Article

Well-Posedness for a Degenerate Hyperbolic Equation with Weighted Initial Data

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DOI: 10.47344/bgz8em14

Abstract

The focus of this study is an initial-boundary value problem associated with the degenerate hyperbolic equation $t\partial_{tt}u + \frac{1}{2}\partial_t u - \Delta u = g$ in a bounded domain. Due to the singularity at $t = 0$, standard initial conditions lead to an ill-posed problem. To achieve solvability of the problem, we introduce a ”modified” Cauchy problem using weighted initial conditions for this degeneracy. The main result of the study is the proof of the well-posedness of this problem within the framework of classical Sobolev spaces, as well as the obtaining of a priori estimates of the solution. Furthermore, the general boundary conditions for the one-dimensional equation were derived by using the restriction and extension theory.

Keywords: degenerate hyperbolic equation, weighted initial condition, well-posed problem, spectral decomposition, weighted Sobolev space

I. Introduction

Degenerate partial differential equations are a significant and challenging area of mathematical physics [1]. Among them, degenerate hyperbolic equations, characterized by change of type or loss of strict hyperbolicity in certain domains or at certain moments in time, are of particular interest [2], [3]. Such equations are often found in mathematical models of various physical processes, especially in fluid and gas dynamics, and they arise naturally in classical elasticity and differential geometry.

The theory of strictly hyperbolic equations provides a clear and well-developed framework for the well-posedness of the Cauchy problem. However, the study of degenerate hyperbolic equations is associated with significant difficulties, see [4]–[7]. This complexity arises when a hyperbolic equation degenerates when the coefficients associated with lower-order terms within the hyperbolic equation become singular (see [8], [9]).

In general, obtaining well-posed solutions to the Cauchy problem for degenerate cases requires either imposing conditions on the coefficients or considering a ”modified” initial condition. As noted in classical works [10], [11], the standard Cauchy problem for such equations may not be well-posed without appropriate modifications. Therefore, a natural approach is to study a weighted

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Received: December 19, 2025. Reviewed: December 25, 2025. Accepted: December 26, 2025.

Cauchy problem, in which the initial data are specified in a weighted form. This weighted formulation is essential to compensate for the singularity of the operator at $t = 0$ and to ensure that the solution remains bounded. For more details, see [12], [13] and the references therein.

In this work, we investigate a modified Cauchy problem for a degenerate hyperbolic equation considered in the cylindrical domain $D = (0, T) \times \Omega$, where $\Omega \subset R^n$ and $T > 0$. In addition, we provide a characterization of all possible regular boundary value problems associated with the corresponding singular ordinary differential equation by applying the restriction and extension theory, specifically relying on Otelbaev's abstract theorem.

II. Preliminaries

A. Inhomogeneous linear ODE with singular coefficient

Let us consider the following ODE

$$ly := ty''(t) + \frac{1}{2}y'(t) + \lambda y(t) = f(t), \quad t \in (0, T), \tag{1}$$

where $\lambda > 0$ is a fixed constant and f(t) is a given function.

First, we begin with the corresponding homogeneous equation

$$ty''(t) + \frac{1}{2}y'(t) + \lambda y(t) = 0. \tag{2}$$

It is well known that a fundamental system of solutions to this equation is given by the functions

$$y_1(t) = \cos 2\sqrt{\lambda t}, \quad y_2(t) = \sin 2\sqrt{\lambda t}.$$

To construct the general solution of (1), we apply the method of variation of parameters. We seek a solution in the form

$$y(t) = C_1(t)y_1(t) + C_2(t)y_2(t), \tag{3}$$

where $C_1(t)$ and $C_2(t)$ are functions to be determined. By imposing the standard condition $C_1'(t)y_1 + C_2'(t)y_2 = 0$, we arrive at the following linear system for the unknown derivatives $C_1'(t)$ and $C_2'(t)$:

$$\begin{cases} C_1'(t)y_1(t) + C_2'(t)y_2(t) = 0, \\ C_1'(t)y_1'(t) + C_2'(t)y_2'(t) = \dfrac{f(t)}{t}. \end{cases} \tag{4}$$

Substituting the explicit forms expressions for y_1 and y_2 and their derivatives we obtain

$$\begin{cases} C_1'(t)\cos 2\sqrt{\lambda t} + C_2'(t)\sin 2\sqrt{\lambda t} = 0, \\ -C_1'(t)\frac{\sqrt{\lambda}}{\sqrt{t}}\sin 2\sqrt{\lambda t} + C_2'(t)\frac{\sqrt{\lambda}}{\sqrt{t}}\cos 2\sqrt{\lambda t} = \dfrac{f(t)}{t}. \end{cases} \tag{5}$$

We determine the derivatives $C_1'(t)$ and $C_2'(t)$ from the linear system (5) by using Cramer's rule. We start by calculating the Wronskian determinant of the fundamental system

$$W = \det\begin{pmatrix} \cos 2\sqrt{\lambda t} & \sin 2\sqrt{\lambda t} \\ -\sqrt{\dfrac{\lambda}{t}}\sin 2\sqrt{\lambda t} & \sqrt{\dfrac{\lambda}{t}}\cos 2\sqrt{\lambda t} \end{pmatrix},$$

$$\begin{aligned} W &= \sqrt{\frac{\lambda}{t}}\cos 2\sqrt{\lambda t}\cdot\cos 2\sqrt{\lambda t} - \left(-\sqrt{\frac{\lambda}{t}}\sin 2\sqrt{\lambda t}\right)\cdot\sin 2\sqrt{\lambda t} \\ &= \sqrt{\frac{\lambda}{t}}\left[\cos^2 2\sqrt{\lambda t} + \sin^2 2\sqrt{\lambda t}\right] = \sqrt{\frac{\lambda}{t}}. \end{aligned} \tag{6}$$

Next, we compute the auxiliary determinants $W_1(t)$ and $W_2(t)$

$$W_1 = \det\begin{pmatrix} 0 & \sin 2\sqrt{\lambda t} \\ \dfrac{f(t)}{t} & \sqrt{\dfrac{\lambda}{t}}\cos 2\sqrt{\lambda t} \end{pmatrix} = -\frac{f(t)}{t}\sin 2\sqrt{\lambda t}, \tag{7}$$

Received: December 19, 2025. Reviewed: December 25, 2025. Accepted: December 26, 2025.

$$W_2 = \det \begin{pmatrix} \cos 2\sqrt{\lambda t} & 0 \\ -\sqrt{\dfrac{\lambda}{t}} \sin 2\sqrt{\lambda t} & \dfrac{f(t)}{t} \end{pmatrix} = \frac{f(t)}{t} \cos 2\sqrt{\lambda t}. \tag{8}$$

Consequently, the derivatives of the parameters are

$$C_1'(t) = \frac{W_1}{W} = -\frac{\sin 2\sqrt{\lambda t}}{\sqrt{\lambda t}} f(t), \tag{9}$$

$$C_2'(t) = \frac{W_2}{W} = \frac{\cos 2\sqrt{\lambda t}}{\sqrt{\lambda t}} f(t). \tag{10}$$

Integrating these expressions over the interval $(0, t)$, we find the functions $C_1(t)$ and $C_2(t)$

$$C_1(t) = \int_0^t -\frac{\sin 2\sqrt{\lambda \xi}}{\sqrt{\lambda \xi}} f(\xi) d\xi + c_1, \tag{11}$$

$$C_2(t) = \int_0^t \frac{\cos 2\sqrt{\lambda \xi}}{\sqrt{\lambda \xi}} f(\xi) d\xi + c_2, \tag{12}$$

where c_1 and c_2 are integration constants.

Inserting the obtained functions $C_1(t)$ and $C_2(t)$ into the general solution yields

$$\begin{aligned} y(t) = &-\cos 2\sqrt{\lambda t} \int_0^t \frac{\sin 2\sqrt{\lambda \xi}}{\sqrt{\lambda \xi}} f(\xi) d\xi + \sin 2\sqrt{\lambda t} \int_0^t \frac{\cos 2\sqrt{\lambda \xi}}{\sqrt{\lambda \xi}} f(\xi) d\xi \\ &+ c_1 \cos 2\sqrt{\lambda t} + c_2 \sin 2\sqrt{\lambda t}. \end{aligned} \tag{13}$$

To simplify the expression for the general solution (13), we group the terms under a single integral,

$$\begin{aligned} y(t) = &\int_0^t \frac{f(\xi)}{\sqrt{\lambda \xi}} \left[-\cos 2\sqrt{\lambda t} \sin 2\sqrt{\lambda \xi} + \sin 2\sqrt{\lambda t} \cos 2\sqrt{\lambda \xi} \right] d\xi \\ &+ c_1 \cos 2\sqrt{\lambda t} + c_2 \sin 2\sqrt{\lambda t}. \end{aligned} \tag{14}$$

Applying the sine difference formula, the general solution admits the compact representation:

$$y(t) = \int_0^t \frac{f(\xi)}{\sqrt{\lambda \xi}} \sin 2\sqrt{\lambda}(\sqrt{t} - \sqrt{\xi}) d\xi + c_1 \cos 2\sqrt{\lambda t} + c_2 \sin 2\sqrt{\lambda t}. \tag{15}$$

We next introduce the Cauchy problem associated with the operator l. To determine modified initial conditions, we investigate the behavior of the solution (15) as $t \to 0$. First, we examine $y(0)$

$$y(0) = \lim_{t \to 0} y(t) = c_1 \cdot 1 + c_2 \cdot 0 = c_1.$$

Hence, the requirement $y(0) = 0$ forces

$$c_1 = 0.$$

Let us next compute the derivative $y'(t)$. Differentiating expression (15) and simplifying yields

$$y'(t) = \int_0^t \frac{f(\xi)}{\sqrt{\xi t}} \cos\Big(2\sqrt{\lambda}(\sqrt{t} - \sqrt{\xi})\Big) d\xi \; + \; c_2 \, \frac{\sqrt{\lambda}}{\sqrt{t}} \cos 2\sqrt{\lambda t}. \tag{16}$$

Because the term $c_2 \frac{\sqrt{\lambda}}{\sqrt{t}} \cos 2\sqrt{\lambda t}$ is unbounded as $t \to 0$, the derivative $y'(t)$ itself cannot be prescribed directly at $t = 0$. Instead, we analyze the limit of the weighted expression $\sqrt{t} \frac{d}{dt} y(t)$:

$$\lim_{t \to 0} \sqrt{t} \, \frac{d}{dt} y(t) = 0 + c_2 \sqrt{\lambda}.$$

For this weighted limit to remain finite, we must take $c_2 = 0$. This choice provides the second initial condition

$$\lim_{t \to 0} \sqrt{t} \, \frac{d}{dt} y(t) = 0.$$

Received: December 19, 2025. Reviewed: December 25, 2025. Accepted: December 26, 2025.

As a result, the conditions defining a well-posed problem for the differential operator (1) are given below

$$y(0) = 0, \quad \lim_{t\to 0} \sqrt{t}\, \frac{d}{dt} y(t) = 0.$$

We can now state the full Cauchy problem as

$$\begin{cases} ty''(t) + \frac{1}{2} y'(t) + \lambda y(t) = f(t), \\ y(0) = 0, \ \lim_{t\to 0^+} \sqrt{t} \frac{d}{dt} y(t) = 0. \end{cases} \tag{17}$$

And finally, the solution to the Cauchy problem is as follows

$$y(t) = \int_0^t \frac{f(\xi)}{\sqrt{\lambda \xi}} \sin 2\sqrt{\lambda}(\sqrt{t} - \sqrt{\xi}) d\xi. \tag{18}$$

B. Eigenvalue problem for the Laplace operator

In this section, we provide the definition of the eigenvalue problem for the Laplace operator with homogeneous Dirichlet boundary conditions, followed by the associated lemma and theorem:

$$\begin{cases} -\Delta\varphi(x) = \lambda\varphi(x), & x \in \Omega, \\ \varphi(x) = 0, & x \in \partial\Omega. \end{cases} \tag{19}$$

The eigenfunctions $\{\varphi_k(x)\}_{k=1}^{\infty}$ of the self-adjoint problem (19) form a complete orthonormal basis for $L^2(\Omega)$, with eigenvalues satisfying $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$ (see, e.g., [14]).

Lemma II.1 (Orthogonality and simplicity [14])**.** *The eigenfunctions $\{\varphi_k(x)\}_{k=1}^{\infty}$ corresponding to the eigenvalues λ_k form an orthonormal system in $L^2(\Omega)$, i.e.,*

$$(\varphi_k, \varphi_m)_{L^2(\Omega)} = \int_\Omega \varphi_k(x)\varphi_m(x)\, dx = \delta_{km},$$

where δ_{km} is the Kronecker delta.

Theorem II.2 (Spectral decomposition and completeness [14])**.** *The system of eigenfunctions $\{\varphi_k(x)\}_{k=1}^{\infty}$ is complete in $L^2(\Omega)$.*

III. Main results

A. Formulation of the Modified Cauchy Problem in a Bounded Domain

Let $\Omega \subset R^n$ be a bounded domain with a sufficiently smooth boundary, specifically $\partial\Omega \in C^2$. We introduce the cylindrical domain defined by $D = (0, T) \times \Omega$. Given a source function $g \in L^2(D)$, we seek a function u that satisfies

$$Lu = tu_{tt} + \frac{1}{2} u_t - \Delta u = g(t, x), \quad (t, x) \in D. \tag{20}$$

The equation is supplemented by the following initial conditions

$$u(0, x) = 0, \qquad \lim_{t\to 0} \sqrt{t} u_t(t, x) = 0, \quad x \in \Omega, \tag{21}$$

and the homogeneous Dirichlet boundary condition

$$u(t, x)\big|_{x\in\partial\Omega} = 0, \quad t \in [0, T]. \tag{22}$$

To solve problem (20)–(22) we employ the method of spectral decomposition. Consider the complete orthonormal system $\{\varphi_k(x)\}_{k=1}^{\infty}$ in $L_2(\Omega)$, consisting of eigenfunctions of the spectral problem (19), and let $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding eigenvalues.

According to Theorem II.2, both the solution and the source function can be expanded into Fourier series

$$u(t, x) = \sum_{k=1}^{\infty} \varphi_k(x) y_k(t), \qquad g(t, x) = \sum_{k=1}^{\infty} \varphi_k(x) g_k(t), \tag{23}$$

Received: December 19, 2025. Reviewed: December 25, 2025. Accepted: December 26, 2025.

where the expansion coefficients are given by

$$y_k(t) = \int_\Omega \varphi_k(x)u(t,x)\,dx, \qquad g_k(t) = \int_\Omega \varphi_k(x)g(t,x)\,dx. \tag{24}$$

From the initial conditions (21), it follows that for the coefficients $y_k(t)$ satisfy

$$y_k(0) = 0, \qquad \lim_{t\to 0} \sqrt{t}y_k'(t) = 0.$$

Substitution of the series representation (23) into equation (20) reduces the partial differential equation to a one-dimensional singular Cauchy problem for the coefficient y_k

$$l_k y_k := t y_k''(t) + \frac{1}{2} y_k'(t) + \lambda_k y_k(t) = f_k(t), \tag{25}$$

$$y_k(0) = 0, \qquad \lim_{t\to 0} \sqrt{t}y_k'(t) = 0. \tag{26}$$

As established in the previous sections, the homogeneous equation $l_k y_k = 0$ possesses solutions spanned by trigonometric functions giving the general homogeneous solution as

$$y_{k,h}(t) = C_1 \cos(2\sqrt{\lambda_k t}) + C_2 \sin(2\sqrt{\lambda_k t}). \tag{27}$$

Employing the variation of parameters method and applying the initial conditions (26) provides the unique solution to the singular Cauchy problem (25)–(26):

$$y_k(t) = \int_0^t \frac{f_k(\xi)}{\sqrt{\lambda_k \xi}} \sin\left[2\sqrt{\lambda_k}(\sqrt{t} - \sqrt{\xi})\right] d\xi. \tag{28}$$

Finally, substituting expression (28) back into (23) leads to the following representation of the solution to (20)–(22):

$$u(t,x) = \sum_{k=1}^{\infty} \varphi_k(x) \left(\int_0^t \frac{f_k(\xi)}{\sqrt{\lambda_k \xi}} \sin\left[2\sqrt{\lambda_k}(\sqrt{t} - \sqrt{\xi})\right] d\xi \right). \tag{29}$$

B. Sobolev Regularity for the Singular ODE

We introduce the weighted Sobolev space $W^2_{2,t}(0,T) = \{y : y \in L^2(0,T) \text{ and } ty'' \in W_2^1(0,T)\}$ with the norm

$$\|y\|_{W^2_{2,t}(0,T)} := \left\| t\frac{d^2}{dt^2}y \right\|_{L^2(0,T)} + \left\| \frac{d}{dt}y \right\|_{L^2(0,T)} + \|y\|_{L^2(0,T)}. \tag{30}$$

Lemma III.1. *Let $f_k \in L^2(0,T)$. Then the solution $y_k \in W^2_{2,t}(0,T)$ of the one-dimensional singular equation* (25)–(26) *satisfies the estimate*

$$\|y_k\|_{W^2_{2,t}(0,T)} \le c\|f_k\|_{L^2(0,T)},$$

where c is a constant.

Proof. Let $v = \sqrt{t}$ and define $w(v) = y(v^2)$. Then $y(t) = w(\sqrt{t})$, and computing derivatives yields

$$w''(v) + 4\lambda w(v) = 4f(v^2).$$

The solution (28) has the form

$$w(y) = \frac{2}{\sqrt{\lambda}} \int_0^v \sin\left(2\sqrt{\lambda}(v-s)\right) g(s)ds,$$

with derivative

$$w'(v) = 4\int_0^v \cos\left(2\sqrt{\lambda}(v-s)\right) g(s)ds.$$

Define $g(v) = f(v^2)$. Since $f \in L^2(0,T)$, we have

$$\int_0^{\sqrt{T}} v|g(v)|^2 dy < \infty.$$

Received: December 19, 2025. Reviewed: December 25, 2025. Accepted: December 26, 2025.

We show $y_k \in W_2^1(0,T)$ by proving

$$\int_0^T |y_k(t)|^2 dt < \infty \quad \text{and} \quad \int_0^T |y_k'(t)|^2 dt < \infty.$$

Using the substitution $t = v^2$, these become

$$\int_0^{\sqrt{T}} v|w(v)|^2 dv < \infty \quad \text{and} \quad \int_0^{\sqrt{T}} \frac{|w'(v)|^2}{v} dv < \infty.$$

From the solution representation, we have the bounds

$$|w(v)| \le \frac{2}{\sqrt{\lambda}} \int_0^v |g(s)| ds, \quad |w'(v)| \le 4 \int_0^v |g(s)| ds.$$

Using the bound on w, Cauchy–Schwarz, and Fubini's theorem, we obtain

$$\int_0^{\sqrt{T}} v|w(v)|^2 dv \le \frac{16}{5\lambda_k} T^{5/4} \int_0^{\sqrt{T}} s|g(s)|^2 ds < \infty.$$

Similarly, with the bound for w', we obtain

$$\int_0^{\sqrt{T}} \frac{|w'(v)|^2}{v} dv \le 64 T^{1/4} \int_0^{\sqrt{T}} s|g(s)|^2 ds < \infty.$$

Using the transformation $t = v^2$, we obtain the following L^2 bounds

$$\|y_k\|_{L^2(0,T)} \le \frac{4}{\sqrt{5\lambda_k}} T^{5/8} \|f_k\|_{L^2(0,T)}, \tag{31}$$

and

$$\|y_k'\|_{L^2(0,T)} \le 4T^{1/8} \|f_k\|_{L^2(0,T)}. \tag{32}$$

Which concludes

$$\int_0^T |y_k(t)|^2 dt < \infty \quad \text{and} \quad \int_0^T |y_k'(t)|^2 dt < \infty.$$

Equation (25) can be rearranged to express the second derivative term:

$$t y_k''(t) = f_k(t) - \frac{1}{2} y_k'(t) - \lambda_k y_k(t),$$

From this, it can be concluded that

$$\begin{aligned} \|t y_k''\|_{L^2(0,T)} \le & \|f_k\|_{L^2(0,T)} + \left\|\frac{1}{2} y_k'\right\|_{L^2(0,T)} + \|\lambda_k y_k\|_{L^2(0,T)} \\ \le & c\|f_k\|_{L^2(0,T)}. \end{aligned} \tag{33}$$

C. Solution estimates for general case

We define the $(-\Delta_x)^{\frac{1}{2}}$ acting on a function $g \in L^2(D)$ by the following rule

$$(-\Delta_x)^{\frac{1}{2}} g = \sum_{k=1}^{\infty} g_k(t) \sqrt{\lambda_k} \varphi_k(x), \tag{34}$$

with the norm $\|(-\Delta_x)^{\frac{1}{2}} g\|^2_{L^2(D)} = \sum_{k=1}^{\infty} \lambda_k |g_k(t)|^2$.

Let $W^{2,2}_{2,t}(D)$ be the weighted Sobolev space with the norm

$$\|u\|_{W^{2,2}_{2,t}} := \left\| t \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(D)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(D)} + \|\Delta u\|_{L^2(D)} + \|u\|_{L^2(D)}. \tag{35}$$

Received: December 19, 2025. Reviewed: December 25, 2025. Accepted: December 26, 2025.

Theorem III.2. *Assume that $g\in L^2(D)$ and that the condition $\sum\limits_{|m|=1}^{\infty} \lambda_m \left|g_m\left(\xi\right)\right|^2 < \infty$ holds. Then there exists a unique solution $u \in W_{2,t}^{2,2}(D)$ of the problem* (20)- (22) *that satisfies the following inequality*

$$\left\|t\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(D)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(D)} + \|\Delta_x u\|_{L^2(D)} + \|u\|_{L^2(D)} \le c\|g\|_{L^2(D)} + c_0 \left\|(-\Delta_x)^{\frac{1}{2}} g\right\|_{L^2(D)},$$

with constants c and c_0 depending only on T.

Proof. By Parseval's identity, we have

$$\|u\|_{L^2(D)}^2 = \sum_{k=1}^{\infty} |y_k(t)|^2\, dt \le c_1 \int_0^T \sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k}|g_k(t)|^2\right) dt.$$

By (31) and $\sum\limits_{k=1}^{\infty} \frac{1}{\lambda_k}\|g_k(t)\|^2 \le \frac{1}{\lambda_1}\sum\limits_{k=1}^{\infty} \|g_k(t)\|^2$, we obtain

$$\|u\|_{L^2(D)}^2 \le c_1 \sum_{k=1}^{\infty} \|g_k\|_{L^2(0,T)}^2 = c_1\|g\|_{L^2(D)}^2.$$

Similarly, by (32) we get

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^2(D)}^2 \le c_2\|g\|_{L^2(D)}^2.$$

Similar considerations apply to $\Delta_x u$, from (31), we have

$$\|\Delta_x u\|_{L^2(D)} = \sum_{k=1}^{\infty} |\lambda_k y_k(t)|^2 \le c\|(-\Delta_x)^{\frac{1}{2}} g\|_{L^2(D)}^2.$$

Finally, rewriting equation (20) as

$$t\frac{\partial^2 u}{\partial t^2} = g(t,x) - \frac{1}{2}\frac{\partial u}{\partial t} + \Delta u.$$

Now it is easily seen that

$$\left\|t\frac{\partial^2}{\partial t^2}u\right\|_{L^2(D)} \le c\|g\|_{L^2(D)} + c_0\left\|(-\Delta_x)^{\frac{1}{2}} g\right\|_{L^2(D)}. \tag{36}$$

This proves the theorem.

IV. Regular boundary value problem for the second-order equation

The aim of this section is to obtain general boundary conditions for equation (1) in the one-dimensional case. Our approach relies on extension and restriction theory for differential operators, and in particular on the abstract theorem of Otelbaev [15].

Defining the correct boundary conditions requires deriving the conjugate problem for the operator l (1). The calculation of the scalar product yields

$$\langle ty'' + \frac{1}{2}y' + \lambda y, w\rangle = \langle y, tw'' + \frac{3}{2}w' + \lambda w\rangle.$$

Thus, the conjugate operator is

$$l^* w = t\,w'' + \frac{3}{2}\,w' + \lambda w.$$

Consequently, the Cauchy problem admits the following conjugate formulation

$$\begin{cases} tw''(t) + \frac{3}{2}w'(t) + \lambda w(t) = \phi(t), \\ w(1) = 0,\ w'(1) = 0. \end{cases} \tag{37}$$

The homogeneous problem

$$tw''(t) + \frac{3}{2}w'(t) + \lambda w(t) = 0,$$

Received: December 19, 2025. Reviewed: December 25, 2025. Accepted: December 26, 2025.

has a general solution of the following form

$$w(t) = -q_1\sqrt{\frac{\lambda}{t}}\sin 2\sqrt{\lambda t} + q_2\sqrt{\frac{\lambda}{t}}\cos 2\sqrt{\lambda t}. \tag{38}$$

where q_1, q_2 are arbitrary constants.

Now let us return to our general solution (15), where constants c_1 and c_2 that depend continuously and linearly on f; that is,

$$c_1 = c_1(f), c_2 = c_2(f).$$

By the Riesz representation theorem, these functionals can be expressed as

$$c_1 = \int_0^1 \sigma_1(t)f(t)dt,\ c_2 = \int_0^1 \sigma_2(t)f(t)dt,$$

where σ_1 and σ_2 belong to the kernel of the operator (37) (see [15]). We choose them in the form

$$\sigma_1(t) = -q_1\sqrt{\frac{\lambda}{t}}\ \sin 2\sqrt{\lambda t}, \quad \sigma_2(t) = q_2\sqrt{\frac{\lambda}{t}}\ \cos 2\sqrt{\lambda t}.$$

Consequently, we obtain

$$c_1 = -q_1\int_0^1\sqrt{\frac{\lambda}{t}}\sin 2\sqrt{\lambda t}f(t)dt,\ c_2 = q_2\int_0^1\sqrt{\frac{\lambda}{t}}\cos 2\sqrt{\lambda t}f(t)dt.$$

Substituting these integral expressions into (15), we derive the following expression for $y(t)$:

$$\begin{aligned} y(t) = &\int_0^t \frac{f(\xi)}{\sqrt{\lambda\xi}}\sin 2\sqrt{\lambda}(\sqrt{t}-\sqrt{\xi})d\xi - \\ &- q_1\cos 2\sqrt{\lambda t}\int_0^1\sqrt{\frac{\lambda}{t}}\sin 2\sqrt{\lambda t}f(t)dt + q_2\sin 2\sqrt{\lambda t}\int_0^1\sqrt{\frac{\lambda}{t}}\cos 2\sqrt{\lambda t}f(t)dt. \end{aligned} \tag{39}$$

Evaluating the integrals in (39) using integration by parts and rearranging the terms, we arrive at the corresponding boundary value problem for equation (1):

$$\begin{cases} -y(0) + q_1\left(-y'(1)\sqrt{\lambda}\sin(2\sqrt{\lambda}) + y(1)\cos(2\sqrt{\lambda}) - \lambda y(0)\right) = 0, \\ \lim\limits_{t\to 0^+}\dfrac{y'(t)\sqrt{t}}{\sqrt{\lambda}} + q_2\left(y'(1)\sqrt{\lambda}\cos(2\sqrt{\lambda}) + \lambda y(1)\sin(2\sqrt{\lambda})\right) = 0. \end{cases} \tag{40}$$

It is not difficult to observe that in the special case when the free constants q_1, q_2 are zero, we obtain the Cauchy problem. Writing this boundary condition in matrix form

$$\begin{pmatrix} -1 - aq_1 & aq_1\cos 2\sqrt{a} & 0 & -\sqrt{a}q_1\sin 2\sqrt{a} \\ 0 & a\sqrt{a}q_2\sin 2\sqrt{a} & -1 & aq_2\cos 2\sqrt{a} \end{pmatrix}\begin{pmatrix} y(0) \\ y(1) \\ \lim_{t\to 0^+}\sqrt{t}y'(t) \\ y'(1) \end{pmatrix} = 0. \tag{41}$$

The obtained results allow us to state the theorem

Theorem IV.1. *The differential equation* (1) *has a unique solution satisfying the boundary condition* (41) *for all* $f \in L^2(0,1)$ *and every* $q_1, q_2 \in R$.

V. Conclusion

In this work, we investigated the initial-boundary value problem for a degenerate hyperbolic equation with a singularity at $t = 0$ by introducing a modified Cauchy problem with weighted initial conditions. We proved the well-posedness of this problem in the weighted Sobolev space $W^{2,2}_{2,t}(D)$, derived the necessary a priori estimates for the solution, and using the theory of operator extension, characterized the general regular boundary conditions for the corresponding one-dimensional singular ordinary differential equation.

Received: December 19, 2025. Reviewed: December 25, 2025. Accepted: December 26, 2025.

Acknowledgements

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP22785364).

References

[1] Bitsadze, A. V., *Some Classes of Partial Differential Equations*, Nauka, Moscow, 1981 (in Russian).

[2] Krasnov, M. L., Mixed boundary value problems for degenerate linear hyperbolic differential equations of second order, *Matematicheskii Sbornik*, 91:1 (1959), 29–77 (in Russian).

[3] Bitsadze, A. V., Nakhushev, A. M., On the theory of degenerate hyperbolic equations in multidimensional domains, *Doklady Akademii Nauk SSSR*, 204 (1972), 1289–1291

[4] O. A. Olêinik, *On the Cauchy problem for weakly hyperbolic equations*, Comm. Pure Appl. Math., 23 (1970), 569-586.

[5] V.N. Vragov, *Boundary value problems for nonclassical equations of mathematical physics*, Novosibirsk, NGU, 1983. (in Russian)

[6] S. A. Tersenov, *Introduction to the Theory of Equations that are Degenerate on the Boundary*, Novosibirsk Univ. Press, Novosibirsk, 1973 (in Russian).

[7] R. W. Carroll and R. E. Showalter, *Singular and Degenerate Cauchy Problems*, Academic Press, New York, 1976.

[8] Smirnov, M. M., *Degenerate hyperbolic equations*, Minsk, 1977 (in Russian).

[9] Colombini, F., Spagnolo, S., An example of a weakly hyperbolic Cauchy problem not well posed in C^{∞}, *Acta Math.*, 148 (1982), 243–253.

[10] Oleinik, O. A., On the Cauchy problem for weakly hyperbolic equations, *Communications on Pure and Applied Mathematics*, 23 (1970), 569–586.

[11] F. Colombini, E. Jannelli, and S. Spagnolo, *Well-posedness in the Gevrey classes of the Cauchy problem for a non-strictly hyperbolic equation with coefficients depending on time*, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 10 (2) (1983), 291-312.

[12] Kakharman, N., Kal'menov, T., Mixed Cauchy problem with lateral boundary condition for noncharacteristic degenerate hyperbolic equations, *Boundary Value Problems*, 2022:1 (2022), 35.

[13] Kakharman, N., Kalmenov, T. Sh., A Modified Cauchy Problem With Nonlocal Boundary Condition for a Degenerate Hyperbolic Equation, *Mathematical Methods in the Applied Sciences*, 48:8 (2025), 9313–9319.

[14] Henrot, A., *Extremum Problems for Eigenvalues of Elliptic Operators*, Springer, 2006.

[15] B. K. Kokebaev, M. Otelbaev, A. N. Shynybekov, *On the theory of contraction and extension of operators I*, Izv. Akad. Nauk Kazakh SSR, Ser. Fiz.-Mat., 5 (1982), 24–26.

Received: December 19, 2025. Reviewed: December 25, 2025. Accepted: December 26, 2025.