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About kernel of inverse operator L^{-1} to third-order derivative's $Ly \equiv -y''' + q(x)y$ operator

In this article we studied Kolmogorov's crosses of set $M = \{y : \|Ly\|_2 + \|y\|_2 \le 2\}$

That relates to the domain of the third-ordered differential operator ,and the kernels of operators that lie in the basis of properties s-numbers

This work proves kernel of inverse operator L^{-1} to operator $Ly \equiv -y''' + q(x)y$ using the problem

$$Ly(x) \equiv -y''' + q(x)y(x) = f(x) \in L_2[-\pi;\pi] \qquad (1)$$

$$y^{(i)}(-\pi) = y^{(i)}(\pi), \quad i = 0,1,2. \qquad (2)$$

Before showing final results we look for helpful definitions and facts needed for proof.

Definition 1. If for $\forall \{\varphi_i\} \subset H$ there exists an inner sequence in $\{L\varphi_i\}$, then operator L is called partially continuous.

Definition 2. If for function $\forall f \in H$ condition $< Lf, f > \ge 0$ holds, then limited linear operator L is called nonnegative, and shown as $L \ge 0$.

Suppose that L is any limited operator, then $L^* L \ge 0$. Because: $< L^* Lf, f > = < Lf, Lf > = \|Lf\|^2 \ge 0.$

We define the operator $|L|$ for any partially continuous linear operator L as written below:

$$|L| = \sqrt{L^* L}.$$

Definition 3. Local values of operator $|L|$ are called s-numbers of operator L (Local values by Schmidt).

Let's number s-numbers' coefficients different from zero by ascending order. as $s_i(L) = \lambda_i(|L|), i = 1,2,3,\ldots$ where s-numbers are positive and real numbers.

Definition 4. If compact operator L satisfies the following condition

$$\sum_i s_i(L) < \infty,$$

then operator L is called kernel operator.

Assume that H_1, H_2 are Banakh spaces and space H_1 is inner of space H_2 and set F is unit sphere in space H_1.

Definition 5. For set F taken with metric of space H_1 of dimension $m \le k$ $M_m \subset H_2$ the minimal value of linear mappings deviation is defined by $d_k(H_1, H_2) = \inf_{m \le k} \delta(F, M_m)(k \ge 1)$, where H_2 includes H_1 is called Kolmogorov's crosses, by definition

$$d_k = \inf_{m \le k} \sup_{f \in F} \inf_{q \in M} \|f - q\|_{H_2}$$

here $\{M_k\}$ is the set of possible linear varieties.

$d_0(H_1, H_2) = \|E\|$, where E is embedding operator.

From definition we directly get the following properties of crosses:

1. $d_0 \ge d_1 \ge \ldots$
2. $d_k(D) \le d_k(C), D \subset C$
3. $d_k(nM) = nd_k(M), n > 0$ where $nM = \{x = nx', x' \in M\}$

Similarly to the second definition of Kolmogorov's crosses the following theorem will hold.

Theorem 1. Assume that L is any partially continuous operator, then number $s_{k+1}(L)$ is equal to k-th Kolmogorov's cross in set $M = LS$, so $s_{k+1}(L) = d_k$, where $S = \{x \in H, \|x\| \leq 1\}$ is unit sphere.

In most cases the last definition of Kolmogorov crosses is more useful than the first one.

Let's add the following function:

$N(\lambda) = \sum_{d_k > \lambda} 1$ for all values larger than λ.

This function is called cross spreading function or countable function. There exist following connection between k-th cross and cross spreading function $d_k = \inf\{\lambda > 0 : N(\lambda) \leq k\}, \forall k > 0$.

The last equation shows that we can evaluate crosses' by evaluating cross spreading function.

Previously we showed that for operator $Ly \equiv -y''' + q(x)y$ there exists limited operator L^{-1}. Let's show main result.

Theorem 2. Assume that $q(x) \geq 1$ is continuous function in range $[-\pi;\pi]$, then L^{-1} is kernel operator.

To prove this theorem we use the following conclusions.

Lemma 1. Suppose that theorem 2 is true, then the inequality below is true

$$\|-y'''\|_2 + \|q(x)y\|_2 \leq C\|Ly\|_2 \qquad (3)$$

Proof: We know that $q(x)$ is continuous and

$$\|Ly\|_2 \geq \|y\|_2$$

From this we get:

$$\|q(x)y\|_2 \leq \max_{x \in [-\pi;\pi]} q(x)\|y\|_2 = C_0\|y\|_2 \leq C_0\|Ly\|_2$$

And also from triangle inequality we get

$$-\|y'''\|_2 = \|Ly - q(x)y\|_2 \leq \|Ly\|_2 + \|q(x)y\|_2 \leq C_1\|Ly\|_2$$

The sum of last inequalities gives usinequality (3.3.3). *Lemma is proved.*

The solution of (1)-(2) lies in space $W_2^3[-\pi;\pi]$. It means that we can acquire properties of eigen values by Schmidt's method from evaluating Kolmogorov's crosses in this space.

Let's strengthen inequality (3). Then:

$$\|-y'''\|_2 + \|q(x)y\|_2 \leq C(\|Ly\|_2 + \|y\|_2) \qquad (4)$$

We will use it below.

Let's look at the following sets:

$$M_2 = \{y \in L_2 : \|Ly\|_2 + \|y\|_2 \leq 2\}$$

$$\overline{M} = \{y \in L_2 : \|-y'''\|_2 + \|q(x)y\|_2 \leq 1\}$$

$$\overline{M}_{2C} = \{y \in L_2 : \|-y'''\|_2 + \|q(x)y\|_2 \leq 2C\}$$

Let's denote this sets' Kolmogorov's crosses as $d_k, \overline{d_k}, q_k$. Using the above mentioned properties of crosses we can easily prove that $q_k = 2C\overline{d_k}$.

Lemma 2. Assume that theorem2 holds, thenfor crosses the following is true:

$$\overline{d_k} \leq d_k \leq 2Cd_k \qquad (5)$$

Proof: Suppose that $y \in M_2$ then from inequality 3.3.4

$$\|-y'''\|_2+\|q(x)y\|_2\le C(\|Ly\|_2+\|y\|_2)\le 2C.$$

So $M_2\subseteq \overline{M}_{2C}$. And also suppose that $y\in\overline{M}$, then

$$\|Ly\|_2+\|y\|_2\le\|-y'''\|_2+2\|q(x)y\|_2\le 2(\|-y'''\|_2+\|q(x)y\|_2)\le 2.$$

So, $\overline{M}\subseteq M_2\subseteq\overline{M}_{2C}$.

Further proof of lemma is proved by properties of crosses. *Lemma is proved.*

Let's take two-side values of cross as spreading function of set $\overline{M}$. This result is known and can be written as followed.

Lemma3. Assume that theorem is true, then for $N(\lambda)$ following is true:

$$C^{-1}\lambda^{-\frac{1}{3}}\le N(\lambda)\le C\lambda^{-\frac{1}{3}};$$

For cross of inequality the following is true

$$C^{-1}i^{-3}\le\overline{d}_i\le Ci^{-3} \qquad (6)$$

Proof of theorem: Considering $s_{i+1}=d_i$ and the properties of crosses

$$\sum_{i=0}^{\infty}S_{i+1}(L^{-1})=\sum_{i=1}^{\infty}d_i\le 2C\sum_{i=1}^{\infty}\overline{d}_i \qquad (7)$$

And by applying 7 to ratio 6 we get

$$\sum_{i=0}^{\infty}S_{i+1}(L^{-1})\le 2C\sum_{i=1}^{\infty}\overline{d}_i\le C\sum_{i=1}^{\infty}\frac{1}{i^3}<\infty.$$

It defines the kernel of operator L^{-1}. Theorem is proved.

USED MATERIALS

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Түйіндеме

Бұл мақалада үшінші ретті дифференциалдық оператордың Колмогоров бойынша M жиынының екі жақты бағалаулары және S-сандарының қасиеті қарастырылды.

Rezume

In this article we studied Kolmogorov's crosses of set $M=\{y:\|Ly\|_2+\|y\|_2\le 2\}$ *that relates to the domain of the third-ordered differential operator , and the kernels of operators that lie in the basis of properties s-numbers.*

Özet

Makale S-numaraları ve Kolmogorov set M, üçüncü dereceden bir diferansiyel operatörün iki taraflı tahminlerinin özelliklerini tartışır.