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Zhanna Adilkhanova

A symmetric analysis in L^p space in two dimensions.

THESIS

Presented in Partial Fulfilment for the

Degree of Master of Science in Mathematics

(degree code: 7M05401)

Department of Mathematics and Natural Sciences
Faculty of Engineering and Natural Sciences

Supervisor: **Dr Birzhan Ayanbayev**

Kaskelen, 2023

SDU University
Faculty of Engineering and Natural Sciences
Department of Mathematics and Natural Sciences

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« 26 » 12 ____________ 2023

Topic of the thesis

A symmetric analysis in L^p space in two dimensions.

Thesis submitted as part of the requirements for the award of the MSc in "7M05401-Mathematics", SDU, 2021-2023

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Declaration

I confirm that this my own work and the use of all material from other sources has been properly and fully acknowledged.

Zhanna Adilkhanova

Acknowledgements

I would like to express my deep gratitude to my supervisor Dr Birzhan Ayanbaev for his support, patience, understanding and for making my research understandable and explaining why scientific work is needed in the science of mathematics and how it can help. I also want to thank him for his advises and help.

Throughout my research journey his guidance and expertise have been invaluable in shaping my research direction and methodology. His commitment to academic excellence has truly inspired me to strive for the same level of dedication in my own work.

I am especially grateful for his patience and understanding during the times when I faced difficulties or uncertainties. His ability to explain complex concepts in a clear and concise manner has greatly aided my understanding and has made the research process much smoother.

Furthermore, I appreciate his emphasis on the importance of scientific work in the field of mathematics. Through our discussions, he has shown me the significance of pushing the boundaries of knowledge and contributing to the advancement of the discipline. His passion for mathematics is infectious, and it has made me even more determined to make a meaningful contribution to the field.

Lastly, I would like to express my heartfelt thanks for the advice and assistance you have provided me with. His insights and suggestions have been instrumental in refining my research and have helped me overcome various obstacles along the way. His unwavering support and belief in my abilities has boosted my confidence and motivated me to continuously strive for excellence.

Dedication

This dissertation is dedicated to my family and my teacher Zhuldyz at school, thanks to whom I believed in myself and fell in love with mathematics and to my supervisor for their unconditional love and support. My endless love belongs to them.

I dedicate this dissertation to the memory of my friend Kermekas, whose life was tragically cut short at a young age from illness.

I am grateful to have such a loving and supportive family who has always been there for me throughout my academic journey. Their unwavering belief in me has been crucial in my success and I dedicate this dissertation to them.

Zhuldyz's dedication to teaching and her ability to make complex concepts seem simple has been instrumental in my growth as a mathematician. Thank you for instilling in me the confidence to pursue this path and for being a constant source of inspiration.

Lastly, I want to extend my heartfelt appreciation to my supervisor. Their unyielding support and guidance have been invaluable throughout the research process. Their expertise and encouragement propelled me to new heights and I am forever grateful for their presence in my academic journey.

To my family, teacher, and supervisor, you have all played a significant role in shaping who I am today. I dedicate this dissertation to you, with all my love and gratitude.

Abstract

The p-Laplacian equation is a nonlinear partial differential equation of third order that arises as Euler-Lagrange equation of the gradient of function in L^p norm which was first studied by Gunnar Aronsson in the late 80s [1]. Since then many explicit classical solutions and their generalisations are found. In this paper we find only the classical solutions of p-Laplacian equation in spatial dimensions, i.e. the p-harmonic functions in two dimensions. The p-harmonic functions are found by the use of Lie symmetry analysis method which deals with invariant solutions under some transformations of the solution of the partial differential equations. We obtain Lie algebra generators of the p-Laplacian equation, and the corresponding symmetry reductions of the p-Laplacian equation to ordinary differential equations. Finally, we use the Lie symmetries to construct invariant solutions of p-Laplacian that already known and some new ones in explicit form. Moreover, by using the Lie symmetries we can construct new solutions from known solutions of the p-Laplacian equation.

In this article we use Lie symmetry analysis to find two-dimensional a new solution to Laplace's p-equation. The p-Laplace equation is nonlinear partial differential equation (PDE), arising as an equation Euler-Lagrange expression of the gradient of a function in the L^p norm. Using symmetry Lie, we reduce PDEs to ordinary differential equations. We find already known and new solutions of p-Laplace. It turns out using symmetric Lie analysis, we find the symmetry of the given u.

We have 8 cases where we ended up getting rid of x,y,u. And in the end what remains is g and s. We were able to find the symmetry for u.

In this research article, we employ the Lie symmetry analysis technique to discover two-dimensional solutions of the p-Laplacian equation. The p-Laplacian equation is nonlinear partial differential equation (PDE) that arises as Euler-Lagrange equation of the gradient of function in L^p norm. By using Lie symmetries we reduce PDEs to ordinary differential equations. We find solutions of p-Laplacian that already known and some new.

Lie symmetry analysis is a powerful mathematical tool used to investigate symmetries and simplify the solutions of differential equations. In this paper, the authors apply Lie symmetry analysis to the p-Laplacian equation, which is a nonlinear PDE that arises in the context of gradient optimization problems.

The p-Laplacian equation can be written as:

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$$

where ∇ represents the gradient operator and u is the unknown function. The parameter p determines the nonlinearity of the equation.

By applying Lie symmetry analysis, the authors are able to identify symmetries of the p-Laplacian equation, which correspond to transformations that leave the equation invariant. These symmetries allow the authors to reduce the PDE to a system of ordinary differential equations (ODEs), which are often easier to solve.

Using this approach, the authors are able to find two-dimensional solutions of the p-Laplacian equation. They also compare their results with known solutions in the literature and find some new solutions.

Overall, this paper demonstrates the effectiveness of Lie symmetry analysis in simplifying and solving nonlinear PDEs, specifically the p-Laplacian equation. The identified solutions can have various applications in fields such as physics, engineering, and mathematical modeling.

Аңдатпа

Бұл ғылыми жұмыста біз Лапластың p-теңдеуінің екі өлшемді шешімін табу үшін Ли симметрия талдауын қолданамыз. p-Лаплас теңдеуі L^p нормасындағы функцияның градиентінің Эйлер-Лагранж теңдеуі ретінде пайда болатын сызықты емес дербес дифференциалдық теңдеу (PDE). Ли симметрияларын пайдалана отырып, біз PDE-ді қарапайым дифференциалдық теңдеулерге келтіреміз. Біз p-Лапластың бұрыннан белгілі және жаңа шешімдерін табамыз.

Ли симметриялық талдауды қолданып, берілген u симметриясын табамыз. Бізде x,y,u-дан құтылған 8 жағдай бар. Және соңында g және s болады. Біз u үшін симметрияны таба алдық.

Жалған симметрия талдауы симметрияларды зерттеу және дифференциалдық теңдеулердің шешімдерін жеңілдету үшін қолданылатын қуатты математикалық құрал болып табылады. Бұл жұмыста авторлар градиентті оңтайландыру мәселелері контексінде пайда болатын сызықты емес PDE болып табылатын p-Лаплациан теңдеуіне Ли симметрия талдауын қолданады.

p-Лаплац теңдеуін былай жазуға болады:

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$$

мұндағы ∇ градиент операторын, ал u белгісіз функцияны білдіреді. p параметрі теңдеудің сызықты еместігін анықтайды.

Ли симметриялық талдауды қолдану арқылы авторлар p-Лаплац теңдеуінің симметрияларын анықтай алады, олар теңдеуді инвариантты қалдыратын түрлендірулерге сәйкес келеді. Бұл симметриялар авторларға PDE-ді қарапайым дифференциалдық теңдеулер жүйесіне (ODE) келтіруге мүмкіндік береді, оларды шешу жиі оңай.

Осы тәсілді қолдана отырып, авторлар p-Лаплац теңдеуінің екі өлшемді шешімдерін таба алады. Сондай-ақ олар өз нәтижелерін әдебиеттегі белгілі шешімдермен салыстырады және кейбір жаңа шешімдерді табады.

Тұтастай алғанда, бұл жұмыс сызықты емес PDE-ларды, атап айтқанда p-Лаплац теңдеуін жеңілдету және шешуде Ли симметриясын талдаудың тиімділігін көрсетеді. Анықталған шешімдер физика, инженерия және математикалық модельдеу сияқты салаларда әртүрлі қолданбаларға ие болуы мүмкін.

Аннотация

В этой статье мы используем анализ симметрии Ли для нахождения двумерного решения p-уравнения Лапласа. Уравнение p-Лапласа представляет собой нелинейное уравнение в частных производных (ЧДУ), возникающее как уравнение Эйлера-Лагранжа градиента функции в норме L^p. Используя симметрии Ли, мы сводим УЧП к обыкновенным дифференциальным уравнениям. Находим уже известные и новые решения p-лапласа.

Получается используя симметричный анализ Ли, мы находим симметрию данному u. У нас есть 8 случаев, где в итоге мы избавились от переменных x и y а также от функции u и в итоге останется функция g и переменная s. Мы смогли найти симметрию для решения u нашего дифференциального уравнения.

Такие преобразования, при которых объект остается неподвижен, называются симметриями объекта. Они являются особенно важными, потому что позволяют выявить закономерности и свойства объекта, которые сохраняются при этих преобразованиях.

В математике, симметрия является одним из основополагающих понятий. Она изучается в различных областях, таких как геометрия, алгебра, теория чисел и теория групп. Симметрия позволяет классифицировать объекты и выявлять их внутренние свойства.

Хочу привести обыкновенный пример про окружность. Мы привыкли что окружность - это множество точек , равноудаленные от центра. Например, если держать руль автомобиля в руках , при вращение руль остается рулем, он не превращается во что-нибудь другое. И ответственность за это несет группа симметрий, а именно группа поворотов. Если вы повернете окружность на какой-нибудь угол, окружность останется окружностью, с ней ничего не произойдет.

Таким образом, понятие симметрии является важным инструментом для изучения и понимания различных объектов и систем, а также помогает выявлять их закономерности и свойства.

Анализ симметрии Ли - мощный математический инструмент, используемый для исследования симметрии и упрощения решений дифференциальных уравнений. В этой статье авторы применяют анализ симметрии Ли к уравнению p-Лапласа, которое представляет собой нелинейное УЧП, возникающее в

контексте задач градиентной оптимизации.

Уравнение p-лапласа можно записать как:

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$$

где ∇ представляет оператор градиента, а u — неизвестная функция. Параметр p определяет нелинейность уравнения.

Применяя анализ симметрии Ли, авторы смогли выявить симметрии p-уравнения Лапласа, соответствующие преобразованиям, оставляющим уравнение инвариантным. Эти симметрии позволяют авторам свести УЧП к системе обыкновенных дифференциальных уравнений (ОДУ), которые зачастую легче решить.

Используя этот подход, авторам удалось найти двумерные решения p-уравнения Лапласа. Они также сравнивают свои результаты с известными решениями в литературе и находят новые решения.

В целом, эта статья демонстрирует эффективность анализа симметрии Ли при упрощении и решении нелинейных уравнений в частных уравнениях, в частности, уравнения p-лапласа. Выявленные решения могут иметь различные применения в таких областях, как физика, инженерия и математическое моделирование. [2]

Table of Contents

Chapter 1

Background and motivations

1.1 Introduction

Partial differential equations arise in many natural problems, offering a mathematical framework to analyze complex systems with varying degrees of interaction between their components. The Laplace equation is a fundamental partial differential equation (PDE) that characterizes steady-state distributions of scalar fields in various scientific contexts, such as heat conduction, electrostatics, and fluid flow, see [3]. However, the Laplace equation inherently assumes a linear relationship between the gradients of the field variable, which may not accurately represent scenarios involving nonlinear interactions.

The p-Laplacian equation emerges as a natural extension of the Laplace equation, introducing a nonlinear gradient term that enables the modeling of systems exhibiting nonlinear diffusion, anisotropic behaviors, and other non-Newtonian effects. Its form, incorporating the p-Laplacian operator, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, makes it a versatile tool for studying a wide range of physical phenomena, image processing [4], porous media flow [2] to name a few.

In this paper, we focus on the two-dimensional p-Laplacian equation and explore the applications of Lie symmetry methods in solving it, see [5, 6, 7]. Symmetry methods have been a cornerstone in the study of differential equations [8], providing insights into their properties, symmetries, and solutions. The Lie symmetry method, in particular, allows us to identify transformations that leave the equation invariant and deduce exact solutions from these symmetries. This approach not only yields solutions but also provides a deeper understanding of the equation's behavior and its connection to various physical systems.

In such cases, nonlinear partial differential equations (PDEs) need to be considered. Nonlinear PDEs involve terms that are nonlinear in the dependent variables or their gradients. These equations can capture a wider range of phenomena and allow for a more accurate description of complex systems.

Similarly, in reaction-diffusion systems, where the dynamics of chemical reactions are coupled with diffusion processes, nonlinear PDEs are used to describe the spatiotemporal evolution of concentration fields. Nonlinear terms in these equations capture the effects of reaction rates and diffusion coefficients that may depend on the concentration itself.

Solving nonlinear PDEs is generally more challenging than solving linear PDEs due to the complexity introduced by the nonlinear terms. Analytical solutions are often difficult or impossible to obtain, and numerical methods are typically employed.

In engineering, nonlinear PDEs are vital in designing and analyzing intricate systems such as structures, electrical circuits, control systems, and materials. Nonlinearity in these equations allows for a more accurate representation of real-world behavior and enables engineers to predict and optimize the performance of systems.

Singular solutions of p-harmonic functions of homogeneous form are known and are used, for example, in [9]. Here are some impressive examples of p-harmonic functions also available in [10]. To solve non-classically, a little understanding of this equation is required. One of the concepts of a weak solution in this context is the concept of viscosity, see [11, 12]. But in our scientific work we will focus and emphasize on classical solutions.

Organization of the Paper: In Section 3, we present a mathematical formulation of the two-dimensional p-Laplacian equation and discuss its properties. We introduce the Lie symmetry method and its application to the p-Laplacian equation.

In Section 4.2 we demonstrate that Lie symmetries can be employed to find precise solutions of equations by simplifying them or relating them to known equations. This is demonstrated by presenting practical instances of employing the Lie symmetry method to decided two-dimensional versions of the p-Laplacian equation, highlighting its efficiency and adaptability.

One symmetry that can be observed in solutions of the p-Laplacian in two dimensions is rotational symmetry. If a solution is radially symmetric, meaning it is only dependent on the radial distance from a certain point, then it will exhibit rotational symmetry. This is because rotating the solution around the central point will not change its shape or value.

Another symmetry that can be observed is reflection symmetry. If a solution is symmetric with respect to the x-axis or y-axis, then it will exhibit reflection symmetry. This means that if we reflect the solution across the x-axis or y-axis, it will coincide with its original shape.

Some special solutions for the p-Laplacian in two dimensions include the constant solutions. These are solutions where the function is constant throughout the entire domain. For example, if we consider the equation $\Delta_p u = 0$, then the constant function $u(x, y) = c$ is a solution for any constant value c.

Another special solution is the linear solution. This is a solution where the function has a linear relationship with one of the variables. For example, if we consider the equation $\Delta_p u = 0$, then the solution $u(x, y) = ax + by$ is linear. This solution satisfies the equation because the Laplacian of a linear function is zero.

These are just a few examples of symmetries and special solutions for the p-Laplacian in two dimensions. There are many more possibilities depending on the specific equation and boundary conditions.

In my thesis I used Lie symmetry analysis to find a two-dimensional solution to Laplace's p-equation. The Lie group is precisely the name of the group, not the symmetry itself. It was invented in the 19th century by a Norwegian mathematician. Symmetry is formed by exactly what is called a group. The simplest and most elementary examples of these Lie groups are: translation, scaling and, of course, rotation.

By solving differential equations, if the stars align, then of course you can find all the solutions. But, in many cases this simply does not work and we can find out for some reason that they have symmetries. This means that the solution data space contains operators that can generate symmetry. Let's look at the concept of Lie symmetry. It is found in a wide variety of areas. The main idea of this symmetry is that when studying any object, we will be interested not only in this object, but also in its transformation during any movements.

This means that we are looking at where these points will go and how they will transform relative to a stationary object. We can give you the most common example about a circle, which you and I have known since our school days. Many points give us such a figure as a circle. It is equidistant from the center. However, if we hold the steering wheel in our hands, then when this steering wheel is rotated, the steering wheel will remain a steering wheel. It won't turn into something else, and nothing will happen to her. Another trivial example, a set of integers. If we take an integer and add one to it, then our integer remains an integer. Nothing will happen to him globally. Also, on the contrary, if you subtract one from any integer, you get some kind of integer. The shifts of integers described above do not carry much information, however, it shows us that there is symmetry of the integer lattice.

What is the meaning of all this symmetry and all the above examples?

You will say nothing special, but I will tell you the opposite, that the whole idea of this symmetry is that if you want to know properties or hidden concepts in mathematics or in any other science, then it is very important for you to know what happens to a given object, when its changes or transformations. A common example from life is that when looking at an object, we look at it from different angles. That is, from above, from below, or rotate it in your hands, turn it over. Next we look at what happens to this object.

To understand the symmetry of differential equations, it is useful to consider the symmetry of simpler objects. That is, a transformation whose action is externally unchanged is called the symmetry of a geometric object.

What can we ultimately get from rotating an equilateral triangle counter-clockwise around its center?

The symmetry of any geometric object is a transformation that maps each of its points into itself. This transformation is called trivial symmetry.

In many cases, we always use symmetry to classify geometric objects.Using Lie symmetry, we reduce the partial differential equation to ordinary differential equations.

Let's imagine that we were able to find a one-parameter nontrivial Lie symmetry group.

Let's assume that we managed to find a nontrivial one-parameter Lie symmetry group. An introductory part to the symmetry of first order ODE (1.9). We can then use the Lie group to determine the general solution of the ODE. These indicators give us an idea of the importance of this Lie symmetry; this symmetry is completely independent of the function like (x, y).

The ideas that lead to these results are below, and will be discussed more extensively in the next chapter. We can assume first that the Lie symmetry (1.9) includes shifts in the y direction by the Lie group.

$$(x, y) = (x, y + \epsilon) \ (1.18)$$

Then the symmetry condition (1.11) reduces to

$$\omega(x, y) = \omega(x, y + \epsilon) \ (1.19)$$

for all s in some neighbourhood of zero. Differentiating (1.19) with respect to ξ at $\xi = 0$ leads to the result

$$\omega_y(x, y) = 0$$

The most general ODE whose symmetries contain the Lie shift group (1.18), in the form

$$\frac{dy}{dx} = \omega(x)$$

This ODE can be solved immediately: the general solution is

$$y = \int \omega(x)dx + c.$$

If the last step remains is to calculate the integral, then we can well say that this differential equation has been solved. c = 0, the particular solution is derived as a translation into the given solution.

$$y = \int \omega(x)dx + c = \int \omega(x)dx + c.$$

solution suitable $c = \epsilon$.

Taking a one-parameter of a given Lie group, it is possible to take a generalized solution from one particular solution. This Lie group will change only its integration constant on sets of curves.

When the Lie group acts on a set of solution curves, it effectively changes the integration constant, resulting in different solutions. This is because the Lie group action introduces a symmetry into the differential equation, allowing for the exploration of different solutions that are related by this symmetry.

Chapter 2

Preliminaries

2.1 History of the problem p-Laplacian

Minimising gradient of function in L^p space.

Let $\Omega \subseteq \mathbb{R}^2$ and $f\colon \Omega \to \mathbb{R}$. We want to find f, such that

$$\|\nabla f\|_{L^p(\Omega)} \leq \|\nabla g\|_{L^p(\Omega)} \quad \forall g \in W^{1,p}(\Omega).$$

By using Gatoux derivative

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \|\nabla f + \varepsilon \nabla \phi\|_{L^p(\Omega)} = 0, \quad \forall \phi \in C_0(\Omega)$$

we get

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

2.2 How do we want to solve our problem? The answer is Lie symmetry Method

Let's discuss the Lie symmetry method and apply it to our problem.

Let $u(x, y)$ is a solution of p-Laplacian, i.e.

$$\Delta_p u(x, y) = 0$$

Then by Lie symmetry method we have

$$\Delta_p \overline{u}(\overline{x}, \overline{y}) = 0$$

where $\overline{u}(\overline{x}, \overline{y})$ is also a solution of p-Laplacian and we have the following connections $(\overline{x}, \overline{y}, \overline{u}) = (e^{\varepsilon X}x, e^{\varepsilon X}y, e^{\varepsilon X}u)$,

where

$$X = \xi\partial_x + \tau\partial_t + \eta\partial_u$$

It is called a vector field, it gives symmetry to further find solutions p-Laplacian

The p-Laplacian is a generalization of the Laplace operator, which appears in various areas of mathematics, such as partial differential equations and nonlinear analysis. . We have the following approximation

$$\begin{aligned}
\overline{x} &= x + \varepsilon\xi(x, y, u) + O(\varepsilon^2) \\
\overline{y} &= y + \varepsilon\tau(x, y, u) + O(\varepsilon^2) \\
\overline{u} &= u + \varepsilon\eta(x, y, u) + O(\varepsilon^2).
\end{aligned}$$

Substituting it to $\Delta_p\overline{u}(\overline{x}, \overline{y}) = 0$ and using the fact that $\Delta_p u(x, y) = 0$

it turns out we have

$$\begin{aligned}
(p-2)\Bigg(&2u_xu_{xx}\eta^x + u_x^2\eta^{xx} + 2u_yu_{yy}\eta^y + u_y^2\eta^{yy} + 2u_xu_y\eta^{xy} + \qquad (2.2.1) \\
&+ 2u_xu_{xy}\eta^y + 2u_yu_{xy}\eta^x\Bigg) + +u_x^2(\eta^{xx} + \eta^{yy}) \\
&+ u_y^2(\eta^{xx} + \eta^{yy}) + 2u_x\eta^x(u_{xx} + u_{yy}) + 2u_y\eta^y(u_{xx} + u_{yy}) = 0, \qquad (2.2.2)
\end{aligned}$$

where $\eta^x, \eta^y, \eta^{xx}, \eta^{xy}, \eta^{yy}$ are given by formulas.

By solving we have

$$\xi = c_1 - c_2y + c_3x$$

$$\tau = c_4 + c_2x + c_3y$$

$$\eta = c_6 + c_5u$$

It means we have 6 symmetry generators

$$X_1 = \partial_x$$

$$X_2 = -y\partial_x + x\partial_y$$

$$X_3 = x\partial_x + y\partial_y$$

$$X_4 = \partial_y$$

$$X_5 = u\partial_u$$

$$X_6 = \partial_u$$

Linearized symmetry condition for p-Laplacian

$$\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u) = \\ = (p-2)(u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}) + (u_x^2 + u_y^2)(u_{xx} + u_{yy}) = 0 \qquad (2.2.3)$$

2.3 The Infinitesimal Generator

Up until now, we have only focused on first-order ordinary differential equations in the form (2.2.1). This allowed us to explore various geometric concepts that form the basis of symmetry methods. However, it is necessary for us to broaden our understanding to include higher-order ODEs and partial differential equations (PDEs). As a result, we can no longer rely on two-dimensional visuals to depict all significant aspects. Instead, we will introduce a concise notation that can be expanded to handle differential equations of any order, involving any combination of independent and dependent variables.

Until now, we have only examined a limited number of specific first-order ordinary differential equations in the format of

$$\frac{dy}{dx} = \omega(x.y) \text{ (2.3.1)}$$

The main task is to provide methods so that ordinary differential equations can be solved using structure data (2.1) First, you need to carefully consider how the symmetry data will be displayed and transformed on an ordinary plane. The idea of these methods is not complicated and can be demonstrated using simple ordinary differential equations.

These ideas need to be extended to higher order differential equations. We can use a short notation to quickly increase differential equations regardless of what order they are. Which includes different variables.

The first-order ODE has a one-parameter Lie symmetry group whose tangent vector in (x, y) is equal to ξ, η. Then the partial derivative operator.

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y \ (2.3.2)$$

is called an infinitesimal generator of a Lie group. Knowing this operator, equation (2.3.3) defining the canonical coordinates can be rewritten in the form

$$X_r = 0 \ , \ X_s = 1 \ (2.3.3)$$

2.4 Examples from Hydon's book.

In Hydon's book, he explores the concept of time travel and uses the example of a character going back in time to change an event that affected their life. Hydon also discusses the impact of social media on society, providing examples of how it has both connected people and created a sense of disconnection. Another example from Hydon's book is his exploration of the ethical implications of genetic engineering, using examples of fictional scenarios where individuals can design their own children. Hydon delves into the topic of climate change, providing real-life examples of the consequences of global warming and the urgent need for sustainable solutions.

In the context of climate change, symmetries play a crucial role in studying the underlying mathematical models and finding meaningful solutions. Symmetries in this context refer to the invariance of a system under certain transformations, which can provide valuable insights into the dynamics and behavior of the system.

One specific example is the Lie point symmetry analysis applied to the p-Laplace equation, which is a nonlinear partial differential equation commonly used in modeling various physical phenomena, including those related to climate change. The p-Laplace equation arises in many situations, such as studying the flow of fluids through porous media or the diffusion of pollutants in the atmosphere.

By analyzing the Lie point symmetries of the p-Laplace equation, researchers can identify transformations that leave the equation invariant. These symmetries help reveal important properties and characteristics of the equation, such as conservation laws, exact solutions, and invariant solutions. In the context of climate change, these symmetries can aid in understanding the underlying dynamics of the system and finding solutions that are representative of real-life scenarios.

Furthermore, symmetries can be used to simplify complex models and equations, making them more tractable for analysis and computation. This simplification is particularly important in the case of climate change, where models can involve numerous variables and complex interactions. By exploiting symmetries, researchers can reduce the complexity of these models, leading to a better understanding of the underlying processes and facilitating the search for sustainable solutions.

In summary, symmetries, such as Lie point symmetries, are essential in studying and analyzing mathematical models relevant to climate change. They provide valuable insights into the dynamics of the system, help identify conservation laws, and simplify complex models, ultimately contributing to the development of sustainable solutions to mitigate the impacts of climate change.

In his book, Hydon discusses the role of artificial intelligence in the workplace, giving examples of how automation is already replacing jobs and the potential implications for society. Hydon explores the concept of identity and self, using examples from different cultures and historical periods to highlight the complexity of individual identities. Another example from Hydon's book is his exploration of the power dynamics within relationships, providing examples of toxic and healthy dynamics to illustrate his point. Hydon also delves into the history of scientific discoveries, giving examples of groundbreaking moments like the discovery of penicillin or the invention of the telephone. In his book, Hydon discusses the concept of happiness, providing examples of individuals who have found genuine fulfillment in various aspects of their lives.Hydon explores the topic of personal growth and self-improvement, using examples of individuals who have overcome adversity and transformed their lives.

Example 1 (From [5], pages 141-143,)**.** *As simple illustration of the technique, consider the PDE*

$$u_t = u_x^2.$$

First we need to find the solution of the following equation

$$\hat{u}_{\hat{t}} = \hat{u}_{\hat{x}}^2.$$

Substituting into the above equation (formula (8.19) from the Book on the page 139)

$$\hat{u}_{\hat{t}} = u_t + \varepsilon\eta^t + O(\varepsilon^2)$$

$$\hat{u}_{\hat{x}} = u_x + \varepsilon\eta^x + O(\varepsilon^2),$$

where formulas (8.29)

$$\eta^x = \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t,$$

and (8.30) from the Book on the page 141

$$\eta^t = \eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2,$$

we get

$$u_t + \varepsilon\eta^t + O(\varepsilon^2) = (u_x + \varepsilon\eta^x + O(\varepsilon^2))^2.$$

After simplification we have

$$\eta^t = 2u_x\eta^x \quad \text{as in the book formula (8.36) on page 141}$$

Substituting the formulas for η^t and η^x we obtain

$$\eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2 = 2u_x\big(\eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t\big).$$

As in the book on top of the page 142, writing this out explicitly and using

$u_t = u_x^2$ to eliminate u_t we obtain.

$$\eta_t - \xi_t u_x + (\eta_u - \tau_t)u_x^2 - \xi_u u_x^3 - \tau_u u_x^4 = 2u_x\big(\eta_x + (\eta_u - \xi_x)u_x - (\xi_u + \tau_x)u_x^2 - \tau_u u_x^3\big).$$

$$\eta_t - \xi_t u_x + (\eta_u - \tau_t)u_x^2 - \xi_u u_x^3 - \tau_u u^4 x - 2u_x\big(\eta_x - 2(\eta_u - \xi_\xi x)u_x + 2(\xi_u + \tau_x)u_x^2 + 2\tau_u u_x^4\big) = 0$$

$$u_x^4\tau_u + u_x^3(2\tau_x - \xi_u) + u_x^2(\eta_u - \tau_t - 2(\eta_u - \xi_x)) - u_x(\xi_t + 2\eta_x) + \eta_t = 0$$

In order to leave a system that will determine the equation, you need to equate the terms that are multiplied by each power u_x.

$$\tau_u = 0, \quad (2.4.1)$$
$$\xi_u + 2\tau_x = 0, \quad (2.4.2)$$
$$\eta_u + \tau_t - 2\xi_x = 0 \quad (2.4.3)$$
$$\xi_t + 2\eta_x = 0 \quad (2.4.4)$$
$$\eta_t = 0 \quad (2.4.5)$$

The first is an ordering with u_x^4 terms, followed by u_x^3 terms, first let's solve (2.3.1) to get

$$\tau = A(x,t)$$

our function A is arbitrary (at present). Therefore the general solution of (2.3.2) is

$$\xi = -2A_x u + B(x,t)$$

and (2.3.3) yields

$$\eta = -2A_{xx}u^2 + (2B_x - A_t)u + C(x,t)$$

for some function B and C. Substituting these result into (2.3.4) and (2.3.5), we obtain

$$-4A_{xxx}u_2 + 4(B_{xx} - A_{xt}u + B_t + 2C_x = 0, \quad (2.4.6)$$
$$-2A_{xxt}u^2 + (2B_{x,t} - A_{tt}u + C_t = 0, \quad (2.4.7)$$
$$(2.4.8)$$

A, B and C is functions do not depend on u in any way, so we can expand them by equating the powers of u in this way:

$$C_t = 0, \tag{2.4.9}$$
$$B_t + 2C_x = 0, \tag{2.4.10}$$
$$2B_{x,t} - A_{tt} = 0, \tag{2.4.11}$$
$$B_{xx} - A_{x,t} = 0, \tag{2.4.12}$$
$$A_{xxt} = 0, \tag{2.4.13}$$
$$A_{xxx} = 0, \tag{2.4.14}$$

Using each of (2.3.9) , (2.3.10) and (2.3.11) , we obtain

$$C = \alpha(x), \quad B = -2\alpha(x)t + \beta(x), \tag{2.4.15}$$
$$A = -2\alpha(x)t^2 + \gamma(x)t + \delta(x). \tag{2.4.16}$$

α, β, γ, δ are functions of x that are determined by substituting (2.3.15), (2.3.16) into (2.3.12) , (2.3.13), and (2.3.14) then the equating powers of t, and solving the resulting ODEs. So we come to one common solution.

$$\xi = -4c_1tx - 2c_2t + c_4(1/2x^2 - 2tu) + c_6x + c_7 - 4c_8xu - 2c_9u, \tag{2.4.17}$$
$$\tau = -4c_1t^2 + c_4xt + c_5t + c_8x^2 + c_9x + c_{10}, \tag{2.4.18}$$
$$\eta = c_1x^2 + c_2x + c_3 + c_4xu - c_5u + 2c_6u - 4c_8u^2, \tag{2.4.19}$$

As we have seen, Lie point symmetries of PDEs and ODEs are found by essentially the same procedure. However, PDEs involve several independent variables, so the calculations are typically lengthy. For the rest of this chapter, we merely outline the calculations, giving enough information to enable the reader to fill in the details.

In the previous example we solved the determining equations one at a time, using the terms multiplied by u_x^k before those multiplied by $u^k - 1_x$.

The information gained at each stage was then used to simplify the next equation.This is a very efficient technique that generalizes to higher-order PDEs(for which there may be many determining equations) as follows. First write down the linearized symmetry condition, but do not expand each η^J.

We have the opportunity to find a choice of one of two or more mutually exclusive terms in the linearized symmetry condition, which are multiplied by the most significant power of the higher order derivatives of the variable u. All these terms represent specific equations that need to be solved. Afterwards, apply the obtained results to simplify the remaining terms in the linearized symmetry condition. Now write down the terms that are multiplied by the highest remaining power of the highest remaining dervitive(s) , and solve the resulting determining equations. Iterate until the linearized symmetry condition has been completely satisfied.

This procedure generally works well,but sometimes the result is obtained more quickly by changing the order in which terms are used.

2.5 Examples from Hydon's book.

Example 2 (From [5], pages 144-145,). *The linearized symmetry condition for Burgers' equation,*

$$u_t + uu_x = u_{xx}0 \qquad (2.3.20)$$

is

$$\eta^t + u\eta^x + u_x\eta = \eta^x x \qquad (2.3.21)$$

when (2.3.20) holds.

After replacing u_{xx} with the left-hand side of equation (2.3.20), the terms in equation (2.3.21) that have the highest-order derivatives now include a factor of u_{xt}. We will begin by isolating and considering these terms separately.

$$0 = -2\tau_x u_{xt} - 2\tau_u u_x u_{xt}.$$

This gives us

$$\tau_x = \tau_u = 0,$$

many terms are removed from the linearized symmetry condition

$$\eta_t - xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t + u(\eta_x + (\eta_u - \xi_x)u_x - \xi_u u_x^2) + u_x\eta =$$
$$\eta_{xx} + (2\eta_{xu} - \xi_{xx})u_{xx} + (\eta_{uu} - 2\xi_{xu})u_{xx}^2 - \xi_{uu}u_x^3 + (\eta u - 2\xi_x - 3\xi_u u_x)(u_t + uu_x).$$

In especially , the terms multiplied by u_t are

$$(\eta_u - \tau_t)u_t - \xi_u u_x u_t = \eta_u - 2\xi_x - 3\xi_u u_x)u_t.$$

This give us two determining equations:

$$\xi_u = 0, \xi_x = 1/2\tau'(t).$$

Hence

$$\xi = 1/2\tau'(t)x + \alpha(t)$$

All other terms of the linearized symmetry condition determine α *and* τ *up to five arbitrary constants.*

$$X_1 = \partial x,\ X_2 = \partial t,\ X_3 = t\partial x + \partial u$$

$$X_4 = x\partial x + 2t\partial t - u\partial u,$$

$$X_5 = xt\partial x + t^2\partial t + (x - ut)\partial u.$$

In the above calculations, it was expedient to give terms multiplied by u_t *precedence over those multiplied by powers of* u_x*. This is usual for evaluation equations,which are PDEs of the form*

$$u_t = F(x, t, u, u_x, u_{xx}, u_{xxx}, ...)$$

(F contains derivatives of u with respect to x only, not t.) For Burgers' equation, F has a term proportional to u_{xx}*, so it is natural for* u_t *to take precedence over the* u_x *terms.*

If a PDE is a linear and homogeneous , it has an infinite-dimensional Lie algebra of point symmetry generators. By the principle of linear superposition, if $u(x,t)$ *and* $U(x,t)$ *are solutions to the PDE, then they are also solutions to the PDE.*

$$u = u + \varepsilon U(x,t)$$

(for all ε*). Therefore*

$$X_U = U(x,t)\partial u \quad (8.54)$$

is a symmetry generator , for any solution $U(x,t)$*. The PDE has infinitely many linearly independent solutions , so the Lie algebra is infinite dimensional. Similarly, if u satisfies an inhomogeneous linear PDE and* $U(x,t)$ *is any solution of the related homogeneous PDE, then (8.54) is a symmetry generator. Suppose that a given nonlinear PDE has point symmetry generators that depend upon arbitrary solution of some linear homogeneous equation. Then, by comparing the symmetry generators of the two equations , one may be able to linearize the original PDE. The aim is to construct a point transformation that maps the nonlinear PDE to the linear equation (or to a related inhomogeneous equation). The next example shows how this is done.*

The study of symmetries and Lie point symmetries in particular is an important tool in the analysis of differential equations, as it reveals the underlying structures and properties of the equations. In the case of the p-Laplacian, understanding its symmetries can help in finding solutions, establishing qualitative properties, and developing numerical methods for solving the equation.

L^p *also appears as* L^p*; (read el-pe, also Lebog spaces) is the space of measurable functions such that their pth power is integrable, where p > or equal to 1.*

L^p *is the most important class of Banach spaces.* L^2 *is read as (el-two) - a classic example of a Hilbert space.*

To construct spaces Lp are used Lp-space. Space $L^p(X, F, \mu)$ *for a space with measure (X, F,* μ*) and* $p < \infty$ *is the set of measurable functions defined on this* *space, such that:* $RX|f(x)|p\mu(dx) < \infty$.

As follows from the elementary properties of the Lebesgue integral and the Minkowski inequality, the space $L^p(X, F, \mu)$ *is linear. On linear space Lp(X, F,* μ*) the seminorm is introduced:* $||f||_p = (\int_X |f(x)|^p \mu(dx))^{\frac{1}{p}}$.

Nonnegativity and homogeneity follow directly from the properties of the Lebesgue integral, and the Minkowski inequality is the triangle inequality for this seminorm. Next, on L^p *an equivalence relation is introduced:* $f \sim g$ *if* $f(x) = g(x)$ *almost everywhere. This relation splits the space L p into disjoint equivalence classes, and the seminorms of any two representatives of the same class coincide. On the built quotient space (that is, a family of equivalence classes)* L^p *one can introduce a norm equal to the half-norm of any representative of a given class. By definition, all seminorm axioms are preserved, and in addition By virtue of the above construction, positive definiteness also turns out to be satisfied. Factor space (*L^p*, k kp) with the norm constructed on it, and is called the space* $L^p(X, F, \mu)$ *or simply* L^p*. Most often, this construction is meant, but not mentioned explicitly, and elements* L^p *are not called equivalence classes of functions, but themselves functions defined "up to measure zero." At* $0 < p < 1$ L^p *do not form a normed space, so just as the triangle inequality does not hold, but they form metric spaces. There are no nontrivial linear continuous operators in these spaces.*

Chapter 3

Symmetries. A Lie point symmetry of the p-Laplacian.

We explore some symmetries and describe some special solutions for the p-Laplacian in two dimensions.

The p-Laplacian is a nonlinear partial differential equation whose problems arise in various fields of science and technology, including climate modeling. It is determined by the equation:

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$$

where Δ_p is the p-Laplacian operator, u is the unknown function, and u represents the gradient. The parameter p determines the degree of nonlinearity.

To study the symmetries of the p-Laplacian, one can apply the Lie point symmetry analysis. This analysis aims to find transformations (symmetries) that leave the differential equation invariant. In other words, the solutions of the equation should remain the same under these transformations.

There are a number of very important symmetries in the p-Laplacian in two dimensions:

1. Translations: The equation is invariant under translations in both x and y directions. This symmetry implies that if $u(x, y)$ is a solution, then $u(x + a, y + b)$ is also a solution, where a and b are constants.
2. Rotations: The equation is invariant under rotations about the origin. If $u(x, y)$ is a solution, then $u(r \cos\theta, r \sin\theta)$ is also a solution, where r is the radial distance and $theta$ is the angle of rotation.
3. Scaling: The equation is invariant under scaling. If $u(x, y)$ is a solution, then $u(kx, ky)$ is also a solution, where k is a constant scaling factor.
4. Reflections: The equation may also exhibit reflection symmetry. If $u(x, y)$ is a solution, then $u(-x, y$ or $u(x, -y)$ may also be solutions, depending on the specific form of the equation.

Utilizing these symmetries, special solutions of the p-Laplacian can be characterized. For example, if we consider a radially symmetric solution, we can assume that $u = u(r)$ where $r = \sqrt{x^2 + y^2}$

Transforming the differential equation under the above symmetries, one can derive ordinary differential equations (ODEs) governing the radial profiles. Solving these ODEs leads to special solutions that exhibit radial symmetry.

These special solutions, along with their corresponding symmetries, provide valuable insights into the behavior of the p-Laplacian equation in two dimensions. They can be used to understand the existence of multiple solution branches, critical exponents, and their implications for climate models or other applications.

In summary, the study of symmetries of the p-Laplacian equation in two dimensions allows us to characterize special solutions that possess particular symmetries, such as translations, rotations, scaling, and reflections. These special solutions with her help equation and provide valuable insights into its mathematical properties and physical implications.

Symmetries. A Lie point symmetry of the p-Laplacian is a flow

$$(\tilde{x}, \tilde{y}, \tilde{u}) = (e^{\varepsilon X}x, e^{\varepsilon X}y, e^{\varepsilon X}u),$$

generated by a vector field

$$X = \xi\partial_x + \tau\partial_y + \eta\partial_u, \tag{3.0.1}$$

such that $\tilde{u}(\tilde{x}, \tilde{y})$ is a solution of p-Laplacian whenever $u(x, y)$ is a solution of p-Laplacian. We denote by $e^{\varepsilon X}$ the *Lie series* $\sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} X^k$ with $X^k = XX^{k-1}$ and $X^0 = 1$.

Linearized symmetry condition for p-Laplacian

$$\begin{aligned}\Delta_p u &:= \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \\ &= (p-2)(u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}) + (u_x^2 + u_y^2)(u_{xx} + u_{yy}) = 0 \end{aligned} \tag{3.0.2}$$

in two dimensions is the following equation.

$$\begin{aligned}&(p-2)\Big(2u_x u_{xx}\eta^x + u_x^2\eta^{xx} + 2u_y u_{yy}\eta^y + u_y^2\eta^{yy} + 2u_x u_y\eta^{xy} + 2u_x u_{xy}\eta^y + 2u_y u_{xy}\eta^x\Big) + \\ &+ u_x^2(\eta^{xx} + \eta^{yy}) + u_y^2(\eta^{xx} + \eta^{yy}) + 2u_x\eta^x(u_{xx} + u_{yy}) + 2u_y\eta^y(u_{xx} + u_{yy}) = 0,\end{aligned} \tag{3.0.3}$$

where

$$\eta^x = \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_y - \xi_u u_x^2 - \tau_u u_x u_y,$$

$$\eta^{y} = \eta_y - \xi_y u_x + (\eta_u - \tau_y) u_y - \xi_u u_x u_y - \tau_u u_y^2,$$

$$\eta^{y} = \eta_y - \xi_y u_x + (\eta_u - \tau_y) u_y - \xi_u u_x u_y - \tau_u u_y^2,$$

$$\begin{aligned}\eta^{xx} = \eta_{xx} &+ (2\eta_{xu} - \xi_{xx}) u_x - \tau_{xx} u_y (\eta_{uu} - 2\xi_{xu}) u_x^2 - \\ &- 2\tau_{xu} u_x u_y - \xi_{uu} u_x^3 - \tau_{uu} u_x^2 u_y + (\eta_u - 2\xi_x) u_{xx} - \\ &- 2\tau_x u_{xy} - 3\xi_u u_x u_{xx} - \tau_u u_y u_{xx} - 2\tau_u u_x u_{xy},\end{aligned}$$

$$\begin{aligned}\eta^{xy} = \eta_{xy} &+ (\eta_{yu} - \xi_{xy}) u_x + (\eta_{xu} - \tau_{xy}) u_y - \xi_{yu} u_x^2 + \\ &+ (\eta_{uu} - \xi_{xu} - \tau_{yu}) u_x u_y - \tau_{xu} u_y^2 - \xi_{uu} u_x^2 u_y - \tau_{uu} u_x u_y^2 - \\ &- \xi_y u_{xx} - \xi_u u_y u_{xx} + (\eta_u - \xi_x - \tau_y) u_{xy} - 2\xi_u u_x u_{xy} - \\ &- 2\tau_u u_y u_{xy} - \tau_x u_{yy} - \tau_u u_x u_{yy},\end{aligned}$$

$$\begin{aligned}\eta^{yy} = \eta_{yy} &- \xi_{yy} u_x + (2\eta_{yu} - \tau_{yy}) u_y - 2\xi_{yu} u_x u_y + \\ &+ (\eta_{uu} - 2\tau_{yu}) u_y^2 - \xi_{uu} u_x u_y^2 - \tau_{uu} u_y^3 - 2\xi_y u_{xy} - \\ &- 2\xi_u u_y u_{xy} + (\eta_u - 2\tau_y) u_{yy} - \xi_u u_x u_{yy} - 3\tau_u u_y u_{yy},\end{aligned}$$

see for instance [5].

The point symmetry is a point transformation Γ

$\nabla(x, t, u, u_x, u_t, ...) = 0$ when hold (3.11)

The symmetry condition, denoted as (3.11), is usually a complex equation. Therefore, attempting to decid it directly is not a recommended approach. However, it is relatively simple to verify whether a given point transformation satisfies the symmetry condition and can be considered a symmetry of a specific partial differential equation (PDE).

From the Lie symmetry data one obtains by differentiating the symmetry conditions (3.11) with respect to ϵ for $\epsilon = 0$. That is, in the end we can obtain a linearized symmetry condition $X\Delta = 0$ when $\Delta = 0$

By applying the restriction (3.10) in equation (3.14), we can eliminate the term u_σ . Next, we divide all the remaining terms based on their dependence on the derivatives u. This allows us to take a linear system of defining equations for ξ, τ and η.

It is important to note that the vector space L of all Lie point symmetry generators for a specific partial differential equation (PDE) forms a Lie algebra. However, it should be noted that this Lie algebra may not be finite-dimensional.

The aim is to find classical solutions to the following equation (3.02) in two dimensions is the following equation

To find the symmetries of the equation (3.0.2), first we solve the equation (3.0.3), which is an infinitesimal condition for the invariance of the vector field (3.0.1). We will use X [7] as a continuation. The infinitesimal symmetry condition decomposes into a large overdetermined system of linear PDEs for ξ, τ and η, known as the defining equation. The three proposals give an overdetermined system, a general form of the defining equations, and Lie algebra generators. These results were proven using the SYM package [13, 14].

Lie point symmetries are important in the study of differential equations as they provide insights into the underlying properties and structures of the equations. In the case of the p-Laplacian, a nonlinear partial differential operator, the Lie point symmetries can reveal valuable information about its behavior.

To discuss the Lie point symmetry of the p-Laplacian, let's consider a PDE of the form:

$$F(x, u, u_x, u_t) = 0$$

where F is a nonlinear function of the dependent variable u, its spatial derivatives u_x, and possibly its time derivative u_t. This equation represents a general form of a p-Laplacian equation, where p is a positive constant determining the degree of nonlinearity.

The Lie point symmetries of this p-Laplacian equation correspond to infinitesimal transformations of the independent and dependent variables that leave the equation invariant. Mathematically, this can be represented by the following infinitesimal generator:

$$\xi = \xi_x \partial_x + \xi_y \partial_u + \xi_t \partial_t$$

where ξ_x, ξ_u, and ξ_t are the transformation parameters associated with the infinitesimal generator.

To find the Lie point symmetry of the p-Laplacian equation, we need to solve the determining equations, which are given by:

$$\xi_t F_t + \xi_u F_u + \xi_x F_x + (\xi_t u_t + \xi_u u + \xi_x u_x) F_u$$

$$\xi_t F_t + \xi_u F_u + \xi_x u_x + (\xi_u u_x + \xi_t u_t) F_{ux}$$

Solving these determining equations will yield the specific form of the Lie point symmetry generator for the given p-Laplacian equation.

It's worth noting that due to the nonlinear nature of the p-Laplacian, finding explicit Lie point symmetries can be challenging in most cases. However, general techniques for finding Lie point symmetries, such as the Lie symmetry method or group classification methods, can be employed to obtain symmetries for specific p-Laplacian equations or certain classes of these equations.

In summary, the Lie point symmetries of the p-Laplacian represent transformations that leave the equation invariant. Finding these symmetries can provide valuable insights into the properties and behavior of the equation. However, the explicit determination of the Lie point symmetries for the general p-Laplacian equation can be a challenging task due to its nonlinearity.

Proposition 1 (Infinitesimal invariance)**.** *The infinitesimal invariance condition is equivalent to the following system of 19 equations:*

$$\xi_{uu} = \tau_{uu} = \xi_u = \tau_u = \eta_{xy} = \eta_x = \eta_y = 0 \tag{3.0.4}$$

$$\eta_{uu} - 2\xi_{ux} = \xi_{uy} + \tau_{ux} = \eta_{uu} - 2\tau_{uy} = \xi_{uy} + \tau_{ux} = \tau_y - \xi_x = \tau_x + \xi_y = 0 \tag{3.0.5}$$

$$(1-p)\tau_{yy} - \tau_{xx} + 2(p-1)\eta_{uy} = (1-p)\xi_{xx} - \xi_{yy} + 2(p-1)\eta_{ux} = 0 \tag{3.0.6}$$

$$(p-1)\eta_{xx} + \eta_{y,y} = (p-1)\eta_{yy} + \eta_{x,x} = 0 \tag{3.0.7}$$

$$2(p-1)\eta_{ux} - \xi_{xx} - 2(p-2)\tau_{xy} - (p-1)\xi_{yy} = 0 \tag{3.0.8}$$

$$2(p-1)\eta_{uy} - \tau_{yy} - 2(p-2)\xi_{xy} - (p-1)\tau_{xx} = 0 \tag{3.0.9}$$

Redefined systems of linear PDEs give an algebra of symmetry generators (3.0.1) of the p-Laplacian. The form of a PDE can vary depending on the specific problem being modeled. Some common types of PDEs include the diffusion equation, wave equation, and Laplace's equation. Each type of PDE has its own particular solution methods and properties.

Solving PDEs often involves finding a function that satisfies the equation subject to some boundary conditions. These conditions specify how the solution behaves at the boundaries of the domain in which the PDE is defined. There are various techniques for solving PDEs, such as separation of variables, method of characteristics, and numerical methods.

Proposition 2 (Determining equations)**.** *The determining equations can refer to different types of equations depending on the context. Provide more specific information or equations to determine the general solution. The general solution of the determining equations* (3.0.4)-(3.0.9) *is given by*

$$
\begin{aligned}
\xi &= c_3 x - c_2 y + c_1 \\
\tau &= c_2 x + c_3 y + c_4 \\
\eta &= c_6 + c_5 u
\end{aligned}
\tag{3.0.10}
$$

Proposition 3 (Lie algebra generators)**.** *It follows that the solution* (3.0.10) *defines a six dimensional Lie algebra of generators where a basis is formed by the following vector fields*

$$X_1 = \partial_x, \quad X_2 = -y\partial_x + x\partial_y, \quad X_3 = x\partial_x + y\partial_y, \quad X_4 = \partial_y, \quad X_5 = u\partial_u, \quad X_6 = \partial_u.$$

$$c_1 : X_1 = c_1\partial_x = \partial_x$$

$$c_2 : X_2 = -c_2 y\partial_x + c_2 x\partial_y + 0\partial_u = -y\partial_x + x\partial_y$$

$$c_3 : X_3 = c_3 x\partial_x + c_3 y\partial_y = x\partial_x + y\partial_y$$

$$c_4 : X_4 = \partial_y$$

$$c_5 : X_5 = u\partial_u$$

$$c_6 : X_6 = \partial_u$$

It turns out that the Lie algebra of symmetry point generators is elongated

$$X_1 = \partial_x, \quad X_2 = -y\partial_x + x\partial_y, \quad X_3 = x\partial_x + y\partial_y, \quad X_4 = \partial_y, \quad X_5 = u\partial_u, \quad X_6 = \partial_u.$$

These vector fields give rise to the fields that form the basis of Lie algebra; any vector field can be expressed as a linear combination of these basis vectors.

A vector space equipped with a binary operation called a Lie bracket is called a Lie algebra, which defines the algebraic structure of the vector space. That is, a set of vectors covering the vector space, as well as other algebra vectors, are written as a linear combination of these basis vectors.

If the vector fields in question form the basis of a Lie algebra, this means that they span the vector space of all vector fields in the algebra. This means that any other vector field in algebra can be expressed as a linear combination of these basis vectors.

3.1 Concept about partial differential equation

A partial differential equation (PDE) is a mathematical equation that describes how a physical quantity or a system evolves in space and time. Unlike ordinary differential equations (ODEs) that involve one independent variable, PDEs involve multiple independent variables.

One concept related to PDEs is the notion of solutions. A solution to a PDE is a function that satisfies the equation when substituted into it. PDEs can have various types of solutions, such as explicit solutions, implicit solutions, exact solutions, or numerical solutions obtained through approximation methods.

Another concept is the classification of PDEs. PDEs can be classified based on their order, linearity, and type. The order of a PDE refers to the highest order of derivatives involved in the equation. Linearity distinguishes between linear and nonlinear PDEs, with linear equations having superposition properties. The type of a PDE categorizes it into elliptic, parabolic, or hyperbolic, depending on the nature of the equation and its properties.

Furthermore, PDEs are often used to model physical phenomena in various scientific fields, such as physics, engineering, and finance. For example, the heat equation is a PDE that describes how heat is transferred in a physical system, while the wave equation governs the propagation of waves. PDEs can also be used to describe fluid flow, diffusion processes, electromagnetic fields, and quantum mechanics.

Solving PDEs can be a challenging task due to their complexity and dependence on initial and boundary conditions. Different techniques and methods can be used, including separation of variables, Fourier series, Laplace transforms, finite difference methods, finite element methods, and numerical simulations.

3.1.1 Partial Differential Equations Applications

Applications of partial differential equations involve solving equations that involve multiple variables and their partial derivatives. These equations are used in various branches of science and engineering to model and analyze a wide range of physical phenomena. Some common applications include analyzing heat conduction, fluid flow, electromagnetic fields, and quantum mechanics. These equations help in understanding and predicting the behavior of complex systems and provide valuable insights into many real-world problems.

Partial differential equations find applications in various scientific fields, including physics and engineering. Some specific uses of these equations include:

- For equations to describe the propagation of heat, partial differential equations are used. Below we present the form of this equation $u_{xx} = u_t$

- To describe the propagation of light and dynamics we need wave equations. The equation has the form of the second order $u_{xx} - u_{yy} = 0$.
- To build financial models we need to use an equation called the second-order Black-Scholes equation.

Important Notes on Partial Differential Equations

Derivatives with only one variable are called ODEs, while PDEs include derivatives with respect to several variables. In the case of ODEs, we can often consider only one independent variable as a time variable, so ODEs control the motion or flow of an object through time.

- An equation consisting of an unknown function of many variables and its partial derivatives is called a partial differential equation There are different types of partial differential equations. For example, quasilinear partial differential equations of the first and second order, as well as homogeneous partial differential equations.
- Parabolic, hyperbolic and elliptic are types of second order partial differential equations

  They are differential equations that contain derivatives. In ODE we have derivatives of one variable, and in PDE we have derivatives with respect to different variables. In the case of an ODE, we can often consider only the time variable; on the contrary, in an ODE we are also determined by the movement.

3.1.2 What does ODE and PDE stand for?

To find out how models are transformed, we must take an ODE that will help us cope with this task. When we need to transform this or that model, it is very important for us to know about the ODE and use it correctly.

The first type that is often found in differential equations is the equation that we know and use, that is, an ordinary first-order differential equation.

There are different types of ordinary differential equations which are separable ODEs, linear ODEs with constant coefficients and systems of ODEs. Scientists have studied types of first-order ODEs and now there is no problem for us when studying this type of equation, we know very well about their properties and applications, and they are all relatively easy and quick to solve.

By using equation reduction, we can find solutions to second-order nonlinear differential equations. ODEs with constant coefficients are simple ODEs but of a higher order; they are also linear.

We have the opportunity in these types of types of ODEs of heterogeneous and homogeneous.

In our considered ODE, all derivatives have only one variable, but PDEs, on the contrary, have variables that have one or more variables. In connection with this, we can consider our simple ODE as a more serious differential PDE equation

ODEs involve derivatives with respect to only one variable, whereas PDEs involve derivatives with respect to multiple variables. Therefore, all ODEs can be considered as PDEs.

When solving differential equations, we need to take into account all factors and use different methods to solve them. In the world where we live, we are surrounded by differential equations everywhere; it is quite difficult and sometimes impossible to find solutions. In this case, we will need to use different methods and find the correct classification of these equations. The main task after solving is to check whether the solution we found actually exists or not. In my scientific master's thesis, the main goal was to translate the PDE into a simpler form of differential equation, that is, into an ODE.

Let's look at whether there is a significant difference between ordinary differential equations and partial differential equations

Partial differential equations. Relatively simple partial differential equation: ux(x, y)=0

In general solutions there are so-called arbitrary functions; we write the general form of the equation given above as: u(x, y)=f(y) .

That is, our ordinary differential equation will be written in the form u'(x)=0. We write the general solution to the above equation in the form: u(x)=c.

The general solution includes arbitrary constants, that is, they are not changeable.

Classification of ODE Equations

This equation is called an ordinary differential equation $\frac{dx}{dy} = x$

This equation is called an partial differential equation $\frac{df}{dx} + \frac{df}{dy} = 0$

We divide ordinary differential equations into two types according to certain properties. The first is linear, the second type is called nonlinear. We call an ODE linear if it does not have products of the dependent variable and its derivatives. If there are no functions, for example, cotangent, tangent, e y, arcsin, and other types of dependent variable or its derivatives.

We know that there is some kind of symmetry, then we will find out what it should be. It turns out that using Lie symmetric analysis, we find the symmetry of this and. We will also find out u.

We have 8 cases where we ended up getting rid of x,y,u. And in the end, g and s remained, and we were able to find u^- the symmetry that we know from theory.

A partial differential equation that establishes the relationship between the various partial derivatives of a multivariate function

PDE partial differential equation how does it work? Used to describe various phenomena in science and engineering. They describe the dependence of some function on its partial derivatives, and not just on its arguments.

In equations that include various independent ones in the form of x, y, z, t and others

A function that will depend on u and on these variables are also partial derivatives of the dependent function u with respect to independent variables, in the form $F(x, y, z, t, ..., u_x, u_y, u_z, u_t, u_{xx}, u_{yy}, ..., u_{xy}, ...) = 0$ is called a partial differential equation.

If we have some kind of function with different, that is, several different variables, when solving them we need to use a partial differential equation to form a solution to the given task. In the form of dynamics, heat, liquids. We have partial differential equations (PDE).

The first is called the linear three-dimensional heat equation:

$$u_t = k(u_{xx} + u_{yy} + u_{zz})$$

Name of the second Laplace equation in three dimensions:

$$u_{xx} + u_{yy} + u_{zz} = 0$$

The third equation is a linear three-dimensional wave equation:

$$u_{tt} = c_2(u_{xx} + u_{yy} + u_{zz})$$

The fourth equation is the nonlinear one-dimensional Burger equation:

$$u_t + uu_x = \mu u_{xx}$$

Fifth equation linear one-dimensional heat equation:

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial (x)^2}$$

Let's look at different PDE examples:

In different fields like science and technology we use PDE. We consider some of the most striking and necessary examples in mathematical physics of the main three types There are three fundamental equations in mathematical physics.

The first is the well-known equation called Laplace

We often use this equation and come across it when we study applied sciences and their phenomena $\triangle u = 0$. All solutions of this type are called a harmonic function. Let us give an example of this function: the equilibrium position of an ideally elastic membrane is a harmonic function, since the velocity potential of a homogeneous and isotropic body is a harmonic function, and in this case Laplace's equation is a diffusion equation that does not depend on any time in space.

The second equation, called thermal conductivity, is a partial differential equation that describes the uniform distribution of the field that gives us heat in a given body over time. But with all this, precise knowledge of the temperature field is impossible. The so-called conductivity law or Fourier's law.

We call the third equation the wave equations. The matter that we know, can touch and see does not carry any wave. An impulse, any information, energy of various types all carry with them one or another wave that we do not see, but feel.

An equation of this kind, which takes the form of a partial differential equation, completely describes the spatial and temporal evolution of waves before we began to study them.

This wave equation is linear, which means that whatever linear combination of any two solutions is also a solution to the wave equation. To write down the most general solution, we take a linear combination of possible solutions to a given type of equation.

Chapter 4

Invariant solutions through symmetry reductions

4.1 Group-Invariant Solutions

Now knowing the different methods from chapter three, we now have the ability to find Lie symmetry points. In the fourth chapter we will describe exactly how Lie symmetry points can be used to construct different solutions that will ultimately be accurate. We associate these exact solutions with PDEs and those with the same or different ODEs. To find a first-order solution to a PDE that will be general and the solution will be based on the integration of this equation. There is one problem that in many cases of PDE it is impossible to write down a "general solution" and in such cases we have only one choice - to rely on various hypotheses. There are different ways to find a given solution such as traveling waves, separable solutions, and the like. The methods listed above give us the assumption that in order to find a solution, we need invariants with respect to a certain group of symmetries. Let's take an example for PDE u(x, t), which includes symmetry generators

Other methods involve seeking solutions that are invariant under scaling transformations, reflections, or combinations of these symmetries. These methods can be used to construct self-similar solutions, symmetric solutions about specific axes, or solutions with other geometric properties.

The construction of exact solutions using symmetries often involves reducing the original PDE to a system of ODEs. This is done by assuming a specific form for the solution and substituting it into the PDE. The resulting ODEs can then be solved using techniques known for solving ODEs.

Variable separation methods are one of the most common methods we use in PDE. These solution methods give us a hypothesis when solving PDEs can be written as a product of functions, where any function will only depend on one variable that we know. If we take a PDE, then the equation contains second-order partial derivatives with two variables in the form $u(x, y)$, the method of qualification of variables gives us the hypothesis that when solving they can be written in the form $u(x, y) = X(x)Y(y)$.

The next step is to substitute this assumed form of the solution into the PDE. After all these transformations, we end up with two ordinary equations, this $X(x)$ and $Y(y)$. These ODEs can then be solved separately using techniques known for solving ODEs, such as separation of variables, integrating factors, or various numerical methods.

We can also construct a general solution to the PDE by combining the solution to the ODE and use them further. This can be done by taking linear combinations of the solutions and incorporating any boundary or initial conditions that may be known.

We should also note that this form of solution, which we assume may not always be known to us, in such cases we need to identify various trial and error methods or we can use symmetries again. Additionally, there are cases where exact solutions cannot be obtained using these techniques and numerical methods may need to be employed instead.

Indeed, the construction of exact solutions using symmetries is a powerful technique in solving partial differential equations (PDEs). By assuming a specific form for the solution and substituting it into the PDE, we can reduce the problem to a system of ordinary differential equations (ODEs). This reduction is possible due to the symmetries present in the PDE, which allow us to take advantage of the invariance properties to simplify the equations.

Assuming a specific form for the solution involves introducing symmetry transformations to the PDE. These transformations generally depend on a set of parameters, which can be determined by matching coefficients or boundary conditions. By substituting the assumed solution into the PDE, we obtain a system of ODEs involving the unknown function(s) and the parameters. The original PDE is then replaced by this reduced system of ODEs.

Solving the resulting system of ODEs can be done using well-established methods and techniques for solving ODEs. These methods include separation of variables, integrating factors, variation of parameters, or using specialized techniques such as the power series method, Frobenius method, or Laplace transforms.

The solution of the ODE system provides the exact solution of the original PDE, satisfying the given symmetry assumptions. This approach is particularly useful when exact solutions for a particular PDE are not readily available or when studying the behavior of the PDE under certain conditions.

It is worth noting that while the method of symmetry reductions is a powerful tool, it may not always be possible to find exact solutions using this approach, especially for complex or highly nonlinear PDEs. In such cases, numerical methods or other approximation techniques may be required.

Overall, the construction of exact solutions using symmetries and reducing the PDE to a system of ODEs is an effective and widely used approach in the study of PDEs. It allows us to exploit the underlying symmetries and simplify the equations, enabling the use of well-established techniques for solving ODEs and obtaining exact solutions.

You also need to know that with exact solutions that were obtained using symmetry methods, we cannot say for sure that for any solutions there may be parameter values or initial conditions for everyone. They represent special cases that satisfy specific symmetry conditions. In any case, we receive information that helps us determine how a given PDE changes, and can help us to approximate a general solution.

In summary, the use of symmetries allows us to construct exact solutions for PDEs by relating them to one or more ODEs. Various methods involve seeking solutions that are invariant under specific groups of symmetries, such as translations, rotations, scalings, and reflections. These methods can generate solutions with desired properties, such as travelling waves, symmetric or self-similar solutions, among others.

Constructing exact solutions using symmetry methods usually involves reducing the original PDE to a system of ODEs, which we can solve using methods that have already been studied.

$$X_1 = \partial_x,$$

$$X_2 = \partial_t$$

traveling wave type in many cases

$$u = F(x - ct) \text{ (4.1)}$$

These solutions are invariant for groups

$$X = cX_1 + X_2 = c\partial_x + \partial_t \text{ (4.2)}$$

u, and $x - ct$ they are invariant. It is the same as PDE with scale symmetry

We can generalize the data that is presented with PDEs that have any Lie symmetry groups.

$$\nabla u = 0(4.3)$$

For scalar PDEs that have two independent variables. This solution will be $u = u(x, t)$ generated by a formula that will be invariant under this group

$$X = \xi\partial_x + \tau\partial_t + \eta\partial_u$$

this will be under the condition that the characteristic will vanish at this solution. In each invariant solution, the condition must be satisfied, which will be the invariant surface that we used

$$Q \equiv \eta - \xi u_x - \tau u_t = 0 \text{ (4.4)}$$

By solving this (4.4), we detect out what solutions (4.3) can be. Giving an example about a group that is generated by (4.2), will have the given initial

$$Q = -cu_x - u_t. \text{ (4.5)}$$

The traveling wave ansatz (4.1) is a general solution to the given surface condition of the invariant Q $= 0$. First, we can identify the hypothesis that for ξ and r will not be equal to zero at the same time. Under such conditions that we have set, the invariant surface will represent a quasi-linear PDE of the first order, which we can solve using the method of characteristics. Below are the equation data:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\tau} \ (4.6)$$

$r(x, t, u)$ and $v(x, t, u)$ are two functionally independent first integrals (4.6), for each invariant group the function will be r and v. By allowing one invariant to be the dependent variable. $v_u \neq 0$ if written in this form, then our general solution to the invariant surface condition will be written as

$$v = F(r) \ (4.7)$$

We substitute the given solution of these PDEs into (4.6) to find the function in the form F. With r and v depending on u, we will need to clarify whether this PDE will have a solution of any kind at all

$$r = c \ (4.8)$$

We obtain one solution to the invariant surface condition, which does not have the form (4.7). If r is a function of only independent variables x and t, then (4.8) cannot give the solution $u = u(x, t)$.

Example 4.2 The examples that were above, there we were not looking for invariant scaling solutions that are generated

$$X = u\partial_u$$

Where ξ and τ they are equal to zero, but the invariant surface condition has a solution, namely

$$y = 0$$

Although travelling waves and scale-invariant (similarity) solutions will be familiar to most readers, there is nothing that sets these transformations apart from other symmetries. For any group, no matter what they are, to find group-invariant solutions we use the same method, no matter what group it is.

When solving differential equations, many different methods and methods are used. But with all this, there is a high probability that they will work only for one or more than two classes of tasks. Here the question arises: will we be able to solve types of differential equations that we have not seen before and know nothing about.

To solve an equation of an unfamiliar form, it is easier for us to first find its symmetry and apply it to find a solution that will be exact. Finding the symmetry of differential equations is much easier. Symmetry can also be used to solve problems.The effectiveness and importance of symmetry that I used in my scientific work is undeniable. Since it is precisely this that gives us the simplification and solution of nonlinear equations. All these solutions have many applications in various areas of our daily life.

4.2 Invariant solutions through symmetry reductions

Let us formulate our solutions when the result of reducing p-Laplace to ODE symmetry arises thanks to the algebra generators given in Proposition 3. We consider each generator seperately and examine some examples of solutions from each generator.

Invariant solutions refer to solutions of a mathematical problem that remain unchanged under a certain transformation or symmetry. In the context of partial differential equations (PDEs), symmetry reductions are used to simplify and classify solutions by reducing the number of independent variables.

Symmetry reductions are often applied to PDEs that possess certain symmetries, such as translational, rotational, or scaling symmetries. These symmetries allow for the reduction of the number of independent variables, which can lead to a simpler form of the PDE and potentially facilitate solving or analyzing the problem.

Through symmetry reductions, invariant solutions can be obtained. These are solutions that remain unchanged under the specific symmetry transformation applied. Invariant solutions are useful because they can be used as building blocks to construct more general solutions or provide insight into the behavior of the system.

Invariant solutions can be obtained through symmetry reductions, which involve finding transformations that leave the equations invariant. These transformations typically involve changes in coordinates or variables.

Symmetry reductions are a powerful technique for simplifying complex systems of equations. By identifying symmetries and reducing the number of independent variables, it is often possible to find simplified equations that are easier to solve or analyze.

To obtain invariant solutions through symmetry reductions, the following steps can be taken:

1. Identify the symmetries: Determine the transformation that leaves the equations unchanged. These symmetries can be translations, rotations, scaling, or other transformations.

2. Apply the symmetry reduction: Apply the identified transformation to the equations to obtain a reduced set of equations with fewer independent variables.

3. Solve the reduced equations: Solve the reduced equations to obtain solutions that are invariant under the identified symmetry.

4. Back-transform the solutions: Once the solutions of the reduced equations are obtained, they can be back-transformed to obtain solutions of the original equations.

By identifying and utilizing symmetries, invariant solutions can often be found more easily. This is particularly useful in physical problems where symmetries play a crucial role, such as in conservation laws or invariance principles.

The process of obtaining invariant solutions through symmetry reductions involves finding the appropriate coordinate transformation that preserves the symmetry of the problem. This transformation is then applied to the original PDE, resulting in a reduced PDE with fewer independent variables. Solving this reduced PDE can lead to invariant solutions specific to the given symmetry.

Overall, symmetry reductions play a crucial role in simplifying and understanding the solutions of PDEs by taking advantage of the inherent symmetries of the problem. Invariant solutions obtained through symmetry reductions provide valuable insights into the behavior of the system and can be used as a starting point for further analysis or more general solution constructions.

Case 1. Solutions of (3.0.2) they are invariant under the ratio of the given generated symmetryby $X_1 = \partial_x$ are of the form $u = g(y)$. From these data we find out that g is a solution to the trivial ODE that we know $g'(y)^2 g''(y) = 0$ and thus it follows that $u = c_1 y + c_2$.

Given the equation (3.0.2) can allow the variables x and y to be interchanged, and contains second order derivatives, we can check that the linear polynomial in x and y, $u(x, y) = c_1 x + c_2 y + c_3$ is also a solution.

Case 2.

$$X = \xi\partial_x + \tau\partial_y + \eta\partial_u$$

is a vector field that will give "symmetry" to find the solution p-Laplacian.

$$\frac{dx}{\xi} = \frac{dy}{\tau} = \frac{du}{\eta}$$

$$\frac{dx}{-y} = \frac{dy}{x}$$

Instead of ξ I put $-y$, and instead of τ I put x. Using the proportion, x was multiplied by dx, and $-y$ was multiplied by du. We take both sides into the integral and replace the constant with g(s).

$$xdx = -ydy$$

We took the integral after replacing ξ and τ, x and $-y$

$$\frac{x^2}{2} = \frac{-y^2}{2} + C$$

$$x^2 + y^2 = C$$

Equated $x^2 + y^2$ to a constant C.

$$u(x, y) = g(C) = g(x^2 + y^2)$$

In our case of the generator $X_2 = -y\partial_x + x\partial_y$ we got rotationally invariant solutions of the form $u(x, y) = g(s)$, where $s = x^2 + y^2$. Omitting the argument g

for simplicity, we can find the derivatives of u as follows:

$$u_x = 2xg_s \quad u_y = 2yg_s \quad u_{xx} = 4x^2 g_{ss} + 2g_s \quad u_{xy} = 4xyg_{ss} \quad u_{yy} = 4y^2 g_{ss} + 2g_s$$

and replace these derivatives into equation (3.0.2) to obtain,

$$(p-2)\Big(4x^2 g_s^2 (4x^2 g_{ss} + 2g_s) + x^2 y^2 g_{ss}^2 g_{ss} + 4y^2 g_s^2 (4y^2 g_{ss} + 2g_{ss})\Big) + \\ + (4x^2 g_s^2 + 4y^2 g_s^2)(4x^2 g_{ss} + 2g_s + 4y^2 g_{ss} + 2g_s) = 0.$$

After simplifying above expression we get

$$8sg_s^2\Big((p-2)(2g_{ss}s + g_s) + 2sg_{ss} + 2g_s\Big) = 0. \tag{4.2.1}$$

That the polynomial can be expressed as a product of linear factors.

$$u(x,y) = g(x^2 + y^2) = (x^2 + y^2)^{\frac{p-2}{2p-2}}$$

and solution u is linear corresponds to the first factor of (4.2.1).

Case 3.

The quantities u and $s = xy^{-1}$ are algebraic invariants of the Lie group, which came from the formula $X_3 = x\partial_x + y\partial_y$.

The hypothesis is that $u(x,y) = g(s)$, and we will get this differential equation for g, as in the previous case. Partial derivatives of u:

$$u_x = \frac{1}{y} g_s, u_y \quad = -\frac{x}{y^2} g_s, u_y = -\frac{x}{y^2} g_s, u_{yy} \quad = \frac{x}{3y^3} g_{ss}, u_{xy} = \frac{1}{y} g_{ss} - \frac{1}{y^2} g_s, u_{xx} \quad = \frac{1}{y^2} g_{ss}$$

and substituting these derivatives into equation (3.0.2) we obtain,

$$(p-2)\big(g_s^2 g_{ss} + 2s^3 g_s^2 g_{ss} + 2g_s^3 s^2\big) + 2g_s^2 g_{ss} s^4 + 4g_s^3 s^3 + 2g_{ss} g_s^2 s^2 + g_s^2 g_{ss} + 2g_s^3 s = 0.$$

The general solution invariant under the symmetry generated by X_3 is of form

$$u(x,y) = g(s) = \int e^{-\int f(s)ds} ds,$$

where function f is given by following formula

$$f(s) = \frac{2s - 4s^2 + 2ps^2 + 4s^3}{-1 + p + 2s^2 - 4s^3 + 2s^3 p + 2s^4}.$$

Case 4. $\alpha X_1 + X_5 = \alpha\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$, the most general form of the invariant solution is equal to $u(x,y) = e^{\frac{x}{\alpha}}F(y)$. Hence the partial derivatives are equal to

$$u_x = \frac{1}{\alpha}e^{\frac{x}{\alpha}}F(y) \qquad u_y = e^{\frac{x}{\alpha}}F'(y) \quad u_{xx} = \frac{1}{\alpha^2}e^{\frac{x}{\alpha}}F(y)$$
$$u_{xy} = \frac{1}{\alpha}e^{\frac{x}{\alpha}}F'(y) \qquad u_{yy} = e^{\frac{x}{\alpha}}F''(y)$$

and reduced equation of (3.0.2) becomes

$$(p-2)(F^3(y)+2\alpha^2F(y)F'^2(y)+\alpha^4F'^2(y)F''(y))+(\alpha^2F^2(y)+\alpha^4F'(y)^2)(\alpha^2F(y)+\alpha^4F''(y)) = 0$$

The solution of above equation is $F(y) = e^{\int f(y)dy}$ where f is given implicitly by the following expression

$$y = -\int \frac{(p-2)\alpha^4f^2 + \alpha^6(1+\alpha^2f^2)}{(p-2+\alpha^4)(1+\alpha^2f^2)^2}df.$$

Case 5. In the case of the generator $\alpha X_2 + X_5 = \alpha(-y\partial_x + x\partial_y) + u\partial_u$ the invariants are $s = x^2+y^2$ and $r = \arctan\left(\frac{y}{x}\right) + \alpha \ln u$. The most general solution invariant under the symmetry generated by $\alpha X_2 + X_5$ is of the form

$$u = e^{\frac{g(x^2+y^2)}{\alpha} - \frac{1}{\alpha}\arctan\left(\frac{y}{x}\right)}.$$

This abbreviated ODE that we obtained for $g(s)$ is too complex to solve or write in any form.

Case 6. For the generator $X_3 + \alpha X_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \alpha u\frac{\partial}{\partial u}$ we have $s = xy^{-1}$ and $r = ux^{-\alpha}$, hence $u(x,y) = x^\alpha g(s)$ and

$$\begin{aligned}
u_x &= x^{\alpha-1}(\alpha g + sg_s)\\
u_y &= -x^{\alpha-1}s^2g_s\\
u_{xx} &= x^{\alpha-2}(\alpha(\alpha-1)g + 2\alpha sg_s + s^2g_{ss})\\
u_{xy} &= -x^{\alpha-2}((1+\alpha)s^2g_s + s^3g_{ss})\\
u_{yy} &= x^{\alpha-2}(2s^3g_s + s^4g_{ss}).
\end{aligned}$$

Thus the reduced differential equation is equal to

$$\begin{aligned}
&(p-2)\Big((\alpha g + sg_s)^2(\alpha(\alpha-1)g + 2\alpha sg_s + s^2g_{ss}) + 2(\alpha g + sg_s))s^2g_s((1+\alpha)s^2g_s + s^3g_{ss})\\
&+ s^4g_s^2(2s^3g_s + s^4g_{ss})\Big) + ((\alpha g + sg_s)^2 + s^4g_s^2)(\alpha(\alpha-1)g + 2\alpha sg_s + s^2g_{ss} + 2s^3g_s + s^4g_{ss}) = 0,
\end{aligned}$$

which is too hard to handle.

Case 7. For the generator $\alpha X_1 + \beta X_2 + X_3 = (\alpha - \beta y + x)\partial_x + (\beta x + y)\partial_y$ we have to consider the two subcases $\alpha = 0$ and $\alpha \neq 0$. When $\alpha = 0$ the generator is

$\beta X_2 + X_3$ with $\beta \in \mathbb{R}\ \{0\}$ and we can verify that u and

$$s = \arctan\left(\beta\frac{x}{y}\right) - \frac{\beta}{2}\ln(x^2 + y^2)$$

they are invariant under the corresponding Lie group. The ansatz $u = g(s)$ leads to an ODE that is too difficult to include.

Similarly, when $\alpha \neq 0$ the invariants are u and

$$s = \arctan\left(\frac{\beta x + y}{x - \beta y + \alpha}\right) - \frac{\beta}{2}\ln\left((\beta x + y)^2 + (x - \beta y + \alpha)^2\right)$$

The ansatz $u = g(s)$ leads to an ODE, its inclusion here is very difficult.

Case 8. Unfortunately reductions for generator $\alpha X_1 + \beta X_2 + \gamma X_3 + X_5 = \alpha\partial_x + \beta(-y\partial_x + x\partial_y) + \gamma(x\partial_x + y\partial_y) + u\partial_u$ are too difficult to handle.

In difficult situations, it is important to recognize that challenges are a natural part of life and can provide opportunities for growth and learning.

We have the opportunity to break down the solution to these problems into small manageable tasks. This will help us reduce the feeling of depression and give us the opportunity to take action further.

Difficult situations can present challenges and require careful consideration and problem-solving skills. Some situations may feel overwhelming or unmanageable at first, but with the right approach and mindset, they can be handled effectively. It may be helpful to break down the situation into smaller, more manageable tasks and seek support or advice from others. Maintaining a positive attitude, practicing self-care, and staying resilient can also aid in managing difficult situations.

Chapter 5

Conclusions and future work

5.1 Conclusion

Our main goal of this paper has been to find a new solutions of the p-Laplacian in two dimensions by using the Lie point symmetries. Of course some of the found solutions are already known but to the best knowledge of the authors some solutions are new as well.

It is very interesting to see how these solutions are changing as $p \to \infty$ and more importantly if the limiting solutions are solutions of the ∞-Laplacian [15]. Also if the explicit ∞-harmonic solutions (see for example [16, 17]) can be approximated as a p-harmonic solutions as $p \to \infty$.

The task of the scientific work was to find new solutions of the p-Laplacian in two dimensions using the symmetries of the Lie point. I agree that some of the solutions found are already known, but, as far as the authors know, some solutions are also new.

It is very interesting to see how these solutions are changing as $p \to \infty$ and more importantly if the limiting solutions are solutions of the ∞-Laplacian.

It is very useful to study the properties and behavior of these solutions under different boundary conditions or in the presence of some external forces. This could help reveal a deeper understanding of Laplace's p-equation and its applications in various fields such as physics and engineering.

Furthermore, exploring the stability and regularity of these solutions would be crucial in determining their practicality and reliability in real-world scenarios. This would involve studying the behavior of the solutions under small perturbations and analyzing the convergence of the solutions as the parameters of the equation vary.

I would like to look at research into higher dimensions and study the existence and uniqueness of solutions to Laplace's p-equation in three or more dimensions. Then there would be a complete understanding of the equation in spaces of different dimensions and would potentially reveal new phenomena or patterns in the behavior of solutions.

Overall, the findings and insights from this paper provide a solid foundation for future research in the field of p-Laplacian equations and open up numerous avenues for further exploration and discoveries.

This symmetric analysis in L^p space can be useful in various applications where the symmetry properties are important, such as partial differential equations with symmetric boundary conditions, harmonic analysis on symmetric domains, etc.

In addition to finding new solutions of the p-Laplacian in two dimensions using Lie point symmetries, our main goal in this paper is to investigate how these solutions change as p approaches infinity. We are particularly interested in whether the limiting solutions are solutions of the ∞-Laplacian.

While some of the solutions we have found are already known, we believe that we have also discovered new solutions. These findings contribute to the existing knowledge in this field.

However, there are still many unanswered questions and avenues for future research. Exploring how the solutions evolve and behave as p tends to infinity is a fascinating and complex topic. It is unclear whether the limiting solutions will exhibit similar behavior to solutions of the ∞-Laplacian. Investigating this relationship and understanding the implications of the limiting solutions is an important area of future work.

In conclusion, our paper aims to contribute new solutions to the p-Laplacian in two dimensions using Lie point symmetries. The investigation of how these solutions change as p approaches infinity, and whether they are solutions of the ∞-Laplacian, will be the focus of future research in this field.

Additionally, studying the properties of solutions of the ∞-Laplacian itself and how they differ from solutions of the p-Laplacian for finite p values is another interesting avenue of research.

Furthermore, investigating the stability of these solutions and their sensitivity to initial conditions is another important aspect that can be explored. Understanding how small perturbations in the initial conditions affect the behavior and properties of the solutions as p approaches infinity can provide valuable insights into the overall dynamics of these equations.

Moreover, considering higher dimensions and extending the analysis to the p-Laplacian in three or more dimensions can offer new challenges and opportunities for exploration. The behavior of solutions in higher dimensions may differ significantly from the two-dimensional case, and understanding these differences can expand our understanding of the p-Laplacian equation.

Overall, our paper provides a starting point for further investigations into the behavior of solutions of the p-Laplacian as p tends to infinity. By uncovering new solutions and studying their properties, we contribute to the existing knowledge in this field and open up new avenues for future research.

By investigating the Lie point symmetries of the p-Laplacian in two dimensions, our main objective in this paper has been to uncover novel solutions. While some of these solutions are already established, we have also identified previously unknown solutions.

A particularly intriguing aspect of our findings is observing how these solutions evolve as p approaches infinity. Furthermore, it is crucial to determine if these limiting solutions are indeed solutions of the ∞ p-Laplacian.

In our future work, we aim to extend our analysis to the p-Laplacian in three or more dimensions. Investigating the behavior of solutions in higher dimensions will provide new challenges and opportunities for exploration. We expect that the solutions in higher dimensions may differ significantly from the two-dimensional case, and understanding these differences will expand our overall understanding of the p-Laplacian equation.

Extending the analysis to the p-Laplacian equation in three or more dimensions is indeed an exciting prospect. The p-Laplacian equation is a nonlinear generalization of the Laplace equation, and it exhibits rich and complex behavior in higher dimensions.

Investigating of the behavior of solutions in higher dimensions and information about the influence of dimension on the properties of solutions.

In two dimensions, there are certain features and structures that may arise, such as the formation of singularities, blow-up phenomena, or the existence of special solution patterns. Understanding how these features change or evolve in higher dimensions can give us a more comprehensive understanding of the p-Laplacian equation.

One particular challenge in higher dimensions is the increased complexity of the mathematical analysis. The dimensionality of the problem introduces new intricacies and difficulties in solving the equation analytically. It may require the development of new techniques or the adaptation of existing methods to handle the higher-dimensional case.

Moreover, exploring the behavior of solutions in higher dimensions can lead to the discovery of new phenomena or patterns that are not present in lower dimensions. This can open up new avenues for research and provide opportunities for further exploration and investigation.

Overall, extending the analysis to the p-Laplacian in three or more dimensions is a valuable endeavor that will deepen our understanding of the equation and its properties. It will allow us to explore new challenges, uncover new phenomena, and contribute to the broader field of nonlinear partial differential equations.

One specific aspect we will explore is the behavior of solutions as p tends to infinity. By uncovering new solutions and studying their properties, we will contribute to the existing knowledge in this field and open up new avenues for future research. In particular, we will investigate the existence and properties of limiting solutions as p approaches infinity for both established and previously unknown solutions.

Additionally, we will continue investigating the Lie point symmetries of the p-Laplacian in higher dimensions. By finding and analyzing these symmetries, we can gain further insights into the equation and its solutions.

Finally, we will address the question of whether the limiting solutions identified in our study are solutions of the infinity-Laplacian. Understanding the relationship between the p-Laplacian and the infinity-Laplacian in the limiting case of p approaching infinity is an intriguing aspect that requires further investigation. Overall, our paper serves as a starting point for future investigations into the behavior of solutions of the p-Laplacian, particularly as p tends to infinity. We believe that the questions raised in our study open up new and intriguing research directions, and we look forward to further exploring these topics in our future work.

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