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PROVING SOME INEQUALITIES BY JENSEN'S INEQUALITY

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Түйін

Генсен теңсіздігінің дәлелі ашық түрде алгебралық формада көрсетілген. Бұл теңсіздік басқа да атақты теңсіздіктерді дәлелдеуге және де сол теңсіздіктерге байланысты сұрақтарға жауап ретінде қолданыла алады. Бұл жұмыстың негізгі идеясы, осы теңсіздіктің басқа да теңсіздіктерді қалай дәлелдей алатынын көрсету.

Кілт сөздер: Генсен теңсіздігі, дөңес функция, ойыс функция.

Резюме

Докозание Неравенство Генсена показано в открытой алгебрической форме. Это неравенство может применятся для докозания других известных неравенств а также ответить на некоторые вопросы касательно этих неравенств. Основной идеи данной работы является показать как с помощью этого неравенства можно доказать другие неравенства.

Ключевые слова: Неравенство Генсена, выпуклая функция, вогнутая функция.

Abstract. The proof of Jensen inequality is shown in explicit algebraic form. The inequality can be effectively used in the proof of some other popular inequalities and intricate questions which requires a lot of operations. The main idea of this method is to show how the inequality is widely used to prove the inequalities.

1. Introduction:

JENSEN'S INEQUALITY

Let f be a convex function on the interval I

If $x_1, x_2, \ldots, x_n \in I$ and $t_1, t_2, \ldots, t_n$ are nonnegative real numbers such that $t_1 + t_2 + \ldots + t_n = 1$, then $f(\sum_{i=1}^{n} t_i x_i) \le \sum_{i=1}^{n} t_i f(x_i)$

Let f be a concave function on the interval I. If $x_1, x_2, \ldots, x_n \in I$ and $t_1, t_2, \ldots, t_n$ are nonnegative real numbers such that $t_1 + t_2 + \ldots + t_n = 1$,then $f(\sum_{i=1}^{n} t_i x_i) \ge \sum_{i=1}^{n} t_i f(x_i)$

Let's prove when the function is convex.

Thus to apply Jensen's Inequality, we have three conditions:

1. The function f must be a convex function on the Interval I.
2. $x_1, x_2, \dots, x_n \in I$ And $t_1, t_2, t_3, \dots, t_n$ are non negative real numbers.
3. $t_1 + t_2 + \dots + t_n = 1$

Jensen's Inequality can be easily proved by mathematical induction. So let's prove it.

Step 1 Let's prove for $n = 2$

$$f\left(\sum_{i=1}^{2} t_i x_i\right) \le \sum_{i=1}^{2} t_i f(x_i)$$

$$f(t_1 x_1 + t_2 x_2) \le t_1 f(x_1) + t_2 f(x_2) \quad , \begin{Bmatrix} t_1 + t_2 = 1 \\ t_2 = 1 - t_1 \end{Bmatrix}$$

$$f(t_1 x_1 + (1 - t_1) x_2 \le t_1 f(x_1) + (1 - t_1) f(x_2) \qquad (1)$$

Note that, inequality (1) gives the definition of convexity of a function. Since our function is convex, for $n = 2$ it is true.

Step 2 Assume that for $n = k$ it is true.

Step 3 Show for $n = k + 1$

We must show

$$f\left(\sum_{i=1}^{k+1} t_i x_i\right) \le \sum_{i=1}^{k+1} t_i f(x_i)$$

$$f\left(\sum_{i=1}^{k+1} t_i x_i\right) = f\left(t_{k+1} x_{k+1} + \sum_{i=1}^{k} t_i x_i\right)$$

$$= f\left(t_{k+1} x_{k+1} + (1 - t_{k+1}) \frac{1}{(1 - t_{k+1})} \sum_{i=1}^{k} t_i x_i\right)$$

By the definition of convex.

$$\le t_{k+1} f(x_{k+1}) + (1 - t_{k+1}) f\left(\frac{1}{1 - t_{k+1}} \sum_{i=1}^{k} t_i x_i\right)$$

$$= t_{k+1} f(x_{k+1}) + (1 - t_{k+1}) f\left(\sum_{i=1}^{k} \frac{t_i}{1 - t_{k+1}} x_i\right)$$

$$\le t_{k+1} f(x_{k+1}) + (1 - t_{k+1}) \sum_{i=1}^{k} \frac{t_i}{1 - t_{k+1}} f(x_i)$$

$$= t_{k+1} f(x_{k+1}) + (1 - t_{k+1}) \frac{1}{(1 - t_{k+1})} \sum_{i=1}^{k} t_i f(x_i)$$

$$= \sum_{i=1}^{k+1} t_i f(x_i)$$

$$f\left(\sum_{i=1}^{k+1} t_i x_i\right) \le \sum_{i=1}^{k+1} t_i f(x_i)$$

Then is true when the function f is convex.

Now let's prove some known inequalities.

1. Proving Arithmetic- Geometric Mean Inequality

Theorem: If $x_1, x_2, \ldots, x_n$ are nonnegative numbers ,then

$$\sqrt[n]{x_1 \cdot x_2 \cdots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Proof: Let f(x) = lnx, $f''(x) = \frac{-1}{x^2} < 0$, f is concave

For $t_1 = t_2 = \cdots = t_n = \frac{1}{n}$, by the Jensen's inequality:

$$f(\sum_{i=1}^{n} t_i x_i) \ge \sum_{i=1}^{n} t_i f(x_i)$$

Hence, $\ln(t_1 x_1 + t_2 x_2 + \cdots + t_n x_n) \ge t_1 \ln x_1 + t_2 \ln x_2 + \cdots + t_n \ln x_n$

$$\ln(\frac{1}{n} x_1 + \frac{1}{n} x_2 + \cdots + \frac{1}{n} x_n) \ge \frac{1}{n} \ln x_1 + \frac{1}{n} \ln x_2 + \cdots + \frac{1}{n} \ln x_n$$

$$\ln(\frac{x_1 + x_2 + \cdots + x_n}{n}) \ge \frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n}$$

$$\ln(\frac{x_1 + x_2 + \cdots + x_n}{n}) \ge \frac{\ln (x_1 \cdot x_2 \cdots x_n)}{n}$$

$$\ln(\frac{x_1 + x_2 + \cdots + x_n}{n}) \ge \ln (x_1 \cdot x_2 \cdots x_n)^{\frac{1}{n}}$$

$$(\frac{x_1 + x_2 + \cdots + x_n}{n}) \ge (x_1 \cdot x_2 \cdots x_n)^{\frac{1}{n}}$$

$$(\frac{x_1 + x_2 + \cdots + x_n}{n}) \ge (x_1 \cdot x_2 \cdots x_n)^{\frac{1}{n}}$$

$$\sqrt[n]{(x_1 \cdot x_2 \cdots x_n)} \le (\frac{x_1 + x_2 + \cdots + x_n}{n})$$

2. Proving Holder's Inequality

Theorem: For $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots y_n$ are nonnegative numbers ,

if $\frac{1}{p} + \frac{1}{q} = 1$ then

$(x_1 y_1 + x_2 y_2 + \ldots + x_n y_n) \le (x_1^p + x_2^p + \ldots + x_n^p)^{\frac{1}{p}} \cdot (y_1^q + y_2^q + \ldots + y_n^q)^{\frac{1}{q}}$ Proof:

Let

$$\Rightarrow\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)\le\left(x_1^p+x_2^p+\ldots+x_n^p\right)^{\frac{1}{p}}\cdot\left(\frac{1}{n^q}+\frac{1}{n^q}+\ldots+\frac{1}{n^q}\right)^{\frac{1}{q}}$$

$$\Rightarrow\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)\le\left(x_1^p+x_2^p+\ldots+x_n^p\right)^{\frac{1}{p}}\cdot\left(\frac{n}{n^q}\right)^{\frac{1}{q}}$$

$$\Rightarrow\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)\le\left(x_1^p+x_2^p+\ldots+x_n^p\right)^{\frac{1}{p}}\cdot\left(\frac{n^{\frac{1}{q}}}{n}\right)$$

$$\Rightarrow\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)\le\left(x_1^p+x_2^p+\ldots+x_n^p\right)^{\frac{1}{p}}\cdot\left(\frac{1}{n^{1-\frac{1}{q}}}\right)$$

$$1$$

$$\Rightarrow 1-\frac{1}{q}=\frac{1}{p}$$

$$\Rightarrow\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)\le\left(x_1^p+x_2^p+\ldots+x_n^p\right)^{\frac{1}{p}}\cdot\left(\frac{1}{n^{\frac{1}{p}}}\right)$$

$y_1=y_2=y_n=\frac{1}{n}$

$$\Rightarrow\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)\le\left(\frac{x_1^p+x_2^p+\ldots+x_n^p}{n}\right)^{\frac{1}{p}}$$

$$\Rightarrow\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^p\le\left(\frac{x_1^p+x_2^p+\ldots+x_n^p}{n}\right)$$

Let $f(x)=x^p\Rightarrow f''(x)=p(p-1)x^{p-2}>0$, then f is convex.

By Jensen's inequality $f(\sum_{i=1}^{k+1}t_ix_i)\le\sum_{i=1}^{k+1}t_if(x_i)$

For $t_1=t_2=\ldots=t_n=1/n$

$$f(\sum_{i=1}^{k+1}\frac{x_i}{n})\le\sum_{i=1}^{k+1}\frac{1}{n}f(x_i)$$

Hence $\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^p\le\left(\frac{x_1^p+x_2^p+\ldots+x_n^p}{n}\right)$

3. Proving Arithmetic- Quadratic Mean Inequality

Theorem: If $x_1, x_2,\ldots,x_n$ are nonnegative numbers ,then

$$\sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}}\ge\frac{x_1+x_2+\cdots+x_n}{n}$$

Proof: Let $f(x)=x^2$, $f''(x)=2>0$, f is convex

For $t_1=t_2=\cdots=t_n=\frac{1}{n}$, by the Jensen's inequality:

$$f(\sum_{i=1}^{k+1}t_ix_i)\le\sum_{i=1}^{k+1}t_if(x_i)$$

$$t_1 f(x_1)+t_2 f(x_2)+\cdots+t_n f(x_n)\geq f(t_1x_1+t_2x_2+\cdots+t_nx_n)$$

$$t_1x_1^2+t_2x_2^2+\cdots+t_nx_n^2\geq (t_1x_1+t_2x_2+\cdots+t_nx_n)^2$$

$$\frac{1}{n}x_1^2+\frac{1}{n}x_2^2+\cdots+\frac{1}{n}x_n^2\geq\left(\frac{1}{n}x_1+\frac{1}{n}x_2+\cdots+\frac{1}{n}x_n\right)^2$$

$$\sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}}\geq\frac{x_1+x_2+\cdots+x_n}{n}$$

4. Discussion and conclusion

APPLICATIONS OF JENSEN'S INEQUALITY IN PROBLEMS
Example1

If A,B,C are the interior angles of a triangle then, prove that

$$SinA+SinB+SinC\leq\frac{3\sqrt{3}}{2}$$

Proof:

Let $f(x)=\sin x$

$$f''(x)=(\sin x)''=-\sin x$$

$$-Sinx<0 \text{ When } x\in(0,\pi)$$

As $f(x)$ is concave, we can apply the Jensen's Inequality.

If $t_1+t_2+.....+t_n=1$, then

$$f(t_1x_1+t_2x_2+......+t_nx_n)\geq t_1f(x_1)+t_2f(x_2)+.....+t_nf(x_n)$$

And Let $f(x)=\sin x$

$$f(\frac{A+B+C}{3})\geq\frac{1}{3}f(A)+\frac{1}{3}f(B)+\frac{1}{3}f(C)$$

$$Sin(\frac{180}{3})\geq\frac{1}{3}(SinA+SinB+SinC)$$

$$Sin(60^\circ)=\frac{1}{3}(SinA+SinB+SinC)$$

$$\frac{3\sqrt{3}}{2}\geq SinA+SinB+SinC$$

Example2

$a_1, a_2, \ldots\ldots, a_n \in R^+ \cup \{0\}$ And all a_i's are not 0 directly.

If $A = \sum_{i=1}^{n} a_i$, $B = \sum_{i=1}^{n} i.a_i$ and R is the positive real root of the $x^n - a_1.x^{n-1} - \ldots\ldots - a_{n-1}.x - a_n = 0$, then prove that $A^A \leq R^B$

Proof:

For R being the real root

So $R^n - a_1.R^{n-1} - \ldots\ldots - a_{n-1}R - a_n = 0$------------ (1)

$$\Rightarrow 1 = \frac{a_1}{R} + \frac{a_2}{R^2} + \ldots\ldots + \frac{a_n}{R^n}$$

Consider the function $f(x) = -\ln x$

$f''(x) = \frac{1}{x^2} > 0$ So the f function is a convex when function in the interval $(0, \infty)$.

If we take $t_i = \frac{a_i}{A}$ $(i = 1,2,\ldots\ldots,n)$

Then $t_1 + t_2 + \ldots. + t_n = \frac{a_1 + a_2 + \ldots.. + a_n}{A} = 1$

By taking $x_i = \frac{A}{R^i}$ $(i = 1,2,\ldots\ldots,n)$

We can apply the Jensen's Inequality.

$$t_1(-\ln x_1) + t_2(-\ln x_2) + \ldots. + t_n(-\ln x_n) \geq -\ln(t_1x_1 + t_2x_2 + \ldots. + t_nx_n)$$

$$\Rightarrow \sum_{i=1}^{n} t_i.\left(-\ln \frac{A}{R^i}\right) \geq -\ln\left(\sum_{i=1}^{n} t_i \frac{A}{R^i}\right)$$

$$\Rightarrow \sum_{i=1}^{n} \frac{a_i}{A}(-\ln A + \ln R^i) \geq -\ln\left(\sum_{i=1}^{n} \frac{a_i}{A} \cdot \frac{A}{R^i}\right)$$

$$\Rightarrow \sum_{i=1}^{n} \frac{a_i}{A}(-\ln A + \ln R^i) \geq -\ln 1 \qquad , (-\ln 1 = 0)$$

$$\Rightarrow \sum_{i=1}^{n} \frac{a_i}{A} . \ln R^i \geq \sum_{i=1}^{n} \frac{a_i}{A} \ln A$$

$$\Rightarrow \ln R \sum_{i=1}^{n} i a_i \geq \ln A . \sum_{i=1}^{n} a_i$$

$$\Rightarrow B . \ln R \geq A . \ln A$$

Consequently $\Rightarrow R^B \geq A^A$

Acknowledgments

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