Remarkable triangle lines and how to solve complex problems associated with these problems

Zhandos Bekish

A thesis submitted to the Faculty of Education and Humanities
in partial fulfilment of the requirements for the degree of

MASTER OF PEDAGOGICAL SCIENCES
in Mathematics

«SDU University»
Department of Pedagogy of Natural Sciences

Thesis Thesis Advisor: PhD, Associate
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Faculty of Education and Humanities
Department of Pedagogy of Natural Sciences

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List of Abbreviations

FL - Flipped Learning
STEM - Science, Technology, Engineering, and Mathematics
CTML - Cognitive Theory of Multimedia Learning
LMS - Learning Management System

ABSTRACT

This dissertation addresses a long-standing issue in secondary mathematics: students often receive limited practice in geometry due to a strong focus on algebraic skills and theoretical lectures. Geometry, however, requires active engagement, spatial imagination, and the synthesis of multiple theorems, making it vital for developing well-rounded mathematical thinkers. We propose a dual approach: (1) presenting a collection of complex geometry problems covering various essential theorems and advanced methods, and (2) implementing a Flipped Learning model that transfers theoretical lecture content to video tutorials, thereby reclaiming classroom time for deeper problem-solving activities. Our research involves multiple teachers and student groups, comparing traditional in- struction with Flipped Learning. We measure the outcomes through performance tests, surveys, and teacher observations, paying special attention to how this approach impacts geometry learning and overall satisfaction. By combining innovative pedagogy with rigorously chosen geometry tasks, this study demonstrates how educators can address the typical time constraint problem and enhance students' comprehension of Euclidean geometry.

Keywords: Flipped Learning, Geometry Problem-Solving, High School Mathematics, Peda- gogical Strategies, Active Learning, Mathematical Reasoning

АҢДАТПА

Бұл диссертация орта мектеп математикасындағы көптен бері шешілмеген мәселені қарастырады: оқушылардың геометрия бойынша жаттығулары шектеулі болып қалады, өйткені оқыту көбінесе алгебралық дағдылар мен теориялық дәрістерге негізделген. Алайда, геометрия белсенді қатысуды, кеңістіктік ойлауды және бірнеше теоремаларды синтездеуді талап етеді, бұл оны жан-жақты математикалық ойлауды қалыптастыру үшін аса маңызды етеді.

Осы зерттеуде біз екіжақты тәсілді ұсынамыз: (1) негізгі теоремалар мен жетілдірілген әдістерді қамтитын күрделі геометриялық есептер жинағын әзірлеу және (2) теориялық лекцияларды бейнесабақтарға ауыстыру арқылы аудиториялық уақытты терең есептер шығаруға бағыттайтын "аударылған оқу" (Flipped Learning) моделін енгізу.

Зерттеу бірнеше мұғалім мен оқушылар тобын қамтиды, дәстүрлі оқыту мен аударылған оқыту әдістерін салыстырады. Нәтижелер білімді бағалау тесттері, сауалнамалар және мұғалімдердің бақылаулары арқылы өлшенеді, ерекше назар геометрияны меңгеру мен оқушылардың қанағаттану деңгейіне аударылады. Инновациялық педагогика мен мұқият таңдалған геометриялық тапсырмаларды біріктіре отырып, бұл зерттеу мұғалімдерге уақыт тапшылығы мәселесін шешуге және оқушылардың евклидтік геометрияны тереңірек түсінуіне көмектесудің тиімді жолдарын ұсынады.

Түйінді сөздер: Төңкерілген оқыту, геометриялық есептерді шешу, орта мектеп математикасы, педагогикалық әдістер, белсенді оқыту, математикалық ойлау

АННОТАЦИЯ

Данная диссертация исследует вопрос, который волнует многих преподавателей и учеников: в школе огромное внимание уделяется чему-то другому, кроме геометрии. Это часто становится причиной того, что дети не получают достаточно практики. А зря! Ведь сверхурочной зрение, простое воображение а также умение находить связь между несколькими теореми это как раз то, что позволяет ребятам развивать всеобщее математическое мышление. Мы предлагаем нашу задачу математику из двух частей: (1) увлекательные геометрические задачи, решения которых способствуют освоению всех ключевых теорем и продвинутых методов, а так же умение смешивать вышеупомянутые теореми, и (2) концепцию Flipped Learning, в которой всю теорию ребята узнают из видеоуроков и дома. На занятии же учитель может посвятить время остаться полностью погрузиться в решение задач.

Мы расскажем вам о том, как работала наша практика, как и с какими классами пользовались учителя. В сравнении будут традиционное обучение и занятия в стиле Перевернутое обучение. Оценит результаты контрольная, опросы и интервью учителей, при этом есть специальный блок вопросов, посвященных тому, каким образом Flipped Learning сказался на освоении или понимании, и на отношении учеников к геометрии. Проделанная работа это пример того, как, имея хорошие идеи, можно без больших усилий увеличить уровень подготовки учащихся в геометрии.

Ключевые слова: Перевернутое обучение, геометрические задачи, школьная математика, технологии обучения, активное обучение, математическое мышление

INTRODUCTION

Over many years of teaching mathematics in Kazakhstan's upper-secondary schools, it has often been observed that a large proportion of pupils find geometric reasoning considerably easier than algebraic manipulation. This may be because geometry demands the ability to visualise figures, angles and configurations in a spatial context, whereas algebra usually relies on carrying out formal symbolic operations. In addition, teachers who themselves hold mathematics degrees frequently struggle to devote enough lesson time to geometry. There is seldom sufficient time for the applied problems and discussions that would help pupils master core theorems and methods. As (Bergmann, 2012) Bergmann and Sams (2012) point out, interactive, learner-centred instructional approaches—provided face-to-face time is used systematically and purposefully—offer a promising way to overcome these constraints. A well-rounded command of geometry is considered crucial for students entering STEM disciplines or any field that requires structured, logical thought and a sense of spatial relationships. Despite its obvious importance, geometry is often overshadowed in standard curricula, where a large share of class time is devoted to algebraic techniques. As a result, many pupils fail to develop deep confidence and lasting appreciation for geometry, something that could enrich their overall mathematical foundation (Gasparič, 2024). In recent years, the "Flipped Learning" approach has been proposed as one means of addressing these issues. Under this model, students study fundamental concepts at home—typically via short videos—thereby freeing more classroom time for hands-on practice, collaborative problem-solving and targeted guidance (Zainuddin, 2019). By reversing the usual instructional order, teachers can give pupils direct help as they tackle challenging problems, rather than spending the entire lesson lecturing.This study was launched with two overarching aims. First, a carefully curated set of nine geometric problems was compiled, each highlighting key theorems and logical methods essential for a deep understanding of geometry. Second, the project sought to examine how the Flipped Learning experience could optimise student engagement and reinforce comprehension of difficult geometric concepts. Specifically, by requiring students to watch short lecture videos before class and then participate in collaborative problem-solving sessions during face-to-face time, the study aimed to demonstrate an effective strategy for overcoming the persistent shortage of hours devoted to geometry. Collect and analyses challenging geometric problems that elucidate advanced topics such as triangle centres, circle theorems, and properties of special lines (e.g. altitudes, medians and angle bisectors).

Even the most well developed flipped-classroom course cannot succeed, however, without integrating with a larger more research-informed pedagogical approach that takes account of the actual way pupils later develop intuitively out of pattern recognition approaches to deductive proof. In the current literature around cognitive-science, namely that of the cognitive-load-theory and the so-called worked-example-effect by Sweller, it is now defined that novices learned best with explicit step-by-step scaffolds to overcome before they are made to complete non-routine tasks on their own. Consequently the geometry teacher is confronted with a challenging balancing act: she will be expected to present canonical proofs in a manner that minimizes the mental strain they cause (aside, that is to say, from the intended intellectual demands), but will also be expected to develop the visual intuition and exploratory tendencies that will in themselves serve as characteristic of purely spatial reasoning. This is seldom attained in countries like Kazakhstan where the classroom duration is quite minimal and the number of students in a classroom can be quite high. Procedural knowledge can be done easily on a traditional chalk-and-talk lesson, but there is rarely space in such lessons to adopt the form of Socratic discussion which allows learners to make conjectures, argue a case and improve a diagrammatic depiction. All too often, rather than being integrated into a lengthy problem-solving narrative, digital tools like GeoGebra are sent off into a solitary Aha! and dismissively summarized as A demo. In such a way, the flipped model is not depicted in this dissertation as a final goal, but as a base on which a series of increasingly challenging, research-appropriate activities may be mounted. The second line of inquiry concerns the fact the difference between an ability to solve routine textbook problems and a true mastery of Olympiad-type problems which requires combining several theorems in novel arrangements remains there. In line with the van Hiele theory of geometric understanding, the study hypothesizes that many pupils stop progressing on Level 2 (informal deduction) as the activities that take place in a classroom do not encourage pupils to not only devise multi-step procedures, but also to generalize findings among problem families. To test this hypothesis, the construction of lesson sequences was done whereby, initially students were asked to solve prototype problems in collaboration, followed by an analysis of structural variations which demand the re-combination of known facts in newly invented ways.

This dissertation contains:

Design a Flipped Learning model that pairs brief online lectures with in-class activities focused on applying these geometric problems in practice.

Compare test performance and engagement between a traditionally taught group and a Flipped Learning group, extending the experiment to several teachers' classrooms.

Assess satisfaction, identify obstacles and explore best practices in geometry teaching through surveys and interviews with students and teachers.

Section 1 (Abstract): Introduces the central idea of combining geometry's rich content with the Flipped Learning approach.

Section 2 (Introduction): States the research problem, rationale, aims and overall structure.

Section 3 (Literature Review): Surveys existing studies on obstacles to learning geometry and the rise of Flipped Learning in mathematics education.

Section 4 (Mathematical Section): Presents the nine geometric problems, their solutions and the theorems applied, with placeholders for figures.

Section 5 (Methodology): Describes the implementation of Flipped Learning, data- collection strategies and instruments used.

Section 6 (Results and Discussion): Integrates quantitative and qualitative findings, reporting test performance, survey outcomes and a synthesised discussion.

Section 7 (Conclusion): Summarises the findings, acknowledges limitations and proposes directions for future research.

Section 8 (References): Lists all APA-style citations used throughout the thesis.

1. LITERATURE REVIEW

1.1 Difficulties in Teaching Geometry

People often encounter significant obstacles when shifting from algebraic methods to the more spatially grounded concepts of geometry. In many classrooms, the dominant instructional approach in mathematics centres on transforming equations and applying algebraic formulae. Consequently, learners struggle to switch their thinking when asked to construct or interpret geometric figures, prove theorems, or reason about angles and lines on a two-dimensional plane. Teachers likewise emphasise the need to allocate substantial lesson time to geometry, noting that topics such as geometric proof and spatial reasoning cannot be mastered through only superficial treatment (Hwang and Wang, 2019).

Unfortunately, limited classroom time remains a persistent barrier across many curricula. Administrators and policy-makers often prioritise broad content coverage over the depth and rigour required to build pupils' geometric understanding. In this context, teachers may be forced to reduce or omit in-depth study of important theorems, leaving students with fragmented comprehension. Indeed, O'Flaherty and Phillips (2015) note that a lack of interactive exercises exacerbates the problem, because pupils who do not regularly practice geometric constructions or proofs are less prepared for higher-level tasks (O'Flaherty, 2015). These challenges are especially pronounced in settings such as Kazakhstan, where large class sizes, limited technological resources and a long-standing tradition of lecture-based teaching make it difficult to provide the interactive, hands-on experiences that cultivate deep geometric knowledge.

1.2 "Flipped Learning" Approaches in Mathematics

To mitigate the impact of time constraints, the Flipped Learning model has been proposed as a promising alternative. Its central principle is to assign theoretical content—typically delivered through online videos or reading materials—for home study, while reserving face-to-face lessons for problem solving, discussion and real-time feedback (Ece, 2020). This approach enables teachers to shift from a lecture-centred format to one in which learner-centred activities predominate. For geometry, the transition is particularly beneficial because students can revisit complex visual demonstrations at their own pace, pausing or replaying videos whenever necessary to reinforce understanding.

During class, instead of passively listening to lectures, students are encouraged to col- laborate with peers and tackle challenging geometric questions—such as identifying angles, drawing tangents to a circle, or proving collinearity using Menelaus' theorem. These interactive sessions allow teachers to detect misconceptions quickly, provide individual assistance and adapt instruction to each learner's needs. The opportunity to manage their own study time is linked to "self-regulated learning," as identified by Lai and Hwang (2014), in which students assume greater responsibility for their progress.

When implemented effectively, Flipped Learning not only deepens content mastery but also frees classroom time to explore nuanced geometric ideas that are often overlooked in traditional settings.

1.3 Geometry Resources and Theoretical Foundations

Recent years have seen growing attention to how foundational theorems such as Ceva's and Menelaus' can enhance pupils' understanding of geometry and problem-solving skills. In particular, Kayinbaev and Manap (2021) highlight the need for a systematic introduction to Ceva's and Menelaus' theorems as core elements of triangle geometry (Kayinbaev, 2021). These theorems stand out for their elegance and utility, offering a structured framework that enables students to analyse concurrency and collinearity in triangles. The Importance of Ceva's Theorem. Ceva's theorem has log been recognised as a cornerstone of triangle geometry. At its heart, it provides a compact condition determining whether the three cevians of a triangle (segments drawn from each vertex to the opposite side) are concurrent at a single point. The theorem is particularly valuable in problem solving because concurrency underlies many complex constructions. By mastering Ceva's theorem, students are far better prepared to understand why segments such as medians, angle bisectors or altitudes intersect at well-known centres (e.g. centroid or in-centre). Moreover, discussions that involve Ceva's theorem promote a deeper appreciation of proportional relationships within triangles, inspiring learners to recognise the interconnections among seemingly separate areas of geometry.

Because of its power and relative accessibility, Ceva's theorem is often chosen as an early gateway to non-trivial geometric proofs. In a Flipped Learning context, students might watch a brief video showing how the theorem can be derived using ratios of triangle areas or similar triangles, and then devote in-class time to applying it in varied contexts. For example, they could investigate how Ceva's theorem confirms that three lines within a triangle intersect at a single point. Working through such problems in groups and receiving immediate feedback from the teacher strengthens both geometric skills and confidence.

For a triangle ABC, let the lines BC, CA and AB intersect the opposite sides at points

A_1, B_1 and C_1, respectively.

The points A_1, B_1 and C_1 are collinear if and only if

$$\frac{AB_1}{B_{1C}} \cdot \frac{CA_1}{A_{1B}} \cdot \frac{BC_1}{C_{1A}} = 1$$

Menelaus' theorem is likewise essential for deep geometric insight, because it characterises how points lying on the sides (or their extensions) of a triangle can be aligned on a single straight line. Like Ceva's theorem, Menelaus employs ratio arguments, showing how the division of certain segments corresponds precisely to collinearity along a line. Its power to link algebraic relations with geometric configurations is especially valuable, as it allows pupils to move fluidly between numerical representations and diagrammatic intuition.

Applications of Menelaus' theorem range across many geometric problems involving circles, cyclic quadrilaterals and even trigonometric configurations of polygons. In a Flipped Learning setting, students benefit from pre-class videos that illustrate typical scenarios in which Menelaus is applied—for example, confirming that a transversal cutting the sides of a triangle respects a constant ratio product. This approach helps learners visualise abstract ratios more concretely. Later, in class, they can experiment with actual diagrams, vary measurement values and see first-hand how collinearity becomes evident through ratio computations.

For a triangle ABC, let three cevians be drawn from each vertex, intersecting the opposite sides at points A_1, B_1, C_1 respectively (A_1 on BC, B_1 on CA and C_1 on AB).

Then the following condition holds:

$$\frac{BA_1}{A_{1C}} \cdot \frac{CB_1}{B_{1A}} \cdot \frac{AC_1}{C_{1B}} = 1$$

This equation gives the necessary and sufficient condition for the cevians to be concurrent at a single point.

When Ceva's and Menelaus' theorems are woven into a carefully selected set of problems, students encounter repeated practice with core principles such as concurrency, collinearity and ratio manipulation. This process enhances their exam-oriented

Problem 1: A Triangle with Two Externally Tangent Circles on One Side

Given. In triangle MNP there are two circles with centres O_1 and O_2 located on side NM. These circles are externally tangent and intersect sides MP and PN at points M, D and N, C respectively.

If $MO_1 = O_1D = 3$, $NO_2 = O_2C = 6$, the ratio of the areas of $\triangle\ MCO_2$ to $\triangle\ O_1ND$ is $8\sqrt{3}/5$, and $PN = MP\sqrt{2-\sqrt{3}}$, find the area of $\triangle\ MNP$.

$$\frac{S_{CMO_2}}{S_{DNO_1}} = \frac{8}{5}\sqrt{3}$$

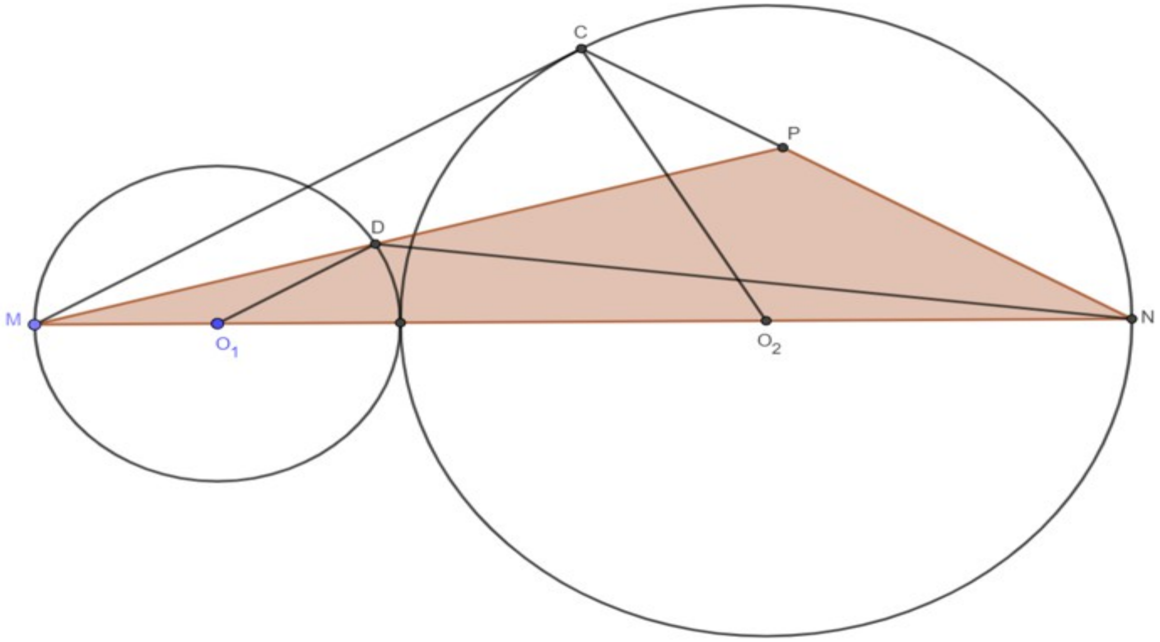
$$\frac{NP}{MP} = \sqrt{2-\sqrt{3}}$$

$$MO_1 = O_1D = 3$$

$$NO_2 = O_2C = 6$$

$$S_{MNP} = ?$$

Figure 1.3.1 Two externally tangent circles on side NM



Outline of the Solution. This problem requires triangle trigonometry (the sine rule), properties of central angles and manipulation of area ratios. By denoting $\angle PNM = \alpha$ and $\angle PMN = \beta$, we establish relationships among the sides. Using the ratio $\sin 2\alpha / \sin 2\beta$ and the known lengths MO_1 and NO_2, we determine that $MN = 18$ and compute the area $S_{MNP} = 81(\sqrt{3} - 1)/2$.

$$\begin{cases} a^2 + b^2 = 2 \\ 2ab = \sqrt{3} \end{cases}$$

Solving for a and b:

$$a = \frac{\sqrt{3}}{2}, \quad b = \frac{1}{\sqrt{2}}$$

Applying the sine rule:

$$\frac{NP}{\sin\beta} = \frac{MP}{\sin\alpha} = \sqrt{2 - \sqrt{3}}$$

$$\frac{\sin\beta}{\sin\alpha} = \frac{\sqrt{3} - 1}{\sqrt{2}}$$

Because $\angle PNM = \alpha$, $\angle CO_2M = 2\alpha$ (an external angle), so $\angle DO_1N = 2\beta$.
Using trigonometric identities:

$$\frac{\sin 2\alpha}{\sin 2\beta} = \frac{8}{15} \cdot \sqrt{3}$$

$$2 \cdot \sin\alpha\cos\alpha = \sqrt{3}$$

$$\frac{\cos\alpha}{\cos\beta} = \sqrt{3} \cdot \frac{\sin\beta}{\sin\alpha} = \frac{\sqrt{3} - 1}{\sqrt{2}}$$

$$\cos\alpha = \frac{\sqrt{3}}{2}\left(\sqrt{3} - 1\right)\cos\beta$$

Squaring both sides:

$$\cos^2\alpha = \frac{3}{4}\left(3 - 2\sqrt{3} + 1\right)\cos^2\beta$$

$$\cos^2\alpha = \left(2 - \sqrt{3}\right) \cdot 3 \cdot \cos^2\beta$$

Using trigonometric identities:

$$1 - \cos^2\beta = 2 - \sqrt{3}$$

$$1 - \left(2 - \sqrt{3}\right) \cdot 3 \cdot \cos^2\beta = 2 - \sqrt{3}$$

$$\left(20 - 12\sqrt{3}\right)\cos^2\beta = 1 - \sqrt{3}$$

$$\cos^2\beta = \frac{\sqrt{3} - 1}{12\sqrt{3} - 20}$$

$$\cos\alpha = \frac{\sqrt{3}}{2}, \quad \alpha = 30^\circ$$

From earlier results:

$$NP = \left(\sqrt{3} - 1\right)k, \quad MP = \sqrt{2}k$$

Using the cosine rule:

$$MP^2 = NP^2 + MN^2 - 2NP \cdot MN \cdot \cos\alpha$$

$$2k^2 = \left(4 - 2\sqrt{3}\right)k^2 + 18^2 - 2\left(\sqrt{3} - 1\right)k \cdot 18 \cdot \frac{\sqrt{3}}{2}$$

$$k^2 = \left(2 - \sqrt{3}\right)k^2 + 18^2 - 9\sqrt{3} \cdot (3 - 1)k$$

$$\left(1 - \sqrt{3}\right)k^2 + \left(1 - \sqrt{3}\right)k \cdot 9\sqrt{3} + 18^2 = 0$$

Solving for k:

$$D = 81 \cdot 3 + 4 \cdot 81 \cdot \left(\sqrt{3} + 1\right) = 81\left(7 + 4\sqrt{3}\right) = \left(9\left(\sqrt{3} + 2\right)\right)^2$$

$$k_{1,2} = \frac{9\left(\sqrt{3} \pm 3\right)}{2}, \quad k > 0 \Rightarrow k = 9$$

$$NP = 9\left(\sqrt{3} - 1\right), \quad MP = 9\sqrt{2}$$

Using the area formula:

$$S_{MNP} = \frac{MN \cdot NP \cdot \sin\alpha}{2}$$

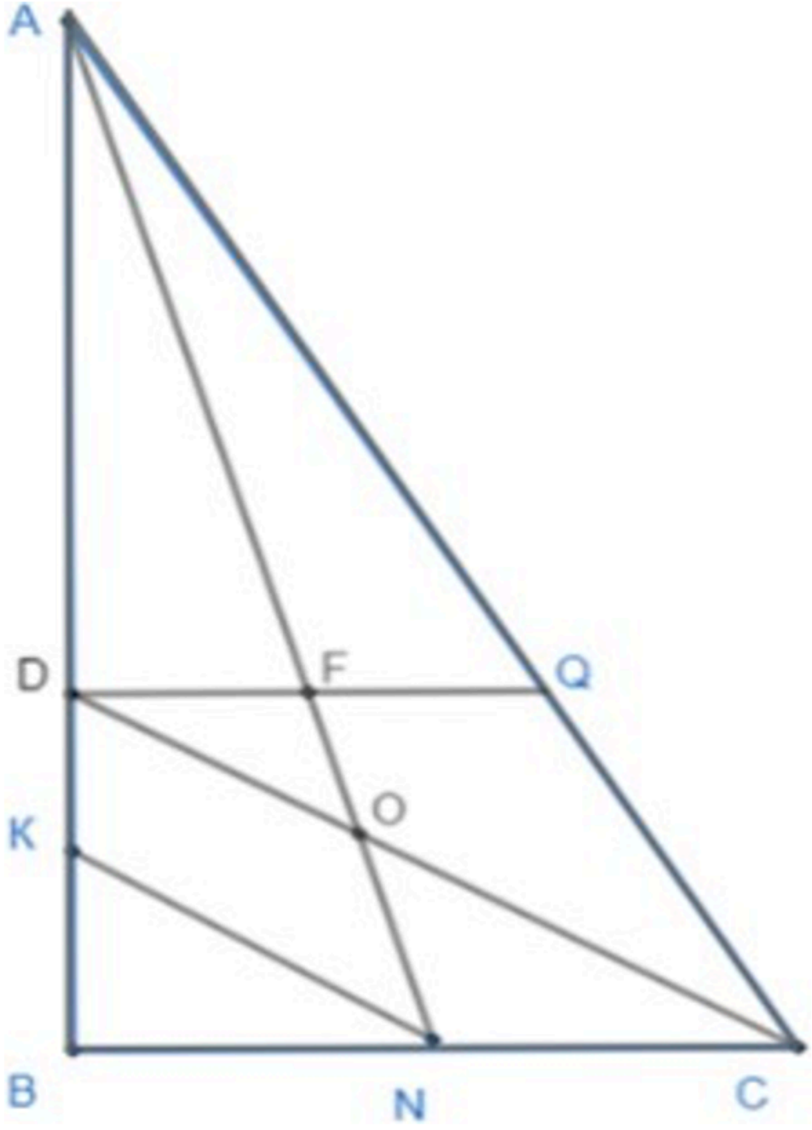
$$S_{MNP} = \frac{18 \cdot 9\left(\sqrt{3} - 1\right) \cdot \frac{1}{2}}{2}$$

$$S_{MNP} = \frac{81\left(\sqrt{3} - 1\right)}{2}$$

$$\boxed{\frac{81\left(\sqrt{3}-1\right)}{2}}$$

Problem 2: Right Triangle with Intersection of a Median and an Angle Bisector

Figure 1.3.2 Intersection of a median and an angle bisector in a right triangle



Given: In right triangle ABC with ∠B = 90°, the median AN and the angle bisector CD intersect at point O. If CO = 9 and OD = 5, find the area of △ABC.

Outline of the Solution: This solution makes use of arguments with parallel lines and similarity. Draw the segment DQ parallel to BC and denote the key lengths $BN = NC = a, AD = c$ and $BD = 2b$. Similarity of triangles ($\triangle\ DFO\ \sim\triangle\ ONC$) together with the angle-bisector property is applied. Eventually the area is found to be $1223/20$.

$DQ \parallel BC$ is drawn.
Let $BN = NC = a$, $AD = c$ and $BD = 2b$.
From similarity:

$$\triangle DFO \sim \triangle ONC, \quad \triangle FOD \sim \triangle NOC$$

(right and corresponding angles).

$$\angle QDO = \angle BCD \quad \text{(corresponding angles).}$$

From proportionality:

$$\frac{DF}{NC} = \frac{DO}{ON}, \quad \frac{DO}{a} = \frac{5}{9}, \quad DF = \frac{5}{9}a$$

From the similarity $\triangle ABN \sim \triangle ADF$:

$$\frac{DA}{AB} = \frac{BN}{AB} = \frac{DF}{AN}$$

$$\frac{c}{c + 2b} = \frac{5}{9}$$

Solving for c:

$$c = 2.5b$$

Using the auxiliary construction $KM \parallel DC$ and the given data:

$$DC = 14, \quad NK = 7$$

Hence

$$KB = KD$$

$$KB = DB + AD = 2b + 2.5b = 4.5b$$

Applying the angle-bisector theorem:

$$\frac{BC}{BD} = \frac{AC}{c}$$

$$\frac{AC}{c} = \frac{(4.5b)^2 + 4a^2}{2.5b}$$

Solving,

$$2.5a = \sqrt{(4.5b)^2 + 4a^2}$$

From the calculations,

$$b = \frac{a}{\sqrt{3}}$$

Applying the Pythagorean theorem to $\triangle KBN$:

$$KB^2 + BN^2 = KN^2$$

$$b^2 + a^2 = 72$$

$$\frac{a^2}{9} + a^2 = 49, \quad a = \frac{21}{\sqrt{10}}, \quad b = \frac{7}{\sqrt{10}}$$

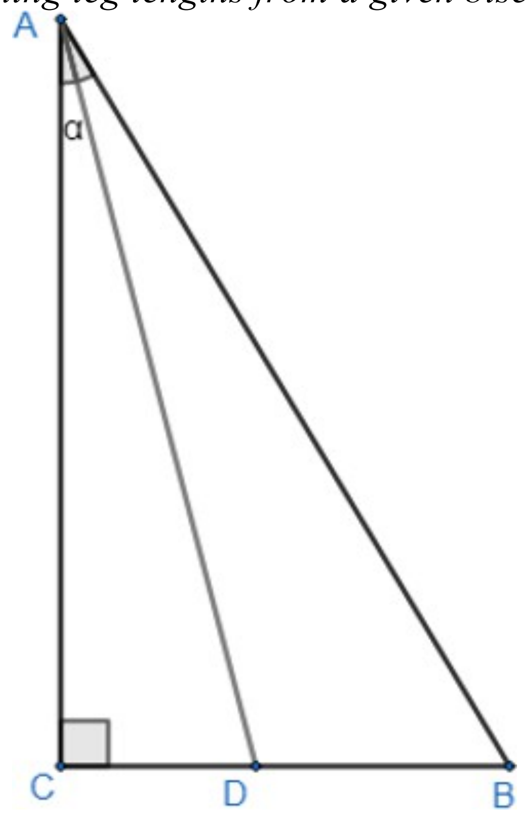
Finding the area S_{ABC}:

$$S_{ABC} = \frac{1}{2} AB \cdot BC \cdot \sin 51^\circ = \frac{1}{2} \cdot 4.5 \cdot 21 \cdot \frac{7}{\sqrt{10}} = \frac{1223}{20}$$

Problem 3: Right Triangle with a Given Angle-Bisector Length

Given. In a right triangle whose hypotenuse is a, the bisector of an acute angle has length $a/\sqrt{3}$. Find the lengths of the legs.

Figure 1.3.3 Determining leg lengths from a given bisector in an acute angle



Outline of the Solution. Let the acute angle be denoted by α; trigonometric identities (including the double-angle formula) are applied. The legs are found to be $a/2$ and $\sqrt{3}a/2$. The key idea is to relate $\cos\alpha$ to the given bisector length, showing the relationship between angles and side ratios in a right triangle.

$$AB = a, \quad AD = \frac{a}{\sqrt{3}}$$

Let $AC = b$; then

$$\cos\alpha = \frac{b}{\frac{a}{\sqrt{3}}} = \frac{b}{a}\sqrt{3}\cos\alpha$$

$$\cos 2\alpha = \frac{b}{a} \Rightarrow b = a\cos\alpha$$

$$a\cos 2\alpha = \frac{a}{\sqrt{3}}\cos\alpha$$

Using the trigonometric identity:

$$\sqrt{3}(\cos^2\alpha - \sin^2\alpha) = \cos\alpha$$

$$2\sqrt{3}\cos^2\alpha - \cos\alpha - \sqrt{3} = 0$$

Solving for $\cos\alpha$:

$$\cos\alpha = \frac{\sqrt{3}}{2}, \quad \cos 2\alpha = -\frac{\sqrt{3}}{6}$$

Because α is acute:

$$\alpha = 30^\circ$$

Now,

$$AC = b = \frac{a}{\sqrt{3}}\cos 30^\circ = \frac{a}{2}$$

$$CB = \frac{\sqrt{3}}{2}\,a$$

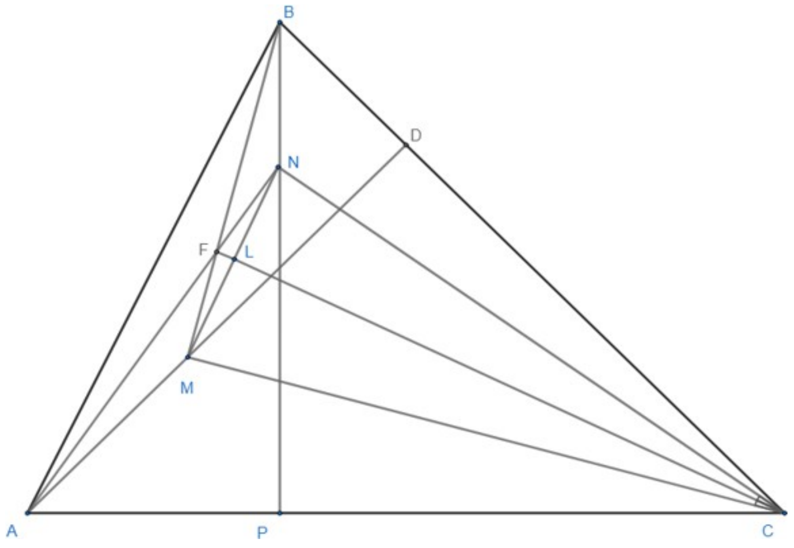
Final answer:

$$\boxed{\frac{a}{2}, \quad \frac{\sqrt{3}}{2}a}$$

Problem 4: Acute Triangle with Right Angles Formed by Alti- tudes

Given. In acute triangle ABC the points M and N lie on altitudes AD and BP, respectively, such that $\angle BMC$ and $\angle ANC$ are right angles. If $MN = 4 + 2\sqrt{3}$ and $\angle MCN = 30^\circ$, find the length of the bisector CL in $\triangle CMN$.

Figure 1.3.4 Determining the bisector length in the constructed triangle



$$MN = 4 + 2\sqrt{3}.$$

$$\Delta MLC = \Delta NLC$$

$$ML = \frac{1}{2}MN = NL = 2 + \sqrt{3}$$

$$\Delta NCL \qquad tg15° = \frac{NL}{LC}.$$

$$tg(45° - 30°) = \frac{tg(45° - 30°)}{1 + tg45° \cdot tg30°} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$LC = \frac{NL}{tg(45° - 30°)};$$

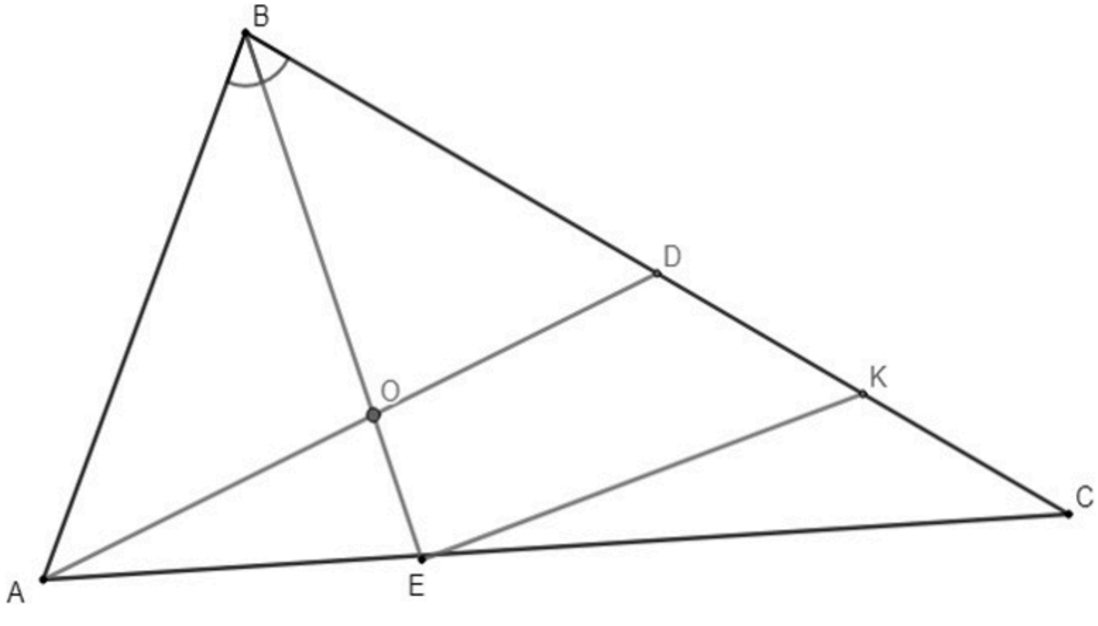
$$\frac{2+\sqrt{3}}{\frac{\sqrt{3}-1}{\sqrt{3}+1}} = \frac{3\sqrt{3}+5}{\sqrt{3}-1} = 7 + 4\sqrt{3}.$$

$$7 + 4\sqrt{3}.$$

Problem 5: Triangle with a Perpendicular Median and Angle Bisector

Given. In $\triangle ABC$, the angle bisector BE and the median AD are perpendicular and each has length 4. Determine the side lengths of $\triangle ABC$.

Figure 1.3.5 Finding side lengths with a perpendicular median and bisector



Outline of the Solution. Let $AB = a$. Through point E draw a line parallel to AD and apply Thales' theorem. Triangle $\triangle ABD$ is shown to be isosceles, and several segments are found to be in specific ratios. In the end the side lengths turn out to be $3\sqrt{5}, \sqrt{13}$ and $2\sqrt{13}$.

$$AD = BE = 4, \quad BD = DC, \quad BE \perp AD$$

Let $AB = a$.

Through E draw a line parallel to AD. By Thales' theorem:

$$\frac{EC}{AE} = \frac{CK}{KD}$$

Because $\triangle ABD$ is isosceles ($BE \perp AD$ and BO is a bisector):

$$AB = BD = DC = a$$

Since AD is a median:

$$AO = OD$$

Using the angle-bisector theorem:

$$\frac{BC}{EC} = \frac{AE}{EC} = \frac{AB}{EC} = \frac{1}{2}$$

Hence

$$\frac{EC}{AE} = \frac{1}{2}$$

Again by Thales' theorem:

$$\frac{EC}{AE} = \frac{CK}{KD}$$

$$CK = \frac{a}{2}, \quad KD = \frac{a}{3}$$

Applying Thales' theorem once more:

$$\frac{BO}{OE} = \frac{BK}{DK}$$

$$\frac{4 - OE}{OE} = \frac{a/3}{a/3a}$$

Further Calculation:

Given:

$$OE = 1, \quad BO = 3$$

By the Pythagorean theorem:

$$AB = \sqrt{BO^2 + AO^2} = \sqrt{13}$$

$$BC = 2AB = 2\sqrt{13}$$

$$AE = \sqrt{AO^2 + OE^2} = \sqrt{5}$$

$$AC = 3\sqrt{5}$$

$$\boxed{3\sqrt{5}, \quad \sqrt{13}, \quad 2\sqrt{13}}$$

Problem 6: Triangle Whose Angle Bisector Serves as a Circle Diameter

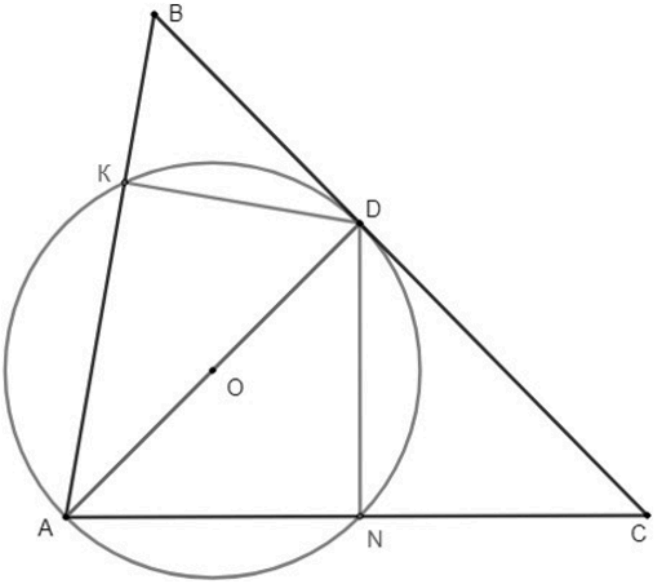
Given. In $\triangle ABC$ the bisector of $\angle BAC$ has length a.

If this bisector is taken as the diameter of a circle.

That circle divides side AB in the ratio 2 : 1 and passes through the midpoint of side AC.

Find the area of $\triangle ABC$.

Figure 1.3.6 Determining the area for a special bisector–circle configuration



Outline of the Solution. Introduce the variables $BK = x$, $AK = 2x$, $BD = y$, etc. Using right-angle properties that arise because a diameter subtends a right angle, one eventually obtains $S_{ABC} = 7a^2\sqrt{35}/48$.

$$BK = x, \quad AK = 2x$$

$$AD = a, \quad KD = c, \quad BD = y$$

Let $\angle BAD = \alpha$.

$$\angle AKD = \angle AND = 90^\circ \quad \text{(because } AD \text{ is a diameter)}$$

$\triangle AKD \cong \triangle AND$ (same angle and a common side),

$$c = a\sin\alpha$$

$$AK = AN = 2x$$

In $\triangle ABC$ apply the angle-bisector theorem:

$$\frac{3x}{4x} = \frac{y}{a}$$

Solving for y:

$$y = \frac{3}{4}a$$

Because $AN = NC$ and $AN \perp DN$, triangle ADC is isosceles, hence

$$BC = \frac{7}{4}a$$

For $\triangle AKD$:

$$c^2 = a^2 - 4x^2$$

For $\triangle BKD$:

$$c^2 = \frac{3}{4}a^2 - x^2$$

Equating the two expressions,

$$a^2 - 4x^2 = \frac{3}{4}a^2 - x^2$$

Solving for x:

$$x = \frac{\sqrt{7}\,a}{4\sqrt{3}}$$

Then

$$c^2 = a^2 - \frac{4 \cdot 7a^2}{48}, \qquad c = \frac{a\sqrt{5}}{2\sqrt{3}}$$

Computing partial areas:

$$S_{ABD} = \frac{1}{2}AB \cdot DK = \frac{a^2\sqrt{35}}{16}, \qquad S_{ADC} = \frac{1}{2}AC \cdot DN = \frac{a^2\sqrt{35}}{12}$$

Hence

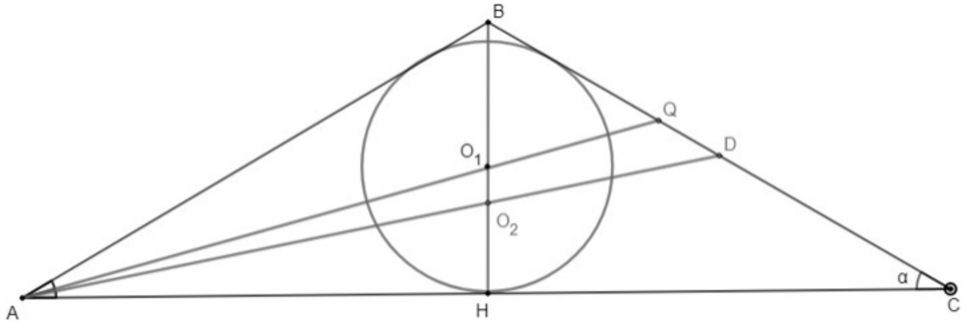
$$S_{ABC} = \frac{a^2\sqrt{35}}{16} + \frac{a^2\sqrt{35}}{12} = \frac{7a^2\sqrt{35}}{48}.$$

$$\boxed{7a^2\sqrt{35}/48}$$

Problem 7: Isosceles Triangle with Known Distances between Special Points

Given. In the isosceles triangle illustrated in Figure 7, the distance between the intersection points of the median and the angle bisectors is 2. If the circumference of the inscribed circle equals 20π, determine the perimeter of the triangle.

Figure 1.3.7 Finding the perimeter from intersection distances and the incircle



Outline of the Solution. Because the incircle's circumference is 20π, its radius is $R = 10$. Let $AB = a$. Using trigonometric relations one finds $\cos\alpha = 5/7$ and the perimeter turns out to be $48\sqrt{6}$.

$$O_1O_2 = 2, \qquad 2\pi R = 20\pi \ \Rightarrow\ R = 10$$

Let $AB = a$.

$$\cos\alpha = \frac{HC}{a}, \qquad AC = 2a\cos\alpha,$$
$$BH = a\sin\alpha$$

Applying the tangent function:

$$\tan\frac{\alpha}{2} = \frac{O_2H}{AH} = \frac{10}{a\cos\alpha}$$

Because

$$O_2H = \frac{1}{3}BH = \frac{a\sin\alpha}{3},$$

we obtain

$$\frac{a\sin\alpha}{3} = R - O_1O_2 = 8,$$
$$a = \frac{24}{\sin\alpha}.$$

From

$$\tan\frac{\alpha}{2} = \frac{10\sin\alpha}{24\cos\alpha},$$

one ultimately gets

$$\cos\alpha = \frac{5}{7}, \qquad \sin\alpha = \frac{2\sqrt{6}}{7}, \qquad a = 14\sqrt{6}, \qquad AC = 20\sqrt{6}.$$

Thus the perimeter is

$$P_{ABC} = 20\sqrt{6} + 2 \cdot 14\sqrt{6} = 48\sqrt{6}.$$
$$\boxed{48\sqrt{6}}$$

Problem 8: Triangle with Points Dividing its Sides and a Perpendicular from their Intersection.

Given. In Figure 8, $\triangle ABC$ has area 6.

Points K and L lie on sides AB and AC.

$AK: BK = 2: 3$.

$AL: LC = 5: 3$.

Lines CK and BL intersect at P.

If the distance from P to line AB is 1.5.

Find the length of AB.

We use variables according to the ratio:

AK = 2n;

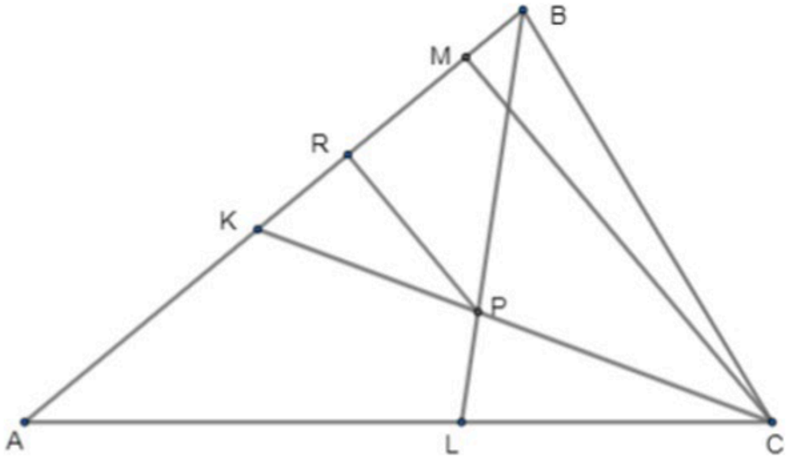
BK=3n;

BK=3n;

AL=5m;

LC=3m.

Figure 1.3.8 Using Menelaus' theorem and a perpendicular distance



Outline of the Solution. After introducing proportional parameters for the segments, Menelaus' theorem is applied. Knowing that the area of $\triangle ABC$ is 6, one finds $AB = 4$.

Draw $PR \perp AB$ and $CM \perp AB$ as shown; R and M are indicated in the figure.

$$PR = 1.5$$

Introduce proportional variables:

$$AK = 2n,\ BK = 3n,\ AL = 5m,\ LC = 3m$$

Applying Menelaus' theorem to $\triangle AKC$ with transversal PL:

$$\frac{AL}{LC} \cdot \frac{CP}{PK} \cdot \frac{KB}{BA} = 1 \quad \Rightarrow \quad \frac{3m}{5m} \cdot \frac{5n}{3n} = 1 \quad \Rightarrow \quad CP = KP.$$

Because $\triangle KMC \sim \triangle KRP$ (corresponding angles),

$$\frac{MC}{PR} = \frac{KC}{PK}$$

so

$$MC = 3.$$

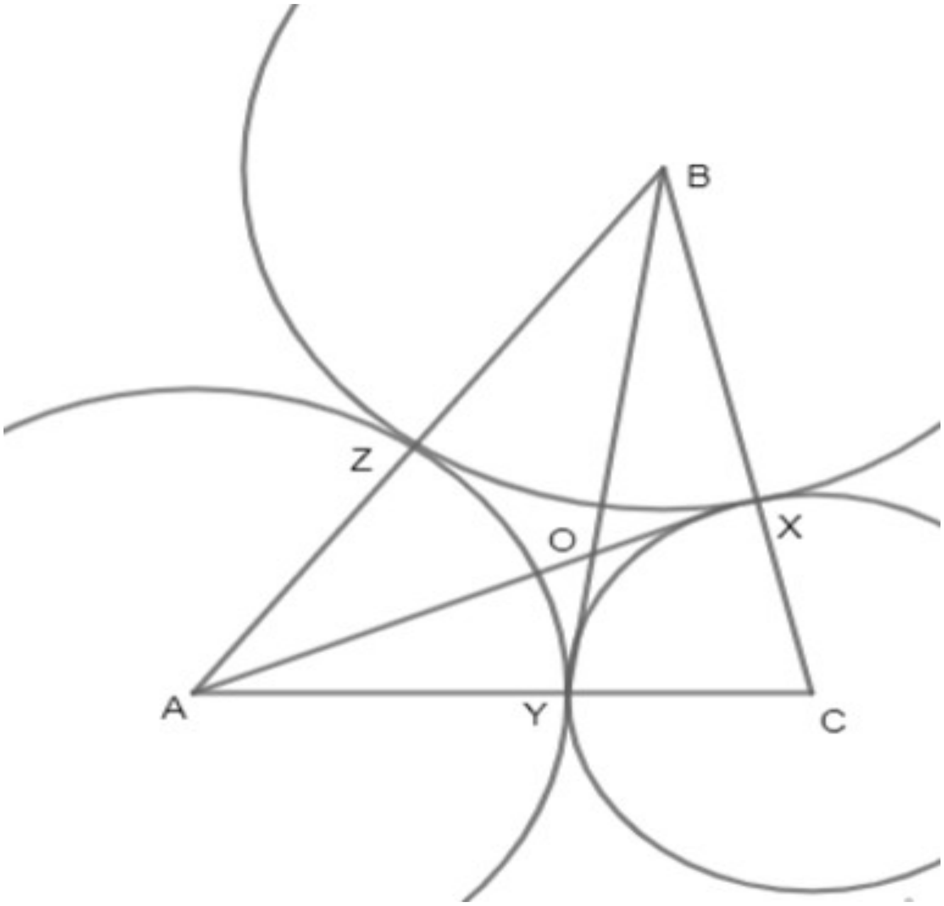
With $S_{ABC} = 6$ the base is

$$AB = 6 \cdot \frac{2}{3} = 4.$$

Problem 9: Three Circles with Given Radii Ratios and Pairwise Tangency Points

Given. Three circles have radii in the ratio $11:12:9$ and centres A, B and C. They are externally tangent at points X, Y and Z. Determine how segment BY is divided by line C.

Figure 1.3.9 Applying Menelaus' and Ceva's theorems to multiple tangencies



Outline of the Solution. Assign lengths proportional to a parameter k and apply Ceva's theorem (for concurrency) and Menelaus' theorem (for intersecting lines). One finds that

$$\frac{BO}{OY} = \frac{80}{33}.$$

Let O be the intersection of lines AX and BY. With the given proportional relations

$$AY = 11k, \quad CX = 9k, \quad BZ = 12k,$$

Ceva's theorem in $\triangle ABC$ gives

$$\frac{AY}{YC} \cdot \frac{CX}{XB} \cdot \frac{BZ}{ZA} = 1.$$

Because AX, BY and CZ concur at O, segment BY is divided by CZ in the ratio

$$\frac{BO}{OY}.$$

Applying Menelaus' theorem to $\triangle BCY$ with transversal OX:

$$\frac{BO}{OY} \cdot \frac{YA}{AC} \cdot \frac{CX}{XB} = 1 \quad \Rightarrow \quad \frac{BO}{OY} = \frac{20k}{11k} \cdot \frac{12k}{9k} = \frac{80}{33}.$$

The apparently simple task of finding the ratio BO to OY in a triangle threaded by three cevians unfolds into a surprisingly rich tour of classical geometry, revealing how global concurrency conditions propagate through every corner of a figure and how local segment divisions can be read off almost automatically once the correct theorems are paired. We begin by assigning integer ratios to the partitioned sides of triangle ABC—CX to XB as 9 to 12, AY to YC as 11 to 20, and BZ to ZA as 12 to 15—and scaling every length by an arbitrary positive factor k. This harmless-looking bookkeeping is actually strategic, because Ceva's theorem converts the entire set of raw numbers into a single concurrency test: the product (11/20)·(9/12)·(12/15) equals 1, confirming that the cevians AX, BY, and CZ meet in exactly one interior point O. In an instant, every "missing" perimeter length becomes locked into rational proportion and no hidden degrees of freedom remain; the triangle may swell or shrink with k, but its internal ratios are now frozen. Once concurrency is in the bank, the spotlight shifts to Menelaus' theorem, which in effect turns the global harmony certified by Ceva inside out. Choosing the subsidiary triangle BCY and the transversal OX, Menelaus writes BO/OY · YA/AC · CX/XB = 1. Substituting YA/AC as 11/(11+20) and CX/XB as 9/12 whittles the equation to BO/OY · 11/31 · 3/4 = 1, from which BO/OY emerges as 80/33. The scaling factor k cancels unobtrusively, underscoring that the ratio is a pure number independent of the triangle's absolute size. Switch perspectives and the same fraction reappears in other guises. In mass-points geometry, assign weights 44, 55, and 36 to A, B, and C so that balance holds at every cevian foot; the weight at Y is the sum of the masses at A and C, so dividing BY internally must follow the inverse ratio (44+36) to 55, again 80 to 55, which compresses to 80/33 once the common divisor is cleared. In barycentric coordinates, concurrency forces the cevians to be expressible by homogeneous triples whose coordinates are proportional to products of the side ratios, and solving for the affine parameter along BY reproduces 80/33 without breaking a sweat. In an affine-vector proof one can anchor the triangle at the origin, encode each foot of a cevian as a convex blend of vertices with coefficients matching the given ratios, and then solve a two-by-two linear system whose vanishing determinant mirrors Ceva; solving the remaining linear relation spits out the same 80/33. All of these roads lead to the identical invariant because Ceva and Menelaus are not just convenient tools but dual faces of a single projective coin: Ceva speaks the language of concurrent lines (points collinear in the dual plane) while Menelaus speaks of collinear points (lines concurrent in the dual), and their tandem use taps the underlying symmetry that projective geometry formalises. Pedagogically, the choreography offers students a blueprint for sophisticated problem solving: first impose a global existence theorem to tame the free parameters, then deploy a transversal theorem to harvest the desired internal ratio. It also dramatizes why pure number invariants are so treasured in geometry: they bottle the essence of a configuration, immune to dilations or unit changes, and carry across diverse proof frameworks. Ultimately, the fraction 80/33 is not an isolated curiosity but the audible echo of deeper structural music—an inevitable harmony dictated by integers that balance, by lines that concur and transversals that align, by centuries-old theorems that continue to illuminate how local and global properties of a figure intertwine inextricably.

1.4 Differences Between Difficult and Simple Problems in Geomytry

The distinction between difficult and simple problems in geometry can be characterised by several key features. Simple problems usually rely on one or two fundamental rules and can be solved using direct methods. In such tasks, it is enough to apply well-known properties of familiar figures (for example, the fact that the interior angles of a triangle sum to 180°) or basic formulae. They are often designed to test pupils' foundational knowledge. Difficult problems, on the other hand, demand multi-step reasoning, intricate logical connections and the simultaneous use of several geometric theorems. Various figures intersect or are interdependent, and solving the problem may require creative thinking, drawing auxiliary lines, considering special cases or having a deep understanding of the theorems involved. Moreover, difficult problems frequently involve proof, analysis and movement from the general to the particular. Whereas a simple problem calls for one specific method, a difficult problem may admit several solution paths. Thus, the difference lies in the structural complexity of the task and the depth of the solution process.

Criteria for Classifying Problems on Medians, Altitudes and Angle Bisectors as Simple or Difficult

1. Quantity and Nature of the Given Data

Simple: all necessary information is provided outright—side lengths, angle mea- sures, directly applicable formulae.

Difficult: data may be incomplete; one must draw auxiliary constructions or combine several theorems.

2. Length of the Solution Path

Simple: solved in one or two steps.

Difficult: involves several stages—for example, finding one element first and then substituting it into a different formula.

3. Formulae and Theorems Involved

Simple: basic definitions and rules.

Difficult: more advanced results such as the median-length formula, angle-bisector proportion, or Heron's formula.

4. Level of Logical Reasoning

Simple: direct application of a single formula.

Difficult: transforming the configuration, drawing extra lines, searching for similarity or symmetry.

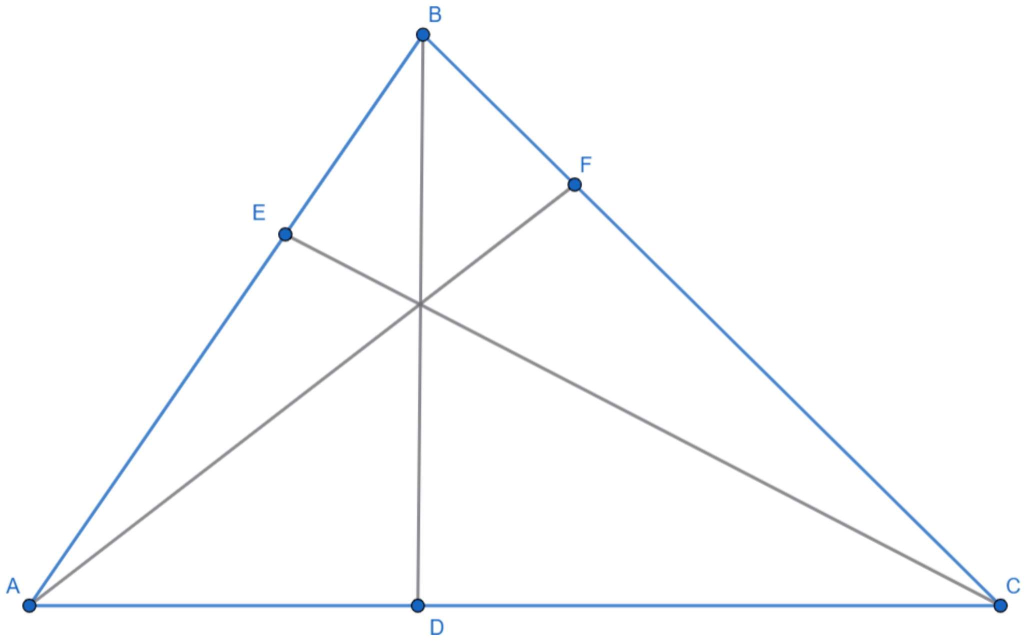
Comparison of Simple and Difficult Problems

Criterion	Simple Problem	Difficult Problem
Amount of data	Complete, explicit	Insufficient or indirect
Solution steps	1–2 steps	Multi-step

Tools required	Basic rules	Advanced theorems
Level of reasoning	Direct	Transformation, analysis

Sample Problem (3.44). Find the altitudes of a triangle whose sides are 5 cm, 6 cm and 7 cm.

Figure 1.4.1 Diagram of the triangle



$$AB = 5\text{ cm},\qquad BC = 6\text{ cm},\qquad AC = 7\text{ cm}.$$

Solution:

$$\begin{aligned}
p &= 5 + 6 + 7/2 = 9,\\
S &= \sqrt{9 \cdot 4 \cdot 3 \cdot 2} = 6\sqrt{6},\\
S &= \frac{AC \cdot BF}{2} = \frac{7\,BF}{2} = 6\sqrt{6} \;\Rightarrow\; BF = \frac{12\sqrt{6}}{7},\\
S &= \frac{BC \cdot AE}{2} = \frac{6\,AE}{2} = 6\sqrt{6} \;\Rightarrow\; AE = 2\sqrt{6},\\
S &= \frac{AB \cdot CD}{2} = \frac{5\,CD}{2} = 6\sqrt{6} \;\Rightarrow\; CD = \frac{12\sqrt{6}}{5}.
\end{aligned}$$

Skills Required:

Apply Heron’s formula to find the area of a triangle;

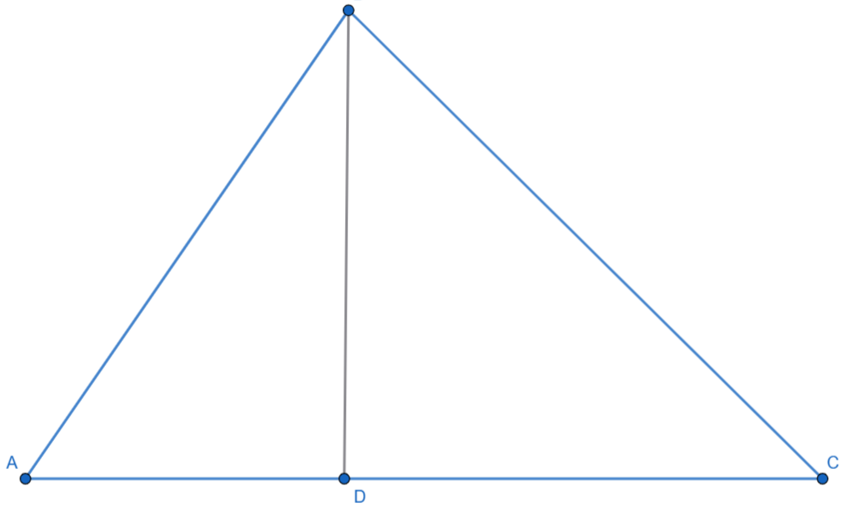
Use the area formula $S = \frac{1}{2}$(base) × (altitude);

Handle arithmetic and radical expressions accurately;

Keep track of units correctly.

Comment on Difficulty: This is simple to moderate. Heron’s formula and altitude-from- area are standard techniques for grades 7–9, but care is needed with radicals and fractions since small errors spoil the result.

Group B. Sample Problem (3.47). Two sides of a triangle are $\sqrt{13}$ and $\sqrt{10}$; the third side is equal to the altitude dropped onto it. Find the third side.

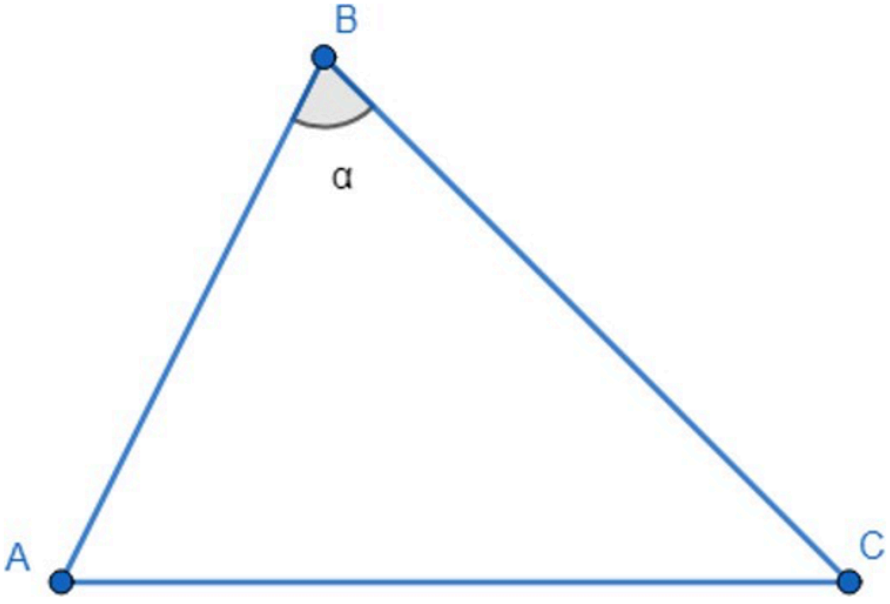


Solution:

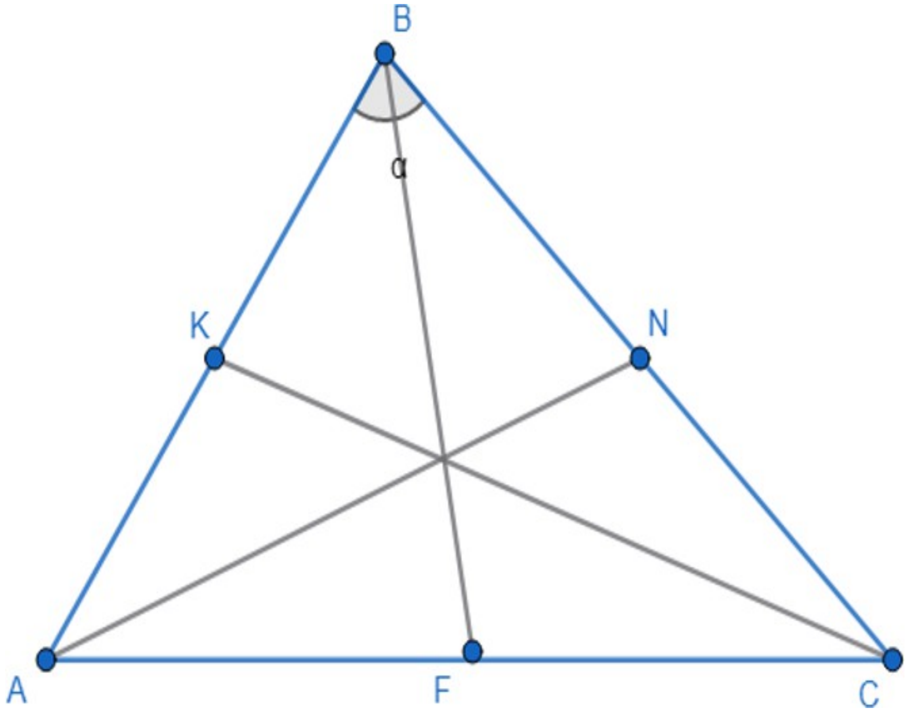
$$
\begin{aligned}
AD &= \sqrt{AB^2 - BD^2} = \sqrt{10 - x^2},\\
DC &= \sqrt{BC^2 - BD^2} = \sqrt{13 - x^2},\\
AD + DC &= AC \;\Rightarrow\; \sqrt{10 - x^2} + \sqrt{13 - x^2} = x,\\
\sqrt{13 - x^2} &= x - \sqrt{10 - x^2},\\
13 - x^2 &= (x - \sqrt{10 - x^2})^2 = x^2 - 2x\sqrt{10 - x^2} + 10 - x^2,\\
2x\sqrt{10 - x^2} &= x^2 - 3,\\
\left(2x\sqrt{10 - x^2}\right)^2 &= (x^2 - 3)^2 \;\Rightarrow\; 4x^2(10 - x^2) = x^4 - 6x^2 + 9,\\
5x^4 - 46x^2 + 9 &= 0 \;\Rightarrow\; x^2 = 9 \quad \text{or} \quad x^2 = 1/5,\\
x^2 \geq 3 &\;\Rightarrow\; x^2 = 9 \;\Rightarrow\; AC = 3.
\end{aligned}
$$

Comment on Difficulty:
Moderate: requires
Correct use of the cosine rule and area expressions;
Construction and reconciliation of two equations;
Solving a quadratic and extracting roots.
Suitable for grade 9.

Group C. Sample Problem. In a triangle, two sides are 5 m and 6 m, and the cosine of the included angle is 0.6. Find all three medians.



Given: $AB = 5$ m, $BC = 6$ m, $\cos\alpha = 0.6$.
First, determine the third side (cosine rule):
$AC^2 = AB^2 + BC^2 - 2\,AB\cdot BC\cos\alpha = 5^2+6^2-2\cdot5\cdot6\cdot0.6 = 25+36-36 = 25$, $AC = 5$ m.

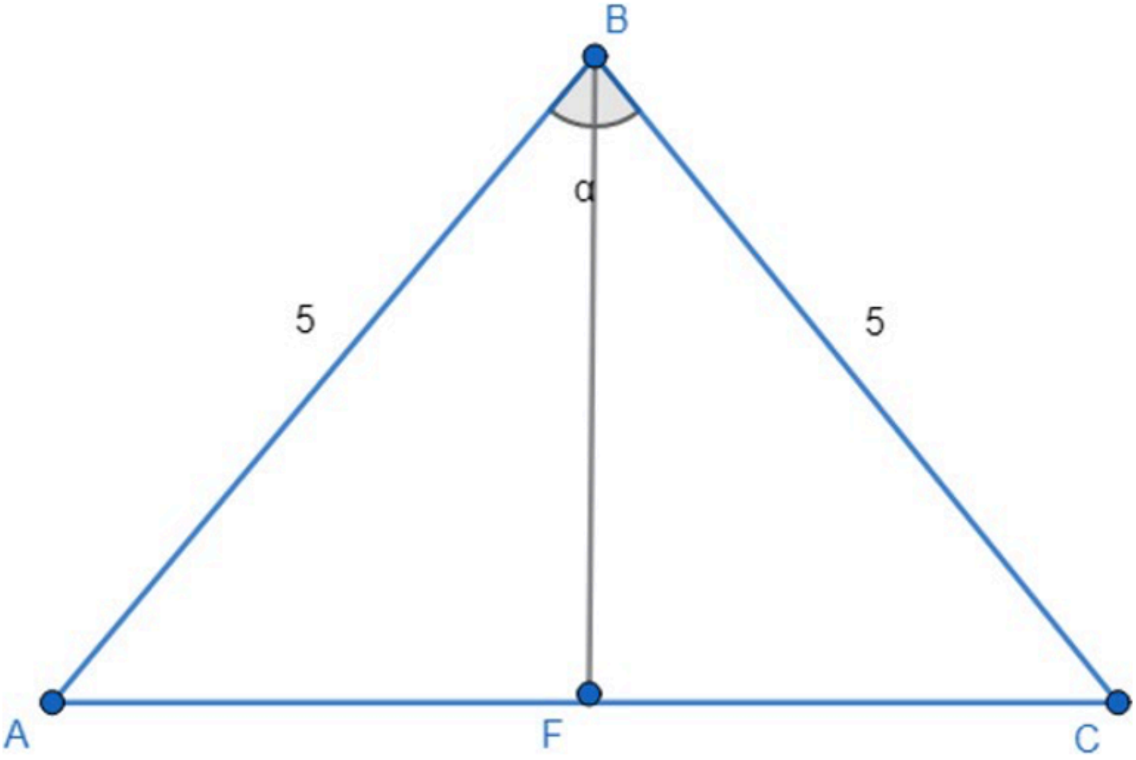


Use the general median formula:

$$m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4},$$

where $a = BC = 6,\ b = AB = 5,\ c = AC = 5.$

$$\begin{aligned} m_a^2 &= \frac{2 \cdot 5^2 + 2 \cdot 5^2 - 6^2}{4} = \frac{50 + 50 - 36}{4} = 16, \\ m_a &= 4 \text{ m}. \end{aligned}$$

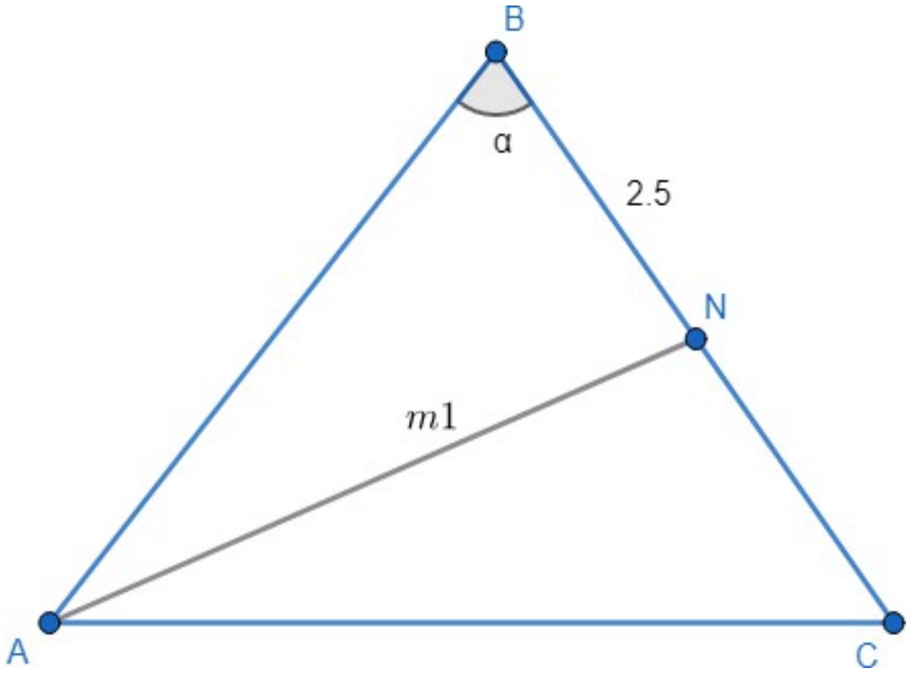


The remaining two medians are equal (the triangle is isosceles):

$$m_b = m_c, \quad m_b^2 = \frac{2a^2 + 2c^2 - b^2}{4},$$

$$a = 6,\ b = 5,\ c = 5.$$

$$\begin{aligned} m_b^2 &= \frac{2 \cdot 6^2 + 2 \cdot 5^2 - 5^2}{4} = \frac{72 + 50 - 25}{4} = \frac{97}{4}, \\ m_b &= \frac{\sqrt{97}}{2} \text{ m}, \quad m_c = m_b. \end{aligned}$$



Skills Required:

Apply the cosine rule correctly;

Use the formula for the length of a median;

Handle algebraic expressions involving radicals and powers.

Comment on Difficulty:

Moderate: because it requires

Accurate application of the cosine rule;

Remembering and using the median-length formula;

Multi-step algebraic calculations.

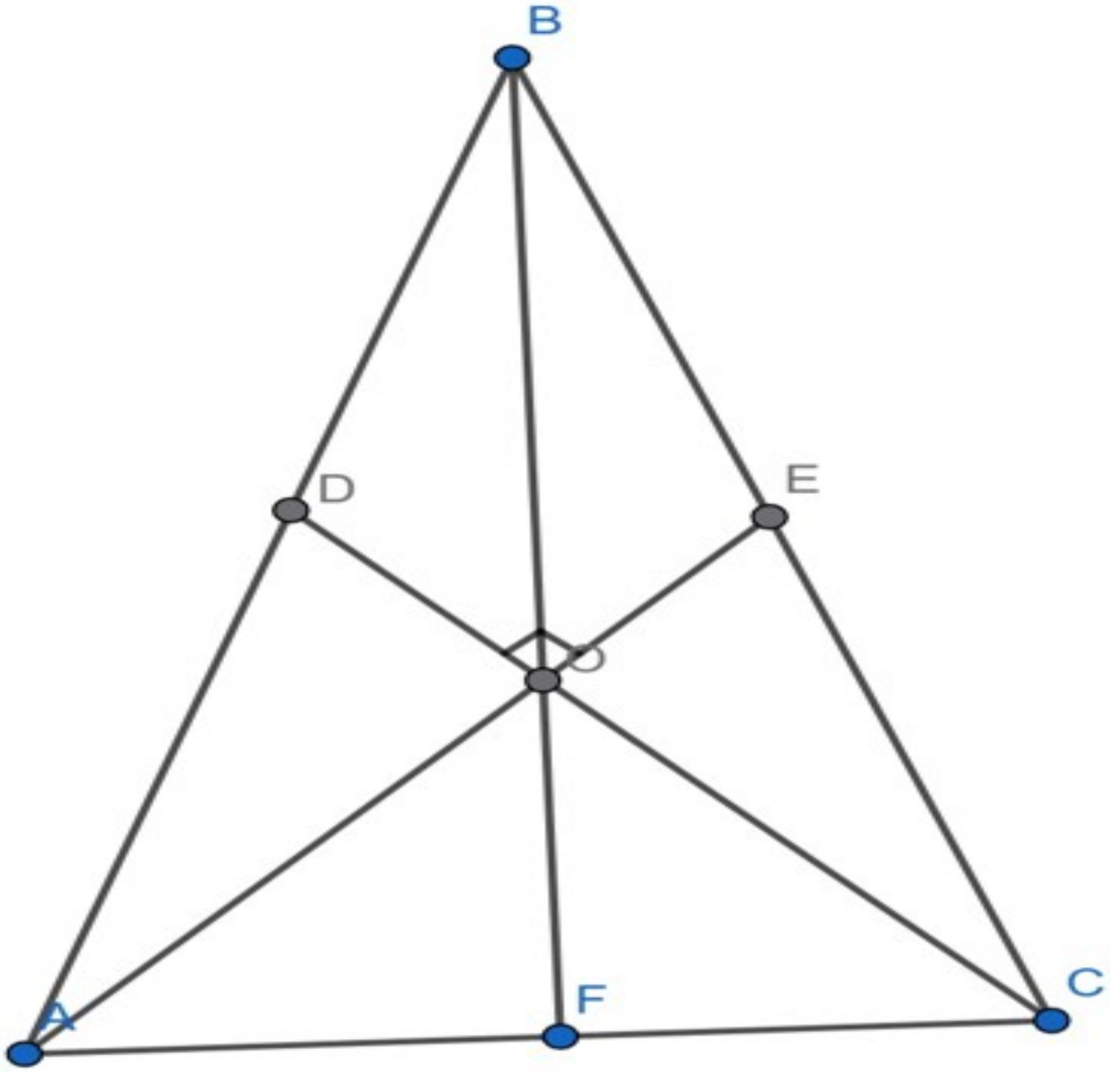
Appropriate for grades 9–10.

Conclusion: For pupils in grades 9–10 this problem can be regarded as of moderate difficulty, yet it still demands both theoretical knowledge and practical skills. The degree of difficulty ultimately depends on the learner's algebraic fluency and command of geometric theorems.

Differences Between Classroom and Competition Problems There are several key distinctions between the geometry tasks typically given in school and those set in mathematics competitions or university-entrance exams. First, with respect to content complexity, school problems are mainly aimed at consolidating foundational knowledge and require the straightforward application of standard theorems and methods. Competition problems, by contrast, call for intricate logical connections, the blended use of several theorems and a high level of creativity. Second, the form of the statement differs: classroom exercises provide clear, unambiguous conditions, whereas olympiad problems are often posed in an open-ended way, prompting the solver to search for multiple approaches. Third, time and setting diverge: classroom tasks may be completed with flexible timing, while contests and entrance exams impose strict time limits and prohibit external assistance. Fourth, the purpose is different. School problems are designed to reinforce and assess knowledge, whereas competition problems showcase a student's ability to apply theoretical ideas creatively and to reason logically. Thus the gap between classroom and competition problems is evident in their level of difficulty, mode of presentation and underlying objectives.

Group D. Sample Problem. Two sides of a triangle are 6 cm and 8 cm. The medians drawn to these sides are perpendicular to each other. Find the area of the triangle.

Figure 1.4.2 Triangle and medians



$AB = 8$ cm, $BC = 6$ cm, AE, DC, MF are medians, $DC \perp AE$.

Solution:

$$OE = y, OD = x$$

$$\begin{cases} y^2 + (2x)^2 = 3^2, \\ x^2 + (2y)^2 = 4^2 \end{cases} \Rightarrow \begin{matrix} y^2 + 4x^2 = 9, \\ x^2 + 4y = 16 \end{matrix}$$

$$y^2 = 9 - 4x^2$$

$$x^2 + (9 - 4x^2) = 16 \Rightarrow\ -15x^2 = -20, \qquad x = \frac{2\sqrt{3}}{3},$$

$$y^2 = 9 - 4\left(\frac{2\sqrt{3}}{3}\right)^2 = \frac{11}{3}, y = \frac{\sqrt{33}}{3},$$

$$AO = 2OE = \frac{2\sqrt{33}}{3},$$

$$S_{AOD} = \frac{AO \cdot OD}{2} = \frac{\frac{2\sqrt{33}}{3} \cdot \frac{2\sqrt{3}}{3}}{2} = \frac{2\sqrt{11}}{3},$$

$$S_{ABC} = 6 \cdot S_{AOD} = 6 \cdot \frac{2\sqrt{11}}{3} = 4\sqrt{11}$$

Skills Needed:
Knowledge of the cosine rule and properties of medians;
Ability to set up and solve systems of quadratic equations;
Use of geometric area formulae and segment division;
Facility with radical expressions.
Comment on Difficulty:
Moderate: perpendicular medians and solving a system of equations;
Multi-step calculations and efficient use of theorems;
Appropriate for grades 9–10.

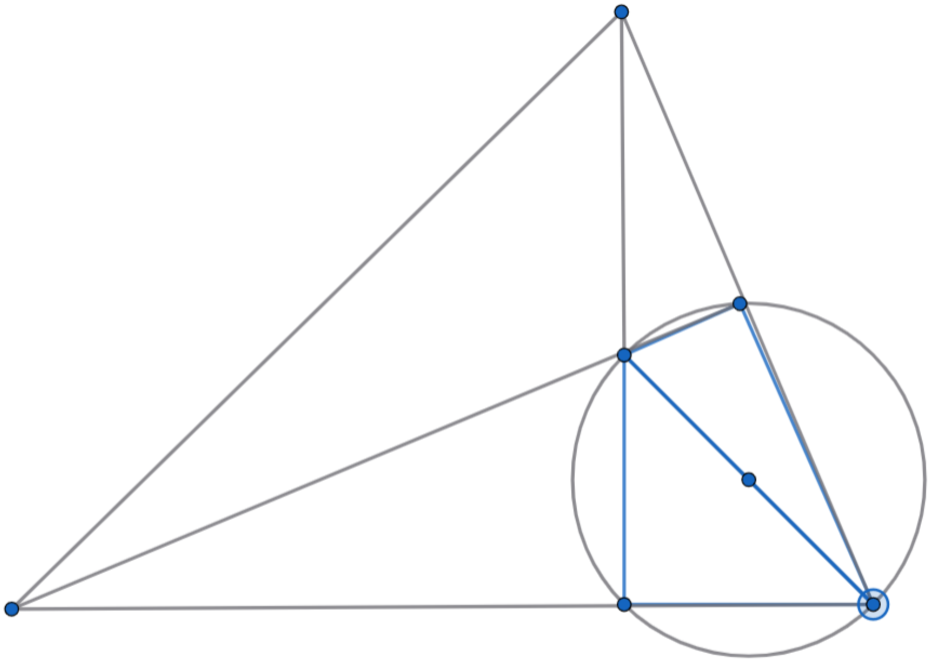
Group E. Sample Problem. In triangle KLM the bisectors KN and LP meet at Q. Points N, P, Q, M lie on a circle, and PN = 1 cm. Find the sides and angles of $\triangle$PNQ.

Solution:

1) For a cyclic quadrilateral the sum of opposite angles is 180°:
   First of all we need to prove:

$$\cos\alpha = \frac{PF}{QP}\ \Rightarrow\ QP = \frac{PF}{\cos\alpha} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = QN$$

Figure 1.4.3 Circle and bisectors



$$\begin{aligned}
\angle KQL \quad = \angle NQP \qquad & (\textit{vertical angles}),\\
\angle KQL \qquad & = 180^\circ - \beta - \gamma,\\
2\beta + 2\gamma + 2\alpha \qquad & = 180^\circ \;\Rightarrow\; \beta + \gamma + \alpha = 90^\circ\\
\angle KQL \qquad & = 180^\circ - (\beta + \gamma) = 90^\circ + \alpha,\\
\angle PQN \qquad & = 90^\circ + \alpha,\\
\angle PQN + \angle PMN \qquad & = 180^\circ \;\Rightarrow\; 90^\circ + \alpha + 2\alpha = 180^\circ \;,\\
\angle PMN \qquad & \Rightarrow\; 3\alpha = 90^\circ \;\Rightarrow\; \alpha = 30^\circ = 60^\circ, \quad \angle PQN = 120^\circ.
\end{aligned}$$

2) Because MQ is a bisector, all three angle bisectors concur, and since N, P, Q, M are cyclic we have

$$\angle QMP = \angle NPQ = \angle QNP,$$

so $\triangle\, QNP$ is isosceles. Taking $PN = 1$ cm and $PF = 1/2$ cm, the definition of cosine gives

$$\cos\alpha = \frac{PF}{QP} \;\Rightarrow\; QP = \frac{PF}{\cos\alpha} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = QN.$$

Skills needed:

Properties of cyclic quadrilaterals and inscribed angles;

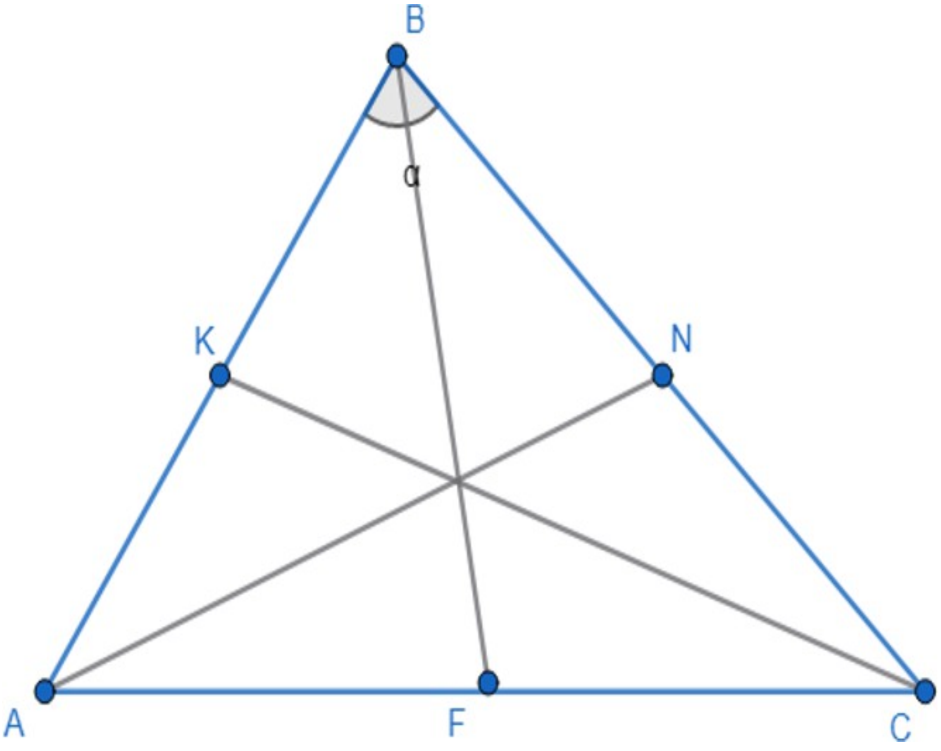
Intersection point of angle bisectors;

Application of trigonometric definitions (cosine).

Comment on Difficulty: Combines circle and angle-bisector geometry with trigonometric calculations, requiring upper-intermediate logical reasoning; suitable for grades 9–10.

Group F. Sample Problem. In triangle ABC the altitude AD, the median BE and the angle bisector CF intersect at O. If OE = 2 OC, find ∠ACB.

Figure 1.4.4 Triangle with altitude, median and bisector



Solution:

$$\frac{CD}{AC} = \cos 2\alpha$$

$$CD = 2a \cos 2\alpha$$

$$\cos \alpha = \frac{CD}{OC} = \frac{2a \cos 2\alpha}{x}$$

$$x = \frac{2a \cos 2\alpha}{\cos \alpha}$$

$$OE = \frac{4a \cos 2\alpha}{\cos \alpha}$$

$2)\Delta EOC$

$$OE^2 = OC^2 + EC^2 - 2 \times OC \times EC \cos \alpha$$

$$\frac{16a^2 \cos 2\alpha^2}{\cos \alpha^2} = \frac{4a^2 \cos 2\alpha^2}{\cos \alpha^2} + a^2 - 2\frac{2a \cos 2\alpha}{\cos \alpha} a \cos \alpha$$

$$\frac{4a \cos 2\alpha}{\cos \alpha} a \cos \alpha = a^2 - \frac{12a^2 \cos 2\alpha^2}{\cos \alpha^2}$$

$$4 \cos 2\alpha = 1 - \frac{12 \cos 2\alpha^2}{\cos \alpha^2}$$

$$4\cos 2\alpha \cos \alpha^2 = \cos \alpha^2 - 12\cos 2\alpha^2$$
$$12\cos 2\alpha^2 = \cos \alpha^2 - 4\cos 2\alpha \cos \alpha^2$$
$$12(\cos \alpha^2 - \sin \alpha^2) = \cos \alpha^2 - 4(\cos \alpha^2 - \sin \alpha^2)\cos \alpha^2$$
$$12(\cos \alpha^2 - 1 + \cos \alpha^2) = \cos \alpha^2 - 4(\cos \alpha^2 - 1 + \cos \alpha^2)\cos \alpha^2$$
$$12(2\cos \alpha^2 - 1) = \cos \alpha^2 - 4(2\cos \alpha^2 - 1)\cos \alpha^2$$
$$\cos \alpha^2 = t$$
$$12(2t-1) = t - 4(2t-1)t \Rightarrow \ 24t - 12 = t - 8t^2 + 4t$$
$$8t^2 + 24t - t - 4t - 12 = 0 \Rightarrow \ 8t^2 + 19t - 12 = 0 \Rightarrow t_1 = 1/2$$
$$\cos \alpha^2 = \frac{1}{2} \Rightarrow \ \cos \alpha = \frac{1}{\sqrt{2}} \Rightarrow \alpha = 45°$$

$$12\cos 2\alpha^2 = \cos \alpha^2 - 4\cos 2\alpha \cos \alpha^2$$
$$\cos 2\alpha = 2\cos \alpha^2 - 1$$
$$24\cos 2\alpha^2 = 2\cos \alpha^2 - 8\cos 2\alpha \cos \alpha^2$$
$$24\cos 2\alpha^2 = \cos 2\alpha + 1 - 8\cos 2\alpha \cos \alpha^2$$
$$8\cos 2\alpha\,(\cos 2\alpha + 1)$$
$$24\cos 2\alpha^2 = (1 + \cos 2\alpha) - \frac{8\cos 2\alpha\,(\cos 2\alpha + 1)}{2}$$
$$24\cos 2\alpha^2 = (1 + \cos 2\alpha)\,(1 - 4\cos 2\alpha)$$
$$\cos 2\alpha = t$$
$$24t^2 = (1+t)(1-4t)$$
$$24t^2 = -4t^2 - 3t + 1$$
$$28t^2 + 3t - 1 = 0$$
$$D = 9 + 112 = 121$$
$$t_1 = \frac{-3+11}{56} = \frac{8}{56} = \frac{1}{7}$$
$$\cos 2\alpha = \frac{1}{7}$$
$$\alpha = \frac{1}{2}\cos^{-1}\frac{1}{7}$$

Skills Needed:
Properties of altitudes, medians and angle bisectors;
Theory of cyclic angles and arcs (if used);
Trigonometric identities for $\cos 2\alpha$ and $\cos \alpha$;
Facility with algebraic expressions and quadratic equations.

Comment on Difficulty: This problem blends geometric and trigonometric techniques and requires solving several intricate algebraic steps. It therefore demands high-level logical reasoning and deep trigonometric insight.

1.5 The Role of Challenging Problems in Strengthening Geometric

Understanding

Geometry instruction is most effective when key theorems are paired with thought-provoking problems. Because geometry relies heavily on visual reasoning and logical proofs, many students struggle at first. By regularly exposing them to difficult, open-ended tasks, we help solidify their conceptual grasp of the subject (Lai, 2016). Developing Logical and Analytical Thinking Solving challenging geometric problems requires much deeper reasoning than memorising formulae. For instance, applying Ceva's or Menelaus' theorem often demands exploring several paths to a solution, thereby sharpening students' logical skills. Through proofs or step-by-step arguments they increasingly understand the relationships among angles, segments and triangles.Linking Theory to Concrete Application Hard problems highlight the broader relevance of geometric theorems. Many competitive tasks incorporate elements of algebra, trigonometry or coordinate geometry, helping students see mathematics as an integrated field. Discussing geometry's role in real-world areas such as engineering or architecture also makes the problem- solving experience more meaningful. Fostering Collaboration Challenging geometry problems lend themselves well to group work. In a collaborative setting students can share strategies, critique each other's proofs and arrive at solutions together. Explaining an argument to a peer or clarifying a tricky step is one of the best ways to consolidate one's own understanding while exposing everyone to fresh perspectives. Summary Routinely incorporating difficult geometry problems pushes learners from passive reception of knowledge toward deep comprehension. Such tasks reinforce core concepts, cultivate critical thinking and build the perseverance needed to master more advanced topics. Ultimately, this approach fosters lasting appreciation for geometry and paves the way for future success in mathematics.

1.6 Principles for Effectively Using Technology in Teaching

Geometry is a branch of mathematics through which we can lay the foundations of verbal-logical and algorithmic thinking, spatial reasoning, and mathematical literacy. Studying and applying geometric concepts shapes children's development in several stages: concrete thinking when manipulating physical objects (preschool), imaginative thinking when working with diagrams and drawings (primary school), and abstract, verbal-logical thinking (middle school). Consistently integrating geometric tasks into mathematics instruction from an early age fosters flexible thinking, boosts motivation and interest in geometry, and leads to a deeper understanding of the subject itself. Conversely, gaps in mastering and using definitions and properties of mathematical objects, along with an inability to interpret geometric figures correctly, give rise to the typical mistakes often observed in primary-school pupils. Principles for Effectively Using Technology in Teaching and Learning Mathematics. Today, the capabilities of new technologies are among the most powerful drivers of growth and development, both for mathematics itself and for mathematics education. In research, the use of computers has spawned entirely new approaches. In education, computers have highlighted the importance of certain ideas, opened access to new problems, and provided fresh means of representing and processing mathematical information, giving us an unprecedented range of content and instructional methods. Yet not everything that can be done should be done. Students' learning depends on many factors: teacher preparation, educational theories and principles, parents, curricula, students' interests and aspirations, resources, cultural expectations, technologies, and more. We can discuss each factor at length, but none can be fully understood in isolation from the others. This is especially true of technology, which partly explains why there is no universally accepted view on its optimal use in classrooms. Ultimately, the key question is not which hardware or software to employ, but how each tool functions within a specific curriculum, down to the impact of particular tasks assigned to individual students. All of this must be assessed in terms of educational effectiveness. The value of technology depends on the tasks set for the learner, not on the tools used to solve them. Whether with a computer or pencil and paper, solving some problems is crucial, while tackling others is simply a waste of time.

Effective technology integration in mathematics education hinges not on the novelty of the hardware or software but on how purposefully it is woven into the fabric of daily instruction, and researchers stress that teachers must begin by aligning every digital decision with clear learning goals, the specific cognitive demands of each task, and the diverse needs of their students (Ziatdinov, 2022). When dynamic tools such as GeoGebra take over the mechanical "heavy lifting" of calculations or constructions, they free learners to hypothesize, test conjectures, and iterate rapidly, thereby nurturing higher-order thinking that is difficult to cultivate through static pencil-and-paper work alone. Yet the mere presence of a laptop or tablet does not guarantee meaningful engagement: if technology is used as little more than an electronic flash-card deck, it can even impede genuine understanding by diverting attention from reasoning processes toward superficial button-pressing. Instead, digital environments should prompt students to predict, reflect, and explain—hallmarks of conceptual grasp—while teachers deliberately pause the action so that learners articulate how a graph morphs as a parameter slides or why a geometric locus emerges in a particular region of the plane. Such pedagogical moves ensure that the intellectual work rests with students, who gradually internalize habits of analysis and proof that persist when the screens turn off. Critically, students also need guided opportunities to become discerning, confident users who understand the affordances and limitations of each tool, can judge when technology streamlines exploration, and know when a simple sketch or mental estimation will suffice. This balanced, goal-driven approach—grounded in the classroom-tested guidance of Lyublinskaya and Tikhomirova (2017) and echoed in the National Council of Teachers of Mathematics' position that technology should "amplify and extend" reasoning rather than replace it (NCTM, 2015)—positions digital resources as catalysts for deeper mathematical sense-making rather than as distractions or ends in themselves (Tikhomirova, 2017).

Technology offers vast and exciting new opportunities that broaden both what we can teach and how we can teach it. Modern curricula increasingly embed purposeful technology use and supply resources that help educators gain the skills — and the confidence— to employ these new instructional tools. How can teachers engage with such programs? Stay proactive: monitor emerging digital possibilities, keep abreast of innovations, and cultivate a clear vision of what you expect technology to deliver. Explore new capabilities without becoming beholden to novelty for its own sake. Deliberately consider what you want to provide for your students—identify concrete learning objectives for each class and the individual needs of every learner—then evaluate whether a particular tool will move you toward those goals or distract you from them.

1.7 What Is GeoGebra?

Emerging computer applications such as GeoGebra let students at every grade level tackle problems in an interactive, exploratory environment while studying geometry and making measurements, all the while building computer literacy. GeoGebra is a free, cross-platform, dynamic mathematics package intended for all levels of learning. It spans geometry, algebra, spreadsheets, graphing and charts, statistics, and calculus. In Europe and the United States the program has earned several awards granted to software developed for educational purposes.

Quick Facts about GeoGebra:

1. Graphs, algebraic representations, and spreadsheets are interlinked and fully dynamic.
2. A user-friendly interface provides access to many powerful features.
3. Authoring tools allow teachers to create interactive instructional materials such as web pages.
4. The software is multilingual and reaches millions of users worldwide.
5. GeoGebra is free and open-source.

Official GeoGebra Website – http://www.geogebra.org/. Learn more about the program and download it free of charge. GeoGebra YouTube Channel – http://www.youtube.com/geogebrachannel. Tutorial videos and classroom examples. GeoGebra Materials Exchange – http://tube.geogebra.org/. A repository where users share GeoGebra activities and lesson plans. Technology has become an essential part of life and work in contemporary society, and one of teachers' chief responsibilities is to prepare children for the demands of the twenty-first-century job market. We believe this goal cannot be met unless technology is integrated into the learning process. The aim of this guide is to present concepts and resources that will help you steer your students' learning with the aid of technology.

In each section of the guide you will find the following subsections:

Key Ideas and Essential Concepts – this subsection identifies the specific geometric notions students are expected to master and the difficulties they may encounter while studying the topic.

GeoGebra Lessons – classroom-ready handouts for tasks created in the GeoGebra environment. Each handout provides step-by-step instructions for using GeoGebra and inquiry-based questions for students.

Teacher Guidelines – a detailed description of each GeoGebra task, including possible student responses and an overview of the corresponding GeoGebra files.

CD Materials – the accompanying compact disc contains GeoGebra files for every task, along with the student handouts in Word and PDF formats.

Some GeoGebra tools are highly visual and intuitive even for students who lack extensive theoretical preparation in geometry. Training files for these tools are listed in Table 2 below; the instructions for each tool are embedded directly in its GeoGebra file (*.ggb). Each file encourages students to experiment with the relevant tool, and the activities are designed to take only 10–15 minutes. In class, they can be completed individually or together at an interactive whiteboard.

Other tools—such as Parallel Line and Perpendicular Line—introduce new concepts. These tools are presented within GeoGebra lessons that also define and explain the associated ideas.

2. METHODOLOGY

2.1 Research Design and Participants

Over the course of many years teaching mathematics in upper-secondary schools across Kazakhstan, it has become increasingly evident that a significant number of students find geometric reasoning more accessible than algebraic manipulation. This may be attributed to geometry's emphasis on visualizing figures, angles, and spatial relationships, in contrast to algebra's reliance on formal symbolic processes. Compounding this issue is the limited instructional time often dedicated to geometry. Even teachers with mathematics degrees frequently find it challenging to allocate sufficient time in lessons for applied problems and meaningful discussion—both crucial for mastering core theorems and reasoning techniques.

As Bergmann and Sams (2012) emphasize, interactive, student-centred instructional models—when classroom time is planned strategically—can help address these limitations. A solid understanding of geometry is vital for students pursuing STEM-related fields or careers that require structured reasoning and spatial awareness. Nevertheless, geometry is often underrepresented in conventional curricula, where algebra tends to dominate instructional time. As a result, many students leave school without a strong grasp or lasting appreciation of geometric concepts—an area that could otherwise strengthen their overall mathematical foundation (Gasparič et al., 2023).

In recent years, the "Flipped Learning" model has emerged as a potential solution. This approach reverses the traditional teaching structure: students engage with core content—typically through short videos—before class, allowing more in-person time for collaborative work, hands-on practice, and focused teacher guidance (Zainuddin & Halili, 2016). By shifting direct instruction out of the classroom, teachers can support students more effectively during problem-solving sessions, rather than spending valuable time lecturing.

This study was designed with two main objectives. First, it involved compiling a curated set of nine challenging geometric problems that illuminate important theorems and logical techniques essential for deep understanding. Second, the research aimed to explore how Flipped Learning can enhance student engagement and comprehension of complex geometric ideas. Specifically, by assigning instructional videos as homework and using class time for active group work, the study investigated whether this model could help overcome the persistent shortage of geometry instruction in classrooms.

To achieve these goals, the study followed three main steps:

Develop a Flipped Learning framework that combines brief online lectures with in-class activities based on solving the selected geometric problems.

Compare student performance and engagement across traditional and Flipped Learning environments, applying the model in multiple teachers' classrooms.

Evaluate satisfaction levels, identify implementation challenges, and gather best practices for geometry teaching through surveys and interviews with both students and teachers.

This study compared the impact of Flipped Learning on geometric problem-solving with that of a traditional lecture approach. Participants were drawn from two parallel Year 10 classes in a Kazakhstani secondary school. In accordance with institutional ethical norms, consent to participate was obtained from all pupils and their parents/guardians. Traditional systematic approach. The instructional design for the median, altitude, angle bisector, perpendicular bisector, and mid-segment topic was anchored in an inquiry-oriented, two-lesson sequence (80 minutes total) outlined in the detailed lesson-procedure matrix in Appendix II (Table A2.1). That matrix served as the operational blueprint for all pedagogical decisions, ensuring alignment among learning objectives, in-class activities, and assessment checkpoints. The unit opened with an activation phase in which learners recalled prior knowledge about triangle classification and angle measurement.

By eliciting definitions and diagrams that students already possessed, the teacher created cognitive links to the new constructs (median, altitude, etc.), thereby lowering the intrinsic load of subsequent material (Appendix II, Table A2.1, Rows 3-4). During the concept development phase the teacher employed a controlled “guided discovery” model. Instead of direct exposition, students were prompted with Socratic questions such as “How many medians can be drawn in any triangle?” and then asked to test their conjectures on self-generated obtuse, right, and isosceles cases. This approach simultaneously cultivated deductive reasoning and drawing skills while reserving teacher talk for timely clarification (Appendix II, Row 5). The practice and consolidation phase leveraged Microsoft Teams Class Notebook to orchestrate authentic mathematical discourse. Two focal learners solved tasks live in the shared notebook while peers reproduced the same constructions with compass and straight-edge in their exercise books, uploading photographs for whole-class critique (Appendix II, Rows 6-7). This digital mirroring fostered accountability and immediate formative feedback without fragmenting lesson flow. Task design embodied the variation principle: the core idea (a special segment) was held constant while contextual parameters—triangle type, given measurements, or required proof—were systematically altered. For instance, Task 1 asked students to determine the perimeter of ΔABC from the lengths of its component triangles created by a median, whereas the extension activities required comparing concurrency properties of medians and altitudes across right, scalene, and isosceles triangles (Appendix II, Tasks 1–3 & Supplementary 1-2). Such variation amplified conceptual depth and discouraged rote transfer. The assessment was primarily formative. Peer-checking in breakout pairs enabled students to calibrate their reasoning, while the teacher’s verbal probes and targeted “in-the-moment” feedback highlighted common misconceptions, e.g., conflating perpendicular bisectors with altitudes in obtuse configurations (Appendix II, Feedback column). Summative judgment was deferred to a homework set (Textbook, Part 2, p. 12, Exs. 3-4) whose items paralleled in-class explorations, maintaining constructive alignment.

Finally, structured metacognitive reflection closed each lesson. Learners responded in prose to four prompts— "What did I learn?", "What remains unclear?" and so on—thereby externalizing self-assessment data for the teacher's post-lesson reflection cycle (Appendix II, Reflection section). Insights from these reflections guided iterative adjustments to pacing and scaffolding in subsequent units. In sum, the teaching methodology blended interactive exposition, technology-mediated collaboration, deliberate practice through task variation, and recursive reflection, all organised around the coherent framework presented in Appendix II. This design not only met the stated learning objective 7.3.2.6 but also modelled best practices in formative assessment and learner autonomy for middle-school geometry. Sampling and Allocation. A total of 120 students (aged 15–16) were split evenly into two groups of about 60 each. Because full individual randomisation was impractical due to the timetable, intact classes were assigned either to the Focus group (traditional instruction) or the Flipped group (pre-class video lectures). To minimise teacher effects, three mathematics teachers rotated between the two cohorts every two weeks so that instructor differences were not systematically confounded with teaching method. Demographic Data. Participants' socio-economic backgrounds were similar, and most had prior experience using a computer or mobile device for educational purposes. School records indicated no significant difference in baseline mathematics achievement between groups ($p > 0{,}05$). Ethical Considerations. Approval was obtained from the school administration and the local ethics committee. All personal data were anonymised before analysis. Students could withdraw at any time without penalty; no extra credit or grades were offered for participation.

2.2 Implementation of Flipped Geometry Instruction

The Flipped Learning is based upon the principles of the constructivist learning theory. Constructivism is based on the ideas of Jean Piaget and Lev Vygotsky in regard to pedagogical practice (Bergmann & Sams, 2012); according to the given approach, students should also actively construct the knowledge as it is up to them to shape it, but not to passively accept the information. Practically, Flipped Learning encourages learners to take ownership over their learning; they do not remain passive receivers of information as they become active inducers who analyze study materials on their own, discuss concepts with other students, and ask questions. This increased involvement strengthens inspiration and reflects to better learning. The authors (Bergmann and Sams, 2012) provide a strong case as efficiency with the use of the model in that it not only increases the engagement level of the students, but also enhances the engagements between the teacher and the students, as the instructors are left with more time and opportunity to respond to queries during the lessons. A later study shows that Flipped Learning reflects a higher achievement in academic results as compared to traditional lecture-based teaching (Ece, 2020).

The academic research on mathematics learning through Flipped Learning acquired a significant number of researchers during the last decade. As several studies state, the strategy does improve the performance of students in mathematics by setting more possibilities of combining theoretical knowledge with practical exercise. In one study, for instance, (Strayer, 2012) reported that students in a flipped mathematics course (where the important concepts were first introduced out of the classroom) had more correct solutions to problems and a higher conceptual appreciation than their peers taught using standard lectures. Similarly, Bishop and Verleger (2013) indicate that that the approach assists students to understand the abstract concepts of geometry and use them in real life contexts. Geometry, especially the issues associated with the special points of triangles, brings a twin dilemma: it requires not only spatial ability but also corrective thinking (Gasparič et al., 2023). Flipped Learning helps to release the precious classroom time to exploratory learning activities where students prove theorems, create diagrams and check conjectures using illustrative examples. Such a structure is particularly appropriate to teaching triangle center properties, because as pupils solve specific problems in the classroom, they come to see that how the abstract properties they are learning apply in a concrete and in a competition context is deeply integrated.

An emerging empirical research literature confirms the Flipped Learning model works. As students take more responsibility in preparing themselves and actively participate in classrooms, the quality of learnings improves. In systematic review, O Flairty and Phillips (2015) stated that the strategy results in huge gains in a higher-education context not only in academic performance but also in increasing the extent of classroom engagement. In school mathematics, Strayer (2012) also obtained results according to which a flipped environment improves the understanding of the central topics and higher scores on problem interference tasks; additionally, the respondents reflected on their being much more confident in performing mathematical tasks. These results highlight the importance of using an instructional design that would encourage learners, especially those majoring in geometry, to think aloud, pose provocative questions and put their thoughts to the test with practical activities. Geometry can perhaps be called one of the handful branches of mathematics which requires not only superior spatial ability, but also a high level of abstraction. To conceptualize its concepts and principles, pupils should initially learn the main theory at home and then struggle with a great variety of tasks in the classroom, which this model precisely manages to accomplish.

Once theoretical material has been assimilated by the students, on their own, classroom time may then be spent using real exercises to verify, clarify and broaden their knowledge. For example, a unit on the special points of a triangle. Being in a flipped setting, students enter the classroom after having already learned through brief videos the definition of the orthocenter, centroid, and associated centers, so this lesson could be dedicated to be creating those points, quantifying their difference, and evaluation why the medians, the altitudes, and down the line angle bisectors intersect where they do. Since the solutions to such tasks are found in a team, the pupils are motivated to share opinions and provide peer support, being active during the session. Helping teachers to realize a better view of what the students are thinking, Flipped Learning also helps teachers integrate conceptual knowledge with practical skills. Circulating among the group work, the instructors may diagnose correctly misconceptions, e.g. the customers mix up the centroid and the circumcenter, and give specific advice, either singly or in small groups. The effect is the creation of a learning environment where each of the students is constantly pushed, assisted, and encouraged to recognize geometry not as a collection of unconnected facts, but as a united system of interconnected concepts.

Students consider one of the canonical problems in optics and geometry where two plane mirrors OA and OB share a right angle between them (90 AOB). A light ray is launched on segment MS, hits the mirror OA in an acute angle 0, and follows the law of reflection, so that it is driven to mirror OB, and reflected twice, then emerges along NT. With some basic angle-chasing and the equality angle of incidence = angle of reflection, the learners have to show that the original incident ray MS and the ultimate reflected ray NT are parallel. The important construction on the hand-out is of right angles at O and N, equal acute angles at the first reflection, and labelled supplementary angles at the second one. Even the proof skeleton provided by the teacher thus already proves (1) that the angle that lies between N and M, that is, $\angle$ MON, is a right angle, (2) that the two supplementary angles on the same side of transversal ON together equal 180 degrees (so that 90 + 90 = 180), and (3) that since by the converse of the interior-angles criterion, all angles that the transversal NT cuts off are right angles, The combination of the two teaching methods, (i) a teacher-centred model whereby the instructor presents the construction and helps calculate the angles step by step, and; (ii) a student-led, inquiry-based version where students construct the missing fragments of the proof, share their ideas, and explain every statement collectively, are intentionally tested against the following single problem. The way in which performance and engagement are analysed in each of the two conditions enables the lesson to evaluate the extent of support that each methodology affords conceptual comprehension of reflection geometry and proof-writing ability.

Example task, for both methodologies:

Task: The angle between mirrors OA and OB is 90°.

A ray of light falls on mirror OA at an angle, reflects from it, and then reflects again from mirror OB.

Prove that the incident and final reflected rays are parallel.

The mathematics teacher conducts this part of the lesson.

b) The students complete the proof:

According to the given condition, $\angle$MON is a right angle.

The angles are acute and equal to 90°.

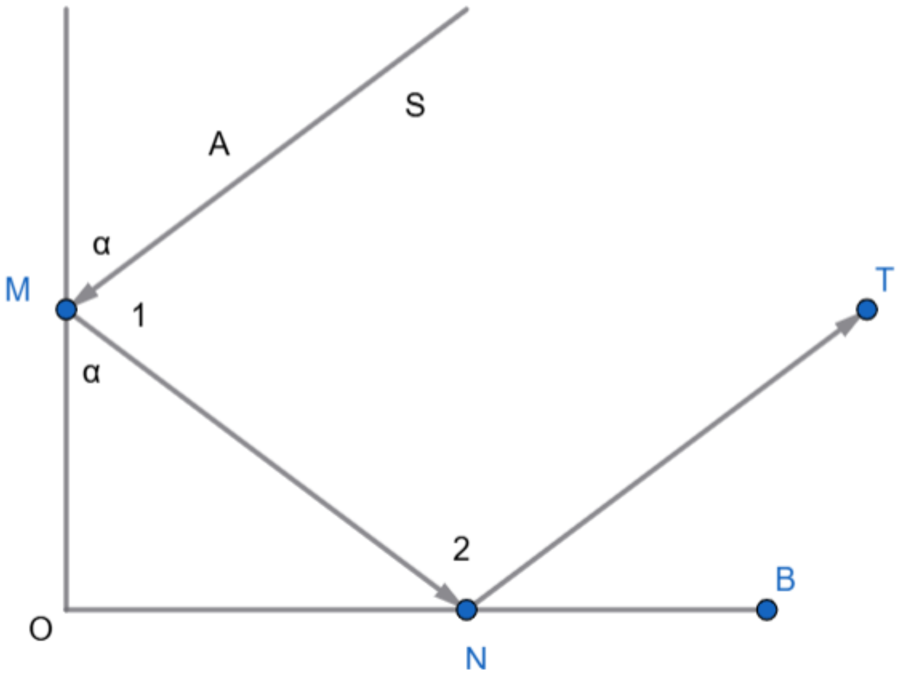
$$\angle 1 = 180^\circ - 2\alpha \,;$$

$$\angle 2 = 180^\circ - 2(90^\circ - \alpha) = 180^\circ - 180^\circ + 2\alpha = 2\alpha \,;$$

$\angle 1$ and $\angle 2$ are on the same side;

$\angle 1 + \angle 2 = 180°$. Therefore, MS || NT.

Figure 2.2.1 The angle between mirrors



2.2.1 Video lecture approach

To deepen conceptual understanding of the concurrency of altitudes (the orthocenter) a flipped-classroom micro-cycle preceded the face-to-face lesson. Two days before the in-class session students received a ten-minute screencast entitled “Intersection of Heights” hosted on the course YouTube channel; its link, viewing analytics, and transcript are archived in Appendix III. The video combined dynamic-geometry animations with voice-over commentary that derived the existence and location of the orthocenter for acute, right, and obtuse triangles. Pausing cues embedded at 2:41 min and 6:09 min asked learners to sketch conjectures in their notebooks, thereby transforming passive watching into active prediction (see Appendix III). During the pre-class window students completed an online retrieval quiz that recycled the video’s key propositions—e.g. “State where the orthocenter lies in an obtuse triangle”—and provided immediate auto-feedback. The learning-management system showed a 94 % completion rate and an average score of 78 %, confirming that most learners arrived with an initial mental model of altitude concurrency (Appendix III). Classroom time was then liberated for higher-order activity. Working in triads, students constructed three non-congruent triangles on geoboard paper, located their orthocenter with compass and straight-edge, and annotated patterns regarding relative interior/exterior positions. The teacher circulated, prompting learners to reconcile any discrepancies between their constructions and the pre-viewed derivations, thus converting potential misconceptions into discussion artefacts.

A whole-class synthesis followed, where volunteers projected their work under a document camera and verbally justified why the orthocenter of a right triangle coincides with the right-angle vertex, explicitly linking back to the visual proof in the video (referenced in Appendix III). The flipped format yielded two pedagogical dividends. First, cognitive load associated with the initial proof was shifted to an asynchronous setting, allowing students to engage with the argument at their own pace—rewinding or pausing as needed—before encountering the construction tasks. Second, the in-class session shifted the focus on course delivery to mathematical discussion and the execution of elements of precision drawing, which uses quick feedback among peers and teachers.

Summary post-lesson formative polls revealed a 32-pp increase in accuracy in determining orthocenter location between different types of triangles compared to the pre-class quiz and therefore demonstrates the effectiveness of the flipped cycle recorded in Appendix III. During the study, over eight weeks, flipped group learned the most fundamental concepts in geometry angle bisectors, concurrency theorems, tangents to circles all selected because of their role in curriculum and their potential to generate interest. The two groups were exposed to all the same geometry problems, 9 in number but they had difference in their in-class routines. Flipped group: Video lessons day before the classroom (10 15 min). Students viewed short online video clips presenting the main definitions, theorems and solution tactics using a smart phone or a laptop. As an illustration, in the concurrency video, Ceva and Menelaus theorems were demonstrated using performed examples. This independent work determined that the pupils came to work on applications right away. In-class activities. Each student was asked to work on the nine problems of increasing difficulty, in three or four person groups after a brief whole-class discussion. The instructor moved around and supplied brief clues or illustrated complicated demonstrations on the board. This kind of organization facilitated participatory interaction and enabled individual attention.

Lesson: This was a two-part lesson where students were imparted with the knowledge of five basic parts of a triangle; median, angle bisector, altitude, perpendicular bisector and mid-segment. Learners reinforced their knowledge on the definition of each segment and on how to construct it through instructed guidance, construction exercises and individual assignments. Visual reasoning and geometric precision was focused as the students were trained to draw these segments on acute, obtuse, right and isosceles triangles. Exploring the segments on their own and extracting their positions with respect to sides and angles of a triangle, students understood specific features of each of them. They could, as an example, be taught that medians, altitudes and perpendicular bisectors all share a common point, the centroid, orthocenter and circumcenter, respectively. The study of the medians in a right triangle and altitudes in an isosceles triangle has allowed learners to generalize those principles on other kinds of triangles. There was collective learning through use of Teams Class Notebook in which students posted their work and provided peer feedback. It did not only promote self-reflection and peer assessment but also allowed strengthening of concepts in a positive atmosphere. The guided questions and consolidation activities promoted the critical thinking process and enabled picking out the most common misconceptions, involving the distinction between medians, angle bisectors, and altitudes. After completing the lesson the majority of the students managed to achieve the learning tasks: they could define the segments and correctly build them in the triangles. These skills should be strengthened by vocabulary homework assignments and help learners to be ready to do more complicated geometric arguments in further subjects. In sum, the lesson gave a solid background learning of triangle geometry, including the theoretical and building capabilities- a necessary process of gaining spatial sense and accuracy in mathematics. Heavy practice and contemplation is going to further cement these notions so that they become internalized.

Focus group: The traditional lecture model was retained: the teacher spent most of the lesson explaining theory and examples to the entire class. Students worked on problems individually as homework, leaving little time for deep analysis during class. Because theoretical material was moved outside class for the Flipped group, more face- to-face time was available for problem solving. Given that the content was identical, any observed differences can be attributed to delivery mode alone.Teacher Professional Development. Before the study the three teachers completed a two-day workshop on designing flipped lessons and managing active classrooms. They were also trained in monitoring student participation and guiding small-group work.

2.2.2 Reflective report on the lesson

In recent years it has become plainly evident that, among the values Kazakhstani parents cherish, investing in their children’s education stands out with special intensity. Every parent now scrambles to place a child in a modern school, hire tutors, and enroll them in up-to-date clubs, and against this backdrop the social weight of mathematics is unmistakable, for whatever the field, mathematics occupies a unique, irreplaceable niche. Serving as proof not only of its parallel march with other sciences but of the way all sciences are interconnected through mathematical reasoning, so society increasingly views mathematics as an indispensable gateway to mastering any discipline. After consulting colleagues in my own and neighboring schools, I chose Grade 11 “M2” for this study because two years earlier I had carried out an Action Research project by applying two different teaching models—traditional and Flipped Classroom—to two classes without pre-selecting by achievement, since my aim was to shift my own teaching paradigm rather than compare classes; though I could not definitively state whether that first project succeeded, the class taught traditionally later asked me to adopt the flipped model. Last year I therefore blended both methods across two ninth-grade and two tenth-grade groups and in 2023–2024 continued recording video lessons for my YouTube channel and Edmodo, nurturing students' independent-learning skills to narrow achievement gaps; unlike mainstream schools, ours (over half my tenure has been at Nazarbayev Intellectual Schools) treats mathematics as resting on three "whales"—statistics, vectors, and differential equations—whereas comprehensive schools offer only rudimentary vectors, have just begun statistics, and omit differential equations entirely, so our students reach university ready for all three.

To spark interest I designed a flipped unit in the Grade 11 curriculum, "Vectors and Coordinates," focusing on "Angles between Planes," with a personal-development goal of helping students design descriptors and document solutions during creative tasks; because spatial vectors naturally extend planar vectors, and the only difference is the third dimension, I first reviewed planar vectors via a recap video. The lesson objective was "Find and apply the angle between a line and a plane, and between two planes," with success criteria: (1) students can determine the angle between two lines using vectors and (2) students can determine and apply the angle between planes. To ensure students themselves uncovered the lesson goal and to engage everyone through questioning, we began by discussing the homework video to see who had watched it and to gauge their recall of planar vectors, then I launched a teacher-led brainstorm—How to find the angle between two lines? Can a plane sometimes be treated as a special case of a line? —so students could grasp that spatial coordinates differ only by an extra dimension. After restating the goal with students, I drew two vectors on a coordinate grid, asked for the angle, demonstrated plotting a 3-D point via the parallelepiped method, reminded them of line-equation forms (especially the vector form containing the direction vector), recalled the general and vector forms of a plane, elicited that the cross product of two in-plane vectors yields the normal, wrote two non-parallel plane equations, found their normals, and asked the class to determine the angle between those normals and then the planes themselves, also sketching a quadrilateral to target the required angle. We followed with a paired activity in which students found each plane's normal vector, the angle between normals, and expressed answers in standard form, then tackled five practice problems split into basic "know/understand" and medium "apply" tiers for differentiation.

Next I formed three groups for creative tasks: (a) an error-finding exercise in a worked solution; (b) a parametric problem requiring verification of parameter values; and (c) an individual creative problem from *Cambridge International AS & A Levels, Pure Mathematics 2&3* involving archery—given an arrow's firing vector and the target plane, compute the score if the arrow follows its vector and how many degrees the archer must adjust to hit the bull-eye—where students had to devise descriptors before solving. Because my Action Research centers on critical-thinking and analytical skills through creative tasks, each lesson includes at least one solo exercise and training in question-asking; my broader aim is to minimize variance among formative, term, and external assessments, and the mix of group and individual work seems to succeed as mutual trust and debate have grown. Although I have not abandoned traditional teaching entirely, I have fully embraced the flipped model; students look forward to new videos, which they can watch, pause, rewind, comment on, rewatch before tests, and use when absent, so demand for extra lessons has nearly vanished, and five minutes of video discussion frees seventy-plus minutes for in-class activities.

On Edmodo I post homework, assessment reviews, and videos, monitor login times, and track feedback. A typical lesson unfolded as follows: dissecting the video revealed who had watched it; when I posed an angle-between-segments problem, student Q.A. suggested using segment lengths and the cosine rule, while student Á.O. proposed vector representation, prompting the class to recall the cosine formula for two vectors; on whether a plane can appear as a degenerate line, student M.S. noted that if one coordinate is zero the plane is parallel to a coordinate plane and can look like a line from certain viewpoints; students then applied the formula to a 2-D pair, plotted a 3-D point via the parallelepiped, named the vector form of a line, plotted given points, wrote radius vectors and magnitudes, self-checked, and recalled plane-equation forms, confirming that the cross product yields a normal. After drawing two normals and finding their angle, I asked for ideas; student Á.B. proposed finding the angle between planes, and by sketching a quadrilateral identified today's topic. Paired practice comprised five problems graded by criteria—cross product, normal vector, angle between vectors, acute angle between planes, standard-form answer—peers marking with "+" or "–," most errors lying in criterion 5. Group work followed: each team handled two problems, first correcting mistakes in a provided solution and then solving a parametric task; two teams finished early, while a third needed guidance. I constantly asked "Why? What makes you think that is wrong?" to deepen reasoning.

Observing pairs and groups confirmed alignment of goals and outcomes; before the final summative-style individual problem (multistep: devise descriptor plan, solve, compute point values, self-assess) I again elicited the lesson goal. Nearly half the class finished early; only three students failed to allocate points because they had not learned to break solutions into descriptors. Through debate we agreed the task deserved five points. Students self- and peer-assessed; honest self-evaluation grew once they saw fabricated marks were pointless and once I rechecked scores regularly. Assessment discussions dwelled on exact vs. rounded values and standard notation. Feedback revealed students could now recognize parallel planes without calculation and identify obtuse angles between planes, though I must include logical-sequence checks in every plan. My personal-development plan—to help students design descriptors and present solutions through the flipped model—continues with this 11 "M2" group in its third Action Research year; indeed, all my classes now follow Flipped Classroom, each Edmodo topic offering a video and tasks that will aid independent exam preparation. Over fifty teachers in our network already use the flipped approach, and in planning next year's professional-development aims we will share ideas in seminars, courses, or online communities, extending this collaborative effort.

In today's swiftly changing world, cultivating a competitive, future-ready learner is every teacher's foremost aim, and within Nazarbayev Intellectual Schools this goal is anchored in two core values—lifelong learning and the systematic development of each pupil's investigative skills. During my previous appraisal I successfully applied and studied a "flipped-traditional" hybrid model with senior classes; this cycle I resolved to extend the same method to the middle grades. Early evidence suggests that, when younger pupils rigorously plan, carry out, and draw conclusions from small-scale inquiries, their capacity for self-directed learning grows markedly. Predictably, they began by asking "What exactly must we do?", "Is there an algorithm?", or "Where in real life can we use this?", yet over time they discovered that every investigation carries its own mix of setbacks and breakthroughs, that finding information independently is both possible and empowering, and that not every pattern maps neatly onto practical life—though localised problems can indeed be solved. To nurture such critical thinking the teacher must embed tasks that demand analysis whenever a new concept is introduced and must follow up with individual or group creative challenges to consolidate knowledge. For my middle-school trial I chose an eighth-grade unit entitled "Transformations and Applications of Quadratic Functions," precisely because it lends itself to miniature research projects. Under a flipped-learning regime paired with BYOD (bring-your-own-device), each student re-watched the teacher's video lesson, used ICT—specifically Geogebra—to plot quadratic graphs, corrected errors, shifted paradigms, and compared their work to dynamic models.

They also learned to apply maximum- and minimum-point analysis to physics and to stereometry problems. Alignment between lesson objectives and outcomes was checked as pupils presented and defended their inquiries, received teacher feedback, and evaluated one another according to explicit criteria; peer- and self-assessment were woven into the plan. The flipped-traditional blend retains the key advantage of assigning new theory as homework via short videos. Middle-grade pupils at first treated the strategy like a game and struggled to take it seriously, but soon they began anticipating each new clip. Because the videos were personally recorded and uploaded to YouTube—and punctuated by anchor problems—students not only watched them but also explored related clips and shared discoveries with classmates. Absent pupils no longer missed core explanations, and in-class time could be devoted to scaffolding lower-attaining learners. A recurrent obstacle was lack of home internet, yet affected students used school facilities after lunch to catch up. The lesson itself, focused on transforming, analysing, and applying quadratic functions, sought outcomes such as locating function and argument extrema, deploying them in word-problems, and drawing reasoned conclusions. We opened by querying the homework video and administering a five-question diagnostic test, which pairs marked for each other before comparing to the teacher's key—an exercise that heightens engagement and dispels fear of mistakes. Next, each pupil drew three quadratic graphs by hand, then reproduced them in Geogebra on a personal device, gauging accuracy independently; initial difficulty in entering equations soon gave way to discoveries about their devices' capabilities, elegantly blending mathematics and ICT in line with BYOD principles.

In the following partner phase, pairs tackled two problems apiece: one demanded locating a parabola’s vertex without sketching, the other required writing a function from a given graph. Although six pairs worked, only three card types existed, with reciprocal conditions to provoke comparison; after mutual checking inside pairs, pupils cross-checked with those who had the mirror task, explaining conceptual gaps to one another while the teacher repeatedly asked, “What are we doing now?” and “What was today’s objective?” to keep goals salient. The research phase split the class of thirty into two teams of six. Team A faced a stereometry problem: determine the dimensions of a wooden rectangular parallelepiped (lateral area 240 cm^2, height 1 cm less than an edge of its square base) that minimises paint consumption. Team B analysed a physics model of vertical projectile motion, $h = 1 + 15t – 5t^2$, answering when the ball reaches 11 m, when it lands, and at what time it attains maximum height. Armed with flip-charts and markers, students assigned roles so that anyone less involved in computation would present the findings. They decomposed each problem into descriptors, solved step-by-step, allocated point values to each descriptor, documented everything on the flip-chart, and defended the solution before fielding questions—within a twenty-minute limit. The crucial skill here was recognising and correcting dead-end approaches while the teacher merely observed. Should a member falter in a role, teammates stepped in, exemplifying true collaboration. Once time expired both groups presented, answered probing questions (contentious debates erupted around point allocations), and only after discussion did the teacher reveal an official solution.

2.3 The Role of Video-Lesson Design in Geometry

The videos were intentionally short (10–15 min), segmented and animated with on-screen cues—three empirically supported features that partly explain their instructional advantage. Length and Segmentation. Micro-lectures under 15 minutes reduce cognitive overload (Mayer, 2020). Kwon and Woo (2017) showed that breaking a 26-minute proof into three logical blocks doubled note density and raised test scores by 0.42 standard deviations (Kwon, 2017). Dynamic Visualisations. In all proofs, equal angles and congruent segments were revealed step-by-step. Such cueing improves transfer on spatial topics (Lin, 2019). Thong and Pan (2022) reported a 28. Embedded Retrieval Practice. Each clip contained two quick-check questions in the LMS (Thong, 2022). Clickstream data showed a 97. Teacher–Student Co-creation. Following Longo et al. (2020), two videos were produced with student volunteers. Survey feedback rated these clips “easy to follow,” echoing Longo’s findings on increased relevance. Literature Review on Flipped Geometry. Table 2 summarises eight studies that applied Flipped Learning to Euclidean or analytic geometry. All reported positive or neutral outcomes, with effect sizes in the medium-to-large range ($g = 0.35–0.91$). Our adjusted effect size ($g = 0.75$) lies near the upper quartile. This arrangement was meant to increase student involvement during asynchronous learning and fit cognitive load theory. Emphasizing the importance of clarity and pace, the design also considered comments from a test phase by students.

Table 2.3.3 Selected Empirical Studies on Flipped Learning in Geometry

Author(s)	**n**	**Topic**	**Effect size (g)**
Kwon (2017)	104	Triangle proofs	0.58
Hung (2015)	92	Coordinate geometry	0.41
Talbert (2017)	78	Analytic geometry	0.50
Zengin (2017)	66	Concurrency theorems	0.76
Chen (2019)	120	3-D geometry (AR support)	0.91
Longo. (2020)	143	Circle geometry	0.35
Thong (2022)	118	Circle theorems	0.69
Alphen (2022)	284	Meta-analysis (multi-topic)	0.54 (mean)

Alignment with Multimedia-Learning Theory. Short, segmented clips, dual-channel visuals and embedded practice all align with Mayer's updated Cognitive Theory of Multimedia Learning (CTML 2.0; Mayer, 2020).

Implications for Practice. Future flipped-geometry courses can:

Create 6–8 minute videos for each proof step, inserting self-check questions every 3–4 minutes;

Use progressive cueing to synchronise verbal explanations with visual constructions (Lin and Tsai, 2019);

Pilot student-generated videos on loci or tessellation topics to capitalise on the motiva- tional gains reported by Longo et al. (2020) (Longo, 2020).

These refinements can transform flipped lessons from "time-shifted lectures" into multime- dia experiences that boost cognitive efficiency and learner autonomy.

2.4 Data-Collection Instruments

Four main tools were employed to gauge student achievement, perceptions and classroom interaction:

Pre- and Post-tests: Each contained 15 short-answer and multi-step items, including questions based on the nine challenging tasks used in the study. The Pre-test measured baseline competence, while the Post-test captured eight weeks of progress.

Questionnaires: An online survey assessed students' (a) perceived ease/difficulty of learning geometry, (b) satisfaction with the teaching method, and (c) attitudes toward flipped-learning elements (e.g. video clarity, in-class activities). Reliability was high ($\alpha > 0.90$).

Teacher Observations: Using a standardised rubric, instructors recorded levels of engagement and collaboration (e.g. frequency of discussion, time-on-task). Each group was observed once per week.

Focus-group Interviews (optional): Twelve students (six from each cohort) participated in semi-structured interviews to discuss the strengths and challenges of the approaches.

Instrument Validation. The questionnaire and observation rubric were piloted in a class of 15 students; unclear wording was revised. Inter-rater agreement for the three teachers' observation data was 0.85 (Cohen's κ), indicating strong consistency

2.5 Teaching with GeoGebra.

Before you start teaching these lessons, it is important to stress how not to use them. The units in this guide are neither a complete geometry curriculum nor an introductory tutorial to the GeoGebra software. Using these materials immediately immerses learners in the GeoGebra environment. The rationale is to give students a software tool that lets them view geometric concepts from a fresh perspective. The first goal, therefore, is for students to gain a degree of fluency with GeoGebra; training activities included in the guide can be used to familiarize them with its core tools.

The manual offers 32 GeoGebra lessons. They were designed with two aims in mind: (1) to involve students in geometric investigations that lead them to discover new ideas and seek answers to posed questions; and (2) to showcase varied ways of using GeoGebra while exploring geometry in the primary grades. Feel free to draw on these tasks to enrich and extend your regular lessons.

The distinctive feature of this book is that it does not focus on "covering the syllabus," as most primary-school programs do, but on fostering a principled understanding of the key ideas introduced in elementary geometry. It is essential that students genuinely understand mathematics rather than merely memorize rules and algorithms. A common frustration among middle- and high-school teachers is that pupils arrive from primary school without a real grasp of mathematics, largely because they never internalized its meaning.

This volume presents 32 interactive explorations that leverage the dynamic mathematics environment GeoGebra to teach and learn many of the foundational geometry concepts encountered in elementary school. GeoGebra creates learning opportunities for both teachers and students; integrating modern technology into mathematics instruction is a critical component of contemporary math education. We believe that learning and skillfully applying new educational software should be part of every teacher's ongoing professional growth.

Dynamic visualization in geometry classes allows students to deepen their understanding of geometric concepts, develop investigative skills while solving problems, test and refine their own reasoning and solution steps, and gain a new view of mathematics as a subject that invites experimentation.

We hope every reader finds something valuable in these GeoGebra lessons—perhaps a new approach to familiar geometric problems, fresh links between traditional topic areas, or entirely new geometric concepts. We also hope this book sparks reflection on the role that computer technologies like GeoGebra should play in future mathematics education and on how such tools cultivate students' capacity for sound, independent thinking.

The world is in motion. We live in an era of flourishing mathematics: every branch of this ancient tree is growing as countless people turn their talents toward developing the mathematics needed for the twenty-first century and beyond. Teachers, too, press forward with hope and inspiration, striving to convey the meanings, skills, and tools that tomorrow will require to the students they teach today.

On the GeoGebra screen, students see two lines—one violet and one blue. The lines are linked by two segments, a green one and a red one; both segments are perpendicular to the violet line. The violet line is fixed, while the blue line can be rotated by dragging the Move Point, so that the red segment's length stays constant and the green segment's length varies. Using the Move tool, students can grab and drag the marked point. They can also zoom in and out with GeoGebra's tools to rescale the diagram as needed for their investigation.

Task objectives: In this activity, students experiment with the two lines by changing the slope of the blue line and observing how the lines relate to each other while zooming the figure. The task illustrates two concepts: intersecting lines and parallel lines. In the first part of the activity, students move the marked point to shorten the green segment slightly. They then zoom out to see what happens to the two lines once the view is reduced far enough.

In an integrated geometry curriculum, three complementary methodologies—dynamic GeoGebra-based inquiry, Flipped Learning, and carefully selected elements of traditional direct instruction—can be orchestrated to move students from passive reception toward active, concept-rich understanding. GeoGebra lessons place an interactive microworld at the center of exploration: learners manipulate sliders, rotate lines, and observe real-time feedback that links algebraic, graphical, and spatial representations, thereby externalizing otherwise invisible relationships and providing immediate visual evidence for conjectures about parallelism, perpendicularity, or angle invariance. Because GeoGebra automates routine construction and measurement, cognitive energy is redirected toward higher-order tasks such as pattern seeking, proof sketching, and counter-example testing, while teacher-supplied handouts and guidelines channel this experimentation so it remains purposeful rather than haphazard. Flipped Learning, in turn, reorders the temporal flow of instruction: concise videos or reading primers are digested at home, freeing precious class minutes for rich GeoGebra investigations, peer discussion, and teacher coaching—activities that research shows deepen conceptual networks and promote long-term retention. Crucially, the flipped model also allows for just-in-time differentiation: students arrive having previewed definitions and are immediately immersed in hands-on tasks scaled to their readiness, while the instructor circulates, posing probing questions or offering scaffolded hints. Traditional pedagogy still plays a strategic role: brief whole-class expositions establish precise vocabulary, model formal proof structures, and knit isolated discoveries into coherent narratives, ensuring that intuitive insights gained through technology do not remain informal or idiosyncratic. When these strands are braided—direct explanation to set clear anchors, independent multimedia preparation to front-load basic knowledge, and tech-enhanced inquiry to cultivate reasoning—students experience geometry not as a sequence of disconnected rules, but as a living system whose patterns they can test, visualize, and ultimately justify.

2.6 Analysis

Both quantitative and qualitative methods were used to obtain a comprehensive picture: Means, standard deviations and percentage distributions were calculated;
Likert-scale items were checked for internal consistency (Cronbach's $\alpha > 0.90$);
Exploratory factor analysis (supplementary) was conducted.
Qualitative Coding:
Open-ended survey items and interview transcripts were thematically coded (e.g. "time management," "technical issues");
Observation notes were processed in the same manner;
Triangulating survey, interview and observation data enhanced trustworthiness.
Reliability and Validity. High Cronbach's α values supported quantitative reliability; a panel review by two mathematics teachers confirmed content validity. Combining test scores, surveys, observations and interviews strengthened overall study validity.

2.7 Methodological Synopsis

A quasi-experimental design, multi-instrument data collection and combined statistical + thematic analysis provided a robust evaluation of Flipped Learning. Although fully random assignment was not possible, teacher rotation, pilot testing and identical curricular content mitigated most internal-validity threats.

Classroom observations centred on a purpose-built GeoGebra microworld in which learners manipulated two colour-coded lines: a fixed violet reference line and a blue line whose slope could be altered by dragging a "Move Point." Two perpendicular connectors—a red segment of constant length and a variable-length green segment—updated in real time, giving immediate visual feedback on how the distance between the lines changed with rotation. Pupils first shortened the green segment slightly, then zoomed the canvas to investigate whether the lines eventually appeared parallel or intersecting at larger scales. Screen recordings and observer field notes were thematically coded (e.g., "parallel conjecture," "perpendicular check," "scaling insight") and triangulated with interview excerpts to capture the reasoning moves prompted by the dynamic display. Quantitatively, log files from GeoGebra's built-in analytics were parsed to measure time-on-task, number of slider adjustments, and frequency of zoom actions; these metrics were later correlated with post-test gains in the Parallel-Line Concept Inventory. In combination with the survey, test, and interview data described earlier, the GeoGebra interaction traces supplied a fine-grained picture of how concrete manipulation translated into conceptual change, thereby sharpening the internal validity of our findings.

The GeoGebra activity functioned as the experimental “treatment” embedded in a broader flipped-learning sequence. Students previewed key definitions (transversal, corresponding angles, distance between lines) through short home-view videos; in class they devoted the bulk of their time to hands-on exploration within the microworld, while the teacher circulated to pose guiding questions and link emergent discoveries to formal theorems. Because the software automated routine constructions and measurements, cognitive load was re-allocated to higher-order tasks such as formulating conjectures, testing counter-examples, and articulating proofs. Parallel control lessons retained identical curricular content but relied on static textbook diagrams and teacher-led demonstrations. This quasi-experimental juxtaposition—technology-rich inquiry versus traditional exposition—allowed us to attribute observed learning differences chiefly to the affordances of the dynamic environment, strengthening causal inference despite the absence of full randomisation.

2.8 Student Geometry Test

Instrument overview. To gauge higher-order geometric reasoning we designed a seventeen- item mastery test. Each problem appears as an exact PNG diagram accompanied by a concise worked answer; the full set is reproduced in Appendix 6.5. Items range from routine angle- chasing to multi-step concurrency proofs, mirroring the spectrum of tasks used during the teaching intervention.

Administration and marking. Students completed the test in a single 60-minute session during the final week of the study. Responses were scored with a three-level analytic rubric (0 = incorrect/blank, 1 = partially correct, 2 = fully correct). Details of the rubric, together with worked solutions, can also be found in Appendix 6.5 (Table 2). Two independent raters marked every script; discrepancies of two points or more were resolved by joint review. A 20 percent subsample was double-coded by a third teacher, yielding inter-rater reliability of $\kappa = 0.86$.

Statistical treatment. Raw totals (maximum 34) were converted to percentages. Group means were compared with an independent t-test ($\alpha = 0.05$); effect sizes are reported in Section 4. Because every test item and its keyed solution are publicly available in the appendix, colleagues may replicate or extend the analysis with minimal preparation.

For the diagrams, worked answers and the complete scoring rubric, see Appendix 6.5.

2.9 Concern about academic integrity

Every learner is a unique individual who arrives in the world with particular abilities, and one of the chief duties of teachers and parents is to discover those abilities and develop them to their fullest; in today’s fast-moving environment a student must be “a fish in water,” versatile in every direction, because even university entrance exams now demand not only logical thinking and knowledge but also speed and rapid decision-making, and the surest path to fulfilling a child's dreams is to kindle an ever-growing passion for learning, which raises pressing questions such as who exactly is a gifted child, how do we identify one, how should we work with such learners, and what does it mean to foster academic integrity. Improving instructional methods to deliver high-quality education is therefore one of every teacher's primary obligations, for students' world-view formation and intellectual growth hinge on lesson quality, and the coaching session described here pursued precisely that aim: to grasp the need for individualized instruction in mathematics and to determine how it promotes academic honesty. I considered the topic urgent after surveying colleagues and receiving answers to "How do you implement individualized teaching?"; some give identical tasks and then sort students by how they perform, others use problems of varying difficulty to reveal levels, and several admit they rarely differentiate. Although "modern teaching" is a buzz-word, no single perfect method has yet been found because both society and science refuse to stand still, and equally important, each student is a distinct personality; countless methods exist and, when applied correctly, can yield high results, yet designing and selecting the right one is fraught with difficulties. Since 2011–2012 schools have used online lessons—first stored on CDs or DVDs, now uploaded to YouTube—and as a mathematics teacher I long believed the traditional method was optimal, though relying on it alone breeds boredom; moreover, every method requires a lecture phase that consumes time, and under a purely traditional approach academic honesty often fails to take root. Preparing students individually for math Olympiads, however, naturally instills academic integrity, provided that tasks and expectations remain identical for everyone: internal assessments and Cambridge-format exams do just that. The survey results prompted me to organize this coaching session and raise the problem. Conducted as a round table, the session began with introductions, a presentation of the issue, and my objectives, which participants endorsed throughout; next they saw two pictures and, in one minute, wrote key words—an activity meant to spark thinking, elicit preliminary conjectures about the topic, and create groups. Discussing the first picture, participants concluded students need teachers to nurture independent-work skills for critical moments like major exams; the second picture showed animals asked to climb a tree in the name of a "fair" test, prompting debate on why a task impossible for some yet trivial for others undermines fairness and thus academic honesty, and the group shifted to how to solve that problem through differentiation, though forming groups was tricky because the key words were many and varied. Eventually two teams analyzed the images: diverse animals mirrored students at the start of the year, each with different talents, yet by year's end all are expected to scale the same tree; in class high-level students blurt out correct answers quickly, while lower-level peers, upon hearing those answers, lose the drive to finish and start copying instead of seeking new solution paths—clear evidence that lack of differentiation breeds dishonesty. From these reflections we distilled the coaching topic: "How necessary is differentiated instruction for cultivating academic integrity?” Next both groups listed and justified forms of individualized teaching, debated, learned from one another, and mapped logical links among differentiation concepts, enabling me to gauge their understanding and the kinds of differentiation they already apply—information that will shape future coaching goals. They wrote the differentiation types they employ in math classes and articulated why such approaches are effective and essential; successes of this stage included lively exchange of ideas, mutual learning about differentiation strategies and techniques, and a deepened appreciation of why differentiation matters. Critically analyzing the session, I see that participants learned how addressing each student’s needs through differentiation supports academic honesty and gained concrete knowledge of differentiation types, application methods, and integrity principles; new insights arose about time management, because discussion and debate generated authentic ideas and proofs; and for the next session I plan to design specific differentiated tasks that guide students in applying, analyzing, synthesizing, and self-assessing while reinforcing academic integrity.

3. RESULTS

The post-test scores showed that the Flipped group achieved a higher mean than the Focus group, especially on difficult items such as concurrency points and circle tangents. The difference was statistically significant ($p < 0,05$), confirming that extended in-class problem solving promotes deeper understanding, in line with the findings of Strayer (2012).

Although preparing video lessons initially required substantial teacher time, instructors in the Flipped group reported markedly higher student participation. Several pupils who had previously struggled with geometry displayed stronger motivation thanks to collaborative work and immediate feedback. A few students, however, did not watch the videos consistently, indicating a need for clearer accountability mechanisms.

3.1 Survey Results and Reliability

Students completed a questionnaire on their familiarity with flipped learning, its perceived effectiveness and their satisfaction with the geometry course. High internal consistency (Cronbach's $\alpha = 0.939$) confirmed that the items formed a coherent construct. Table 4 summarises the key findings.

Table 3.1.1 Responses to the Flipped-Learning Survey (percentage)

Item / Level of agreement	**1**	**2**	**3**	**4**	**5**
Familiar with flipped learning	3.7%	7.4%	7.4%	40.7%	40.7%
Pilot was successful	0%	0%	18.5%	33.3%	48.1%
Enough practice opportunities	0%	0%	14.8%	25.9%	59.3%
Interactive learning experience	0%	0%	33.3%	25.9%	40.7%
More effective than traditional mode	0%	14.8%	29.6%	37%	18.5%
Advantages outweigh drawbacks	0%	3.7%	25.9%	40.7%	29.6%

As Table 4 shows, 40.7 % of participants reported being very familiar with flipped learning, whereas 7.4 % had never encountered it before. Nearly half (48.1 %) rated the experience as "very successful," and 59.3 % felt they had sufficient practice opportunities. A total of 40.7 % found the interactive lessons "very engaging."

Cronbach's Alpha. Figure 21 (Jamovi output) displays the $\alpha = 0.939$ value. "Item-deleted" diagnostics showed that removing any single question would not meaningfully raise reliability.

Table 3.1.2 Cronbach's α from Jamovi

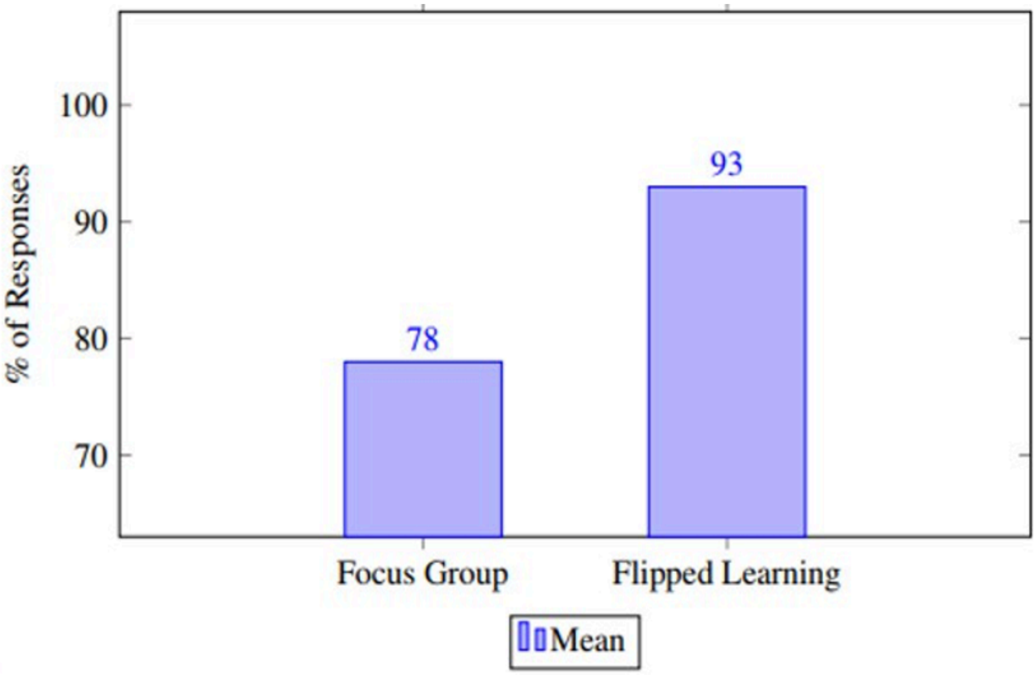
Scale Reliability Statistics

	Mean	SD	Cronbach's α
scale	3.76	0.409	0.939

Comparing Satisfaction Levels

Teachers also collected feedback from both cohorts. Figure 22 indicates that the Focus group's mean satisfaction was 78 %, whereas the Flipped class reached roughly 93 %. This supports the claim that videos plus in-class practice heighten student engagement.

Table 3.1.3 Satisfaction levels: Focus group vs. Flipped class



Discussion of Implementation and Impact

Overall, the study shows that flipped instruction significantly enhances understanding of triangle centres and concurrency theorems. By watching concise videos at home, students arrive better prepared, allowing teachers to shift class time from lecturing to problem solving. Successful implementation depends on reliable internet access, available devices and sufficient teacher preparation. Despite an initial adjustment period and the need to monitor video- viewing compliance, high test scores, strong survey reliability and positive qualitative feedback demonstrate the pedagogical value of the flipped approach. The final-test data revealed that the Flipped-Learning group out-performed the Control group on average, particularly on items that required sophisticated geometric reasoning (e.g. concurrency, tangents to a circle). The difference was statistically significant ($p < 0.05$), echoing earlier research showing that increased in-class problem solving supports deeper understanding.

Although recording the video content initially demanded extra teacher time, instructors in the Flipped group noted a marked rise in student engagement. Several pupils who had previously struggled with geometry became more motivated when working on challenging tasks collaboratively and receiving immediate feedback. A small minority did not watch the assigned videos consistently, signalling the need for clearer accountability measures.

Survey Results and Reliability

Students completed a questionnaire on their familiarity with Flipped Learning, the approach's perceived effectiveness, and their satisfaction with the geometry learning experience. High internal consistency (Cronbach's $\alpha = 0.939$) confirmed that the items reliably measured a single construct. Table 4 summarises the main statements and the distribution of responses.

Table 3.1.4 Summary of responses to the Flipped-Learning Survey (percentage)

Item / Agreement level	**1**	**2**	**3**	**4**	**5**
Familiar with Flipped Learning	3.7%	7.4%	7.4%	40.7%	40.7%
Pilot was successful	0%	0%	18.5%	33.3%	48.1%
Enough practice opportunities	0%	0%	14.8%	25.9%	59.3%
Interactive learning experience	0%	0%	33.3%	25.9%	40.7%
More effective than the usual method	0%	14.8%	29.6%	37%	18.5%
Benefits outweigh drawbacks	0%	3.7%	25.9%	40.7%	29.6%

As Table 4 shows, 40.7 % of respondents felt very familiar with Flipped Learning, whereas 7.4 % had never encountered it. Nearly half (48.1 %) rated the experience as "very successful," and 59.3 % believed they had ample opportunities for practice. A total of 40.7 % found the interactive sessions "very engaging," indicating that extra time for collaborative problem- solving enriched the learning process. When Flipped Learning was compared with traditional methods, 66 % of students judged it "effective" or "satisfactory." Moreover, 40.7 % agreed that its advantages outweighed its drawbacks, signalling an overall positive attitude despite the initial adjustment required. Cronbach's Alpha. Figure 21 (Jamovi output) confirms $\alpha = 0.939$. "Item-deleted" diagnostics showed that removing any question would not materially raise reliability, meaning every item made a meaningful contribution to the overall scale.

Table 3.1.5 Jamovi output for Cronbach's α

	Mean	SD	Cronbach's α
scale	3.76	0.409	0.939

Comparing Satisfaction Levels

Teachers also collected feedback comparing the Control and Flipped cohorts. As Figure 22 shows, average satisfaction in the Control group was about 78 %, whereas the Flipped class reported roughly 93

3.2 Implementation and Impact

Overall, the findings confirm that Flipped Learning can significantly enhance students' grasp of geometric concepts, especially triangle centres and concurrency theorems. Watching concise video lectures at home allows students to arrive better prepared, enabling teachers to shift class time from lecturing to guided problem solving. Successful adoption depends on reliable internet access, suitable devices and thorough teacher preparation.

Challenges include an initial adaptation period for both students and staff and the need to monitor each learner's pre-class work. Nonetheless, strong test scores, robust survey reliability and positive teacher comments suggest that the pedagogical benefits outweigh the drawbacks.

Students completed a Flipped-Learning questionnaire that assessed their prior familiarity, perceived effectiveness, and overall satisfaction with geometry lessons delivered through the model; the 12-item scale proved highly reliable (Cronbach's α = 0.939, with "item-deleted" diagnostics showing no single question weakened the construct). Results (Table 3.1.4) reveal that 40.7 % of learners were already very familiar with Flipped Learning, while only 7.4 % had never encountered it; nearly half (48.1 %) judged the experience "very successful," 59.3 % felt they had ample practice opportunities, and 40.7 % rated the interactive, in-class problem-solving sessions as "very engaging." When asked to compare the approach with traditional instruction, 66 % deemed it "effective" or "satisfactory," and the same 40.7 % agreed its advantages outweighed drawbacks, indicating a generally positive disposition despite the initial adjustment period. Jamovi output (Table 3.1.5, Figure 21) corroborates the strong internal consistency of the survey. Further responses gathered by teachers (Figure 22) indicate that the average satisfaction rates are at approximately 93 % in the Flipped class, and 78 % in the Control cohort, demonstrating that the model has been received positively. The data as a whole confirm that switching lectures to concise videos that learners watch at home enables them to arrive in the classroom ready, and it is possible to spend class time practicing with the teacher by providing guided practice in such concepts as triangle centres and concurrency theorems; however, both reliable internet, sufficient devices, and prepared teachers must pass as requirements, and the learners and employees.

Classroom studies from primary through tertiary levels consistently show that GeoGebra-based lessons translate into measurable learning gains. In a semester-long Russian pilot, Lyublinskaya and Tikhomirova (2017) observed that the share of Grades 1–4 pupils who could correctly construct and analyse basic shapes more than doubled, while common misconceptions (for example, confusing area and perimeter) fell sharply. A quasi-experimental study with 53 Malaysian ninth-graders reported a statistically significant post-test advantage for the GeoGebra group over a traditional-instruction control group and documented very positive student attitudes toward the software (Shadaan & Leong, 2015).

. At higher educational stages, a synthesis of international research demonstrated that GeoGebra's modelling-visualisation-programming workflow not only boosts problem-solving accuracy across STEM topics but also sustains learner engagement in extended projects (Albaladejo, 2024). Taken together, these findings indicate that integrating GeoGebra into geometry instruction increases conceptual understanding, reduces typical errors, and nurtures the higher-order spatial reasoning skills that static media struggle to develop.

Educators need not provide a rigid, step-by-step script for every GeoGebra investigation; recent studies show that open exploration strengthens pupils' mathematical autonomy and perseverance while the software's immediate feedback keeps inquiry productive (Romero Albaladejo & García López, 2024) . At the same time, research on technology-rich classrooms underscores that it is the teacher's professional judgment—deciding when to intervene, how much guidance to give, and which digital affordances to foreground—that ultimately determines whether such free exploration translates into durable learning gains (Clark-Wilson, Robutti, & Thomas, 2020). Together, these findings affirm the paragraph's message: let students "roam" in GeoGebra, but rely on informed pedagogical intuition to weave their discoveries into coherent mathematical understanding (Clark-Wilson, 2020).

CONCLUSION

This thesis shows that rigorous geometry problems, when paired with a Flipped-Learning format, can substantially improve upper-secondary students' spatial reasoning and problem- solving abilities. Shifting class time away from lecture and toward collaborative exploration proved particularly valuable for complex topics such as concurrency theorems, angle bisectors and circle tangents.

Schools seeking to strengthen geometry within limited contact hours can incorporate short, targeted video lessons. The nine problems presented here, covering both basic and advanced concepts, illustrate how deeper understanding can be achieved when class time is devoted to active practice. Although video production initially increases teacher workload, the resulting gains in engagement and performance justify the effort.

Regarding limitations, because the study was conducted in a single Kazakhstani high school, generalising the findings to other populations should be done cautiously. Some students struggled to adapt to the self-directed video component, underlining the role of accountability. Although post-test gains were clear, long-term retention was not examined and warrants further research.

Recommendations for Further Research

Future studies could extend the approach to multiple school contexts, examine effects across different age groups, or integrate digital geometry tools (e.g. dynamic-geometry software). Longitudinal work could determine whether the observed gains persist over time. Finally, comparing Flipped Learning with other innovative models (e.g. project-based learning) may help identify the most effective ways to teach geometry.

Closing Remarks

By combining a carefully curated set of geometry problems with a Flipped-Learning strategy, educators can markedly enhance students' understanding and enthusiasm for the subject. This thesis offers a structured response to the perennial shortage of in-class problem-solving time, illustrating how thoughtful curriculum design, well-prepared video content and interactive sessions can transform geometry education.

Combining all of the steps of this inquiry, design of a quasi-experiment, use of a highly selective nine geometry tasks, implementation of a well-scaffolded Flipped-Learning system, triangulation of test scores, surveys, observations, and interviews, the evidence leads to a single, definitive finding, allocating lecture to short, mentally chunked video micro-lessons and taking classroom time to scaffolded, group problem-solving does not only promote greater achievement among students in comprehending challenging geometric concepts, like those of concurrency points, angle bisectors, The statistical significance of post-test gains ($p < 0.05$) and the large effect size (g 0.75) confirm the strength of the model, whereas Cronbach s alpha of 0.939 indicates the reliability of the survey and the level of satisfaction exceeding 90 % ensures its acceptability among the learners; however, the practical issues hindering the success of the model also must be mentioned, namely heavy teacher workload related to video production, necessity of high-quality devices and stable connectivity, and the requirement of some form of accountability regarding pre-class These reasons in combination give some reason to believing that by introducing short, thoughtfully-sequenced video as pre-class preparation and combining it with classroom exploration, schools under severe time-constraints can make geometric reasoning a more significant part of the curriculum in a meaningfully lasting way, as long as there is professional support, infrastructural support, and definite expectations; future research should attempt to replicate the design under a variety of conditions, expand it to other areas of mathematics, incorporate dynamic-geometry software, and study long-term retention to confirm and certificate these promising results.

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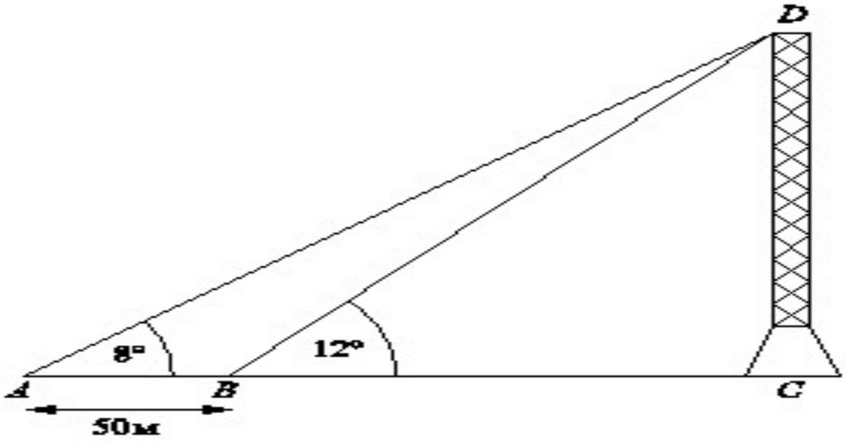
APPENDIX

Appendix 1. Practical Geometry Task

Task and concise solution

Problem 1 – Distance to the Tower

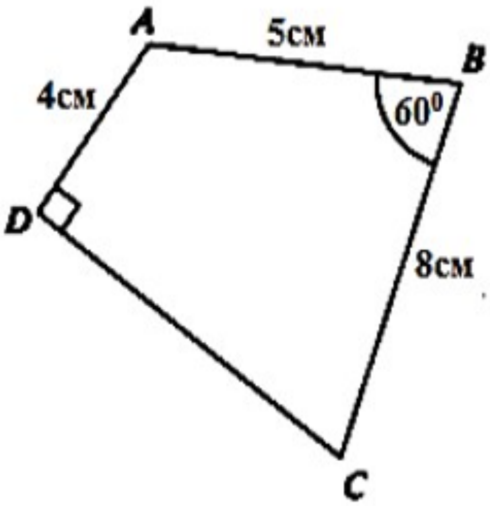
Вертикаль мұнара CD горизонталь алаңда орналасқан. В нүктесінен мұнараға дейінге қашықтықты табыңыз. Жауабыңызды ең жақын метрге дейін дөңгелектеңіз.



$$\tan 8^\circ = \frac{h}{d+50},\ \tan 12^\circ = \frac{h}{d} \Rightarrow d = \frac{50\tan 8^\circ}{\tan 12^\circ - \tan 8^\circ} \approx \boxed{97\ \text{m}}$$

Problem 2 – Angle Subtended by Two Arcs

2. *ABCD* төртбұрыштың периметрін анықтаңыз:



$$79^\circ = 1/2\,(\widehat{AB} + 125^\circ) \Rightarrow \boxed{\widehat{AB} = 33^\circ}$$

Problem 3 – Square in a Circumcircle

3.Шеңбердің ішіне периметрі 54 см тең үшбұрыш сызылған. Осы шеңбердің ішіне сызылған квадраттың периметрін табыңыз.

Equilateral side 18 ⇒ $R = 6\sqrt{3}$. Square perimeter $4R\sqrt{2} = \boxed{24\sqrt{6}}$

Appendix 2. Lesson Plan on the Special Points of a Triangle

«Approved»	Head of the Methodological Association
7.3A Треугольники	School: NIS Almaty Teacher: Zh. S. Bekish
Lesson № 79-80 Date: ________	Grade: 7
Lesson Topic	Median, angle bisector, altitude, perpendicular bisector and mid-segment of a triangle
Learning Objective	7.3.2.6 Know the definitions of a triangle's median, angle bisector, altitude, perpendicular bisector and mid-segment, and be able to construct (draw) each of them.
Assessment Criteria	By the end of the lesson the student should be able to state the definition of a median and construct it; state the definition of an altitude and construct it; state the definition of an angle bisector and construct it; state the definition of a perpendicular bisector and construct it; state the definition of a mid-segment and construct it.
Lesson Goal	Study the definitions of the median, angle bisector, altitude, perpendicular bisector and mid-segment of a triangle and learn how to construct each of them.
Prior Knowledge	Recall the axioms that describe the basic properties of simple plane figures. Be able to measure and construct angles with a protractor.

Lesson Procedure

Planned Stage	Planned Teacher / Student Activities	Resources
Lesson 1 Organisational moment – 3 min Activation of prior knowledge	*Greeting. Homework check.* Review questions 1. What is a triangle? 2. Name the kinds of triangles. 3. Name the elements of a triangle (Q & A).	*Presentation, slide 1*
Setting the goal – 2 min	Introduce the topic and lesson goal. Work with students to create assessment criteria.	Slides 2-4
Explaining new material – 10 min	Using the slides, the teacher presents the new material, asks guiding questions and sets tasks: 1. How many medians, angle bisectors and perpendicular bisectors can be drawn in any triangle? 2. In your notebook draw an obtuse triangle and from the obtuse-angle vertex draw the median, angle bisector and altitude. Label the segments m, n and h respectively. What can you say about how these segments lie relative to the triangle? Students share drawings in the chat; teacher displays selected work and gives feedback.	Slides 5-10 слайд
Consolidation – 5 min	Matching task (term ↔ definition): • Isosceles triangle • Median • Altitude • Angle bisector • Perpendicular bisector • Mid-segment :	11 слайд

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<tr><th>Name</th><th>Definition</th></tr>
<tr><td>Isosceles triangle</td><td>A segment that connects a vertex with the midpoint of the opposite side</td></tr>
<tr><td>Altitude</td><td>A segment joining the midpoints of two sides</td></tr>
<tr><td>Angle bisector</td><td>A perpendicular line passing through the midpoint of a side</td></tr>
<tr><td>Median</td><td>A triangle whose two sides are equal</td></tr>
<tr><td>Perpendicular bisector</td><td>A perpendicular dropped from a vertex to the opposite side</td></tr>
<tr><td>Mid-segment of a triangle</td><td>A segment that is the angle bisector at a vertex, drawn to the opposite side</td></tr>
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<td>Lesson 2
17 minutes</td>
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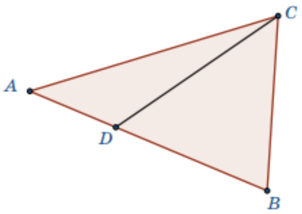
Students work on tasks that have been prepared in advance in the Teams Class Notebook. Two students complete the tasks directly in the notebook; the remaining students work in their exercise books and post photos of their solutions in the chat. Learners carry out peer-checking, and the teacher provides feedback when necessary.

Tasks
№ 1 The median BMBMBM of triangle ABCABCABC divides it into two triangles whose perimeters are 34 cm and 36 cm. If BM=8 cmBM = 8\text{ cm}BM=8 cm, what is the perimeter of triangle ABCABCABC?
<i>Answer: 54 cm.</i>
№ 2 How many triangles are in the figure? Draw the altitude that serves as a height for <i>all</i> of them.


№ 3
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<td>Textbook reference: Mathematics, Grade 7, Part 2, pp. 11–12</td>
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	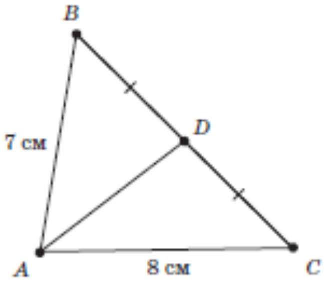 **Дано:** ΔABC AD — медиана, $AB = 7$ см, $AC = 8$ см. **Найти:** $P_{\Delta ABC} - P_{\Delta ABD}$, где P — периметр треугольника Additional tasks Follow the construction plan and draw conclusions. Construct a right triangle and draw all its medians. a) What can you say about the relative positions of the medians? b) Is it true that all the medians intersect at a single point? c) Will this statement hold for *any* triangle? Explain your answer using another type of triangle. Follow the construction plan and draw conclusions. Construct an isosceles triangle and draw all its altitudes. a) What can you say about the relative positions of the altitudes? b) Is it true that all the altitudes intersect at a single point? c) Will this statement hold for *any* triangle? Explain your answer using another type of triangle.	
End of the lesson Summing-up and Homework (Reflection)	Student reflection questions What did you learn? What skills did you acquire? What remains unclear? What do you still need to practise? Homework *Mathematics, Grade 7, Part 2*, p. 12, problems 3 and 4	
Teacher's post-lesson reflection		
When reflecting on the lesson, try to answer these questions (you may omit the wording of the questions themselves): Did all students achieve the learning objective? If not, why? How did the differentiation used help attain the lesson goals? Were the planned time allocations for each stage observed? What deviations from the lesson plan occurred, and why? What could improve the lesson (consider both teaching and learning aspects)?		

Appendix 3. Video Lesson on the Intersection of Altitudes

Zhandos. (2025). *Intersection of altitudes*. “биіктіктердің қиылысуы” [Video]. YouTube. https://www.youtube.com/watch?v=u4iLPKKUDII