}{2|x|^{\alpha+1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]}\right)\left(-|x|^{-\alpha}(\partial_{|x|}f) - \frac{n}{2}|x|^{-\alpha-1}f\right)(x) \\
&= -\frac{1}{2}|x|^{-2\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}f(x) - \frac{n}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x).
\end{aligned}
\tag{3.1.33}
$$

Putting together (3.1.30),(3.1.32) - (3.1.33) in (3.1.31):

$$
\begin{aligned}
&(T_{\alpha,1}T^*_{\alpha,1}f)(x) \\
&= \left(-|x|^{-2\alpha}(\partial^2_{|x|}f) - (n-\alpha-1)|x|^{-2\alpha-1}(\partial_{|x|}f) - \frac{n(n-4\alpha-4)}{4}|x|^{-2\alpha-2}f\right)(x) \\
&+ \frac{n-4\alpha-4}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x) + \frac{1}{2}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2} f(x) \\
&+ \frac{1}{2}|x|^{-2\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}f(x) - \frac{1}{2}|x|^{-2\alpha-1}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1}\partial_{|x|}f(x) \\
&- \frac{n}{4}|x|^{-2\alpha-2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-1} f(x) + \frac{1}{4}\frac{f(x)}{|x|^{2\alpha+2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^2} \\
&= -\left(|x|^{-2\alpha}\partial^2_{|x|}f - (n-\alpha-1)|x|^{-2\alpha-1}(\partial_{|x|}f) - \frac{n(n-4\alpha-4)}{4}|x|^{-2\alpha-2}f\right)(x) \\
&+ \frac{3f(x)}{4|x|^{2\alpha+2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^2} - (\alpha+1)\frac{f(x)}{|x|^{2\alpha+2}\ln\left(\frac{\gamma}{|x|}\right)}.
\end{aligned}
$$

Finally $(T_{\alpha,1}T^*_{\alpha,1}f)(x)$, we have:

$$
\begin{aligned}
(T_{\alpha,1}T^*_{\alpha,1}f)(x) = &-\left(|x|^{-2\alpha}\partial^2_{|x|}f - (n-\alpha-1)|x|^{-2\alpha-1}(\partial_{|x|}f)\right)(x) \\
&- \frac{n(n-4\alpha-4)}{4}|x|^{-2\alpha-2}f(x) \\
&+ \frac{3f(x)}{4|x|^{2\alpha+2}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^2} - (\alpha+1)\frac{f(x)}{|x|^{2\alpha+2}\ln\left(\frac{\gamma}{|x|}\right)}.
\end{aligned}
\tag{3.1.34}
$$

Assuming that (3.1.29) is true for $m \in \mathbb{N}_0$, we will prove it for $m+1$. Before this, let us show the following identity, which will be useful in this proof:

$$
\begin{aligned}
&T_{\alpha,m}\left(\frac{f}{|x|^{\alpha+1}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)(x)+\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\frac{(T^*_{\alpha,m}f)(x)}{|x|^{\alpha+1}}\\
&\qquad=-\frac{2(\alpha+1)f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\\
&\qquad+\frac{2f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\\
&\qquad+\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}. \quad (3.1.35)
\end{aligned}
$$

To show this, we first observe that

$$
\begin{aligned}
&T_{\alpha,m}\left(\frac{f}{|x|^{\alpha+1}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)(x)\\
&=|x|^{-\alpha}\partial_{|x|}\left(\frac{f}{|x|^{\alpha+1}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)(x)\\
&+\frac{n-2\alpha-2}{2}\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\\
&+\frac{1}{2}\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}
\end{aligned}
$$

and,

$$
\begin{aligned}
&\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\frac{(T^*_{\alpha,m}f)(x)}{|x|^{\alpha+1}}=\\
&-\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)^{-1}\right]\frac{(\partial_{|x|}f)(x)}{|x|^{2\alpha+1}}\\
&-\frac{n}{2}\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)^{-1}\right]\\
&+\frac{1}{2}\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}.
\end{aligned}
$$

Thus applying (3.1.18) yields

$$
\begin{aligned}
&J_1 + J_2\\
&= |x|^{-\alpha}\partial_{|x|}\left(\frac{f}{|x|^{\alpha+1}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)(x)\\
&+ \frac{n-2\alpha-2}{2}\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\\
&+ \frac{1}{2}\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\\
&- \prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)^{-1}\right]\frac{(\partial_{|x|}f)(x)}{|x|^{2\alpha+1}} - \frac{n}{2}\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)^{-1}\right]\\
&+ \frac{1}{2}\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\\
&= |x|^{-\alpha}\left[\partial_{|x|}\left(\frac{f}{|x|^{\alpha+1}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right) - \prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)^{-1}\right]\frac{(\partial_{|x|}f)}{|x|^{2\alpha+1}}\right](x)\\
&- (\alpha+1)\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\\
&+ \frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\\
&= |x|^{-\alpha}\left[-(\alpha+1)|x|^{-\alpha-2}\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f + |x|^{-\alpha-2}\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}f\right.\\
&\left.+|x|^{-\alpha-2}\prod_{k=1}^{m}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m-1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f\right](x)\\
&- (\alpha+1)\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}
\end{aligned}
$$

$$+\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}$$

$$=-\frac{2(\alpha+1)f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}$$

$$+\frac{2f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}$$

$$+\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}, \quad (3.1.36)$$

which is (3.1.35).

Again, we return back to the mathematical induction method, that is using (3.1.35) and assuming that (3.1.29) is true for $m\in\mathbb{N}_0$, we prove it for $m+1$ as follows:

$$(T_{\alpha,m+1}T^*_{\alpha,m+1}f)(x)=\left(T_{\alpha,m}+\frac{1}{2|x|^{\alpha+1}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)$$

$$\times\left((T^*_{\alpha,m}f)(x)+\frac{f(x)}{2|x|^{\alpha+1}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)$$

$$=(T_{\alpha,m}T^*_{\alpha,m}f)(x)+\frac{1}{2}T_m\left(\frac{f(x)}{|x|^{\alpha+1}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)$$

$$+\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\frac{(T^*_mf)(x)}{2|x|^{\alpha+1}}+\frac{f(x)}{4|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}$$

$$=(T_{\alpha,m}T^*_{\alpha,m}f)(x)+\frac{1}{2}\left[T_{\alpha,m}\left(\frac{f}{|x|^{\alpha+1}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right)(x)+\right.$$

$$\left.\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\frac{(T^*_{\alpha,m}f)(x)}{|x|^{\alpha+1}}\right]+\frac{f(x)}{4|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}$$

$$=(T_{\alpha,m}T^*_{\alpha,m}f)(x)+\frac{1}{2}\left[-\frac{2(\alpha+1)f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\right.$$

$$+\frac{2f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}$$

$$
\begin{aligned}
&+\frac{f(x)}{2|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}+\frac{f(x)}{4|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}\\
&=-|x|^{-2\alpha}(\partial^2_{|x|}f)(x)-(n-\alpha-1)|x|^{-2\alpha-1}(\partial_{|x|}f)(x)\\
&\left(-\frac{n(n-4\alpha-4)}{4}|x|^{-2\alpha-2}+\frac{3}{4}|x|^{-2\alpha-2}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln\left(\frac{\gamma}{|x|}\right)\right]^{-2}\right)f(x)\\
&-(\alpha+1)|x|^{-2\alpha-2}\sum_{j=1}^{m}\prod_{k=1}^{j}\ln\left[\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\\
&+|x|^{-2\alpha-2}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m-1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}f(x)\\
&-\frac{(\alpha+1)f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}+\frac{f(x)}{2|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}\\
&+\frac{f(x)}{|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\\
&+\frac{f(x)}{4|x|^{2\alpha+2}}\prod_{k=1}^{m+1}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}=-|x|^{-2\alpha}(\partial^2_{|x|}f)(x)\\
&-(n-\alpha-1)|x|^{-2\alpha-1}(\partial_{|x|}f)(x)-\frac{n(n-4\alpha-4)}{4}\frac{f(x)}{|x|^{2\alpha+2}}\\
&+\frac{3f(x)}{4|x|^{2\alpha+2}}\sum_{j=1}^{m+1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}-\frac{(\alpha+1)f(x)}{|x|^{2\alpha+2}}\sum_{j=1}^{m+1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\\
&+\frac{f(x)}{|x|^{2\alpha+2}}\sum_{j=1}^{m+1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}.
\end{aligned}
\tag{3.1.37}
$$

Thus, we have proved (3.1.29). Then, plugging this into (3.1.28) we get

$$
\begin{aligned}
0\le&-\int_\Omega |x|^{-2\alpha}\overline{f(x)}(\partial^2_{|x|}f)(x)dx-(n-\alpha-1)\int_\Omega\frac{\overline{f(x)}}{|x|^{2\alpha+1}}(\partial_{|x|}f)(x)dx\\
&-\frac{n(n-4\alpha-4)}{4}\int_\Omega\frac{|f(x)|^2}{|x|^{2\alpha+2}}dx+\frac{3}{4}\sum_{j=1}^{m}\int_\Omega\frac{|f(x)|^2}{|x|^{2\alpha+2}}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}dx
\end{aligned}
$$

$$
\begin{aligned}
&-(\alpha+1)\sum_{j=1}^{m}\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}dx\\
&+\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m-1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}dx\\
&=\int_{\Omega}|x|^{-2\alpha}|(\partial_{|x|}f)(x)|^2dx-\frac{n(n-4\alpha-4)}{4}\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}dx\\
&+\frac{3}{4}\sum_{j=1}^{m}\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}dx-\sum_{j=1}^{m}\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}dx\\
&+\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m-1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}dx.
\end{aligned}
$$

More concretely,

$$
\begin{aligned}
0\leq&\int_{\Omega}|x|^{-2\alpha}|(\partial_{|x|}f)(x)|^2dx-\frac{n(n-4\alpha-4)}{4}\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}dx\\
&+\frac{3}{4}\sum_{j=1}^{m}\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}dx-\sum_{j=1}^{m}\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}dx\\
&\qquad+\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m-1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}dx,
\end{aligned}
$$

or,

$$
\begin{aligned}
&\int_{\Omega}|x|^{-2\alpha}|(\partial_{|x|}f)(x)|^2dx\geqslant\frac{n(n-4\alpha-4)}{4}\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}dx\\
&-\frac{3}{4}\sum_{j=1}^{m}\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-2}dx+\sum_{j=1}^{m}\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}dx\\
&\qquad-\int_{\Omega}\frac{|f(x)|^2}{|x|^{2\alpha+2}}\sum_{j=1}^{m}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}\sum_{j=1}^{m-1}\prod_{k=1}^{j}\left[\ln_k\left(\frac{\gamma}{|x|}\right)\right]^{-1}dx,
\end{aligned}
$$

yielding (3.1.4).

$\square$

4. Application

4.1 L^2-Caffarelli–Kohn–Nirenberg inequalities

In this section, we establish Caffarelli-Cohn-Nirenberg type inequalities and their proofs. The proof is based on the Hardy inequalities and Holder's inequality.

Let us begin by recalling the classical Caffarelli-Kohn-Nirenberg inequality from [19]:

Theorem 4.1.1. *Let $n \in \mathbb{N}$ and let p, q, r, a, b, d, $\delta \in \mathbb{R}$ such that $p, q \geq 1$, $r > 0$, $0 \leq \delta \leq 1$, and*

$$\frac{1}{p} + \frac{a}{n}, \frac{1}{q} + \frac{b}{n}, \frac{1}{r} + \frac{c}{n} > 0, \tag{4.1.1}$$

where $c = \delta d + (1-\delta)b$. Then there exists a positive constant C such that

$$\||x|_E^c f\|_{L^r(\mathbb{R}^n)} \leq C \||x|_E^a |\nabla f|\|_{L^p(\mathbb{R}^n)}^{\delta} \||x|_E^b f\|_{L^q(\mathbb{R}^n)}^{1-\delta}, \tag{4.1.2}$$

holds for all $f \in C_0^\infty(\mathbb{R}^n)$, if and only if the following conditions hold:

$$\frac{1}{r} + \frac{c}{n} = \delta \left(\frac{1}{p} + \frac{a-1}{n}\right) + (1-\delta)\left(\frac{1}{q} + \frac{b}{n}\right), \tag{4.1.3}$$

$$a - d \geq 0 \quad \text{if} \quad \delta > 0, \tag{4.1.4}$$

$$a - d \leq 1 \quad \text{if} \quad \delta > 0 \quad \text{and} \quad \frac{1}{r} + \frac{c}{n} = \frac{1}{p} + \frac{a-1}{n}. \tag{4.1.5}$$

We actually obtain the Hardy inequality as (3.1.3) and (3.1.4), which plays an important role in obtaining Theorem (4.1.6).

Theorem 4.1.2. *Let $1 < q < \infty$ with $2 + q \geq r$ and $\delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{2}{r}\right]$ and $b, c \in \mathbb{R}$. Assume that $\frac{\delta r}{2} + \frac{(1-\delta)r}{q} = 1$ and $c = -\delta + b(1-\delta)$. Let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{R}^n$. Then for all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ we have the following Caffarelli-Kohn-Nirenberg type inequalities:*

$$\|\omega^c f\|_{L^r(\mathbb{R}^n)} \leq \left\||x|^{-\alpha} \partial_{|x|} f\right\|_{L^2(\mathbb{R}^n)}^{\delta} \left\|\omega^b f\right\|_{L^q(\mathbb{R}^n)}^{1-\delta}, \tag{4.1.6}$$

where,

$$\omega = \frac{\left(\alpha(\alpha-n+2)+\frac{(n-2)^2}{4}+\frac{1}{4}\sum_{j=1}^{m}\prod_{k=1}^{j}\left(\ln_k\frac{\gamma}{|x|}\right)^{-2}\right)^{\frac{1}{2}}}{|x|^{\alpha}}.$$

Case $\delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{2}{r}\right]$. *Recalling* $c = \delta + b(1-\delta)$, *a direct calculation gives*

$$\|\omega^c f\|_{L^r(\mathbb{R}^n)} = \left(\int_{\mathbb{G}} |\omega|^{cr}|f(x)|^r dx\right)^{\frac{1}{r}} = \left(\int_{\mathbb{G}} \frac{|f(x)|^{\delta r}}{|\omega|^{-\delta r}}\cdot\frac{|f(x)|^{(1-\delta)r}}{|\omega|^{-br(1-\delta)}}dx\right)^{\frac{1}{r}}. \tag{4.1.7}$$

We want to use Holder's inequality for $\frac{\delta r}{2}+\frac{(1-\delta)r}{q}=1$ *and* $2+q\geq r$. *Then using Hölder's inequality in* (4.1.7) *we get following:*

$$\|\omega^c f\|_{L^r(\mathbb{R}^n)} \leq \left(\int_{\mathbb{G}} \frac{|f(x)|^2}{|\omega|^{-2}}dx\right)^{\frac{\delta}{2}}\left(\int_{\mathbb{G}} \frac{|f(x)|^q}{|\omega|^{-bq}}dx\right)^{\frac{1-\delta}{q}} = \|\omega f\|^{\delta}_{L^2(\mathbb{R}^n)}\left\|\frac{f}{\omega^{-b}}\right\|^{1-\delta}_{L^q(\mathbb{R}^n)}. \tag{4.1.8}$$

From Theorem (3.1.3) *we have*

$$\left\||x|^{-\alpha}\partial_{|x|}f\right\|_{L^2(\mathbb{R}^n)} \tag{4.1.9}$$
$$\geq \left\|\frac{f(x)}{|x|^{\alpha}}\left(\alpha(\alpha-Q+2)+\frac{(Q-2)^2}{4}+\frac{1}{4}\sum_{j=1}^{m}\prod_{k=1}^{j}\left(\ln_k\frac{\gamma}{|x|}\right)^{-2}\right)^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^n)}. \tag{4.1.10}$$

Recalling

$$\omega(x) = \frac{\left(\alpha(\alpha-Q+2)+\frac{(Q-2)^2}{4}+\frac{1}{4}\sum_{j=1}^{m}\prod_{k=1}^{j}\left(\ln_k\frac{\gamma}{|x|}\right)^{-2}\right)^{\frac{1}{2}}}{|x|^{\alpha}}, \tag{4.1.11}$$

we can rewrite it as

$$\left\||x|^{-\alpha}\partial_{|x|}f\right\|_{L^2(\mathbb{R}^n)} \geq \|\omega f\|_{L^2(\mathbb{R}^n)} \tag{4.1.12}$$

Using (4.1.12) *in* (4.1.8) *gives the desired inequality* (4.1.6).

Theorem 4.1.3. *Let* $1 < q < \infty$ *with* $2+q \geq r$ *and* $\delta \in [0,1]\cap\left[\frac{r-q}{r},\frac{2}{r}\right]$ *and* $b, c \in \mathbb{R}$. *Assume that* $\frac{\delta r}{2}+\frac{(1-\delta)r}{q}=1$ *and* $c = -\delta + b(1-\delta)$. *Let* $|\cdot|$ *be any* *homogeneous quasi-norm on* $\mathbb{R}^n$. *Then for all* $f \in C_0^{\infty}(\mathbb{R}^n \backslash \{0\})$ *we have the following Caffarelli-Kohn-Nirenberg type inequalities:*

$$\|\omega^c f\|_{L^r(\mathbb{R}^n)} \leq \left\||x|^{-\alpha}\partial_{|x|} f\right\|^{\delta}_{L^2(\mathbb{R}^n)} \left\|\omega^b f\right\|^{1-\delta}_{L^q(\mathbb{R}^n)}, \tag{4.1.13}$$

where,

$$\omega = \frac{\left(\frac{n(n-4\alpha-4)}{4} - \frac{3}{4}\sum_{j=1}^m \prod_{k=1}^j \left(\ln_k \frac{\gamma}{|x|}\right)^{-2} + \sum_{j=1}^m \prod_{k=1}^j \left(\ln_k \frac{\gamma}{|x|}\right)^{-1}\right)^{\frac{1}{2}}}{|x|^{\alpha}}$$

$$- \frac{\left(\sum_{j=1}^m \prod_{k=1}^j \left(\ln_k \frac{\gamma}{|x|}\right)^{-1} \sum_{j=1}^{m-1} \prod_{k=1}^j \left(\ln_k \frac{\gamma}{|x|}\right)^{-1}\right)^{\frac{1}{2}}}{|x|^{\alpha}}.$$

Case $\delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{2}{r}\right]$. *Recalling* $c = \delta + b(1-\delta)$, *a direct calculation gives*

$$\|\omega^c f\|_{L^r(\mathbb{R}^n)} = \left(\int_{\mathbb{G}} |\omega|^{cr} |f(x)|^r dx\right)^{\frac{1}{r}} = \left(\int_{\mathbb{G}} \frac{|f(x)|^{\delta r}}{|\omega|^{-\delta r}} \cdot \frac{|f(x)|^{(1-\delta)r}}{|\omega|^{-br(1-\delta)}} dx\right)^{\frac{1}{r}}. \tag{4.1.14}$$

We want to use Holder's inequality for $\frac{\delta r}{2} + \frac{(1-\delta)r}{q} = 1$ *and* $2 + q \geq r$. *Then using Hölder's inequality in* (4.1.14) *we get following:*

$$\|\omega^c f\|_{L^r(\mathbb{R}^n)} \leq \left(\int_{\mathbb{G}} \frac{|f(x)|^2}{|\omega|^{-2}} dx\right)^{\frac{\delta}{2}} \left(\int_{\mathbb{G}} \frac{|f(x)|^q}{|\omega|^{-bq}} dx\right)^{\frac{1-\delta}{q}} = \|\omega f\|^{\delta}_{L^2(\mathbb{R}^n)} \left\|\frac{f}{\omega^{-b}}\right\|^{1-\delta}_{L^q(\mathbb{R}^n)}. \tag{4.1.15}$$

From Theorem (3.1.4) *we have*

$$\left\||x|^{-\alpha}\partial_{|x|} f\right\|_{L^2(\mathbb{R}^n)} \geq \left\| \frac{f(x)}{|x|^{\alpha}} \left(\frac{n(n-4\alpha-4)}{4} - \frac{3}{4}\sum_{j=1}^m \prod_{k=1}^j \left(\ln_k \frac{\gamma}{|x|}\right)^{-2} \right.\right. \tag{4.1.16}$$

$$+ \sum_{j=1}^m \prod_{k=1}^j \left(\ln_k \frac{\gamma}{|x|}\right)^{-1} \tag{4.1.17}$$

$$\left.\left. - \sum_{j=1}^m \prod_{k=1}^j \left(\ln_k \frac{\gamma}{|x|}\right)^{-1} \sum_{j=1}^{m-1} \prod_{k=1}^j \left(\ln_k \frac{\gamma}{|x|}\right)^{-1}\right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^n)} \tag{4.1.18}$$

Recalling

$$\omega = \frac{\left(\frac{n(n-4\alpha-4)}{4} - \frac{3}{4}\sum_{j=1}^m \prod_{k=1}^j \left(\ln_k \frac{\gamma}{|x|}\right)^{-2} + \sum_{j=1}^m \prod_{k=1}^j \left(\ln_k \frac{\gamma}{|x|}\right)^{-1}\right)^{\frac{1}{2}}}{|x|^{\alpha}}$$

$$-\frac{\left(\sum_{j=1}^{m}\prod_{k=1}^{j}\left(\ln_k\frac{\gamma}{|x|}\right)^{-1}\sum_{j=1}^{m-1}\prod_{k=1}^{j}\left(\ln_k\frac{\gamma}{|x|}\right)^{-1}\right)^{\frac{1}{2}}}{|x|^{\alpha}}, \tag{4.1.19}$$

we can rewrite it as

$$\left\||x|^{-\alpha}\partial_{|x|}f\right\|_{L^2(\mathbb{R}^n)} \geq \|\omega f\|_{L^2(\mathbb{R}^n)} \tag{4.1.20}$$

Using (4.1.20) *in* (4.1.15) *gives the desired inequality* (4.1.13).

5. Conclusions and future work

In conclusion, this research has provided a comprehensive study of Hardy inequalities, which have captivated mathematicians for over a century. By utilizing the method of factorizations of differential operators, we have derived improved Hardy-type inequalities, extending the classical results originally introduced by G. H. Hardy in 1918.

Our work unifies sub-critical and critical cases of Hardy inequalities with any homogeneous quasi-norm, both in the Euclidean setting and on general homogeneous Lie groups in [15]. Building upon the advancements by Gesztesy and others, we demonstrated the effectiveness of the factorization method in obtaining refined inequalities and explored its applications in various mathematical and physical contexts. This research not only enhances our understanding of Hardy inequalities but also opens new avenues for further exploration and application in the field of mathematical analysis.

Finally, in this note we have established a unified result that contains both the critical and subcritical Hardy inequalities. Moreover, as a byproduct, a Caffarelli-Kohn-Nirenberg type inequality is obtained. This work is based on the conferences at [20], [21], [22], [23], [24] and was presented at the International Emerging Scholars Conference - 2024 on "Towards Using Generative AI in Academic Research".

In future work, we plan to investigate the applications of the improved Hardy inequality developed in this thesis to more general classes of partial differential equations and operators as in [17], [25]. Specifically, we aim to explore the implications of this inequality for the lower semiboundedness and form boundedness of Schrödinger operators with strongly singular potentials. One promising direction would be to study how the best constant b in the inequality:

$$-\Delta \geqslant \left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2} + bV(x)$$

depends on the geometry and distribution of the singular points. Can sharper constants be obtained by tailoring the inequality to the specific configuration of singularities?

Furthermore, we would like to understand how this inequality can be leveraged to establish essential self-adjointness, spectral properties, and resolvent estimates for the corresponding Schrödinger operators. Another natural extension would be to consider more general nonlocal or fractional differential operators in place of the Laplacian, and to investigate how the improved Hardy inequality and its applications can be generalized to these settings. This could lead to new insights into the interplay between the structure of singular potentials and the properties of nonlocal Schrödinger-type operators. Overall, the techniques developed in this thesis provide a promising foundation for further advancing the mathematical understanding of partial differential equations and operators with strongly singular interactions. We anticipate that exploring these future research directions will yield valuable new theoretical results with potential applications in physics, quantum mechanics, and beyond.

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