МАТЕМАТИКА

Hilbert's Problem 1 and Related Topics in the Theory of Continuous Functions of Several Variables[15]

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Abstract. The paper presents a brief survey of the history, results and problems around the Hilbert's problem 1.

1. D.Hilbert [1], in his celebrated problem 13, deals with the question: how large is the class of functions of several variables which are represented as a finite superposition of functions of two variables?

   As it turned out the answer to the question essentially depends on the class of functions in which this problem is considered. The authors of a lot of paper, a detailed list can be found in surveys [8-10], [14], [17], are dealing mainly with three classes of functions:

(I) – class of analytic functions;
(II) – class of smooth functions;
(III)– class of continuous functions.

For class (II) A.G.Vitushkin ([2]) has shown that there exist smooth functions of several variables which cannot be realized by a finite superposition of smooth functions of two variables.

For class (I) Hilbert ([1]) himself has proved that there are analytic functions of three variables that cannot be obtained by a finite superposition of analytic functions of two variables.

2. Below, by default, we view the Hilbert's question in the class (III).

   Hilbert assumed that any branch of the function f=f(x, y, z) given by the equation

$$f^7+xf^3+yf^2+zf+1=0 \qquad (1)$$

is not represented as a finite superposition of any continuous functions of two variables.

However V.I.Arnol'd [3] has proved that in class (III) any continuous real-valued function of three variables each being from closed unit interval **I**=[0, 1] can be expressed as a superposition of continuous functions of two variables. Thus Hilbert's hypothesis for functions defined on I^3 is false. Earlier A.N.Kolmogorov, in [4], has shown that any continuous functions $\mathbf{I}^n \to R$ of n variables ($n \geqslant 1$) permits a representation as superposition of functions of three variables. These last two results mean: in the class (III) any function of several variables is a finite superposition of functions of two variables. Few later Kolmogorov, in [5], has proved that any continuous function $\mathbf{I}^n \to R$ of n variables ($n \geqslant 2$) permits a representation as a superposition of functions of one variable and addition (function of two variables).

Nevertheless until now Hilbert's question in the class (III) for functions $\mathbf{R}^n \to R$

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is open hitherto. Here it is necessary to point to the R.Doss' [6] result: there are $4n$ functions ψ^p, $p=1,\dots,4n$, ψ^p: $\mathbf{R}^n \rightarrow \mathbf{R}$ such that any function of n variables g: $\mathbf{R}^n \rightarrow$ R permits a representation as a superposition of functions ψ^p, some functions of one variable, depending on g, and addition. Some modification of this result was proposed by S.Demko [21].

In [7], P.Ostrand using Kolmogorov's technique from [4] had proved a theorem which is an analog of Kolmogorov's result with the only difference the variables belong to finite dimensional metric compact spaces. The Ostrand's result is the generalization of Kolmogorov's one. Ostrand's theorem says:

- Let X_1, ..., X_n be finite dimensional metric compact spaces with dimensions d_1, ..., d_n respectively, $m= d_i, \sum_{i=1}^{n}$ and $X=\prod X_i$. Then there exist n(2m+1) functions

$$g(x_1,\dots,x_n) = \sum_{q=1}^{2m+1} \xi^q \left(\sum_{p=1} \varphi^{pg} (x_p)\right) \qquad (2)$$

such that any function gX→R, can be written in the form

here ξ^q :R→R are continuous functions depending on g.
For X_p=I=[0, 1] formula (2) coincides with Kolmogorov's representation from [4].

In [11] A.A.Mal'cev proved an analog of the Ostrand's theorem for the case when the values of functions belong to the circle S^1, or torus $T^k=(S^1)k$; the usual group operation (multiplication) on T^k plays the role of the binary operation (see formula (2)). In particularly, if $X_i=T^k$ then Mal'cev's result means: any function $(T^k)^n \rightarrow T^k$ permits a representation as a superposition of functions $T^k \rightarrow T^k$ and multiplication on T^k.

Similar assertion is true for functions $X^n \rightarrow X$ where X is one of the following spaces:

- m-cube I^m; Cantor discontinuum D^τ; generalized cylinder $I^m \times T^k$; products of the compacts mentioned and a finite abelian group G; a compact abelian Lee group ([11])
- Hilbert space; Hilbert cube $I^{\aleph_0}$; rational numbers Q; irrationals R-Q ([12])
- Integers ([13]).

When X is a finite set (endowed by the discrete topology) then the assertion mentioned above is the well-known Slupecki's theorem from algebra of logic.

Let X be a space. A problem on representation of functions $X^n \rightarrow X$ in terms of superposition of functions of not more than two variables is said to be *the superposition problem for the space X*, briefly SPS(X).

Studying the SPS(X) for non-abelian group $X=S^3$ the author discovered the following phenomena:

- A function $(S^3)^3 \rightarrow S^3$ is a superposition of functions $S^3 \rightarrow S^3$ (of one variable) and two functions $(S^3)^2 \rightarrow S^3$ (of two variables). One of two last functions is the group operation on S^3 ([15]);
- There exist functions of four variables $(S^3)^4 \rightarrow S^3$ that cannot be represented in the form of superposition of functions of fewer variables ([16]). The last state-

ment means that the SPS(S^3) has negative solution.

– SPS(S^7) has also negative solution (Hajiyev's unpublished result, see also [19]).

The analysis of the arguments in [15] and [16] show that for spaces with non-trivial homotopic structure SPS(X) has a homotopic nature. Considering the topological groups (or H-spaces) it is necessary, as a first step, to study a structure of the group of homotopic classes [X^k; X] of the continuous mappings $X^k \to X$. If SPS(X) has the positive solution for the groups [X^k; X] for all $3 \leqslant k \leqslant n-1$, that is, if the generators of the groups [X^k; X] are the homotopic classes of functions of not more than two variables and, in addition, amoung the generators of [X^n; X] there exists a homotopic class of a map essentially depending on n variables then SPS(X) has the negative solution. On other hand, for locally-Euclidean groups the positive solution of the SPS(X) for the group [X^k; X], k=1,1,... lead to the solvability of the SPS(X) itself. Last fact follows from a simple lemma for sphere S3 ([15]) and which is actually true for any locally-Euclidean group.

Note that the study of the sets (groups) of the homotopic classes of the continuous functions of the topological spaces is one of the central problems in algebraic topology. So the study of the sets [X^k; X] is an interesting problem in itself. When studying the structure of the group [$(S^3)^k$; S^3] we actively use the such known classical tools of algebraic topology as Samelson and Whitehead product, the results of J.-P.Serre and P.Hilton on homotopic groups of S^3, and the knowledge of the extension theory for groups.

The interesting ideas related with SPS(X), have been offered in due time by I.M.James [19] and Y.Sternfeld [20].

3. The SPS(X) arises also when studying algebras of functions. Now we face, in brief, to iterative Post algebras of continuous functions ([17], [18]). The set P(X) of all continuous functions $X^n \to X$, n=1,2,...along with the finite number of certain unary and binary operations (signature operations) makes an algebraic system which is called a Post algebra on the space X, denoted $P(X)$. It is well-known (A.I.Mal'cev), that any formula arranged from elements of $P(X)$ can be viewed as a result of applying signature operations to these elements. The question arises. What is the structure of subset that generates whole $P(X)$ or its part? The above statements from the part 2, rephrased in terms of $P(X)$ ([11], [12], [13], [15]), clarify the question.

For locally-compact Hausdorf space X, we endow the algebra $P(X)$ by a natural topology under which all signature operations are continuous. Another interesting question in the theory of topologized Post algebras is the existence problem for finite-generated dense subalgebras of $P(X)$. This problem is closely connected with SPS(X). Some interesting results in the context can be found in [18].

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