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MATHEMATICAL MODELING OF HEAT TRANSFERRING IN ELECTRICAL CONTACTS

Abstract. Research work is about solving Heat equation, exactly, One phaseStefan problem, which, solved with modern method, called Integral Error Function by Hartree. Stefan problem consider heat transferring in electrical contact from one phase to the second. Simple example to the Stefan problem are the melting of ice and the freezing of water. To solve the Stefan problem, which is known problem in numerous industrial and technological applications, such as the manufacture of steel, ablation of heat shields, contact melting in thermal storage systems, ice accretion on aircraft, evaporation of water, used the program «Matlab» by approximation method. Physical meaning of this research is finding melting point by giving heat flow.

Keywords: Integral error function, Stefan problem, Heat equations,temperature,approximation.

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Аңдатпа. Зерттеу жұмысы жылу теңдеуін шешуге арналған, нақтырақ айтқанда, бірфазалы Стефан мәселесі Хартри ойлап тапқан интегралдық қателік функциясы деп аталатын заманауи әдіспен шешу. Стефанның проблемасының мәні электрлік контактідегі жылуды бір фазадан екіншісіне ауыстыру. Стефанның мәселесінің қарапайым мысалы – мұздың еруі және судың қатуы. Көптеген өнеркәсіптік және технологиялық қосымшаларда, мысалы, болат дайындау, жылу қалқандарының абляциясы, жылу сақтау жүйелеріндегі байланыс балқыту, жазықтықта мұз қалыптастыру, судың буланып кетуі сияқты Стефанның мәселесін шешу үшін жуықтау әдісімен шешілетін «Matlab» бағдарламасы қолданылады. Зерттеудің физикалық мағынасы жылу ағыны арқылы балқу нүктесін табу болып табылады.

Кілт сөздер: интегралдық қателік функциясы, Стефан мәселесі, жылу теңдеуі, температура, жуықтау.

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Аннотация. Исследовательская работа посвящена решению уравнения теплопроводности, в точности, одной фазызадача Стефана, которая, решается современным методом, называется интегральной функцией ошибок которогопридумал Хартри. Задача Стефана предусматривает передачу тепла в электрическом контакте от одной фазы ко второй. Простым примером проблемы Стефана являются таяние льда и

замерзание воды. Для решения проблемы Стефана, которая известна в многочисленных промышленных и технологических приложениях, таких как изготовление стали, абляция теплозащитных экранов, контактное плавление в тепловых накопительных системах, нарастание льда на самолете, испарение воды, используется программа «Matlab» методом приближения. Физический смысл этого исследования заключается в нахождении точки плавления путем подачи теплового потока.

Ключевые слова: функция интегральной ошибки, задача Стефана, уравнения теплопроводности, температура, приближение.

General overview

The Stefan type problems

In this chapter we consider the formulation of the Stefan problem as a classical initial boundary value problem for a parabolic partial differential equation. A portionof the boundary of the domain is a priori unknown (the free boundary), and therefore two boundary conditions must be prescribed on it, instead than only one,to obtain a well posed problem. Also this chapter is devoted to introduce specialmethods by the help of which Heat Equations in the domains with fixed and movingboundaries are solved. Heat equations are solved by the help of Integral ErrorFunctions (IEF method) and its properties, which were introduced by Hartree in 1935 and reasonably sometimes called Hartree functions. As it will be shown infurther paragraphs, method can be used to solve first, second and third boundaryvalue problems for Heat Equations with fixed and moving finite, semi-infiniteandinfinite boundaries.

Generalization of integral error function

TheI ntegral Error Functions

The integral error functions determined by recurrent formulas below

$$i^n erfc(x) = \int_x^\infty i^{n-1} erfc(\nu) d\nu, \ n = 1, 2...$$

$$i^0 erfc(x) = erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-\nu^2) d\nu \qquad (2.1.1)$$

were introduced by Hartree in 1935.

We can obtain from (2.1.1)

$$i^n erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty (\nu - x)^n \exp(-\nu^2) d\nu \qquad (2.1.2)$$

These functions satisfy the differential equation

$$\frac{d^2}{dx^2} i^n erfc(x) + 2x\frac{d}{dx} i^n erfc(x) - 2n i^n erfc(x) = 0 \qquad (2.1.3)$$

andrecurrentformulas

$$2n i^n erfc(x) = i^{n-2} erfc(x) - 2x i^{n-1} erfc(x) \qquad (2.1.4)$$

Integral error functions (some times they are called also Hartree functions) are very usefulfor investigation of heat transfer, diffusion and other phenomena which can be described by the equationgivenbelow

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (2.1.5)$$

in a region $D(t > 0, 0 < x < \alpha(t))$ with free boundary $x = \alpha(t)$, however functions

$$u_n(\pm x, t) = t^{\frac{n}{2}} i^n erfc \frac{\pm x}{2a\sqrt{t}}$$

satisfy the equation (2.1.5) as well as their linear combination or even series

$$u(x,t) = \sum_{n=0}^{\infty} \left[A_n u_n(x,t) + B_n u_n(-x,t) \right]$$

for any constants A_n, B_n. We can choose to satisfy the boundary conditions at $x = 0$ and $x = \alpha(t)$, if given boundary functions can be expanded into Taylor series with powers t or $\sqrt{t}$.

Properties of Integral Error Functions

Let us derive new properties of IEF, which are not considered in abovementionedpapers.

1) If is an integer, then

$$i^n erfc(-x) + (-1)^n i^n erfc x = \frac{1}{2^{n-1} n! i^n} H_n(ix) = \frac{1}{2^{n-1} n!} e^{-x^2} \frac{d^n}{dx^n} e^{x^2}$$

with $i = \sqrt{-1}$ and Hermite polynomials $H_n(x)$ in the right side . Indeed , using formula (2.1.2) we can write

$$i^n erfc(-x) + (-1)^n i^n erfc x = \frac{2}{\sqrt{(\pi)}} \frac{1}{n!} \int_{-x}^{\infty} (v+x)^n exp(-v^2) dv +$$

$$+ \frac{(-1)^n 2}{n! \sqrt{\pi}} \int_{x}^{\infty} (v-x)^n exp(-v^2) dv = \frac{2}{n! \sqrt{\pi}} \int_{-\infty}^{\infty} (v+x)^n exp(-v^2) dv = \frac{1}{2^{n-1} n! i^n} H_n(ix)$$

Using formula for Hermitepolynomials we can derive

$$i^n erfc(-x) + (-1)^n i^n erfc x = \sum_{m=0}^{[\frac{n}{2}]} \frac{x^{n-2m}}{2^{2m-1} m! (n-2m)!} \qquad (2.2.1)$$

If n is even, n = 2k, then

$$i^{2k}erfcx + i^{2k}erfc(-x) = \sum_{m=0}^{k} \frac{x^{2(k-m)}}{2^{2m-1}m!(2k-2m)!}$$

In particular

$$erfc(x) + erfc(-x) = 2$$
$$i^2erfcx + i^2erfc(-x) = 1/2 + x^2$$
$$i^4erfcxi^4erfc(-x) = 1/8 + 1/4x^2 + 1/12x^4$$

If n = 2k + 1, then

$$i^{2k+1}erfcx + i^{2k+1}erfc(-x) = \sum_{m=0}^{k} \frac{x^{2(k-m)+1}}{2^{2m-1}m!(2k-2m+1)!}$$

In particular

$$ierfc(-x) - ierfcx = 2x$$
$$i^3erfc(-x) - i^3erfcx = 1/2x + 1/3x^3$$
$$i^5erfc(-x) - i^5erfcx = 1/16x + 1/24x^3 + 1/60x^5$$

2) The proof of the formula

$$i^nerfc(-x) - i^nerfcx = \frac{1}{2^{n-1}n!}e^{-x^2}\frac{d^n}{dx^n}(e^{x^2}erfcx) \qquad (2.2.2)$$

where

$$erfx = 1 - erfcx = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-\nu^2)d\nu$$

can be obtained by mathematical induction method using recurrent formula (2.1.4).

3) Differentiating the right side of formula (2.2.2), we obtain

$$i^nerfc(-x) - (-1)^n i^nerfcx = P_n(x)erfx - Q_n(x)\frac{2}{\sqrt{\pi}}exp(-x^2), \qquad (2.2.3)$$

where polynomials Pn(x) and Qn(x) are defined by formulas

$$P_n(x) = \sum_{m=0}^{[\frac{n}{2}]} \frac{x^{n-2m}}{2^{2m-1}m!(n-2m)!}, \; Q_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^{n-k}H_{n-k-1}(x)}{2^{n-k}(n-k)!}P_k(x)$$

4) From (2.2.3), (2.2.4) we can obtain the explicit expressions for Hartree functions of an integer

index

$$i^n erfcx = \frac{(-1)^n}{2}\left[P_n(x)erfcx + Q_n(x)\frac{2}{\sqrt{\pi}}\exp(-x^2)\right] \qquad (2.2.4)$$

$$i^n erfc(-x) = \frac{1}{2}\left[P_n(x)erfc(-x) - Q_n(x)\frac{2}{\sqrt{\pi}}\exp(-x^2)\right] \qquad (2.2.5)$$

5) Using L'Hospital rule and representation (2.1.1),it is not difficult to show that

$$\lim_{x\to\infty}\frac{i^n erfc(-x)}{x^n} = \frac{2}{n!} \qquad (2.2.6)$$

6) Using property 2 one can derive following formula

$$u(x,t) = \sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n} x^{2n-2m}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t^m\beta_{2n+1,m}\right] \qquad (2.2.7)$$

Where u(x; t) is Heat polynomial which exactly satisfy Heat Equation and

$$\beta_{n,m} = \frac{1}{2^{n+m-1}m!(n-2m)!}$$

Analytical solution of the one phase stefan problem (direct)

Introduction to the problem

The first analytical solution of one phase Stefan problem, which describes the dynamics of soil freezing has been published by Lame and Clayperon. Solution of one phase Stefan problem was represented by Stefan. Solutions of these problems were obtained for $\alpha(t) = \alpha\sqrt(t)$case and some auto model cases.

As for applications: a wide range of electric contact phenomena, in particular, the phenomena occurring at the interaction of electrical arc with electrode can be described in dynamic use of the presented method see e.g., for very short arc duration (nanosecond diapason), when experimental investigation is very difficult. In this study we will try to find solution of one Phase Stefan problem for degenerate domain with $\alpha(t) = \sum_{n=0}^{\infty}\alpha_n t^{\frac{n}{2}}$moving boundary.

Tracking answers of these questions will be organized as following. In the continuation of this section Integral Error Functions and its properties necessary for elaboration of new methods are presented. In subsection 1.4 a test problem is solved by proposed method. In sections two and three one phase Stefan problem, its analytical solution and convergence of series represented. For finding analytical solution we mainly follow the method proposed by

S.N.Kharin in applying Faa Di Brunos formula for Integral Error Functions. For Heat polynomials we utilize Newtons polynomial (generalization of Newtons binomial) and its multinomial coefficients.

Problem formulation

Heat transfer between two bars that have ideal contact is described by the equations:

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial t^2},\ 0 < x < \infty,\ (3.2.1)$$

withtheinitialconditions:

$$U(0,0) = 0 \quad (3.2.2) \quad U(x,0) = f(x) \quad (3.2.3)$$

conditions of conjugations of temperature and heat flux

$$-\lambda U(0,t) = P(t) \qquad (3.2.4)$$
$$U(\alpha\sqrt{t}, t) = U_m \qquad (3.2.5)$$
$$-\lambda \frac{\partial U}{\partial x}\Big|_{x=\alpha\sqrt{t}} = L\gamma \frac{d\alpha}{dt} \qquad (3.2.6)$$

Suggesting that initial function and function of heat flux can be expanded in Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad P(t) = \sum_{n=0}^{\infty} \frac{p^{(n)}(0)}{n!} (\sqrt{t})^n \qquad (3.2.7)$$

we represent the solution in the form

$$U(x,t) = \sum_{n=0}^{\infty} (2a\sqrt{t})^n \left[A_n i^n erfc \frac{x}{2a\sqrt{t}} + B_n i^n erfc \frac{-x}{2a\sqrt{t}} \right] \qquad (3.2.8)$$

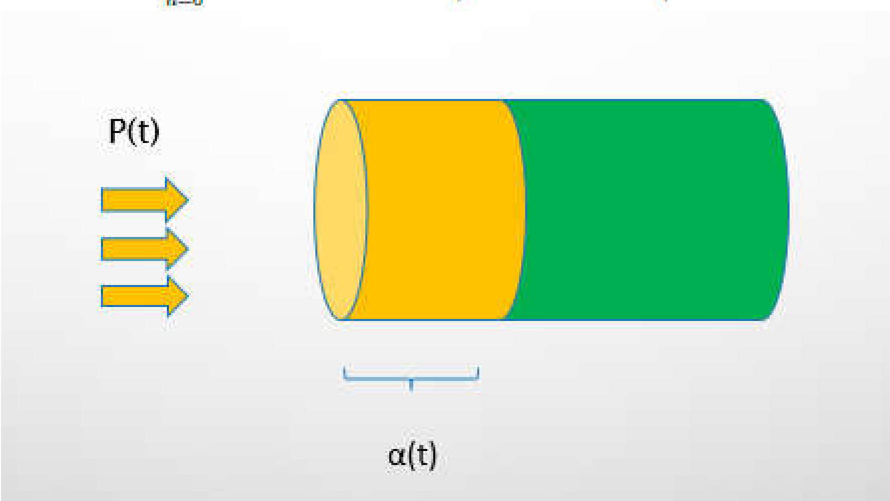

Fig.1. (3.2.1) Boundary between two (liquid and solid) phases

Solution of particular case of One phase Stefan problem

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial U^2}{\partial x^2}, \qquad 0 < x < \alpha(t)$$

$$U(0,0) = 0 \qquad (3.3.1)$$

$$U(0,t) = P(t) \qquad (3.3.2)$$

$$U(\alpha(t),t) = U_m \qquad (3.3.3)$$

$$-\lambda \frac{\partial U}{\partial x}\Big|_{x=\alpha(t)} = L\gamma \frac{d\alpha}{dt} \qquad (3.3.4)$$

Solution:

$$U(x,t) = \sum_{n=0}^{\infty} (2a\sqrt{t})^n \left[A_n i^n erfc \frac{x}{2a\sqrt{t}} + B_n i^n erfc \frac{-x}{2a\sqrt{t}} \right] \qquad (3.3.5)$$

Lemma:

$$1) \lim_{x \to \infty} \frac{i^n erfc(-x)}{x^n} = \frac{2}{n!}$$

$$2) \lim_{t \to 0} (2a_2\sqrt{t})^n i^n erfc \frac{x}{2a_2\sqrt{t}} = 0$$

Using (3.3.5), fromcondition (3.3.2), weget

$$-\lambda U(0,t) = -\lambda \sum_{n=0}^{\infty} (2a\sqrt{t})^n [A_n i^n erfc0 + B_n i^n erfc0] = P(t)$$

$$-\lambda U(0,t) = -\lambda \sum_{n=0}^{\infty} (2a\sqrt{t})^n [A_n + B_n] erfc0 = P(t) = \sum_{n=0}^{\infty} \frac{P^{(n)}(0)}{n!} (\sqrt{t})^n$$

By denoting $P_n = \sum_{n=0}^{\infty} \frac{P^{(n)}(0)}{n!} (\sqrt{t})^n$, we have left

$A_n = -\frac{P_n}{\lambda(2a)^n i^n erfc0} - B_n$, by substituting $\frac{P_n}{\lambda(2a)^n i^n erfc0}$ to the β_n, we get

$$A_n = -\beta_n - B_n \qquad (3.3.6)$$

Atthecondition n=0:

$$A_0 = -\beta_0 - B_0 \qquad (3.3.7)$$

From condition (3.3.3), using $\alpha(t) = \alpha\sqrt{t}$, we have

$$U(\alpha\sqrt{t}, t) = \sum_{n=0}^{\infty} (2a\sqrt{t})^n [A_n i^n erfc \frac{\alpha}{2a} + B_n i^n erfc \frac{-\alpha}{2a}] = U_m$$

n=0:

$$[A_0 erfc \frac{\alpha}{2a} + B_0 i^0 erfc \frac{-\alpha}{2a}] = U_m$$

Byusing (3.3.7) condition, weget

$$(\beta_0 - B_0) erfc \frac{\alpha}{2a} + B_0 i^0 erfc \frac{-\alpha}{2a}] = U_m$$

$$B_0 (erfc \frac{-\alpha}{2a} - erfc \frac{\alpha}{2a}) = U_m + \beta_0 erfc \frac{\alpha}{2a}$$

$$B_0 = \frac{\beta_0 erfc \frac{\alpha}{2a}}{(erfc \frac{-\alpha}{2a} - erfc \frac{\alpha}{2a})} \qquad (3.3.8)$$

From condition (3.3.4), by taking the derivative of (3.3.5), we get

$$-\lambda\sum_{n=0}^{\infty}(2a\sqrt{t})^{n-1}\Big[-A_n i^{n-1}erfc\frac{\alpha}{2a}+B_n i^{n-1}erfc\frac{-\alpha}{2a}\Big]=L\gamma\frac{\alpha}{2\sqrt{t}}$$

Forthecondition n=0:

$$-\lambda\frac{1}{2a}[-A_0+B_0]\frac{2}{\sqrt{\pi}}e^{\frac{-\alpha^2}{4a^2}}=L\gamma\frac{\alpha}{2}$$

By using conditions (3.3.7), we get

$$-\frac{\lambda}{2}[2B_0+\beta_0]\frac{2}{\sqrt{\pi}}=L\gamma\alpha e^{\frac{\alpha^2}{4a^2}}$$

And using condition (3.3.8), we finally get our equation, which consists only from α and coefficients

$$-\frac{\lambda}{2}\beta_0[\frac{2erfc\frac{\alpha}{2a}}{(erfc\frac{-\alpha}{2a}-erfc\frac{\alpha}{2a})}]=L\gamma\alpha e^{\frac{\alpha^2}{4a^2}} \qquad (3.3.9)$$

$a=1$ $\qquad U_m=100$

$\lambda=1$ $\qquad L=50$

$P_0=107$ $\qquad \gamma=2$

Firstly, we find β_0, however it depends only from coefficients as P_0, λ, and a to which we have given numbers. After finding β_0, we come to the last equation (3.3.9) that we obtained. There are two equation which are equal to each other and depends only from α that we should find. We denote left side of equation as $f(\alpha)$, and right side as $g(\alpha)$.By the way we use approximation method by Matlab which code shown in Figure(3.3.1).

```
a=1;
lambda=1;
p0=107;
Um=100;
L=50;
gamma=2;


beta=p0/(lambda);


for i=1:100
    x=i/100;
    y1(i)=-(lambda*2)/(a*sqrt(pi))*((1/(1-erfc(x/2)))*(Um-beta*erfc(x/2))-beta);
    y2(i)=L*gamma*x*exp(x.^2/4);

end

plot(1/100*(1:100),y1,'-r');
hold on
plot(1/100*(1:100),y2, '-b');
grid on
```

Fig.2. (3.3.1) Codeofequation (3.3.9)

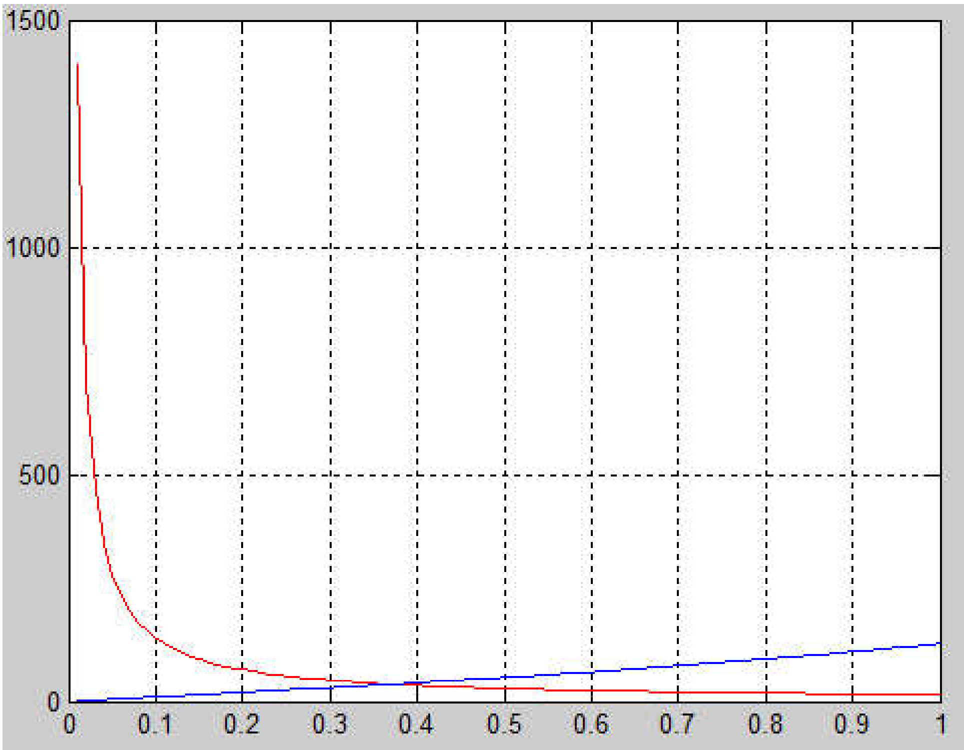

Fig.3. (3.3.1) Graph view of solution of the equation(3.3.9)

We can see that, our α=0.36.

Conclusion

In conclusion, mathematical modeling of heat transferring in electrical contacts has been constructed. In Stefan problem, we have solved analytical solution. Analytical solution of One phase Stefan Problem (direct) and its particular solution in Matlab has been shown .The result from particular case shown in the graph. Faa di Bruno and Leibnizs formulas are used to find solution for analytical solution . By undertaking research, we have an exact result that will allow to economy energy in manufacture, partially in steel manufacture in Kazakhstan where needed solving of this problem. It will give great benefit in the financial plan.

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