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LAGRANGE'S THEOREM AND 2- CONTINUED FRACTION EXPANSION

Abstract. The simple continued fraction theory is a sub-branch of number theory that is well developed. One of the classical results is due to Lagrange which states that the simple continued fraction expansion of a real number has eventually periodic expansion if and only if it is quadratic irrational. Similar results are not available when one considers N-continued fraction expansion which is not so well developed theory. In this article, authors aim to provide computational evidence when a quadratic irrational may not necessarily have eventually periodic 2-continued fraction expansion. Moreover, a proof is provided for a special type of real numbers for which Lagrange's theorem does hold.

Keywords: Lagrange's theorem, N-continued fraction, Mathematica software, quadratic irrational number.

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Аңдатпа. Қарапайым үздіксіз бөлшектер теориясы, сандар теориясының жақсы дамыған қосалқы тармағы болып табылады. Бұл саладағы классикалық нәтижелердің бірі Лагранж теоремасы, ол болса, әр қандай нақты санды үздіксіз бөлшектермен көрсеткен кезде периодты болып шығуы үшін ол сан міндетті түрде квадраттық иррационал болуы керек екендігін айтады. Лагранж теоремасына сәйкес теоремалар N-үздіксіз бөлшектер саласында қол жетімді емес, және бұл N- үздіксіз бөлшектер теориясы әліде толық дамыған теория болып саналмайды. Бұл мақалада авторлар квадраттық иррационал нақты сандардың 2- үздіксіз бөлшектер көрсетуді қарастырады, бұл жағдайда 2- үздіксіз бөлшектер көрсеткен кезде міндетті түрде периодті болып шықпай қалуы мүмкін екенін есептеу техникасын қолдана отырып дәлел келтіреді. Сонымен қатар, Лагранж теоремасы орындалатын ерекше жағдайларды мысылдармен қарастырады.

Кілт сөздер: Лагранж теоремасы, N-үздіксіз бөлшектер, Mathematica жүйесі, квадраттық иррационал сан.

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Аннотация. Простая теория цепных дробей является хорошо развитой ветвью теории чисел. Один из классических результатов связан с Лагранжем, который утверждает, что простое непрерывное дробное разложение действительного числа имеет в конечном итоге периодическое разложение тогда и только тогда, когда оно является квадратичным иррациональным. Подобные результаты недоступны, если рассмотреть обобщения N-цепных дробей, которое не так хорошо развито в теории. В этой статье авторы стремятся предоставить вычислительные доказательства, того что если иррациональное число квадратичная то это число не иметь периодическое представления 2 - цепных дробей. Кроме того, приводится доказательство для специального типа действительных чисел, для которых теорема Лагранжа справедлива.

Ключевые слова: теорема Лагранжа, N- цепных дробей, программное обеспечение Mathematica, квадратичное иррациональное число

Introduction

It is well known that any real number has a simple continued fraction expansion. For a given real number $\alpha \in \mathbb{R}$, its simple continued fraction expansion is written as

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

or simply $\alpha = [a_0; a_1, a_2, \cdots]$, where $a_n \in \mathbb{N}, (n \geq 1)$ and $a_0 \in \mathbb{Z}$.For any natural number n, we consider the following rational numbers, called nth convergents,

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \cdots a_n].$$

Here p_n and $q_n$are relatively coprime and satisfy the recurrence relation:

$$p_n = a_n p_{n-1} + p_{n-2}\,, p_{-1} = 1, \quad p_{-2} = 0,$$
$$q_n = a_n q_{n-1} + q_{n-2}\,, q_{-1} = 0, \quad q_{-2} = 1$$

(see [1, Ch.3]).Then, the above simple continued fraction expansion can be justified by $\alpha = \lim_{n\to\infty} \frac{p_n}{q_n}$. One can generalize the simple continued fraction expansion to *N-continued fraction expansion* [2]for $\alpha \in \mathbb{R}$by

$$\alpha = a_0 + \cfrac{N}{a_1 + \cfrac{N}{a_2 + \cfrac{N}{a_3 + \ddots}}}$$

denote $[a_0; a_1, a_2, \cdots]_N$, where its nth convergent p_n/q_n satisfies the relation:

$$p_n = a_n p_{n-1} + N p_{n-2}\,, \quad p_{-1} = 1, \quad p_0 = a_1 a_0 + b_1,$$
$$q_n = a_n q_{n-1} + N q_{n-2}\,, \quad q_{-1} = 0, \quad q_0 = a_1.$$

If $a_n \geq N$ where $n > 0$, then the expansion is called *proper N-expansion* or *best expansion* due to M. Anselm and S.H. Weintraub [3]. In [4] this best expansion for fixed *N* obtained through map $T_N: [0,N] \to [0,N]$, defined by $T_N(0) = 0$, and

$$T_N(\alpha) = \frac{N}{\alpha} - \left\lfloor \frac{N}{\alpha} \right\rfloor, \alpha \neq 0.$$

By setting $a_1 = a_1(x) = \left\lfloor \frac{N}{\alpha} \right\rfloor$, and $a_n = a_n(x) = a_1(T_N^{(n-1)}(\alpha))$ where $T_N^{(n-1)}(\alpha) \neq 0$, the best expansion can be obtained recursively.

Definition 1.1. A real number $\alpha = [a_0; a_1, a_2, \cdots]$ is *eventually periodic* if there are numbers $N \geq 0$ and $k \geq 1$ with $a_{n+k} = a_k$ for all $n \geq N$. In this case we write

$$[a_0; a_1, a_2, \cdots, a_{N-1}, \overline{a_N, \cdots, a_{N+k}}].$$

Definition 1.2. A real number α is called *quadratic irrational if* $\alpha \notin \mathbb{Q}$ *and there are integers* a, b, c *with* $a\alpha^2 + b\alpha + c = 0$.

The following is a classical result.

Theorem 1.3. *(see [1, Ch.3] Lagrange Theorem) The simple continued fraction expansion of* α *is eventually periodic iff* α *is quadratic irrational.*

In [2] proved that quadratic irrational α can be expressed as a periodic non-simple continued fraction having period length one. That is, showed that quadratic irrational $\alpha = [a_0; a_1, a_2, \cdots, a_{N-1}, \overline{a_N, \cdots, a_{N+k}}]$ as a periodic *N - continued fraction expansion* having period length one, where $N = (p_{k+1} - a_0 q_{k+1})^2$ and p_{k+1}/q_{k+1} is $(k+1)$th convergent of $\overline{a_N, \cdots, a_{N+k}}$. From Theorem 5 in [5] follows if partial quotients of simple periodic continued fraction expansions is bounded then if partial quotients of *N - continued fraction expansions* also bounded, where $N = (p_{k+1} - a_0 q_{k+1})^2$. Note that this is *N - continued fraction expansions* not proper.

In this work we want to study 2-continued fraction expansions and eventually periodic orbits.

Using the above Lagrange's theorem we obtain the following result:

Theorem 1.4. *If 2–continued fraction expansion of a real number is eventually periodic, then so does its simple continued fraction.*

An interesting question is whether the converse of Theorem 1.4 holds, namely, if a real number has eventually periodic simple continued fraction expansion, is it true that its 2-continued fraction expansion will necessarily be eventually periodic?

In view of conjecture in [3] which states that for $N \geq 2$ the proper continued N expansion of a quadratic irrationality is not always periodic, we can say that the answer to the above question may be negative. Nonetheless, we show that in following cases the answer is affirmative.

Theorem 1.5. *For a real number* α, *let* $[a_0; a_1, a_2, \cdots]$ *be its simple continued fraction expansion, if* $a_{2k} > 1$ *for any* $k \in \mathbb{N}$ *then the proper 2-continued fraction expansion of a satisfies*

$$\alpha = [a_0; 2a_1, a_2, 2a_3, a_4, \cdots]_2.$$

In particular, in this case, if α *has eventually periodic simple continued fraction expansion, then so does its 2-continued fraction expansion.*

For the remaining case when $a_{2k} = 1$ for some $k \in \mathbb{N}$, we provide a computational evidence with seemingly unbounded 2-continued fraction expansions of eventually periodic simple continued fraction expansions.

Proof of theorems

In this section we give the prove of theorems. To prove theorem 1.4 it is enough to show that if 2-contined fraction expansion of α is periodic then α is *quadratic irrational.*

Lemma 2.1.*If 2-contined fraction expansion of* α *is periodic then* α *is quadratic irrational.*

Proof: Let $\alpha = [0; \overline{a_1, a_2, \dots a_m}]_2$. That is,

$$\alpha = \cfrac{2}{a_1 + \cfrac{2}{a_2 + \cfrac{2}{\ddots + \cfrac{2}{a_m + \frac{2}{\alpha}}}}},$$

Then for some integers A, B, C, D

$$\alpha = \frac{A\alpha + B}{C\alpha + D}$$

Since, $C\alpha^2 + (D - A)\alpha - B = 0$ for $A, B, C, D \in \mathbb{Z}$, α is *quadratic irration.al.* The same way, we can prove it for $\alpha = [a_0; b_1, \dots, b_k \overline{a_1, a_2, \dots a_m}]_2$.

Proof of Theorem 1.4. Follows from Lemma 2.6. and Theorem 1.3□

Proof Theorem 1.5. Let $\alpha = [a_0; a_1, a_2, \cdots]$ be simple quadratic irrational number, then we can find it's 2-continued fraction expansion in the following way. We multiply numerator and denominator of the first fraction by 2, so we have

$$\alpha = a_0 + \cfrac{2}{2\left(a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}\right)} = a_0 + \cfrac{2}{2a_1 + \cfrac{2}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}.$$

Now, we again multiply the most upper numerator that is not equal to 2 and it's denominator by 2. We get

$$\alpha = a_0 + \cfrac{2}{2a_1 + \cfrac{2}{a_2 + \cfrac{2}{2\left(a_3 + \cfrac{1}{\ddots}\right)}}} = a_0 + \cfrac{2}{2a_1 + \cfrac{2}{a_2 + \cfrac{2}{2a_3 + \cfrac{2}{\ddots}}}}.$$

If we will continue to repeat this process, then we will obtain as result 2-continued fraction expansion

$$\alpha = a_0 + \cfrac{2}{2a_1 + \cfrac{2}{a_2 + \cfrac{2}{2a_3 + \cfrac{2}{\ddots}}}} = [a_0; 2a_1, a_2, 2a_3, a_4, 2a_5 \cdots]_2.$$

Calculations of 2 –continued fraction expansions

In this section we show that for some periodic simple continued fraction expansions with its first 40,000 terms the conjecture in [3]holds. We calculate in Wolfram Mathematica 2-continued fraction expansions from simple continued fraction expansions.

Let be $x = [0; \overline{1,2,3,4,5}]$. Take its first 20,000 terms and find proper 2 – continued fraction of x. Then the maximum number of partial quotients of proper 2 – continued fraction of x is $a_{4296} = 35863$.

In Table 1 we give list of periodic simple continued fractions and calculate its first 20,000 terms of 2-continued fraction expansions. We use the following notations: Max[a_i] - maximum of partial quotients and i its order.

Table 1				
	Simple continued fraction	First 30 terms of proper 2-continued fraction $[\,]_2$	Max a_i	i
1	$[0;\overline{1,2}$	[0; 2,2,7,5,13,3,2,4,3,23,222,5,5,75, 2,11,3,2,2,6, ...]	13	8039
2	$[0;\overline{2,1}$	[0; 5,3,2,2,7,5,13,3,2,4,3,23,222,5,5,75,3,5, 2,28,2,11,3,2, ...]	13	8041

3	[0; $\overline{2,3}$	[0; 4,3,2,2,7,5,13,3,2,4,3,23,222,5,5,75,3,5 2,28,2,11,3,2, ...]	13	804 1
4	[0; $\overline{3,2}$	[0; 6,2,2,3,5,3,2,2,7,5,13,3,2,4,3,23, 78,3,20,2,28, ...]	13	8045
5	[0; $\overline{2,3,4,1}$]	[0; 4,3,9,5,39,3,4,2,2,4,3,3,12,4,33,2,9,95,8 6,2,51,17,53, . .]	16 14	1489 2706 3
6	[0; $\overline{1,2}$	[0; 2,2,6,4,11,5,35,4,2,18,2,9,2,13,4 9,2,5,2,10,572, ...]	3 5863	4295
7	[0; $\overline{2,1,3,4,5}$]	[0; 5,3,2,3,2,2,10,3,2,2,2,4,2,6,8,3,2, 2,17,4,16,13,3, ...]	30	351
8	[0; $\overline{2,3,1,4,5}$]	[0; $\overline{4,3,2,4,10,2,7,3,7,2,4,78,2,15,2,2,6,67,2}$	7 8	12
9	[0; $\overline{2,3,4,1,5}$]	[0; 4,3,9,2,2,9,8,24,2,2,2,2,4,2,12,20,5,6,2, 28,5,2,11,3,7, ...]	1 1916	6388
1 0	[0; $\overline{2,3,4,5,1}$]	[0; 4,3,8,5,2,2,6,4,11,5,35,4,2,18,2,9 99,2,7,9,2,5, ...]	3 5863	4299

From this table we have the following corollary:

Corollary 2.2 *The proper 2-continued expansion of a quadratic irrationality is not always periodic at least 27,063 terms.*

As an example of theorem 1.4 we give list of periodic 2-continued fraction expansions and its simple continued fraction expansions. Also, we see that if we find the simple continued fraction of $[0;\overline{3,4,5,6,7}]_2$,then the length of period of periodic simple continued fraction is equal to 206.

Corollary 2.3. *The length of period of periodic simple continued fraction is not necessary should be equal to the length of period of periodic 2-continued fraction.*

In Theorem 1.4 we proved that if 2-continued fraction expansion is eventually periodic, then its simple continued fraction expansion is necessarily eventually periodic. However, it is not clear how two periodic lengths are related. The Table 2 below provides computational analysis that periods of simple continued fractions may potentially be longer.

Table 2

	2-continued fraction $[\,]_2$	Simple continued fraction	Max	
			a	
	$[0;\overline{1,2,3}]$	$[1;\overline{1,4,9,4,1,1,1}]$	9	
	$[0;\overline{2,1,3}]$	$[3;\overline{3,2,18,2,3,3,}]$	1	
	$[0;\overline{2,3,1}]$	$[2;\overline{2,18,2,3,3}]$	1	
	$[0;\overline{3,2,1}]$	$[3;\overline{1,1,1,1,4,9,4}]$	9	
	$[0;\overline{2,3,4,1}]$	$[2;\overline{1,1,2,3}]$	3	
	$[0;\overline{1,2,3,4,}$	$[1;\overline{1,3,2,6,7,1,12,3,1,25,1,1,52,105,52,1,1,25}]$	1 05	4
	$[0;\overline{2,1,3,4,5}]$	$[3;\overline{3,1,2,1,6,1,1,6,1,2,1,3,3,2,1,2,1,1,1,1,5,2,4,11,1,8,}$ $\overline{5,1,17,2,1,35,1,2,17,1,5,8,1,11,4,2,5,1,1,1,1,2,1,2,3}]$	3	2
	$[0;\overline{2,3,1,4,5}]$	$[2;\overline{2,5,2,2,2,1,2,4,1,5,2,1,1,1,11,1,3,24,1,1,49,99,49,}$ $\overline{1,1,24,3,1,11,1,1,1,2,5,1,4,2,1,2,2}]$	9 9	2
	$[0;\overline{2,3,4,1,5}]$	$[2;\overline{1,1,2,6,1,3,2,1,2,1,7,1,2,1,2,3,1,6,2,1,1,2,3,5,5,1,1,1}$ $\overline{2,11,3,1,22,1,1,46,93,46,1,1,22,1,3,11,2,1,1,1,5,5,3,2}]$	9 3	
0	$[0;\overline{2,3,4,5,1}]$	$[2;\overline{1,1,2,1,2,1,1,3,2,6,7,1,12,3,1,25,1,1,52,105,52,1,}$ $\overline{1,25,1,3,12,1,7,6,2,3}]$	1 05	0
1	$[0;\overline{2,3,4,5,6}]$	$[2;\overline{1,1,2,1,2,6,1,3,2,5,3,2}]$	6	
2	$[0;\overline{3,4,5,6,7}]$	$[3;\overline{2,5,3,7,1,1,2,1,2,6,3,1,3,1,2,1,12,1,1,8,2,26,17,}$ $\overline{4,2,52,1,1,1,1,2,1,25,1,4,6,1,12,2,2,13,1,5,1,2,2,1,}$ $\overline{4,1,2,2,1,5,1,1,1,1,1,5,1,2,2,1,4,3,2,1,5,1,2,2,6,1,2,}$ $\overline{12,1,6,4,1,25,1,2,1,1,1,1,52,2,4,105,1,8,52,1,17,2}$ $\overline{2,1,3,1,3,6,2,1,2,1,1,7,3,5,2,3,3,1,1,1,10,1,6,1,1,4,}$ $\overline{1,1,29,5,3,59,2,1,1,1,118,1,3,238,1,1,477,955,47}$ $\overline{118,1,1,1,2,59,3,5,29,1,1,1,10,14,1,4,1,4,1,1,6,1,1}$]	9 55	72

Especially in the last row we see that a 2-continued fraction with period 5 translates to a simple continued fraction of length 206. So, it seems that there may be no formula to estimate the periodic lengths. Moreover, the same table last column provides the comparison of the maximum valued of partial quotients, which again suggests that there maybe be relation.

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