МАТЕМАТИКА

Some properties of binomial coefficients.

Preface.

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The aim of this project is the researching of some special properties of binomial coefficients. The main results are:

Theorem 2. 1. The following conditions are equivalent:

(a). The number C_n^k is odd.

(b). Natural numbers n, k satisfy the following condition: if we write numbers n and k in binary system and some binary digit of k is equal to one then the same binary digit of n is equal to one.

Theorem 2. 2. For any natural number n and prime number p the following conditions are equivalent:

(a). p is not a divisor of C_n^k for all $k = 0,1,...,n$.

(b). $n = p^s m - 1$, where s is nonnegative integer, m is natural number, $m < p$.

Theorem 2.3. Let natural numbers n, r satisfy the inequality $r+3 \le n$. Then the binomial coefficients $C_n^r, C_n^{r+1}, C_n^{r+2}, C_n^{r+3}$ can not be the sequential terms of any arithmetic progression.

Theorem 2. 4. Let $n > 1$ be a natural number. Then the following conditions are equivalent:

(a). n is a prime number.

(b). For all integers k satisfying the inequalities $1 \le k \le n-1$, n is a divisor of C_n^k.

This project contains two parts. At the first part we consider some definitions and statements which we need for proving of theorems 2.1.-2.4.. At the second part we prove the main results of our project.

I. Definitions and prior statements.

In our project we use the following notations:

N - the set of all natural numbers.

Z - the set of all integers

Z^+ - the set of all nonnegative integers.

R - the set of all real numbers.

$a \in A$ - an element a belongs to a set A.

1.1. Newton's binomial formula and binomial coefficients.

Theorem 1.1. For any $a,b \in R$, $n \in N$ the equality

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k = C_n^0 a^n + C_n^1 a^{n-1} b + ... + C_n^k a^{n-k} b^k + ... + C_n^{n-1} a b^{n-1} + C_n^n b^n \quad (1)$$

is true, where

$$C_n^k = \frac{n!}{k!(n-k)!},$$

$$n! = \begin{cases} 1 \cdot 2 \cdot 3 ... n, & n \in N \\ 1, & n = 0 \end{cases}$$

Definition 1.1. a). The equality (1) is called the *Newton's binomial formula.*

b). C_n^k are called the *binomial coefficients.*

Properties of binomial coefficients.

1. $C_n^0 = C_n^n = 1$.
2. $C_n^k = C_n^{n-k}, \quad 0 \le k \le n$.
3. $C_n^k = C_{n-1}^k + C_{n-1}^{k-1}, \quad 0 < k < n$.
4. $C_n^0 + C_n^1 + ... + C_n^k + ... + C_n^n = 2^n$.
5. $C_n^0 - C_n^1 + ... + (-1)^k C_n^k + ... + (-1)^n = 0$.

1.2. Some definitions and theorems of theory of numbers.

Definition 1.2. Let $n, d \in N$. Then d is called a *divisor of* n if there exists $m \in N$ such that $n = md$.

Definition 1.3. Let $n \in N$, $n > 1$, then a). n is called a *prime number* if the only divisors of n are 1 and n.

b). If n is not prime, it is called *composite number.*

Remark. The number 1 is neither prime nor composite.

Theorem 1.2. (Unique factorization theorem). For any composite natural number n there exist prime numbers $p_1, p_2, \ldots, p_k$ and $s_1, s_2, \ldots, s_k \in \boldsymbol{Z}^+$ such that

$$n = p_1^{s_1} p_2^{s_2} \ldots p_k^{s_k} \qquad (2)$$

and factorization (2) of n is uniquely modulo permutation of multipliers.

Definition 1.4. a). The factorization (2) is called the *prime number factorization of n.*

b). s_i is called a *degree of a prime multiplier* p_i in a prime number factorization of n, $i = 1,2,\ldots,k$. $n \in N$

Theorem 1.3. (Legendre). Let $n \in N$. Then a degree s of a prime multiplier p in a prime number factorization of $n!$ is equal to

$$s = [\frac{n}{p}] + [\frac{n}{p^2}] + \ldots + [\frac{n}{p^k}] + \ldots .$$

For great values of k we have $[\frac{n}{p^k}] = 0$. (For example if $k > \log_p n$, then $[\frac{n}{p^k}] = 0$).

Definition 1.5. Let $k \in N$, $k > 1$. Then a natural number n has *k-digital system representation* $\overline{a_1 a_2 \ldots a_m}$, if

$$n = a_1 k^{m-1} + \ldots + a_{m-1} k^1 + a_m k^0,$$

Where $a_1, \ldots, a_m \in \boldsymbol{Z}^+$, $a_i < k$, $i = 1,2,\ldots,m$, $a_1 > 0$.

The next theorem is a generalization of well-known theorem for decimal system.

Theorem 1.4. Let $k \in N$, $k > 1$. Then for any $n \in N$ there exists k-digital system representation of number n and this representation is unique.

1.3. Functions [x] and {x}.

Definition 1.6. Let $x \in \boldsymbol{R}$.

a). The greatest $z \in \boldsymbol{Z}$ satisfying the inequality $z \le x$ is called the *integer part of* x and denoted as $[\, x \,]$, i.e. $z = [\, x \,]$.

b). The number $\{x\} = x - [x]$ is called the *fractional part of* x.

Remark. It is clear that $[x]$ and $\{x\}$ are functions of real variable x.

<u>Properties of the function $[x]$.</u>

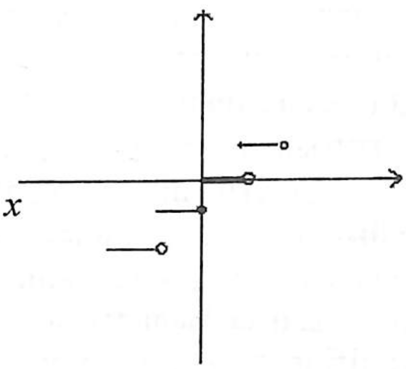
1. $[x]$ is an increasing function.
2. $|x| > |[x]|,\ x > 0$ and $|x| < |[x]|,\ x < 0$.
3. $[x+y] \geq [x] + [y]$.

The function $[x]$ has the following graph

y

1

-2 -1 1 2 x



<u>Properties of the function $\{x\}$.</u>

1. $\{x\}$ is a periodic function with period one.
2. $\{x\} > 0$ for all $x \in \boldsymbol{R}$.

The function $\{x\}$ has the graph

y

1

-2 -1 0 1 x