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# Laguerre polynomials in axisymmetric heat problems with a free boundary

THESIS

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# Abstract

The aim of the thesis is to consider solving heat equation with free boundaries using by heat polynomials method, in particular, using by Laguerre polynomials. There are two problems were considered. It is spherical inverse and direct problems the method of thermal polynomials is appropriate. As exactly as the approximate solutions.

The inverse two-phase spherical Stefan problem for unknown boundary heat flux is solved by the method of the heat polynomials. Side by side with exact solution two methods for the approximate solution, collocation and variational methods, convenient for engineering applications are presented and compared.

# Аңдатпа

Диссертациялық жұмыстың мақсаты жылу полиномдары әдісімен жылжымалы шекараларды қолдану арқылы жылу теңдеуін шешу, дәлірек айтқанда, Лагерр көпмүшесін қолдану арқылы.Осы жұмыста екі мәселе қарастырылды. Бұл сфералық кері және тікелей проблемалары. Олар жылу полиномы әдісі орынды. Шамамен шешімдер сияқты.

Белгісіз шекаралық жылу ағыны үшін кері фазалық сфералық Стефан мәселесі жылу полиномы әдісімен шешіледі. Дәл ерітіндімен бірлесе отырып, жақындастырылған шешім, коллекторлық және вариационды атты екі әдіс ұсынылған, олар инженерлік қолданбаға не.

# Аннотация

Целью дипломной работы является нахождение решения уравнения теплопроводности с движущейся границей с использованием метода тепловых полиномов, в частности, с использованием полиномов Лагерра. Здесь были рассмотрены две проблемы. Для сферических обратных и прямых задач метод термических полиномов является уместным. Точно так же, как приближенные решения.

Обратная двухфазная сферическая задача Стефана для неизвестного граничного теплового потока решается методом тепловых полиномов. Наряду с точным решением представлены и сопоставлены два метода приближенного решения: коллокационный и вариационный, которые удобные для инженерных задач.

# Acknowledgements

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To my family

# Contents

# Nomenclature

| | |
|---|---|
| SDU | Suleyman Demirel University |

# 1. Introduction

The radial heat polynomials and corresponding associated functions are introduced, and their main properties based on the generalized Laguerre polynomials are investigated. It is shown that they satisfy the generalized heat equation and they are biorthogonal with respect to the associated functions.
The inverse Stefan problem for the reconstruction of unknown heat flux on the moving boundary is considered. The solution is found in the form of linear combination of the radial heat polynomials. The unknown coefficients are found from the conditions on the moving boundary.
The method of integral error functions and heat polynomials for the solution of the heat equation in a domain with free boundary enables one to obtain the solution in the form handy for engineering application. The solution of the spherical Stefan problem with the boundary heat flux condition using this method is considered in [3]. It was shown that a given boundary function can be approximated by the linear combination of the system of the integral error functions $i^n erfc(x), n = 0, 1, 2...$, and the first five terms of this combination is sufficient to obtain the error less than 1 percent. It means according to the maximum principle for the heat equation that the error of approximation of the final solution has the same error. Then this approach was successfully applied for the solution of different problems of the Stefan kind. One of the most important problems in the theory of phenomena in electrical contacts is the definition of the arc heat flux entering into electrodes. The experimental measurement the dynamics of this flux is very difficult, and sometimes the mathematical modeling only is capable to obtain required information [8]. The mathematical model describing the process of the interaction of the electrical arc with electrodes and the dynamics of their melting is based on the spherical Stefan problem, and if we want to define the arc heat flux, the inverse spherical Stefan problem should be considered [2].

# 2. Preliminaries

## 2.1 Laguerre polynomials

In this section , lets consider solution of equation

$$xy'' + (1-x)y' + dy = 0 \tag{2.1}$$

on bounded $x \geq 0$, which grows at $x \to \infty$ no faster than some degree $x$. So, lets find solution of (2.1), by substitution

$$y = \sum_{t=0}^{\infty} B_t x^t \tag{2.2}$$

Then, we obtain

$$B_t t(t-1)x^{t-1} + B_t t x^{t-1} - B_t t x^t + dB_t t x^t = 0$$

Equating the coefficients at $x^t$, we get a recurrence relation

$$B_t(t+1)t + B_t(t+1) - B_t t + dB_t = 0$$

$$B_{t+1} = B_t \frac{t-d}{(t+1)^2} \tag{2.3}$$

All coefficients $B_N = 0$, when $N = t+1, t+2 \ldots.$ Because, $B_{t+1} = 0$ at $t = d$. It means that, polynomial of degree $m$ is a solution, which we have tried to find.

$$y = \sum_{t=0}^{d} B_t x^t \tag{2.2}$$

By change index $t$ to $t-1$ in (2.3), we get

$$B_t = B_t \frac{t^2}{t-1-d}$$

at $t = d$,

$$B_{d-1} = B_d \frac{d^2}{d-1-d} = -B_d d^2;$$

at $t = d-1$,

$$B_{d-2} = B_{d-1} \frac{(d-1)^2}{d-1-1-d} = -B_d d^2 \frac{(d-1)^2}{-2} = B_d d^2 \frac{(d-1)^2}{2};$$

at $t = d-2$,

$$B_{d-3} = -B_d d^2 \frac{(d-1)^2}{-2} = B_d d^2 \frac{(d-1)^2 (d-2)^2}{1 \cdot 2 \cdot 3};$$

at $t = d-p$,

$$B_{d-p} = (-1)^p B_d \frac{[d(d-1)(d-2)...(d-p+1)]^2}{p!} = (-1)^p B_d \frac{d!^2}{(d-p)!^2 p!} \quad (2.4)$$

From equality $t = d-p$, can be obtained $p = d-t$. This follows that expression (2.4) takes the form

$$B_t = (-1)^{(d-t)} B_d \frac{d!^2}{(t)!^2 (d-t)!}$$

So, solution can be written in the form

$$y = \sum_{t=0}^{d} B_t x^t = (-1)^d d! B_d \sum_{t=0}^{d} (-1)^t \frac{d! x^t}{(d-t)! t! t!} \quad (2.5)$$

Consider, that

$$B_d^t = \frac{d!}{(d-t)! t!}$$

and suppose

$$B_d = \frac{(-1)^d}{d!}$$

instead of (2.5), we get

$$y = L_d(x) = \sum_{t=0}^{d} (-1)^t \cdot B_d^t \cdot \frac{x^t}{t!}$$

These polynomials are called Laguerre polynomials. They are convenient to find by the formula

$$L_d(x) = \frac{1}{d!} e^{x} \frac{d^d}{dx^d} (x^d e^{-x}).$$

[7]

**Proof.** Using by Leibniz formula, we get

$$[x^d e^{-x}]^{(d)} = \sum_{t=0}^{d} C_d^t (e^{-x})^{(t)} (x^d)^{d-t}$$

Separately calculate derivative $(x^d)^{d-t}$

$$(x^d)^1 = d \cdot x^{d-1},$$

$$(x^d)^2 = d(d-1) \cdot x^{d-2},$$

$$(x^d)^3 = d(d-1)(d-2) \cdot x^{d-3},$$

$$\cdots\cdots\cdots$$

$$(x^d)^p = d(d-1)(x-p+1) \cdot x^{d-p},$$

suppose that $p = d - t$, then we find

$$(x^d)^{d-t} = \frac{d!}{t!} x^t.$$

Consider, that $(e^{-x})^{(t)} = (-1)^t e^{-x}$, we get

$$[x^d e^{-x}]^d = e^{-x} \sum_{t=0}^{d} B_{d^t} (-1)^t \frac{d!}{t!} x^t$$

By substituting this value into formula (2.7), we find

$$L_d(x) = \frac{1}{d!} e^x e^{-x} \sum_{t=0}^{d} B_d^t (-1)^t \frac{d!}{t!} x^t = \sum_{t=0}^{d} (-1)^t B_d^t \frac{x^t}{t!}.$$

## 2.2 Orthogonality of Laguerre polynomials

**Theorem 2.1.** *$e^{-\frac{x}{2}}$ polynomials form a complete and orthogonal system of functions on the half-line $[0, \infty]$. Then,*

$$e^{-x} L_m(x) L_d(x) dx = \begin{cases} 0, & at \quad m \neq d \\ 1, & at \quad m = d \end{cases}$$

[7]

**Proof.** Let $m > d$. Proof, that

$$\int_0^\infty e^{-x} x^d L_m(x) dx = 0 \tag{2.6}$$

Calculate (2.6) by integration by part:

$$\int_0^\infty e^{-x} x^d L_m(x) dx = \frac{1}{m!} \int_0^\infty x^d [e^{-x} x^m]^{(m)} dx$$

Do substation by $x^d = u$, $[e^{-x} x^m]^{(m)} dx = dv$, $d \cdot x^{d-1} \cdot dx = du$, $[e^{-x} x^m]^{(m-1)} = v$. So,

$$\int_0^\infty e^{-x} x^d L_m(x) dx = \frac{1}{m!} \int_0^\infty x^d [e^{-x} x^m]^{(m)} dx =$$

$$= \frac{1}{m!} x^d (e^{-x} x^m)^{(m-1)} \Big|_0^\infty - -\frac{d}{m!} \int_0^\infty x^{d-1} \cdot (e^{-x} x^m)^{(m-1)} dx =$$

$$= (-1)^d \frac{d!}{m!} \int_0^\infty (e^{-x} x^m)^{(m-d)} dx =$$

$$= (-1)^d \frac{d!}{m!} (e^{-x} x^m)^{(m-d-1)} \Big|_0^\infty = 0$$

Let $m \geq 1$. Consider integration

$$\int_0^\infty e^{-x} [L_m(x)]^2 dx =$$

$$= \frac{1}{m!}\int_0^\infty L_m(x)\cdot(e^{-x}x^m)^{(m)}dx = \frac{1}{m!}L_m(x)(e^{-x}x^m)^{(m-1)}\Big|_0^\infty -$$

$$-\frac{1}{m!}\int_0^\infty [L_m(x)]'\cdot(e^{-x}x^m)^{(m-1)}dx... = \frac{(-1)^m}{m!}\int_0^\infty [L_m(x)]^{(m)}\cdot e^{-x}x^m dx$$

$$|[L_m(x)]^{(m)} = (-1)^m| = \frac{1}{m!}\int_0^\infty e^{-x}x^m dx = \frac{1}{m!}\Gamma(m+1) = 1$$

## 2.3 Generalized Laguerre equation

Lets consider equation

$$xy'' + (\alpha + 1 - x)y' + dy = 0 \tag{2.7}$$

The equation below is called Generalized Laguerre equation. We need to find its solution, which is limited at the point $x = 0$, which grows at infinity no faster than some degree $x$. We solve equation exactly as in the chapter above

$$y = \sum_{t=0}^{\infty} B_t x^t \tag{2.8}$$

The coefficients at $x^t$ are related by

$$B_{t+1}(t+1)t + B_{t+1}(\alpha+1)(t+1) - B_t t + mB_t = 0$$

$$B_{t+1} = B_t\frac{t-d}{(t+\alpha+1)^2} \tag{2.9}$$

All coefficients, with starting $B_{t+1}$ are equal to zero, because $B_{t+1} = 0$ at $t = d$. That is why, solution of equation is polynomial in the form

$$y = \sum_{t=0}^{d} B_t x^t$$

By change index $t$ to $t - 1$ in (2.9), we get

$$B_t = B_t\frac{t(t+\alpha)}{t-1-d}$$

at $t = d$,

$$B_{d-1} = B_d \frac{d(d+\alpha)}{d-1-d} = -B_d d(d+\alpha);$$

at $t = d-1$,

$$B_{d-2} = -B_d d(d+\alpha)\frac{(d-1)(d-1+\alpha)}{d-1-d} =$$

$$= B_d \frac{d(d-1)(d+\alpha)(d+\alpha-1)}{2};$$

at $t = d-p$,

$$B_{d-p} = (-1)^p B_d \frac{[d(d-1)(d-2)...(d-p+1)(d+\alpha)...(d+\alpha-p+1)]}{p!} =$$

$$= (-1)^p B_d \frac{d!(d+\alpha)(d+\alpha-1)...(d+\alpha-p+1)}{(d-p)!p!} \tag{2.10}$$

We introduce the notation

$$\binom{a}{t} = \frac{a(a-1)...(a-t+1)}{t!}. \tag{2.11}$$

Formula (2.10) gives a coefficient for $x^{d-p}$, that is, for $x^t$.From equality $t = d-p$, can be obtained $p = d-t$. This follows that expression (2.10) takes the form

$$B_t = (-1)^{(d-t)} B_d \frac{d!(d+\alpha)(d+\alpha-1)...(t+\alpha-1)}{(t)!(d-t)!}$$

$$B_t = d!(-1)^d B_d (-1)^t \binom{d+\alpha}{d-t} \cdot \frac{1}{t!}$$

Thus, the desired solution can be presented as

$$y = (-1)^d d! B_d \sum_{t=0}^{d} (-1)^t \binom{d+\alpha}{d-t} \frac{x^t}{t!} \tag{2.13}$$

Putting in this formula $B_d = \frac{(-1)^d}{d!}$. Finally, we get

$$L_d^\alpha(x) = \sum_{t=0}^{d} (-1)^t \binom{d+\alpha}{d-t} \frac{x^t}{t!} \tag{2.14}$$

At $\alpha = 0$

$$L_d^0(x) = L_d \tag{2.15}$$

Formula (2.14) is called generalized Laguerre function. It is conveniently calculated by the formula

$$L_d^\alpha(x) = \frac{1}{d!}e^x x^{-\alpha}\frac{d^d}{dx^d}(x^{d+\alpha}e^{-x}).$$

**Proof.** Using by Leibniz formula, we get

$$[x^{d+\alpha}e^{-x}]^{(d)} = \sum_{t=0}^{d} C_d^t (e^{-x})^{(t)}(x^{d+\alpha})^{d-t} =$$

$$\sum_{t=0}^{d}\frac{d!}{t!(d-t)!}(-1)^t e^{-x}\cdot(d+\alpha)(d+\alpha-1)...(t+\alpha+1)x^{t+\alpha} =$$

$$= d!e^{-x}x^{\alpha}\sum_{t=0}^{d}(-1)^t\binom{d+\alpha}{d-t}\frac{x^t}{t!}$$

[7]

## 2.4 Heat equation for a solid with variable cross-section.

To elucidate the physical meaning of the equation

$$\frac{\partial\theta}{\partial t} = a^2\left(\frac{\partial^2\theta}{\partial z^2} + \frac{v}{z}\frac{\partial\theta}{\partial z}\right) \tag{2.17}$$

for any $v > 0$ let us consider the surface generated by revolution of a curve $r = y(z, t)$ about $z$-axes. Let us assume that the radial component of the temperature gradient in the solid bounded by above surface is negligible in comparison with the axial component , i.e. the solid can be considered as a bar with a variable cross – section , that has only axial component of heat flux (see Figure 2.1) One can find that the amount of the power $dQ_1$ entering the bridge element between cross–sections $z$ and $z + dz$ with corresponding radius $y$ and $y + dy$ during the

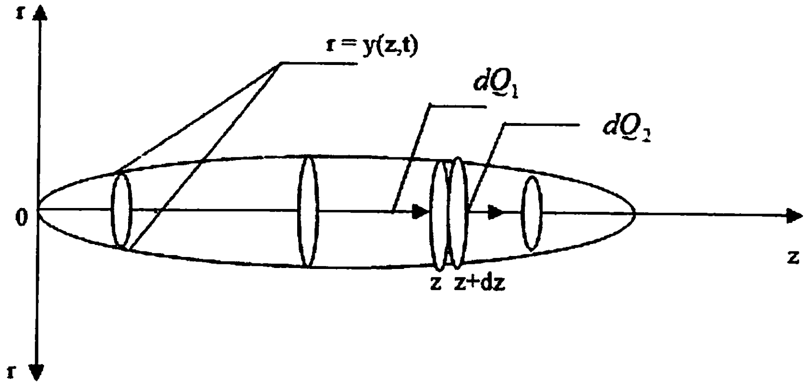

Figure 2.1: A solid with variable cross-section

time $dt$ is

$$dQ_1 = -\lambda \left.\frac{\partial\theta}{\partial z}\right|_z \pi y^2 dt,$$

where $\theta(z,t)$ is the temperature and $\lambda$ is the thermal conductivity, while the amount $dQ_2$ escaping this element from the cross – section $z+dz$ is

$$dQ_2 = -\lambda \left.\frac{\partial\theta}{\partial z}\right|_{z+dz} \pi(y+dy)^2 dt,$$

Using Taylor expansion

$$\left.\frac{\partial\theta}{\partial z}\right|_{z+dz} = \left.\frac{\partial Q}{\partial z}\right|_z + \left.\frac{\partial^2\theta}{\partial z^2}\right|_z dz + 0(dz^2)$$

and taking $(y+dy)^2 = y^2 + 2dy + 0(dy^2)$ one can calculate

$$Q_2 - Q_1 = -\pi\lambda \left.\frac{\partial\theta}{\partial z}\right|_z 2ydydt - \pi\lambda \left.\frac{\partial^2\theta}{\partial z^2}\right|_z y^2dzdt + 0(dz^2, dy^2)$$

The power required for the heating ( or cooling ) of the element from the temperature $\theta$ to the temperature $\theta + d\theta$ during the time $dt$ is

$$dQ_3 = c\gamma\frac{\pi}{3}[(y+dy)^2 + y^2 + y(y+dy)]dzd\theta = \pi c\gamma y^2 dz\frac{\partial\theta}{\partial t}dt + O(dy)$$

If any source of heating with intensity $q(z,t)$ (power per unit volume) is acting inside element, then this additional power is

$$dQ_4 = q(z,t)\pi y^2 dzdt$$

From power balance

$$dQ_2 - dQ_1 + dQ_3 = dQ_4$$

one can derive

$$\frac{\partial \theta}{\partial t} = a^2\left(\frac{\partial^2 \theta}{\partial z^2} + \frac{2y'_z}{y}\frac{\partial \theta}{\partial z}\right) + \frac{1}{c\gamma}q(z,t) \tag{2.18}$$

or in the alternative form

$$\frac{\partial \theta}{\partial t} = \frac{a^2}{y^2(z)}\frac{\partial}{\partial z}\left[y^2(z)\frac{\partial \theta}{\partial z}\right] + \frac{1}{c\gamma}q(z,t) \tag{2.19}$$

The equation (2.17) is the specific case of (2.18) at $y(z,t) = z^{v/2}$ and $q(z,t)$.[5]

# 3. Heat polynomials for the solution of free boundary problems

## 3.1 Basics of the generalized heat equation

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial r^2} + \frac{v}{r}\frac{\partial\theta}{\partial r} \tag{3.1}$$

Equation (3.1) is called The generalized heat equation, which portrays the warm transfer with a variable cross-section in a rod [5].By solving this equation with an initial condition containing a delta function, the Laplace transform is used in the form, fundamental solution can be obtained [11]

$$G(r, r_1, t) = \frac{C_l}{2t}(rr_1)^{-\delta} e^{-\frac{r^2+r_1^2}{4t}} I_\delta(\frac{rr_1}{2t}), \quad \delta = \frac{l-1}{2}, \quad C_l = 2^{-\delta}\Gamma(\delta+1) \tag{3.2}$$

Consider the thermal potentials for this solution.

$$V_{n,l}(r,t) = 2^{-\delta}\Gamma(\delta+1)^{-1}\int_0^\infty G(r, r_1, t) r_1^{2n+v} dr_1 \tag{3.3}$$

Using by integration by part method, we get the explicit equation for the next polynomials:

$$V_{n,l}(r,t) = \sum_{k=0}^{n} 2^{2k} \frac{n!\Gamma(\delta+1)}{k!(n-k)!\Gamma(\delta+1+n-k)} r^{2n-2k} t^k \tag{3.4}$$

For applications, it is more helpful to multiply right and left sides of this equation by $\frac{\Gamma(\delta+1+n)}{\Gamma(\delta+1)}$. Then

$$R_{n,l}(r,t) = \frac{\Gamma(\delta+1+n)}{\Gamma(\delta+1)} Q_{n,l}(r,t) = \sum_{k=0}^{n} 2^{2k} \frac{n!\Gamma(\delta+1+n)}{k!(n-k)!\Gamma(\delta+1+n-k)} r^{2n-2k} t^{k} \tag{3.5}$$

and

$$R_{n,l}(r,0) = r^{2n} \tag{3.6}$$

This could be shown in terms of the influence of the hypergeometric function and the generalized Laguerre polynomials:

$$R_{n,l}(r,t) = n!(4t)^n L_n^{(\delta)}(\frac{-r^2}{4t}) = \frac{\Gamma(\delta+1+n)}{\Gamma(\delta+1)}(4t)^n \Phi(-n,\delta+1;\frac{-r^2}{4t}) \tag{3.7}$$

In particular,

$$R_{0,l}(r,t) = 1,$$

$$R_{1,l}(r,t) = r^2 + \frac{l+1}{2}t,$$

$$R_{2,l}(r,t) = r^4 + 4(l+3)r^2 t + 4(l+1)(l+3)t^2,$$

$$R_{3,l}(r,t) = r^6 + 6(l+5)r^4 t^2 + 72(l+3)(l+5)r^2 t^2 + 8(l+1)(l+3)(l+5)t^3$$

It is easy to check that if $l = 0$, then the function

$$R_{n,0}(x,t) = v_n(r,t)$$

satisfies the equation (3.4), and for $l = 1$ the function

$$\theta(r,t) = \sum_{k=0}^{\infty} A_n R_{2n,1}(r,t) = \sum_{k=0}^{\infty} A_n (4t)^n L_n \left[-\frac{r^2}{4t}\right] = \sum_{k=0}^{\infty} A_n n! \sum_{k=0}^{n} \frac{2^{2k} r^{2(n-k)} t^k}{k![(n-k)!]^2} \tag{3.8}$$

satisfies the equation

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} \tag{3.9}$$

The generating function for the radial polynomials can be written as

$$g_\delta(r,t;z) = (1-4zt)^{-\delta} e^{\frac{zr^2}{1-4zt}} = \sum_{k=0}^{\infty} (4t)^n L_n^{(\delta)} \left[-\frac{r^2}{4t}\right] z^n \tag{3.10}$$

The fundamental source of solution for the generalized heat equation is

$$S_l(r,t) = (4\pi t)^{-\delta} e^{-\frac{r^2}{4t}}, \delta = \frac{l-1}{2} \tag{3.11}$$

Use Appel transform

$$T_{n,l}(r,t) = S_l(r,t) R_{n,l}\left(\frac{r}{t}, -\frac{1}{t}\right) = t^{-2n} R_{n,l}(r,-t) \tag{3.12}$$

We can construct the generating function for radial associated functions

$$g_l(r,t;z) = S_l(r,t+4z) = t^{-2n} R_{n,l}(r,-t) = \sum_{k=0}^{\infty} T_{n,l}(r,t)\frac{z^n}{n!} \tag{3.13}$$

Taking into account the orthogonality of the generalized Laguerre polynomials, it is easy to establish the condition of bi-orthogonality of the radial heat polynomials and their associated functions

$$\int_0^\infty W_l(r) R_{m,l}(r,-t) T_{n,l}(r,t)\,dr = \begin{cases} 0, & m \neq n \\ m!2^{4m}\Gamma(\delta+m+1) & m = n \end{cases} \tag{3.14}$$

where

$$W_l(r) = 2\pi^{\delta+1} r^{v} \tag{3.15}$$

## 3.2 The generalized heat equation with the inverse Stefan problem

The equation (1) is considered in the domain

$$D: \quad 0 \leq r < \eta(t), \quad 0 < t < T \tag{3.16}$$

On the boundary $r = \eta(t)$ the condition

$$u(\eta(t), t) = f(t) \tag{3.17}$$

and the Stefan condition

$$-\lambda\frac{\partial u(\eta(t),t)}{\partial r} = P(t) - L\gamma\frac{d\eta}{dt} \tag{3.18}$$

should be satisfied. At the initial time the domain $D$ collapses into zero, thus the initial conditions are

$$\eta(0) = u(0,0) = f(0) = 0 \tag{3.19}$$

It is required to find the functions $u(r,t)$ and $P(t)$, if $\eta(t)$ and $f(t)$ are given. The solution can be represent by the formula

$$u(r,t) = \sum_{k=0}^{\infty} A_n(4a^2t)^n L_n^{(\delta)}\left(-\frac{r^2}{4a^2t}\right) \tag{20}$$

where $\delta = \frac{l-1}{2}$ and $L_n(x)$ are associated Laguerre polynomials. The function (3.20) satisfies the heat equation (3.1) for arbitrary constants $A_n$. Satisfying the boundary condition (3.17) we get

$$f(t) = \sum_{k=0}^{\infty} A_n(4a^2t)^n L_n^{(\delta)}\left(-\frac{\eta(t)^2}{4a^2t}\right) = \sum_{k=0}^{\infty} A_n\varphi(t) \tag{3.21}$$

where

$$\varphi_n(t) = \sum_{k=0}^{\infty}(4a^2t)^n L_n^{(\delta)}\left(-\frac{\eta(t)^2}{4a^2t}\right) \tag{3.22}$$

Using the property of orthogonality of the associated Laguerre polynomials

$$\int_0^{\infty} t^{\delta}e^{-t}L_n^{(\delta)}(t)L_m^{(\delta)}dt = \begin{cases} 0, & \text{if} \quad n \neq m \\ \frac{\Gamma(n+\delta+1)}{n!}, & \text{if} \quad n = m \end{cases} \tag{3.23}$$

we can expand the function (3.22) into series with respect to the associated Laguerre polynomials

$$\varphi_n(t) = \sum_{k=0}^{\infty} \varphi_{n,k}L_k^{(\delta)}(t) \tag{3.24}$$

where

$$\varphi_{n,k} = \frac{k!}{\Gamma(k+\delta+1)}\int_0^{\infty} t^{\delta}e^{-t}L_k^{(\delta)}(t)\varphi_n(t)dt \tag{3.25}$$

Substituting (3.24) into (3.21) we get

$$f(t)=\sum_{n=0}^{\infty}A_n\sum_{k=0}^{\infty}\varphi_{n,k}L_k^{(\delta)}(t) \tag{3.26}$$

Using the orthogonality property (3.23) we can write the system of algebraic equations for determination of the coefficients $A_n$

$$\sum_{n=0}^{\infty}a_{n,m}A_n=b_m,\quad m=0,1,2,3... \tag{3.27}$$

where

$$a_{n,m}=\frac{1}{m!}\varphi_{n,m}\Gamma(k+\delta+1),\quad b_m=\int_0^{\infty}t^{\delta}e^{-t}L_m^{(\delta)}(t)f(t)dt \tag{3.28}$$

It should be noted that an approximate solution of this problem can be found using the approximation of the function $f(t)$ with a finite sum in expression (3.27). Such approximation can be achieved with an arbitrary small error $\varepsilon$. Then the approximate solution of the problem (3.1), (3.17), (3.18), (3.19) can be obtained by the replacement of the series in the expression (3.20) for the finite sum. Again, this is an exact solution of the heat equation (3.1), and also has an error not exceeding $\varepsilon$ due to the maximum principle for the heat equation.In this case the infinite system of algebraic equations (3.27) becomes to be finite. Thus the required temperature $u(r,t)$ is defined by the expressions (3.20), (3.22), (3.24), (3.25), (3.27). The unknown heat flux $P(t)$ can be found now from the expression (3.18) which can be written in the form

$$P(t)=\lambda\frac{\partial u(\eta(t),t)}{\partial r}-L\gamma\frac{d\eta}{dt} \tag{3.29}$$

where the right side is already found. This method can be effectively applied for the solution of the arc phenomena modeling in electrical contacts [6], [4].

# 4. The two-phase spherical Stefan problem solutions with using heat polynomials

## 4.1 Mathematical model

The inverse Stefan problem consists in the definition of the arc heat flux $P(t)$ and the temperature distribution $\theta(r, t)$ in the molten contact hemisphere $r_0 < r < r + \alpha(t)$ , if $\alpha(t)$ is given from the measurement. If the arc burning period is $0 \leq t \leq t_0$ and the final radius of the molten zone at $t = t_a$ is $r_a$ , then the dynamics of the arc radius increasing at the melting can be approximated by the formula

$$\alpha(t) = r_0 + \alpha_0(t) \quad \alpha_0 = (r_a - r_0)/\sqrt{t_0} \tag{4.1}$$

The heat equation for the melting zone can be written in the form

$$\frac{\partial \theta}{\partial t} = a^2 \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r} \frac{\partial \theta}{\partial r} \right) \quad r_0 < r < \alpha(t), \quad 0 < t < t_a \tag{4.2}$$

The initial and boundary conditions are

$$\theta \big|_{t=0} = 0 \tag{4.3}$$

$$-\alpha \frac{\partial \theta}{\partial r} \bigg|_{r=r_0} = P(t) \tag{4.4}$$

and on the interface of the phase transformation

$$\theta(\alpha(t), t) = \theta_m, \tag{4.5}$$

$$-\alpha \frac{\partial \theta}{\partial r}\bigg|_{r=\alpha(t)} = L\gamma \frac{d\alpha}{dt}, \tag{4.6}$$

where $\theta_m$ is the melting temperature, $\alpha$, $L$, $\gamma$ are the coefficient of the heat conductivity, latent heat of melting and density respectively.

To simplify the calculation we can introduce the new dimensionless time $t_1 = t/t_a$, then the time interval of arcing changes to $0 < t_1 < 1$. Thus we can take $t_a = 1$ at once in the formula (4.2).

This problem for the spherical heat equation can be reduced to the ordinary one-dimensional equation by the substitutions

$$\theta = \frac{u}{r}, \quad r - r_0 = x, \quad \beta(t) = \alpha(t) - r_0 \tag{4.7}$$

Then the problem transforms to the form

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \beta(t), \quad 0 < t < T \tag{4.8}$$

$$\theta\big|_{t=0} = 0 \tag{4.9}$$

$$-\lambda \left[ r_0 \frac{\partial u}{\partial x} - u \right] \bigg|_{x=0} = r_0^2 P(t) \tag{4.10}$$

$$u(\beta(t), t) = U_m \tag{4.11}$$

$$-\lambda \left[ \beta(t) \frac{\partial u}{\partial x} - u \right] \bigg|_{x=\beta(t)} = \beta^2(t) L\gamma \frac{d\beta}{dt} \tag{4.12}$$

The solution of this problem can be represented) in the form:

$$u(x,t) = \sum_{n=0}^{\infty} A_n v_{2n}(x,t) + \sum_{n=0}^{\infty} B_n v_{2n+1}(x,t) \tag{4.13}$$

where

$$v_{2n}(x,t) = \sum_{k=0}^{n} \frac{(2n)! a^{2k} x^{2n-2k}}{k!(2n-2k)!} t^k, \quad v_{2n+1}(x,t) = \sum_{k=0}^{n} \frac{(2n+1)! a^{2k} x^{2n-2k+1}}{k!(2n-2k+1)!} t^k \tag{4.14}$$

are the heat polynomials satisfying the equation (4.8) at arbitrary coefficients $A_n$, $B_n$, which should be chosen to satisfy the boundary conditions. The unknown heat flux we represent in the form

$$P(t) = \sum_{k=0}^{l} P_k t^k \tag{4.15}$$

From the conditions (4.9), (4.10) we have the following system of equations for $A_n, B_n$:

$$\sum_{n=0}^{m} A_n \sum_{k=0}^{n} \frac{(2n)!a^{2k}\alpha_0^{2n-2k}}{k!(2n-2k)!} t^n + \sum_{n=0}^{m} B_n \sum_{k=0}^{n} \frac{(2n+1)!a^{2k}\alpha_0^{2n-2k+1}}{k!(2n-2k+1)!} t^{n+\frac{1}{2}} = U_m \tag{4.16}$$

$$\sum_{n=0}^{m} A_n \sum_{k=0}^{n} \frac{(2n)!a^{2k}\alpha_0^{2n-2k}}{k!(2n-2k)!} t^n + \sum_{n=0}^{m} B_n \sum_{k=0}^{n} \frac{(2n+1)!a^{2k}\alpha_0^{2n-2k+1}}{k!(2n-2k+1)!} t^{n+\frac{1}{2}} =$$

$$-\frac{1}{\lambda} L\gamma\alpha_0^3 \frac{\sqrt{t}}{2} + U_m \tag{4.17}$$

To evaluate unknown coefficients we use two methods, variational and collocation.

## 4.2 Variational method

Similarly, like in [3], we take $m = 5$ , $U_m = 0$ and the basic points $t = t_i = \frac{2i}{10}$, $i = 0, 1, 2, 3, 4, 5$ To satisfy approximately the condition (4.16) we consider the functional:

$$J = \int_0^1 \left( \sum_{n=0}^{5} A_n \sum_{k=0}^{n} \frac{(2n)!a^{2k}\alpha_0^{2n-2k}}{k!(2n-2k)!} t^n + \sum_{n=0}^{5} B_n \sum_{k=0}^{n} \frac{(2n+1)!a^{2k}\alpha_0^{2n-2k+1}}{k!(2n-2k+1)!} t^{n+\frac{1}{2}} \right)^2 dt$$

The minimum of this functional can be found from the equation

$$\frac{\partial J}{\partial A_n} = 2\int_0^1 \left( \sum_{n=0}^{5} A_n v_{2n}(\beta(t), t) + \sum_{n=0}^{5} B_n v_{2n+1}(\beta(t), t) \right) v_{2m}(\beta(t), t)dt = 0$$

where

$$v_{2n}(\beta(t), t) = \sum_{k=0}^{n} \frac{(2n)!a^{2k}\alpha_0^{2n-2k}}{k!(2n-2k)!} t^n,$$

$$v_{2n+1}(\beta(t), t) = \sum_{k=0}^{n} \frac{(2n+1)!a^{2k}\alpha_0^{2n-2k+1}}{k!(2n-2k+1)!} t^{n+\frac{1}{2}}$$

$$\sum_{n=0}^{5} A_n C_{nm} = D_{nm} \tag{4.18}$$

$$C_{nm} = \int_0^1 v_{2n}(\beta(t), t) v_{2m}(\beta(t), t) dt,$$

$$D_{nm} = \int_0^1 \sum_{n=0}^{5} B_n v_{2n+1}(\beta(t), t) v_{2m}(\beta(t), t) dt,$$

$$m = 0, 1 \ldots n$$

Solving the system (4.18) with respect to $A_n$ from (4.18) and substituting the result into the expression (4.17) we get

$$J = \int_0^1 \left( \sum_{n=0}^{5} B_k w(k, t) + f(t) \right)^2 dt \tag{4.19}$$

where

$$w(k, t) = \sum_{n=0}^{5} A_{kn} v_{\bar{2}k}(\beta(t), t) + v_{2\bar{k}+1}(\beta(t), t),$$

$$\bar{v}_{2k}(\beta(t), t) = \sum_{n=0}^{k} \frac{(2k)! a^{2k} (2k - 2m) \alpha_0^{2k-2m}}{m!(2k - 2m)!} t^k$$

$$\bar{v}_{2k+1}(\beta(t), t) = \sum_{n=0}^{k} \frac{(2k + 1)! a^{2k} (2k - 2m + 1) \alpha_0^{2k-2m+1}}{m!(2k - 2m + 1)!} t^{k+\frac{1}{2}}$$

$$f(t) = -\frac{1}{\lambda} L \gamma \alpha_0^3 \frac{\sqrt{t}}{2}$$

From the condition of maximum of (4.19) we have

$$\frac{\partial J}{\partial B_n} = 2 \int_0^1 \left( \sum_{k=0}^{5} B_n w(k, t) + f(t) \right) w(m, t) dt = 0 \tag{4.20}$$

where

$$E_{km} = \int_0^1 w(k, t) w(m, t) dt, \quad F_m = \int_0^1 f(t) w(m, t) dt, \quad m = 0, 1 \ldots k$$

From the expression (4.20) we get the following results:

$$B_0 = -0.784; \quad B_1 = -0.062; \quad B_2 = 0.046 \quad B_3 = -9.712 \times 10^{-3};$$

$$B_4 = 6.788 \times 10^{-4}; \quad B_5 = -1.391 \times 10^{-5}$$

Similarly, from expression (4.16) we obtain:

$$A_0 = 0.058; \quad A_1 = 0.904; \quad A_2 = -0.389; \quad A_3 = 0.071;$$

$$A_4 = -4.716 \times 10^{-3}; \quad A_5 = 9.516 \times 10^{-5}$$

Now we should define the coefficients for the heat flux in the expression (4.15). The corresponding variational functional for the condition (4.10) is

$$J = \int_0^1 \left( \sum_{n=0}^{5} P_n t^n + g(t) \right)^2 dt \tag{4.21}$$

where

$$g(t) = -\frac{\lambda}{r_0^2} \left[ \sum_{n=0}^{5} A_n \left( v'_{2n}(r_0, t) - v_{2n}(r_0 t) \right) + \sum_{n=0}^{5} B_n \left( v'_{2n+1}(r_0, t) - v_{2n+1}(r_0 t) \right) \right]$$

$$v'_{2n}(r_0, t) = \sum_{k=0}^{n} \frac{(2n)! a^{2k} r_0^{2n-2k}}{k!(2n-2k)!} t^k$$

$$v'_{2n+1}(r_0, t) = \sum_{k=0}^{n} \frac{(2n+1)! a^{2k} r_0^{2n-2k+1}}{k!(2n-2k+1)!} t^k$$

$$v_{2n}(r_0, t) = \sum_{k=0}^{n} \frac{(2n)! a^{2k} r_0^{2n-2k}}{k!(2n-2k)!} t^k, \quad v_{2n+1}(r_0, t) = \sum_{k=0}^{n} \frac{(2n+1)! a^{2k} r_0^{2n-2k+1}}{k!(2n-2k+1)!} t^k$$

The minimum of (4.21) gives the equation

$$\frac{\partial J}{\partial P_n} = 2 \int_0^1 \left( \sum_{k=0}^{5} P_n t^n + g(t) \right) t^m dt = 0$$

$$\sum_{n=0}^{5} P_n G_{nm} = H_{nm}. \tag{4.22}$$

where

$$G_{nm} = \int_0^1 t^{n+m} dt, \quad H_m = \int_0^1 g(t) t^m dt, \quad m = 0, 1, 2...n$$

From expression (4.22) we have the following results:

$$P_0 = -0.009; \quad P_1 = 0.085; \quad P_2 = -2.38; \quad P_3 = 6.572;$$

$$P_4 = -7.406; \quad P_5 = 2.872;$$

The results of testing for $a = 1, \quad \alpha_0 = 1, \quad r_0 = 1, \quad L = 1, \gamma = 1, \quad U_m = 1$ depicts in Figure 4.1 the approximated function

$$V(t) = -\frac{\lambda}{r_0^2}\left[\sum_{n=0}^{5} A_n\left(v'_{2n}(r_0, t) - v_{2n}(r_0 t)\right) + \sum_{n=0}^{5} B_n(v'_{2n+1}(r_0, t) - v_{2n+1}(r_0 t))\right]$$

and the exact solution

$$P(t) = \sum_{n=0}^{5} P_n t^n \tag{4.18}$$

which can be obtained by the solution of the direct Stefan problem [9],[10].

One can see the ideal coincidence of exact and approximated solutions.

## 4.3 Collocation Method

Let us take for testing $m = 5$ the basic points $t = t_i = \frac{2i}{10}$, $i = 0, 1, 2, 3, 4, 5$, $a = 1, \quad \alpha_0 = 1, \quad r_0 = 1, \quad L = 1, \gamma = 1, \quad U_m = 0$. Then we get the following values for $A_n$ and $B_n$:

$$A_1 = 0.878; \quad A_2 = -0.051; \quad A_3 = -0.069;$$

$$A_4 = 9.957 \times 10^{-3}; \quad A_5 = -3.073 \times 10^{-4}$$

$$B_1 = -1.448; \quad B_2 = -0.514; \quad B_3 = -0.05;$$

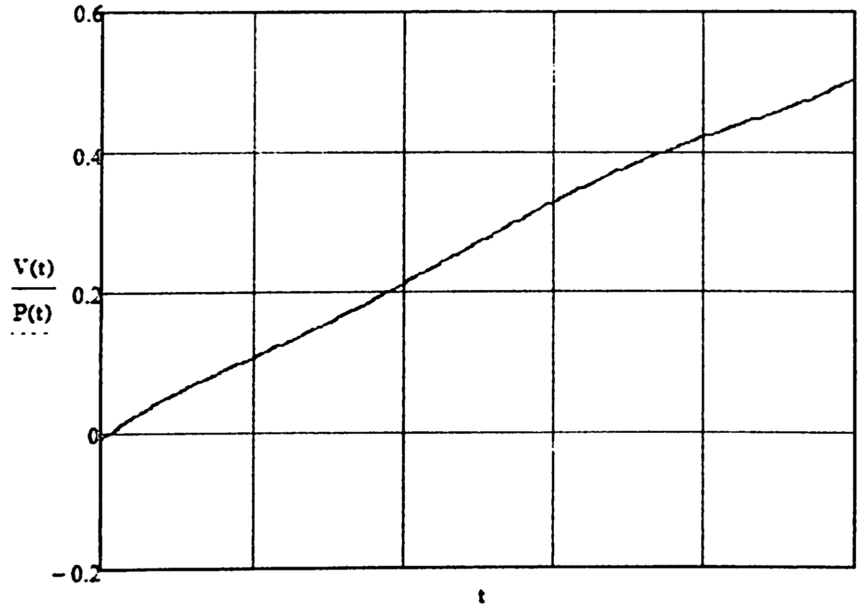

Figure 4.1: Approximated and exact heat fluxes

$$B_4 = -1.041 \times 10^{-3}; \quad B_5 = 1.383 \times 10^{-5}$$

From the condition (4.10) we have the following results:

$$P_1 = 1.212; \quad P_2 = -6.219; \quad P_3 = 18.288;$$

$$P_4 = -21.378; \quad P_5 = 8.597$$

The Figure 4.2 depicts the approximate function

$$V(t) = -\frac{\lambda}{r_0^2}\left[\sum_{n=1}^{5} A_n\left(v'_{2n}(r_0,t) - v_{2n}(r_0 t)\right) + \sum_{n=1}^{5} B_n(v'_{2n+1}(r_0,t) - v_{2n+1}(r_0 t))\right]$$

and original function $P(t) = \sum_{n=0}^{5} P_n t^n$ taking for the corresponding direct Stefan problem

The greatest error of approximation is in the neighborhood of zero and one. The error of approximation is approximately 2 percent.

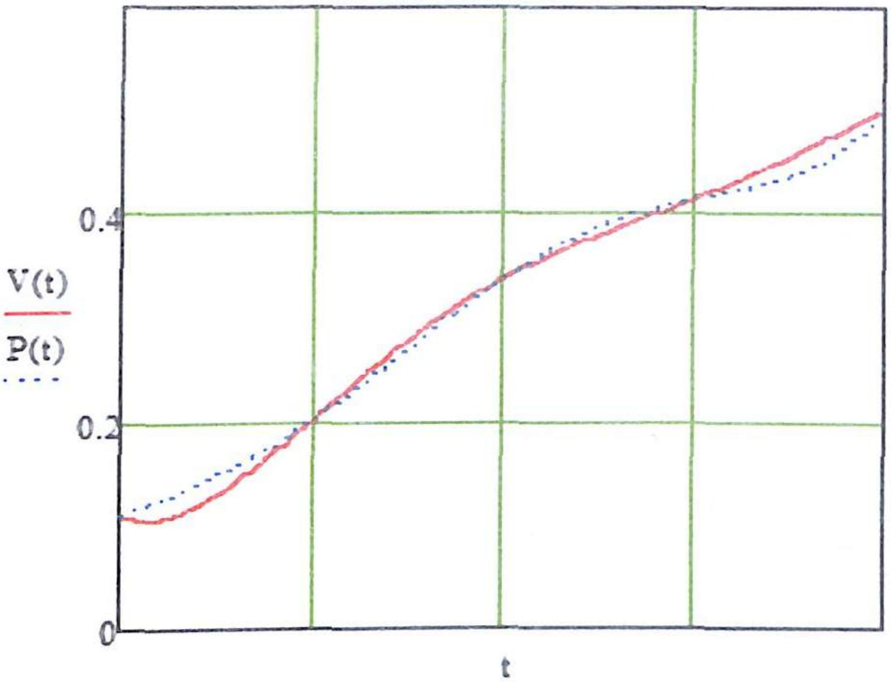

Figure 4.2: Approximated $V(t)$ and exact $P(t)$ heat fluxes

## 4.4 Experimental verification

Let us compare the results of approximation with the exact solution and experimental data presented in [2]. The contact material is $AgCdO$, the initial radius of the arc spot on the contact surface $r_0 = 10^{-4}m$ , the current $I = 1.5kA$ , the voltage $U = 42V$, the arc duration $t_a = 12\mu s$. Then we have the coefficients of original function

$$P_1 = 1.755 \times 10^8; \quad P_2 = -6.17 \times 10^7; \quad P_3 = -8.254 \times 10^8;$$

$$P_4 = -1.673 \times 10^9 \quad P_5 = -9.292 \times 10^8$$

Figure 4.3 shows that the approximation and the original functions are identical everywhere without errors.

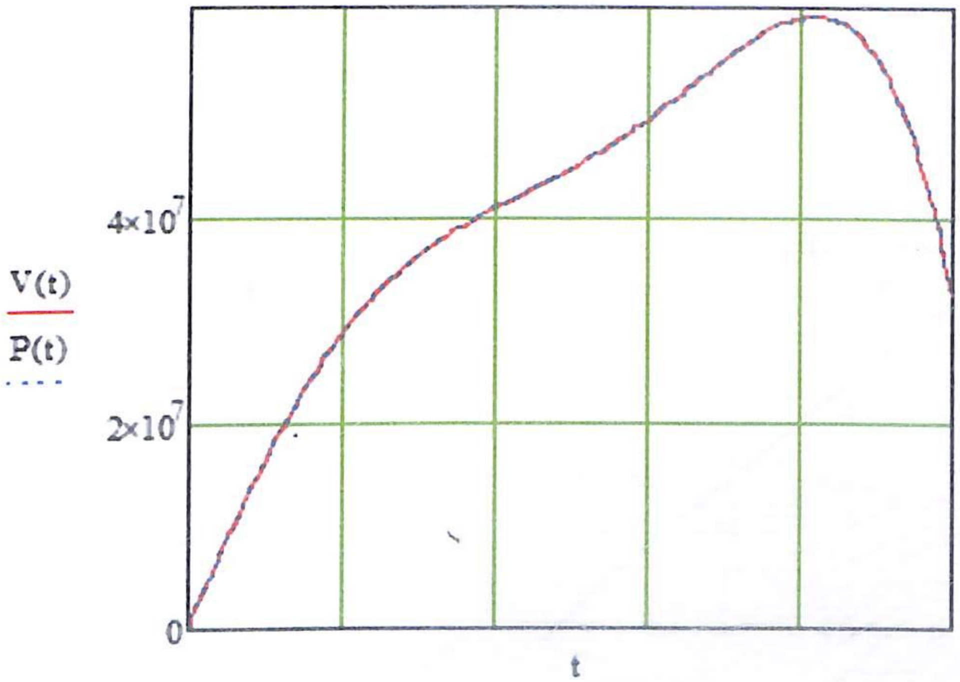

Figure 4.3: the domain $\Omega_{x,t}$ in the case II

# 5. Application

## 5.1 Heat polynomials and their properties

Consider a one-dimensional heat conduction equation in dimensionless co-ordinates

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} \tag{5.1}$$

where $U = U(x,t)$. For a function in space of all continuous functions that converge to 0 to $\infty$, portrayed by the taking after equation

$$U(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3 ..., x_n + dx_n) =$$

$$U = (x_1, x_2, x_3, x_4 ... x_n) + \frac{dU(x_1, x_2, x_3 ..., x_n)}{1!} +$$

$$+\frac{d^2U(x_1, x_2, x_3 ..., x_n)}{2!} + \frac{d^3U(x_1, x_2, x_3 ..., x_n)}{3!} + ... \frac{d^{n-1}U(x_1, x_2, x_3 ..., x_n)}{(n-1)!} + R_n \tag{5.2}$$

where

$$d^kU = \left(\frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 + ... \frac{\partial U}{\partial x_n}dx_n\right)^{(k)}$$

and $R_n$ stands for the $n^{th}$ remainder. Extend $U$ onto Taylor series at $x$ and $t$ are zero. At that point, using the formula (5.1)to dispense with the second-order derivatives with regard to $x$ within the Taylor's series extension, one gets

$$U = U_0 + \frac{\partial U}{\partial x}x + \frac{\partial U}{\partial t}(\frac{x^2}{2!} + t) + \frac{\partial^2 U}{\partial x \partial t}(\frac{x^3}{3!} + xt) + \frac{\partial^2 U}{\partial x \partial t}(\frac{x^4}{4!} + \frac{x^2 t}{2!} + \frac{t^2}{2!}) + ... \tag{5.3}$$

In the equation (5.3) all the derivatives as well as $U_0$ are taken at $x$ and $t$ are zero. The polynomials going with the derivatives fulfill the formula (5.1). The set of polynomials is finished. They are called heat polynomials. The $n^{th}$ heat polynomial has the following form:

$$p_n(x,t) = \sum_{k=0}^{[n/2]} \frac{x^{n-2k} t^k}{(n-2k)!k!}, \quad n = 0, 1, 2... \tag{5.4}$$

Using the formula (5.1) one can eliminate the first-order derivatives with respect to $t$ in the Taylor's series expansion. However, such approach also leads to formula (5.3).

The first polynomials of the sequence $p_n(x,t)_{n=0,1,2...}$

$$p_0(x,t) = 1$$

$$p_1(x,t) = x,$$

$$p_2(x,t) = \frac{x^2}{2!} + t,$$

$$p_3(x,t) = \frac{x^3}{3!} + xt,$$

$$p_4(x,t) = \frac{x^4}{4!} + \frac{x^2 t}{2!} + \frac{t^2}{2t}.$$

With heat polynomials it is conceivable to search for an approximate solution of a direct (initial-boundary) or inverse issue of heat conduction within the shape

$$U \approx \sum_{n=0}^{N} c_n p_n(x,t) \tag{5.5}$$

The right-hand side of (5.5) satisfies the formula (5.1). In order to get the approximate solution one has to find such values of $c_n$-sthat minimize an inaccuracy of the conditions fulfillment.

The general solution of (5.1) in the class of functions bounded for any real $x$ and $0 < t < c$, $c$ is a finite real number, may be expressed as a series

$$U = \sum_{n=0}^{\infty} c_n p_n(x,t) \tag{5.6}$$

convergent in domain $(x,t): 0 < t < c$, One can effectively demonstrate the taking after properties of the heat polynomials

$$\frac{\partial p_n}{\partial x} = np_{n-1}, \quad n = 1, 2...$$

$$\frac{\partial p_n}{\partial t} = n(n-1)p_{n-2}, \quad n = 2, 3... \tag{5.7}$$

$$p_n(x,t) = \frac{x}{n}p_{n-1}(x,t) + \frac{2t}{n}p_{n-2}(x,t) \quad n = 2, 3...$$

In addition, for any real $x$ and positive $t$

$$\lim_{n\to\infty} p_n(x,t) = 0 \tag{5.8}$$

In two dimensional and three dimensional Cartesian geometry the heat polynomials have been determined, using the strategy of the producing functions. Be that as it may, the comes about are the same as those gotten when changing the Taylor series extension. They have the taking after form: In two dimensional:

$$w_{nm}(x,y,t) = p_m(x,t)p_m(y,t), \quad m = 0, 1, 2... \quad \text{and} \quad n = 0, 1, 2... \tag{5.9}$$

In three dimensional:

$$u_{lmn}(x,y,z,t) = p_l(x,t)p_m(y,t)p_n(z,t); \tag{5.10}$$

$$l = 0, 1, 2..., \quad m = 0, 1, 2..., \quad \text{and} \quad n = 0, 1, 2...$$

where $p_l(x,t), p_m(y,t)$ and $p_n(x,t)$ stand for one dimensional heat polynomials In [1] the warm polynomials within the polar framework of arranges have been inferred utilizing the strategy of producing capacities. In this case two sets of the warm polynomials appear. The heat polynomials of the primary kind are characterized as

$$j_n(r,t) = \sum_{k=0}^{n} \frac{(\frac{r}{2})^{2n-2k} t^k}{[(n-k)!]^2 k!}, \quad n = 0, 1, 2... \tag{5.11}$$

The equations for the heat polynomials of the second kind contain $lnr$, hence they ought to or maybe be called solving functions and not polynomials. Besides, they have the taking after form:

$$u_n(r,t) = -j_n(r,t)lnr + q_n r, t \tag{5.12}$$

with

$$q_n(r,t) = \sum_{k=0}^{n} \frac{a_{n-k} \cdot (\frac{r}{2})^{2n-2k} t^k}{[(n-k)!]^2 k!}$$

and $a_n$ given in a recurrent form:

$$a_0 = 0,$$

$$a_1 = 1,$$

and

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \frac{1}{n}, \quad \text{for} \quad n > 1.$$

The set $j_n(r,t), u_n(r,t)_{n=0,1,2\dots}$ is complete within the space of bounded arrangements of the heat conduction equation in polar coordinates.

Within the cylindrical system or coordinates two independent sets of polynomials appear once once more, [1]. The primary kind heat polynomials are characterized as

$$b_{nm}(r,z;t) = j_n(r,t)p_m(z,t); \tag{5.13}$$

and the second kind polynomials:

$$k_{nm}(r,z,t) = u_n(r,t)p_m(z,t); \tag{14}$$

where $j_n$, $u_m$ and $p_m$ are given by (5.11), (5.12) and (5.4), separately. The set of polynomials of the form (5.13) and (5.14) is complete within the space of bounded solutions of the heat conduction equation in cylindrical coordinates.

# 6. Conclusion

The inverse Stefan problem for the reconstruction of unknown heat flux on the moving boundary is considered. The solution is found in the form of linear combination of the radial heat polynomials. The unknown coefficients are found from the conditions on the moving boundary.
The inverse two-phase spherical Stefan problem for unknown boundary heat flux is solved by the method of the heat polynomials. Side by side with exact solution two methods for the approximate solution, collocation and variational methods, convenient for engineering applications are presented and compared. It is shown that both methods give very good approximation even for use of several points only. However, the collocation method gives better result for the initial stage of heating, while the variational method is more preferable for the large values of the Fourier criterion. The estimation of the error of approximation is obtained using the principle of maximum for the heat equation. The application of the obtained results for the calculation of the electrical arc heat flux at the contact opening is presented.

# A. Appendix A

## A.1 Orthogonality of generalized Laguerre polynomials

The generalized Laguerre functions are calculated by the formula

$$L_d^\alpha(x) = \frac{1}{d!}e^{x} \cdot x^{-\alpha}\frac{d^d}{dx^d}(x^{d+\alpha}e^{-x}). \qquad (A.1)$$

**Theorem A.1.** *$e^{-\frac{x}{2}}x^{\frac{\alpha}{2}}L_n^\alpha(x)$ polynomials form an orthogonal system of functions on the half-line $[0,\infty]$. Then,*

$$\int_0^\infty x^\alpha e^{-x}L_n^\alpha(x)L_d^\alpha(x)dx = \begin{cases} 0, & at \quad n \neq d \\ \frac{\Gamma(\alpha+n+1)}{n!}, & at \quad n = d \end{cases} \qquad (A.2)$$

**Proof.** Lets proof at $n > d$

$$\int_0^\infty x^\alpha x^d e^{-x}L_n^\alpha(x)dx = 0 \qquad (A.3)$$

We have

$$\frac{1}{n!}\int_0^\infty e^{-x}x^{\alpha+d}e^{x}x^{-\alpha}[e^{-x}x^{(n+\alpha)}]^{(n)}dx = \frac{1}{n!}\int_0^\infty x^d[e^{-x}x^{(n+\alpha)}]^{(n)}dx \qquad (A.4)$$

by method of integration by part, we have following substitutions

$$x^d = u$$

$$[e^{-x}x^{(n+\alpha)}]^{(n)}dx = dv$$

$$d \cdot x^{d-1}dx = du,$$

$$[e^{-x}x^{(n+\alpha)}]^{(n-1)} = v.$$

So,

$$\frac{1}{n!}\int_0^\infty x^d[e^{-x}x^{(n+\alpha)}]^{(n)}dx = \frac{1}{n!}x^d[e^{-x}x^{(n+\alpha)}]^{(n-1)}\Big|_0^\infty -$$

$$-\frac{d}{n!}\int_0^\infty x^{d-1}\cdot[e^{-x}x^{(n+\alpha)}]^{(n-1)}dx = ...(-1)^d\frac{d!}{n!}\int_0^\infty [e^{-x}x^{(n+\alpha)}]^{(n-d)}dx =$$

$$= (-1)^d\frac{d!}{n!}[e^{-x}x^{(n+\alpha)}]^{(n-d-1)}\Big|_0^\infty = 0$$

We calculate the integral:

$$\int_0^\infty e^{-x}x^\alpha[L_n^\alpha(x)]^2dx = \frac{1}{n!}\int_0^\infty L_n^\alpha(x)\cdot(e^{-x}x^{(n+\alpha)})^{(n)}dx = \frac{(-1)^n}{n!}\int_0^\infty [L_n^\alpha(x)]^{(n)}(e^{-x}x^{(n}$$

$$|[L_n^\alpha(x)]^{(n)} = (-1)^n| = \frac{1}{n!}\int_0^\infty e^{-x}x^{(n+\alpha)}dx = \frac{\Gamma(n+\alpha+1)}{n!} \qquad (A.5)$$

Calculate another integral:

$$\int_0^\infty e^{-x}x^{\alpha+1}[L_n^\alpha(x)]^2dx = \int_0^\infty e^{-x}x^\alpha L_n^\alpha(x)[xL_n^\alpha(x)]dx \qquad (A.6)$$

For Laguerre polynomials $L_n^\alpha(x)$, the recurrence formula is known (H. Bateman, A. Erdelyi):

$$xL_n^\alpha(x) = (\alpha+2n+1)L_n^\alpha(x) - (n+1)L_{n+1}^\alpha(x) - (\alpha+n)L_{n-1}^\alpha(x). \qquad (A.7)$$

Substituting (A.7) into formula (A.6) and taking into account the orthogonality relations (A.5), we find:

$$\int_0^\infty e^{-x}x^{\alpha+1}[L_n^\alpha(x)]^2dx = (\alpha+2n+1)\frac{\Gamma(n+\alpha+1)}{n!} \qquad (A.8)$$

Show that

$$[L_n^\alpha(x)]^{(n)} = (-1)^n \qquad (A.9)$$

To do this, we write the expression $L_n^\alpha(x)$ in a form

$$L_n^\alpha(x)=\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\cdot\frac{x^k}{k!}=\sum_{k=0}^{n}(-1)^k\frac{(n+\alpha)(n+\alpha-1)...(k+\alpha+1)}{(n-k)!}\frac{x^k}{k!}=$$

$$=\sum_{k=0}^{n}(-1)^k\frac{(n+\alpha)(n+\alpha-1)...(k+\alpha+1)}{(n-k)!}\frac{x^k}{k!}=\sum_{k=0}^{n}(-1)^k\frac{(n+\alpha)!}{(n-k)!(k+\alpha)!}\frac{x^k}{k!}.$$

Hence it is clear that the coefficient at $\frac{x^n}{n!}$ is equal to

$$(-1)^n\frac{(n+\alpha)!}{(n-n)!(n+\alpha)!}=(-1)^n$$

Considering now that the derivative

$$\left(\frac{x^n}{n!}\right)^{(n)}=1,$$

we arrive at equality (A.9). [7]

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