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Gröbner-Shirshov bases theory for Zinbiel superalgebras

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SULEYMAN DEMIREL
UNIVERSITY

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Gröbner-Shirshov bases theory for Zinbiel superalgebras

THESIS

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Department of Mathematics and Natural Sciences

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Supervisor: **Bekzat Zhakhayev, PhD**

Kaskelen,2023

Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Meiirbek Kuanysh

Dedication

This thesis is dedicated to:

My students Borte and Aknur.

Abstract

This thesis is a collection of 6 chapters .The Gröbner-Shirshov basis is an important mathematical apparatus in algebra and commutative algebra, which is used to study and analyze polynomials and their ideals. The Gröbner-Shirshov basis has a number of important properties that make it a powerful tool for solving various algebraic problems, such as searching for ideals, solving systems of equations and determining the basic invariants of polynomials. In this paper we will construct a Gröbner-Shirshov basis for Zinbiel algebras. Algebra with the identity $(ab)\,c = a\,(bc) + a\,(cb)$ is called the Zinbiel algebra. In the process of constructing the Grebner-Shirshov basis, two compositions are found and the composition lemma is proved. The method of mathematical induction is used to prove the lemma.

Аннотация

Диссертация состоит из 6 глав.Базис Грёбнера-Ширшова является важным математическим аппаратом в алгебре и коммутативной алгебре, который используется для изучения и анализа многочленов и их идеалов. Базис Грёбнера-Ширшова обладает рядом важных свойств, которые делают его мощным инструментом при решении различных алгебраических задач, таких как поиск идеалов, решение систем уравнений и определение основных инвариантов многочленов. В этой работе мы построим базис Грёбнера-Ширшова для алгебр Цинбиеля. Алгебра с тождеством $(ab)\,c = a\,(bc) + a\,(cb)$ называется алгеброй Цинбиеля. В процессе построения базиса Гребнёра-Ширшова найдены две композиции и доказана лемма о композиции. Для доказательства леммы используется метод математической индукции.

Аңдатпа

Диссертиция 6 тараудан тұрады. Грёбнер-Ширшов базисі алгебра мен коммутативті алгебрадағы маңызды математикалық аппарат болып табылады, ол көп-мүшелер мен олардың идеалдарын зерттеу және талдау үшін қолданылады. Грёбнер-Ширшов базисі әртүрлі алгебралық есептерді шешуде оны қуатты құралға айналдыратын бірқатар маңызды қасиеттерге ие, мысалы, идеалдарды табу, теңдеулер жүйесін шешу және көпмүшелердің негізгі инварианттарын анықтау. Бұл жұмыста біз Цинбиел алгебралары үшін Грёбнер-Ширшов базисінің теориясын құрастырамыз. $(ab)\,c = a\,(bc) + a\,(cb)$сәйкестігі бар Алгебра Цинбиел алгебрасы деп аталады. Грёбнер-Ширшов базисін құру барысында екі композиция табылды және композиция туралы лемма дәлелденді. Лемманы дәлелдеу үшін математикалық индукция әдісі қолданылады.

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Chapter 1

Background and motivations

1.1 Introduction

For the first time, The Theory of the Gröbner-Shirshov basis was developed and applied by A. I. Shirshov [5] (for Lie algebra) and B. Bukhberger [11] (for commutative-associative algebra) in their dissections. Shirshov's work was based on the destruction of the leaders of the Lindon-Shirshov speeches. One of the main advantages of Gröbner-Shirshov basis theory is that it provides a method for converting algebraic geometry problems into algebraic problems. This allows you to convert geometric problems into purely algebraic calculations, which can often be solved algorithmically. Gröbner Shirshov bases method for a class of algebras is based on a Composition-Diamond lemma.Up to now, different versions of Composition-Diamond lemma are known for the following classes of algebras apart those mentioned above: Lie superalgebras [18, 19, 20], tensor product of a free algebra and a polynomial algebra [21], tensor product of two free algebras [8], associative conformal algebras [6], operads [16], modules [14, 15, 17], pre-Lie(right-symmetric) algebras [3], dialgebras [9], Rota-Baxter algebras [7], Lie algebras over a polynomial algebra [10], Gelfand-Dorfman-Novikov (GDN)-algebras [4]. Gröbner bases theory for commutative algebras has been proved to be very useful in different branches of mathematics, including algebraic geometry, see, for example, the books [1, 2, 12] Gröbner bases theory for noncommutative and nonassociative algebras s a powerful tool to solve the following classical problems: normal form, word problem,conjugacy problem; rewritting system; automaton; embedding theorem; expand Poincaré-Birkhoff-Witt theorem; extension; homology; growth function, Dehn function, complexity.

Chapter 2

PRELIMINARIES

2.1 Free Monoid

Definition 2.1. The set S and the binary operation * defined in this set are called monids if they satisfy the following axioms $\forall x, y, z \in M$:

1. $(x * y) * z = x * (y * z)$
2. $.\exists e \in M\,[e * x = x * e = x]$

Let M be a Monoid $X \subset M$. If any element $a \in S$ is expressed by any element $t \geq 0, x_{i1}, \ldots x_{i1} \in X$ with elements $a = x_{i1} \ldots x_{it}$ (we get $t = 0 \Rightarrow a = e$), we call X the generator of S.
Let given two monoids $< s, *_S, e_S >$ and $< T, *_T, e_T >$ be . $\psi : s \rightarrow t$ a mapping is called a homomorphism if for $a, b \in S$:

$$\psi\,(a *_S b) = \psi\,(a) *_T \psi\,(b)$$

and

$$\psi\,(e_S) = e_T$$

Definition 2.2. Let X be a non - empty set. Monoid $A(X)$, which generates a set X, we call the monoid $A(X)$ a free monoid if any $\alpha : X \rightarrow s$ mapping we find a homomorphism ψ, which performs $\psi : A\,(X) \rightarrow s, \forall x \in X, \psi\,(x) = \alpha\,(X)$.

This homomorphism is defined on generators in a unique way, so ψ is unique. Let 's construct a free monoid.Consider the set X^* with the concatenation operation. That is for $u = x_1 \ldots x_n, v = y_1 \ldots y_m, x_i, y_i \in X$ there $u * v = uv = x_1 \ldots x_n y_1 \ldots y_m$. Taking the empty word as corresponding to the set X^*, we see that $< X^*, *, e >$ is a monoid. Let's mark the empty word as 1.
Theorem 2.1. $< X^*, *, 1 >$ will be an free monoid.
Proof. Let S be monoid generated by X and $\alpha : X \rightarrow S$. Let's define the mapping $\psi : X^* \rightarrow S$ as follows:

$$\psi\,(1) = e_S$$

$$\psi\,(x_{i_1}, \ldots x_{i_n}) = \alpha\,(x_{i_1}) \ldots \alpha\,(x_{i_n})$$

then for $u, v \in X^*, u = x_{i_1} \dots x_{i_{|u|}}, v = y_{i_1} \dots y_{i_{|v|}}$

$$\psi(u * v) = \psi\left(x_{i_1} \dots x_{i_{|u|}} y_{i_1} \dots y_{i_{|v|}}\right) = \alpha(x_{i_1}) \dots \alpha\left(x_{i_{|u|}}\right) \alpha(y_{i_1}) \dots \alpha\left(y_{i_{|v|}}\right) =$$
$$= \psi\left(x_{i_1} \dots x_{i_{|u|}}\right) \psi\left(y_{i_1} \dots y_{i_{|v|}}\right) = \psi(u)\, \psi(v)$$

So ψ will be a Homomorphism. In it, the ψ - searched homorphism $< X^*, *, 1 >$ is a free Monoid.

2.2 Free Algebra

Let X be a nonempty set. Consider a vector space over a field k and the basis is all words of the form $x_{i_1} x_{i_2} ... x_{i_n}$ and include empty words e (where $x_{i_j} \in X$). We call such words monomials. In this space , multiplication is defined as concatenation:

$$(x_{i_1} ... x_{i_n})(x_{i_{n+1}} ... x_{i_m}) = x_{i_1} ... x_{i_n} x_{i_{n+1}} ... x_{i_m}$$

this multiplication in $k\{X\}$ defines thė structure of the algebra. In this algebra, the unit will be an empty word.

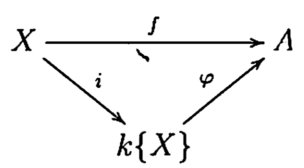
Theorem 2.2. Let X be a nonempty set. A algebra generated with a set X. For any mapping $f : X \to A$ There is a single homomorphism

$$\varphi : k\{X\} \to A$$

such that

$$\varphi(x) = f(x), \quad \forall x \in X$$

.



Proof. To do this, define the φ function as follows

$$\varphi(\emptyset) = 1$$

$$\varphi(x_{i_1} ... x_{i_n}) = f(x_{i_1}) ... f(x_{i_n})$$
$$\varphi(u + v) = \varphi(u) + \varphi(v)$$

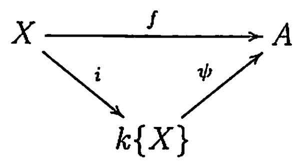
It is clear that ,for $x \in X$

$$\varphi(x) = f(x)$$

We prove that φ is a homomorphism.
For $u = u_1, ... u_n, v = v_1 ... v_m \in k\{X\}$:

$$\varphi(uv) = f(u_1) .... f(u_n) f(v_1) ... f(v_m) = \varphi(u)\varphi(v)$$

Let ψ be a homomorphism satisfying the condition of the theorem.



For $u = u_1 ... u_n \in k\{X\}$

$$\varphi(u) = \varphi(u_1 ... u_n) = f(x_1)...f(x_n) = \psi(x_1)...\psi(x_n) = \psi(x_1...x_n) = \psi(u)$$

That is, the images of the bases are the same, then the images of all elements are the same. This algebra is called Free Algebra

Chapter 3

Gröbner-Shirshov bases for Magmatic algebras

3.1 Composition-Diamond lemma for Magmatic algebras

Let$X = \{x_i : i \in I\}$ be nonempty set and I a well-ordered set. We denote by X^* the set of associative words in alphabet X. Also We denote by X^{**} the set of nonassociative words in alphabet X. We call $X^{**} = M(X)$ magmatic grouppoid.[22] We assume X^* and X^{**} contains an empty word and denote it 1. We donote by $u, v, \ldots$ the elements of the set X^* and denote by $(u), (v), \ldots$ the elements of the set X^{**}.

Let 's define the function $|\cdot| : X^{**} \to N \cup \{0\}$

1. If (x) is empty word $|(x)| = 0$
2. If $(x) \in X \Rightarrow |x| = 1$
3. If $(x) = ((u)(v)) \Rightarrow |(x)| = |(u)| + |(v)|$

kX^{**}-linear space generated by the set X. A linear space kX^{**} equipped with a concatenation operation is called a magmatic (magma) algebra. We will denote this algebra $Mag(X)$.

Let's define the order in the set X^{**} as follows. $\forall (u) \neq (v) \in X^{**}$:

1. Let $|(u)| = 0 \Rightarrow (v) > (u)$
2. Let $|(u)| + |(v)| = 2 \Rightarrow (u) = x_i$, $(v) = x_j$. Then $i > j \Leftrightarrow x_i > x_j$
3. Let $|(u)| + |(v)| > 2$. $(u) > (v)$ is executed only in the following cases.
   (a) $|(u)| > |(v)|$
   (b) If $|(u)| = |(v)|$ and $(u) = ((u_1)(u_2)), (v) = ((v_1)(v_2))$, then $(u_1) > (v_1)$ *or* $(u_1) = (v_1)$ and $(u_2) > (v_2)$.

This order is called deg-lex order.

Theorem 3.1. $(X^{**},<)$ - well-ordered set.

Remark. Any parenthesizing of words is linear basis of $Mag(X)$. Then dimension is

$$C_n = \frac{1}{n+1} \cdot C_{2n}^{n}$$

where n- number of elements the set X.

For any $f \in Mag(X)$ can be written the only way:

$$f = \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_m f_m$$

For each polynomial $f \in Mag(X)$,by $\overline{f}$ we denote the f_1, that is maximal word in f. If $\alpha_1 = 1$ we say the f is monic.

Let $S \subset Mag(X)$ be a set of monic polynomials. We define $S - word\ (u)_s$ by indution:

1. $(s)_s = s$ is a $S - word$ of S-length 1.
2. If $(u)_s$ is an $S - word$ of S-length k and (v) is a non-associative word of length l, then $(u)_s(v)$ and $(v)(u)_s$ are $S - words$ of S-length $k + l$.

Remark. For any $S - word\,(u)_s$:

$$\overline{(u)_s} = \overline{(a\,(s)\,b)} = (a\,(\overline{s})\,b)$$

where $a, b \in X^*$

Let $f, g \in Mag(X)$ and f, g- monic polynomials. We denote by $Id(f,g)$ ideal of $Mag(X)$ generated by f, g. Assume that there are $a, b \in X^*$ such that

$$(\overline{f}) = (a\,(\overline{g})\,b)$$

Let $S \subset Mag(X)$ be a set of monic polynomials. For all $f\ ,\ g\ \in\ S$, we define compositions of S as follows:

$$(f,g)_{\overline{(f)}} = f - (a\,(g)\,b)$$

Then it's easy to see that

$$(f,g)_{\overline{(f)}} \in Id(f,g) \quad , \quad \overline{(f,g)_{\overline{(f)}}} < (\overline{f})$$

We say the inclusion composition $(f,g)_{\overline{f}}$ is trivial modulo $(S, \overline{f})$, if

$$(f,g)_{\overline{f}} = \sum_i \alpha_i\,(a_i\,(s_i)\,b_i)$$

Where $\alpha_i \in k, a_i.b_i \in X^*$, $s_i \in S$, $(a_i\,(s_i)\,b_i)$ is $S - word$, and $(a_i\,(\overline{s_i})\,b_i) < \overline{f}$. If inclusion composition $(f,g)_{\overline{f}}$ trivial modulo $\left(S,\overline{f}\right)$ we write:

$$(f,g)_{\overline{f}} = 0 \ mod \ \left(S,\overline{f}\right)$$

For $x, y \in Mag(X)$, and $(\omega) \in X^{**}$, we write

$$x = y \ mod \ \ (S,(\omega))$$

If and only if

$$x - y = \sum_i \alpha_i \left(a_i \left(s_i\right) b_i\right)$$

Where $\alpha_i \in k, a_i.b_i \in X^*, s_i \in S$, $\left(a_i \left(s_i\right) b_i\right)$ is $S - word$, and $\left(a_i \left(\overline{s_i}\right) b_i\right) < \overline{\omega}$. And we say x is comparable to y modulo $\left(S,\overline{f}\right)$.

Definition 3.1. Let $S \subset Mag(X)$ be a subset of monic polynomials and $<$ a monomial ordering on X^{**}. Then S is called a Gröbner Shirshov basis in $Mag(X)$ if any inclusion composition $(f,g)_{\overline{f}}$ trivial modulo $\left(S,\overline{f}\right)$.

Definition3.2. Let $S \subset Mag(X)$ be a subset of monic polynomials. We define the set $Irr(S)$ as follows

$$Irr(S) = \{(u) \in X^{**} : (u) \neq (a(s)b), \forall a, b \in X^* \& \forall s \in S\}$$

Theorem 3.2.[22]Let $S \subset Mag(X)$ be a subset of monic polynomials and $<$ a monomial ordering on X^{**}. Then the following statements are equivalent:

1. S is Gröbner-Shirshov basis
2. If $f \in Id(S)$, then $\left(\overline{f}\right) = \left(a\left(\overline{s}\right)b\right)$ for some $s \in S$ and $a, b \in X^*, where (a(s)b)$ is a $S - word.$
3. $Irr(S)$ is a linear basis of the algebra $Mag(X|S) = Mag(X)/Id(S)$.

Chapter 4

Linear basis of some algebras

4.1 Linear basis of Associative algebras

Definition 4.1. The free Associative algebra $Ass\,(X) = Mag\,(X|S)$ is the algebra generated X with the defining relations S, where

$$S = \{(ab)c - a(bc) : a, b, c \in X^{**}\}$$

To find the basis, we use the deg-lex order. Let's introduce markup as follows:

$$[a_1, a_2, a_3, ...a_n]_R = (a_1(...(a_{n-2}(a_{n-1}a_n))))$$

Theorem 4.1. Let $A\,(X)$ be the set containing elements of the form

$$[x_1, x_2, \ldots x_n]_R$$

where $x_1, x_2, \ldots, x_n \in X$. Then $Z\,(X)$ is linear basis for $Ass\,(X)$

Proof. To prove the theorem, we use Buchberger's algorithm for magmatic algebras. We need to find the basis of the algebra $Mag\,(X|S)$

$$S = \{(ab)c - a(bc) : a, b, c \in X^*\}$$

For $f, g \in S$ let's look at all the compositions. To facilitate, we use the following letters:

$$f = (xy)z - x(yz)$$

and

$$g = (ab)c - a(bc)$$

In the following cases, the composition may occur:

1. $\overline{f} = (u\overline{g})v$
2. $\overline{f} = u(\overline{g}v)$
3. $\overline{f} = \overline{g}v$

4. $\overline{f} = u\overline{g}$

Let 's look at these compositions

1. Consider the first case: $\overline{f} = (u\overline{g})v$

$$\overline{f} = (xy)z = (u\overline{g})v = (u((ab)c))v$$

Therefore

$$xy = u((ab)c)$$

and

$$z = v$$

From this

$$x = u$$
$$y = (ab)c$$

Consider the compositions $(f,g)_{\overline{f}}$

$$\begin{aligned}
(f,g)_{\overline{f}} &= f - (ug)v \\
(xy)z - x(yz) &- (u((ab)c))v + (u(a(bc)))v \\
&= (u((ab)c))z - u(((ab)c)z) - (u((ab)c))z + (u(a(bc)))z \\
&= -u(((ab)c)z) + (u(a(bc)))z \\
&= -u((a(bc))z) + u((a(bc))z) \\
&\equiv 0 \mod (S, \overline{f})
\end{aligned}$$

$(f,g)_{\overline{f}}$ is trivial.

2. $\overline{f} = u(\overline{g}v)$

$$\overline{f} = (xy)z = u(\overline{g}v) = u((ab)c)v)$$

Therefore

$$xy = u$$

and

$$z = ((ab)c)v$$

Consider the compositions $(f,g)_{\overline{f}}$

$$\begin{aligned}
(f,g)_{\overline{f}} &= f - u(gv) \\
&= (xy)z - x(yz) - u(((ab)c)v) + u((a(bc))v) \\
&= (xy)(((ab)c)v) - x(y(((ab)c)v)) - (xy)(((ab)c)v) + (xy)((a(bc))v) \\
&= -x(y(((ab)c)v)) + (xy)((a(bc))v) \\
&= -x(y((a(bc))v)) + x(y((a(bc))v)) \\
&\equiv 0 \mod (S, \overline{f})
\end{aligned}$$

3. $\overline{f} = \overline{g}v$

$$\overline{f} = (xy)z = \overline{g}v = ((ab)c)v$$

Therefore

$$xy = (ab)c$$

and

$$z = v$$

From this

$$x = ab$$

$$y = c$$

Consider the compositions $(f, g)_{\overline{f}}$

$$\begin{aligned}(f, g)_{\overline{f}} &= f - gv \\ &= (xy)z - x(yz) - ((ab)c)v + (a(bc))v \\ &= ((ab)y)z - (ab)(yz) - ((ab)y)z + (a(by))z \\ &= -(ab)(yz) + (a(by))z \\ &= -a(b(yz)) + a((by)z) \\ &= -a(b(yz)) + a(b(yz)) \\ &\equiv 0 \mod (S, \overline{f})\end{aligned}$$

4. $\overline{f} = u\overline{g}$

$$\overline{f} = (xy)z = u\overline{g} = u((ab)c)$$

$$xy = u$$

and

$$z = (ab)c$$

Consider the compositions $(f, g)_{\overline{f}}$

$$\begin{aligned}(f, g)_{\overline{f}} &= f - gv \\ &= (xy)z - x(yz) - u((ab)c) + u(a(bc)) \\ &= (xy)((ab)c) - x(y((ab)c)) - (xy)((ab)c) + (xy)(a(bc)) \\ &= -x(y((ab)c)) + (xy)(a(bc)) \\ &= -x(y(a(bc))) + x(y(a(bc))) \\ &\equiv 0 \mod (S, \overline{f})\end{aligned}$$

Therefore S-Gröbner-Shirshov basis. Then $Irr(S)$ linear basis. And $Irr(S)$ contains elements in the form $[x_1, x_2, \dots x_n]_R$

4.2 Linear basis of Leibniz algebras

Definition 4.2. The free Leibniz algebra $Leib(X) = Mag(X|S)$ is the non-associative algebra generated X with the defining relations S, where

$$S = \{a\,(bc) - (ab)\,c + (ac)\,b : a, b, c \in X^{**}\}$$

To find the basis, we use the inverse deg-lex order. That is , the order is defined as follows: $\forall\,(u) \neq (v) \in X^{**}$

1. Let $|(u)| = 0 \Rightarrow (v) > (u)$
2. Let $|(u)| + |(v)| = 2 \Rightarrow (u) = x_i\ , (v) = x_j$. Then $i > j \Leftrightarrow x_i > x_j$.
3. Let $|(u)\,| + |\,(v)| > 2.\,(u) > (v)$ is executed only in the following cases
   (a) $|(u)| > |(v)|$
   (b) If $|(u)| = |(v)|$ and $(u) = ((u_1)\,(u_2))\,, (v) = ((v_1)\,(v_2))$, then $(u_2) > (v_2)$ *or* $(u_2) = (v_2)$ and $(u_1) > (v_1)$.

Remark. In this order, the Gröbner-Shirshov basis theorem for magmatic algebras also holds. Because the reverse deg-lex order is also a monomial order. Let's introduce markup as follows:

$$[a_1, a_2, a_3, \ldots a_n]_L = (((a_1, a_2)\,, a_3)\,, \ldots a_n)$$

Theorem 4.2. Let $N\,(X)$ be the set containing elements of the form:

$$[x_1, x_2, \ldots x_n]_L$$

where $x_1, x_2, \ldots, x_n \in X$. Then $N\,(X)$ is linear basis for $Leib\,(X)$
Proof. . To prove the theorem, we use Buchberger's algorithm for magmatic algebras. We need to find the basis of the algebra $Mag\,(X|S)$

$$S = \{a\,(bc) - (ab)\,c + (ac)\,b : a, b, c \in X^{**}\}$$

Let the order be reverse deg-lex. For $f, g \in S$ let's look at all the compositions. To facilitate, we use the following letters

$$f = x(yz) - (xy)z + (xz)y$$

and

$$g = a(bc) - (ab)c + (ac)b$$

The following compositions may occur:

1. $\overline{f} = (u\overline{g})\,v$ for some $u, v \in X^{**}$

$$\overline{f} = (u\overline{g})\,v = x\,(yz) = (u\,(a\,(bc)))\,v$$

Therefore

$$x = u(a(bc))$$

$$yz = v$$

Consider the composition

$$(f,g)_{\overline{f}} = f - (ug)v =$$
$$= x\,(yz) - (xy)\,z + (xz)\,y - (u\,(a\,(bc)))\,v + (u\,((ab)\,c)) - (u\,((ac)\,b))\,v =$$
$$(u(a(bc)))(yz) - ((u(a(bc)))y)z + ((u(a(bc)))z)y$$
$$-(u(a(bc)))(yz) + (u((ab)c))(yz) - (u((ac)b))(yz)$$
$$-((u(a(bc)))y)z + ((u(a(bc)))z)y + (u((ab)c))(yz) - (u((ac)b))(yz)$$
$$= -((u((ab)c))y)z + ((u((ac)b)))y)z + ((u((ab)c))z)y - ((u((ac)b))z)y$$
$$+((u((ab)c))y)z - ((u((ab)c))z)y - ((u((ac)b))y)z + ((u((ac)b))z)y$$
$$\equiv 0 \mod (S, \overline{f})$$

$(f,g)_{\overline{f}}$ is trivial.

2. $\overline{f} = u(\overline{g}v)$

$$\overline{f} = x(yz) = u(\overline{g}v) = u((a(bc))v)$$

Therefore

$$x = u$$

and

$$yz = (a(bc))v$$

From this

$$y = a(bc)$$
$$z = v$$

Consider the composition

$$(f,g)_{\overline{f}} = f - u(gv) =$$
$$x(yz) - (xy)z + (xz)y - u((a(bc))v) + u(((ab)c)v) - u(((ac)b)v)$$
$$= x((a(bc))z) - (x(a(bc)))z + (xz)(a(bc))$$
$$-x((a(bc))z) + x(((ab)c)z) - x(((ac)b)z)$$
$$= -(x(a(bc)))z + (xz)(a(bc)) + x(((ab)c)z) - x(((ac)b)z)$$
$$= -(x((ab)c))z + (x((ac)b))z + (xz)((ab)c) - (xz)((ac)b)$$
$$+(x((ab)c))z - (xz)((ab)c) - (x((ac)b))z + (xz)((ac)b)$$
$$\equiv 0 \mod (S, \overline{f})$$

$(f,g)_{\overline{f}}$ is trivial.

3. $\overline{f} = u\overline{g}$

$$\overline{f} = x(yz) = u\overline{g} = u(a(bc))$$

Therefore

$$x = u$$
$$yz = a(bc)$$

From this

$$y = a$$

$$z = bc$$

Consider the composition

$$\begin{aligned}
(f,g)_{\overline{f}} &= f - u\overline{g} = \\
&= x(yz) - (xy)z + (xz)y - u(a(bc)) + u((ab)c) - u((ac)b) = \\
&= u(a(bc)) - (ua)(bc) + (u(bc))a \\
&\quad -u(a(bc)) + u((ab)c) - u((ac)b) \\
&= -(ua)(bc) + (u(bc))a + u((ab)c) - u((ac)b) \\
&= -((ua)b)c + ((ua)c)b + ((ub)c)a - ((uc)b)a \\
&\quad +(u(ab))c - (uc)(ab) - (u(ac))b + (ub)(ac) \\
&= -((ua)b)c + ((ua)c)b + ((ub)c)a - ((uc)b)a \\
&\quad +((ua)b)c - ((ub)a)c - ((uc)a)b + ((uc)b)a) \\
&\quad -((ua)c)b + ((uc)a)b + ((ub)a)c - ((ub)c)a \\
&\equiv 0 \mod (S, \overline{f})
\end{aligned}$$

$(f,g)_{\overline{f}}$ is trivial.

4. $\overline{f} = \overline{g}v$

$$\overline{f} = x(yz) = \overline{g}v = (a(bc))v$$

Therefore

$$x = a(bc)$$

$$yz = v$$

Consider the composition

$$\begin{aligned}
(f,g)_{\overline{f}} &= f - \overline{g}v = \\
&x(yz) - (xy)z + (xz)y - (a(bc))v + ((ab)c)v - ((ac)b)v \\
&= (a(bc))(yz) - ((a(bc))y)z + ((a(bc))z)y \\
&\quad -(a(bc))(yz) + ((ab)c)(yz) - ((ac)b)(yz) \\
&= -((a(bc))y)z + ((a(bc))z)y + ((ab)c)(yz) - ((ac)b)(yz) \\
&= -(((ab)c)y)z + (((ac)b)y)z + (((ab)c)z)y - (((ac)b)z)y \\
&\quad +(((ab)c)y)z - (((ab)c)z)y - (((ac)b)y)z + (((ac)b)z)y \equiv 0 \mod (S, \overline{f})
\end{aligned}$$

$(f,g)_{\overline{f}}$ is trivial.
Therefore S-Gröbner-Shirshov basis. Then $Irr(S)$ linear basis. And $Irr(S)$ contains elements in the form $[x_1, x_2, \dots x_n]_L$

4.3 Linear basis of Zinbiel algebras

Definition 4.3. The free Zinbiel algebra $Zin(X) = Mag(X|S)$ is the non-associative algebra generated X with the defining relations S, where

$$S = \{(ab)c - a(bc) - a(cb) : a, b, c \in X^{**}\}$$

To find the basis, we use the deg-lex order. Let's introduce markup as follows:

$$[a_1, a_2, a_3, ...a_n]_R = (a_1(...(a_{n-2}(a_{n-1}a_n))))$$

Theorem 4.3. Let $Z(X)$ be the set containing elements of the form

$$[x_1, x_2, \dots x_n]_R$$

where $x_1, x_2, \dots, x_n \in X$. Then $Z(X)$ is linear basis for $Zin(X)$

Proof. To prove the theorem, we use Buchberger's algorithm for magmatic algebras. We need to find the basis of the algebra $Mag(X|S)$

$$S = \{(ab)c - a(bc) - a(cb) : a, b, c \in X^{*}\}$$

For $f, g \in S$ let's look at all the compositions. To facilitate, we use the following letters:

$$f = (xy)z - x(yz) - x(zy)$$

and

$$g = (ab)c - a(bc) - a(cb)$$

In the following cases, the composition may occur:

1. $\overline{f} = (u\overline{g})v$
2. $\overline{f} = u(\overline{g}v)$
3. $\overline{f} = \overline{g}v$
4. $\overline{f} = u\overline{g}$

1. Consider the first case: $\overline{f} = (u\overline{g})v$

$$\overline{f} = (xy)z \doteq (u\overline{g})v = (u((ab)c))v$$

Therefore

$$xy = u((ab)c)$$

and

$$z = v$$

From this

$$x = u$$

$$y = (ab)c$$

Consider the compositions $(f,g)_{\overline{f}}$.

$$(f,g)_{\overline{f}} = f - (ug)v$$
$$= (xy)z - x(yz) - x(zy) - (u((ab)c))v + (u(a(bc)))v + (u(a(cb)))v$$
$$= (u((ab)c))z - u(((ab)c)z) - u(z((ab)c))$$
$$-(u((ab)c))z + (u(a(bc)))z + (u(a(cb)))z$$
$$= -u(((ab)c)z) - u(z((ab)c)) + (u(a(bc)))z + (u(a(cb)))z$$
$$= -u((a(bc))z) - u((a(cb))z) - u(z(a(bc))) - u(z(a(cb)))$$
$$+u((a(bc))z) + u(z(a(bc))) + u((a(cb))z) + u(z(a(cb)))$$
$$\equiv 0 \mod (S,\overline{f})$$

that is , $(f,g)_{\overline{f}}$ is trivial.

2. Consider the case $\overline{f} = u(\overline{g}v)$

$$\overline{f} = (xy)z = u(\overline{g}v) = u(((ab)c)v)$$

Therefore

$$xy = u$$

and

$$z = ((ab)c)v$$

Consider the compositions $(f,g)_{\overline{f}}$

$$(f,g)_{\overline{f}} = f - u(gv)$$
$$= (xy)z - x(yz) - x(zy) - u(((ab)c)v) + u((a(bc))v) + u((a(cb))v)$$
$$= (xy)(((ab)c)v) - x(y(((ab)c)v)) - x((((ab)c)v)y)$$
$$+(xy)(((ab)c)v) + (xy)((a(bc))v) + (xy)((a(cb))v)$$
$$= -x(y(((ab)c)v)) - x((((ab)c)v)y) + (xy)((a(bc))v) + (xy)((a(cb))v)$$
$$= -x(y((a(bc))v)) - x(y((a(cb))v)) - x(((a(bc))v)y) - x(((a(cb))v)y)$$
$$+x(y((a(bc))v)) + x(((a(bc))v)y) + x(y((a(cb))v)) + x(((a(cb))v)y)$$
$$\equiv 0 \mod (S,\overline{f})$$

that is , $(f,g)_{\overline{f}}$ is trivial.

3. Consider the case $\overline{f} = \overline{g}v$

$$\overline{f} = (xy)z = \overline{g}v = ((ab)c)v$$

Therefore

$$xy = (ab)c$$

and

$$z = v$$

From this

$$x = ab$$

$$y = c$$

Consider the compositions $(f, g)_{\overline{f}}$

$$\begin{aligned}(f, g)_{\overline{f}} &= f - gv \\
&= (xy)z - x(yz) - x(zy) - ((ab)c)v + (a(bc))v + (a(cb))v \\
&= ((ab)c)z - (ab)(cz) - (ab)(zc) - ((ab)c)z + (a(bc))z + (a(cb))z \\
&= -(ab)(cz) - (ab)(zc) + (a(bc))z + (a(cb))z \\
&= -a(b(cz)) - a((cz)b) - a(b(zc)) - a((zc)b) \\
&\quad +a((bc)z) + a(z(bc)) + a((cb)z) + a(z(cb)) \\
&= -a(b(cz)) - a(c(zb)) - a(c(bz)) - a(b(zc)) - a(z(cb)) - a(z(bc)) \\
&\quad +a(b(cz)) + a(b(zc)) + a(z(bc)) + a(c(bz)) + a(c(zb)) + a(z(cb)) \\
&\equiv 0 \mod (S, \overline{f})\end{aligned}$$

that is , $(f, g)_{\overline{f}}$ is trivial.

4. Consider last case $\overline{f} = u\overline{g}$

$$\overline{f} = (xy)z = u\overline{g} = u((ab)c)$$

$$xy = u$$

and

$$z = (ab)c$$

Consider the compositions $(f, g)_{\overline{f}}$

$$\begin{aligned}(f, g)_{\overline{f}} &= f - ug \\
&= (xy)z - x(yz) - x(zy) - u((ab)c) + u(a(bc)) + u(a(cb)) \\
&= (xy)((ab)c) - x(y((ab)c)) - x(((ab)c)y) \\
&\quad -(xy)((ab)c) + (xy)(a(bc)) + (xy)(a(cb)) \\
&= -x(y((ab)c)) - x(((ab)c)y) + (xy)(a(bc)) + (xy)(a(cb)) \\
&= -x(y(a(bc))) - x(y(a(cb))) - x((a(bc))y) - x((a(cb))y) \\
&\quad +x(y(a(bc))) + x((a(bc))y) + x(y(a(cb))) + x((a(cb))y) \\
&\equiv 0 \mod (S, \overline{f})\end{aligned}$$

that is , $(f, g)_{\overline{f}}$ is trivial.

All the composition is trivial so S is Gröbner-Shishov basis. Then $Irr(X)$ is the linear basis of the $Zin(X)$ algebra . $Irr(X)$ consists of words that do not begin with the leader S-words that is, words of the following forms:

$$[x_1, x_2, \dots x_n]_R$$

4.4 Linear basis of Perm algebras

Definition 4.4. The free Perm algebra $Perm(X) = Mag(X|S)$ is the non-associative algebra generated X with the defining relations S, where

$$S = \{(ab)c - a(bc), x(yz) - x(zy) : a, b, c, z, y \in X^{**}, y > z\}$$

We need to find the basis of the algebra $Mag(X|S)$

$$S = \{(ab)c - a(bc), x(yz) - x(zy) : a, b, c, z, y \in X^{**}, y > z\}$$

For $f, g \in S$ let's look at all the compositions. To facilitate, we use the following letters:

$$f = (xy)z - x(yz)$$
$$d = (pq)r - p(qr)$$

and

$$g = a(bc) - a(cb), \quad b > c$$
$$h = k(lm) - k(ml), \quad l > m$$

In the following cases, the composition may occur:

1. $\overline{f} = (u\overline{d})v$
2. $\overline{f} = u(\overline{d}v)$
3. $\overline{f} = \overline{d}v$
4. $\overline{f} = u\overline{d}$
5. $\overline{g} = (u\overline{h})v$
6. $\overline{g} = u(\overline{h}v)$
7. $\overline{g} = \overline{h}v$
8. $\overline{g} = u\overline{h}$
9. $\overline{f} = (u\overline{g})v$
10. $\overline{f} = u(\overline{g}v)$
11. $\overline{f} = \overline{g}v$
12. $\overline{f} = u\overline{g}$
13. $\overline{g} = (u\overline{f})v$
14. $\overline{g} = u(\overline{f}v)$
15. $\overline{g} = \overline{f}v$
16. $\overline{g} = u\overline{f}$
17. $\overline{g} = \overline{f}$

Let's look at all these compositions

1. $\overline{f} = (u\overline{d})v$

$$(xy)z = (u((pq)r))v$$

Therefore

$$xy = u((pq)r)$$
$$z = v$$

Then

$$x = u$$
$$y = (pq)r$$

Consider the compositions $(f,d)_{\overline{f}}$

$$\begin{gathered}(f,d)_{\overline{f}} = (xy)z - x(yz) - (u((pq)r))v + (u(p(qr)))v = \\ = (u((pq)r))v - u(((pq)r)v) - (u((pq)r))v + (u(p(qr)))v = \\ -u(((pq)r)v) + (u(p(qr)))v = -u((p(qr))v) + u((p(qr)))v) \\ \equiv 0 \mod (S, \overline{f})\end{gathered}$$

2. $\overline{f} = u(\overline{d}v)$

$$(xy)z = u(((pq)r)v)$$

Therefore

$$(xy) = u$$
$$z = ((pq)r)v$$

Consider the compositions $(f,d)_{\overline{f}}$

$$\begin{gathered}(f,d)_{\overline{f}} = (xy)z - x(yz) - u(((pq)r)v) + u((p(qr))v) \\ = (xy)(((pq)r)v) - x(y(((pq)r)v)) - (xy)(((pq)r)v) + (xy)((p(qr))v) \\ = -x(y(((pq)r)v)) + (xy)((p(qr))v) \\ = -x(y((p(qr))v)) + x(y((p(qr))v)) \\ \equiv 0 \mod (S, \overline{f})\end{gathered}$$

3. $\overline{f} = \overline{d}v$

$$(xy)z = ((pq)r)v$$

Therefore

$$xy = (pq)r$$
$$z = v$$

then

$$x = pq$$
$$y = r$$

Consider the compositions $(f,d)_{\overline{f}}$

$$\begin{aligned}(f,d)_{\overline{f}} &= (xy)z - x(yz) - ((pq)r)v + (p(qr))v \\ &= ((pq)r)v - (pq)(rv) - ((pq)r)v + (p(qr))v \\ &= -(pq)(rv) + (p(qr))v \\ &= -p(q(rv)) + p((qr)v) \\ &= -p(q(rv)) + p(q(rv)) \\ &\equiv 0 \mod (S,\overline{f})\end{aligned}$$

4. $\overline{f} = u\overline{d}$

$$(xy)z = u((pq)r)$$

Thefore

$$xy = u$$

$$z = (pq)r$$

Consider the composition $(f,d)_{\overline{f}}$.

$$\begin{aligned}(f,d)_{\overline{f}} &= (xy)z - x(yz) - u((pq)r) + u(p(qr)) \\ &= (xy)((pq)r) - x(y((pq)r)) - (xy)((pq)r) + (xy)(p(qr)) \\ &= -x(y((pq)r)) + (xy)(p(qr)) \\ &= -x(y(p(qr))) + x(y(p(qr))) \\ &\equiv 0 \mod (S,\overline{f})\end{aligned}$$

5. $\overline{g} = (u\overline{h})v$

$$a(bc) = (u(k(lm)))v$$

Therefore

$$a = u(k(lm))$$

$$bc = v$$

Consider the composition $(g,h)_{\overline{g}}$.

$$\begin{aligned}(g,h)_{\overline{g}} &= a(bc) - a(cb) - (u(k(lm)))v + (u(k(ml)))v \\ &= (u(k(lm))(bc) - (u(k(lm))(cb) - (u(k(lm)))(bc) + (u(k(ml)))(bc) \\ &= -(u(k(lm))(cb) + (u(k(ml)))(bc) \\ &= -(u(k(ml))(cb) + (u(k(ml)))(cb) \\ &\equiv 0 \mod (S,\overline{g})\end{aligned}$$

6. $\overline{g} = u(\overline{h}v)$

$$a(bc) = u((k(lm))v)$$

Therefore

$$a = u$$

$$bc = (k(lm))v$$

then

$$b = k(lm)$$
$$c = v$$

Consider the composition $(g, h)_{\overline{g}}$

$$\begin{aligned}(g, h)_{\overline{g}} &= a(bc) - a(cb) - u((k(lm))v) + u((k(ml))v) \\ &= u((k(lm))v) - u(v(k(lm))) - u((k(lm))v) + u((k(ml))v) \\ &= -u(v(k(lm))) + u((k(ml))v) \\ &= -u(v(k(ml))) + u((k(ml))v)\end{aligned}$$

Here we consider two cases:

(a) If $v > k(ml)$ then

$$\begin{aligned}(g, h)_{\overline{g}} &= -u(v(k(ml))) + u((k(ml))v) \\ &= -u((k(ml))b) + u((k(ml))v) \\ &\equiv 0 \mod (S, \overline{g})\end{aligned}$$

(b) If $v < k(ml)$ then

$$\begin{aligned}(g, h)_{\overline{g}} &= -u(v(k(ml))) + u((k(ml))v) \\ &= -u(v(k(ml))) + u(v(k(ml))) \\ &\equiv 0 \mod (S, \overline{g})\end{aligned}$$

Two cases holds the composition to zero. Therefore $(g, h)_{\overline{g}} = 0$

7. $\overline{g} = \overline{h}v$

$$a(bc) = (k(lm))v$$

Therefore

$$a = k(lm)$$
$$bc = v$$

Consider the composition $(g, h)_{\overline{g}}$

$$\begin{aligned}(g, h)_{\overline{g}} &= a(bc) - a(cb) - (k(lm))v + (k(ml))v \\ &= (k(lm))(bc) - (k(lm))(cb) - (k(lm))(bc) + (k(ml))(bc) \\ &= -(k(lm))(cb) + (k(ml))(bc) \\ &= -(k(ml))(cb) + (k(ml))(cb) \\ &\equiv 0 \mod (S, \overline{g})\end{aligned}$$

8. $\overline{g} = u\overline{h}$

$$a(bc) = u(k(lm))$$

Therefore

$$a = u$$

$$bc = k(lm)$$

Then

$$b = k$$

$$c = lm$$

Consider the composition $(g, h)_{\overline{g}}$

$$\begin{aligned}(g, h)_{\overline{g}} &= a(bc) - a(cb) - u(k(lm)) + u(k(ml)) \\ &= u(k(lm)) - u((lm)k) - u(k(lm)) + u(k(ml)) \\ &= -u((lm)k) + u(k(ml)) \\ &= -u((ml)k) + u(k(ml))\end{aligned}$$

Here we consider two cases:

(a) If $k > ml$ then

$$\begin{aligned}(g, h)_{\overline{g}} = -u((ml)k) + u(k(ml)) &= -u((ml)k) + u((ml)k) \\ &\equiv 0 \mod (S, \overline{g})\end{aligned}$$

(b) If $ml > k$ then

$$\begin{aligned}(g, h)_{\overline{g}} = -u((ml)k) + u(k(ml)) &= -u(k(ml)) + u(k(ml)) \\ &\equiv 0 \mod (S, \overline{g})\end{aligned}$$

Two cases holds the composition to zero. Therefore $(g, h)_{\overline{g}} = 0$

9. $\overline{f} = (u\overline{g})v$

$$(xy)z = (u(a(bc)))v$$

Therefore

$$xy = u(a(bc))$$

$$z = v$$

then

$$x = u$$

$$y = a(bc)$$

Consider the composition $(f,g)_{\overline{f}}$

$$\begin{aligned}(f,g)_{\overline{g}} &= (xy)z - x(yz) - (u(a(bc)))v + (u(a(cb)))v \\ &= (u(a(bc)))v - u((a(bc))v) - (u(a(bc)))v + (u(a(cb)))v \\ &= -u((a(bc))v) + (u(a(cb)))v \\ &= -u((a(cb))v) + u((a(cb))v) \\ &\equiv 0 \mod (S,\overline{f})\end{aligned}$$

10. $\overline{f} = u(\overline{g}v)$

$$(xy)z = u((a(bc))v)$$

Therefore

$$xy = u$$

$$z = (a(bc))v$$

Consider the composition $(f,g)_{\overline{f}}$

$$\begin{aligned}(f,g)_{\overline{f}} &= (xy)z - x(yz) - u((a(bc))v) + u((a(cb))v) \\ &= (xy)((a(bc))v) - x(y((a(bc))v)) - (xy)((a(bc))v) + (xy)((a(cb))v) \\ &= -x(y((a(bc))v)) + (xy)((a(cb))v) \\ &= -x(y((a(cb))v)) + x(y((a(cb))v)) \\ &\equiv 0 \mod (S,\overline{f})\end{aligned}$$

11. $\overline{f} = \overline{g}v$

$$(xy)z = (a(bc))v$$

Therefore

$$xy = a(bc)$$

$$z = v$$

then

$$x = a$$

$$y = bc$$

Consider the composition $(f,g)_{\overline{f}}$

$$\begin{aligned}(f,g)_{\overline{f}} &= (xy)z - x(yz) - (a(bc))v + (a(cb))v \\ &= (a(bc))v - a((bc)v) - (a(bc))v + (a(cb))v \\ &= -a((cb)v) + a((cb)v) \\ &\equiv 0 \mod (S,\overline{f})\end{aligned}$$

12. $\overline{f} = u\overline{g}$

$$(xy)z = u(a(bc))$$

Therefore

$$xy = u$$

$$z = a(bc)$$

Consider the composition $(f, g)_{\overline{f}}$

$$\begin{aligned}
(f, g)_{\overline{f}} &= (xy)z - x(yz) - u(a(bc)) + u(a(cb)) \\
&= (xy)(a(bc)) - x(y(a(bc))) - (xy)(a(bc)) + (xy)(a(cb)) \\
&= -x(y(a(bc))) + (xy)(a(cb)) \\
&= -x(y(a(cb))) + x(y(a(cb))) \\
&\equiv 0 \mod (S, \overline{f})
\end{aligned}$$

13. $\overline{g} = (u\overline{f})v$

$$a(bc) = (u((xy)z))v$$

Therefore

$$a = u((xy)z)$$

$$bc = v$$

Consider the composition $(g, f)_{\overline{g}}$

$$\begin{aligned}
(g, f)_{\overline{g}} &= a(bc) - a(cb) - (u((xy)z))v + (u(x(yz)))v \\
&= (u((xy)z))(bc) - (u((xy)z))(cb) - (u((xy)z))(bc) + (u(x(yz)))(bc) \\
&= -(u((xy)z))(cb) + (u(x(yz)))(bc) \\
&= -(u(x(yz)))(cb) + (u(x(yz)))(cb) \\
&\equiv 0 \mod (S, \overline{g})
\end{aligned}$$

14. $\overline{g} = u(\overline{f}v)$

$$a(bc) = u(((xy)z)v)$$

Therefore

$$a = u$$

$$bc = ((xy)z)v$$

then

$$b = (xy)z$$

$$c = v$$

Consider the composition $(g, f)_{\overline{g}}$

$$\begin{aligned}
(g, f)_{\overline{g}} &= a(bc) - a(cb) - u(((xy)z)v) + u((x(yz))v) \\
&= u(((xy)z)v) - u(v((xy)z)) - u(((xy)z)v) + u((x(yz))v) \\
&= -u(v((xy)z)) + u((x(yz))v) \\
&= -u(v(x(yz))) + u((x(yz))v)
\end{aligned}$$

Here we consider two cases:

(a) If $v > x(yz)$ then

$$(g,f)_{\overline{g}} = -u(v(x(yz))) + u((x(yz))v) = -u((x(yz))v) + u((x(yz))v) \equiv 0 \mod (S, \overline{g})$$

(b) If $v < x(yz)$ then

$$(g,f)_{\overline{g}} = -u(v(x(yz))) + u((x(yz))v) = -u(v(x(yz))) + u(v(x(yz))) \equiv 0 \mod (S, \overline{g})$$

Two cases holds the composition to zero. Therefore $(g,f)_{\overline{g}} = 0$

15. $\overline{g} = \overline{f}v$

$$a(bc) = ((xy)z)v$$

Therefore

$$a = (xy)z$$

$$bc = v$$

Consider the composition $(g,f)_{\overline{g}}$

$$\begin{aligned}(g,f)_{\overline{g}} &= a(bc) - a(cb) - ((xy)z)v + (x(yz))v \\ &= ((xy)z)(bc) - ((xy)z)(cb) - ((xy)z)(bc) + (x(yz))(bc) \\ &= -((xy)z)(cb) + (x(yz))(bc) \\ &\quad -(x(yz))(cb) + (x(yz))(cb) \\ &\equiv 0 \mod (S, \overline{g})\end{aligned}$$

16. $\overline{g} = u\overline{f}$

$$a(bc) = u((xy)z)$$

Therefore

$$a = u$$

$$bc = (xy)z$$

then

$$b = xy$$

$$c = z$$

Consider the composition $(g,f)_{\overline{g}}$

$$\begin{aligned}(g,f)_{\overline{g}} &= a(bc) - a(cb) - u((xy)z) + u(x(yz)) \\ &= u((xy)z) - u(z(xy)) - u((xy)z) + u(x(yz)) \\ &= -u(z(xy)) + u(x(yz))\end{aligned}$$

17. $\overline{g} = \overline{f}$

$$a(bc) = (xy)z$$

Therefore

$$a = xy$$

$$bc = z$$

Consider the composition $(g, f)_{\overline{g}}$

$$\begin{aligned}(g,f)_{\overline{g}} &= a(bc) - a(cb) - (xy)z + x(yz) \\ &= (xy)(bc) - (xy)(cb) - (xy)(bc) + x(y(bc)) \\ &= -(xy)(cb) + x(y(bc)) \\ &= -x(y(cb)) + x(y(cb)) \\ &\equiv 0 \mod (S, \overline{f})\end{aligned}$$

$S^0 = S \cup H_0$

$$H_0 = \{h_0 = -t_0(z_0(x_0y_0)) + t_0(x_0(y_0z_0)) : x_0y_0 > z_0, z_0 > x_0\}$$

18. $\overline{g} = u\overline{h_0}$

$$a(bc) = u(t_0(z_0(x_0y_0)))$$

Therefore

$$a = u$$

$$bc = t_0(z_0(x_0y_0))$$

then

$$b = t_0$$

$$c = x_0(y_0z_0)$$

Consider the composition $(g, h_0)_{\overline{g}}$

$$\begin{aligned}(g,h_0)_{\overline{g}} &= a(bc) - a(cb) - u(t_0(z_0(x_0y_0))) + u(t_0(x_0(y_0z_0))) \\ &= u(t_0(z_0(x_0y_0))) - u((x_0(y_0z_0))t_0) \\ &= -u(t_0(z_0(x_0y_0))) + u(t_0(x_0(y_0z_0))) \\ &= -u((x_0(y_0z_0))t_0) + u(t_0(x_0(y_0z_0))) \\ &= -u((x_0(y_0z_0))t_0) + u((x_0(y_0z_0))t_0) \\ &\equiv 0 \mod (S^0, \overline{g})\end{aligned}$$

19. $\overline{g} = (u\overline{h_0})v$

$$a(bc) = (u(t_0(z_0(x_0y_0))))v$$

$$a = u(t_0(z_0(x_0y_0)))$$

$$bc = v$$

$$\begin{aligned}(g,h_0)_{\overline{g}} &= a(bc)-a(cb)-(u(t_0(z_0(x_0y_0))))v+(u(t_0(x_0(y_0z_0))))v\\ &= (u(t_0(z_0(x_0y_0))))(bc)-(u(t_0(z_0(x_0y_0))))(cb)\\ &\quad -(u(t_0(z_0(x_0y_0))))(bc)+(u(t_0(x_0(y_0z_0))))(bc)\\ &= -(u(t_0(z_0(x_0y_0))))(cb)+(u(t_0(x_0(y_0z_0))))(bc)\\ &= -(u(t_0(x_0(y_0z_0))))(cb)+(u(t_0(x_0(y_0z_0))))(cb)\\ &\equiv 0 \mod (S^0,\overline{g})\end{aligned}$$

20. $\overline{g}=u(\overline{h_0}v)$

$$a(bc)=u((t_0(z_0(x_0y_0)))v)$$

$$a=u$$

$$b=t_0(z_0(x_0y_0))$$

$$c=v$$

$$\begin{aligned}(g,h_0)_{\overline{g}} &= a(bc)-a(cb)-u((t_0(z_0(x_0y_0)))v)+u((t_0(x_0(y_0z_0)))v)\\ &= u((t_0(z_0(x_0y_0)))v)-u(v(t_0(z_0(x_0y_0))))\\ &\quad -u((t_0(z_0(x_0y_0)))v)+u((t_0(x_0(y_0z_0)))v)\\ &= -u(v(t_0(z_0(x_0y_0))))+u((t_0(x_0(y_0z_0)))v)\\ &= -u(v(t_0(x_0(y_0z_0))))+u(v(t_0(x_0(y_0z_0))))\\ &\equiv 0 \mod (S^0,\overline{g})\end{aligned}$$

21. $\overline{g}=\overline{h_0}v$

$$a(bc)=(t_0(z_0(x_0y_0)))v$$

$$a=t_0(z_0(x_0y_0))$$

$$bc=v$$

$$\begin{aligned}(g,h_0)_{\overline{g}} &= a(bc)-a(cb)-(t_0(z_0(x_0y_0)))v+(t_0(x_0(y_0z_0)))v\\ &= (t_0(z_0(x_0y_0)))(bc)-(t_0(z_0(x_0y_0)))(cb)\\ &\quad -(t_0(z_0(x_0y_0)))(bc)+(t_0(x_0(y_0z_0)))(bc)\\ &= -(t_0(z_0(x_0y_0)))(cb)+(t_0(x_0(y_0z_0)))(bc)\\ &= -(t_0(x_0(y_0z_0)))(cb)+(t_0(x_0(y_0z_0)))(cb)\\ &\equiv 0 \mod (S^0,\overline{g})\end{aligned}$$

22. $\overline{g}=\overline{h_0}$

$$a(bc)=t_0(z_0(x_0y_0))$$

$$a=t_0$$

$$b=z_0$$

$$c=x_0y_0$$

$$\begin{aligned}(g,h_0)_{\overline{g}} &= a(bc)-a(cb)-t_0(z_0(x_0y_0))+t_0(x_0(y_0z_0))\\ &= t_0(z_0(x_0y_0))-t_0((x_0y_0)z_0)-t_0(z_0(x_0y_0))+t_0(x_0(y_0z_0))\\ &= -t_0((x_0y_0)z_0)+t_0(x_0(y_0z_0))\\ &\equiv 0 \mod (S^0,\overline{g})\end{aligned}$$

Let 's denote

$$h_1 := -t_1(z_1(x_1y_1))+t_1(x_1(y_1z_1))$$

23. $h_1 = u(h_0v)$

$$t_1(z_1(x_1y_1)) = u((t_0(z_0(x_0y_0)))v)$$

$$t_1 = u$$

$$z_1 = t_0(z_0(x_0y_0))$$

$$x_1y_1 = v$$

$$\begin{aligned}(g,h_0)_{\overline{g}} &= -t_1(z_1(x_1y_1))+t_1(x_1(y_1z_1))\\ &\quad -u((t_0(z_0(x_0y_0)))v)+u((t_0(x_0(y_0z_0)))v)\\ &= u((t_0(z_0(x_0y_0)))(x_1y_1))-u(x_1(y_1(t_0(z_0(x_0y_0)))))\\ &\quad -u((t_0(z_0(x_0y_0)))(x_1y_1))+u((t_0(x_0(y_0z_0)))(x_1y_1))\\ &= -u(x_1(y_1(t_0(z_0(x_0y_0)))))+u((t_0(x_0(y_0z_0)))(x_1y_1))\\ &= -u(x_1(y_1(t_0(z_0(x_0y_0)))))+u(x_1(y_1(t_0(z_0(x_0y_0)))))\\ &\equiv 0 \mod (S^0,\overline{g})\end{aligned}$$

24. $h_1 = (uh_0)v$

$$t_1(z_1(x_1y_1)) = (u((t_0(z_0(x_0y_0)))))v$$

$$t_1 = u((t_0(z_0(x_0y_0)))$$

$$z_1(x_1y_1) = v$$

$$\begin{aligned}(g,h_0)_{\overline{g}} &= t_1(z_1(x_1y_1))-t_1(x_1(y_1z_1))\\ &\quad -(u(t_0(z_0(x_0y_0))))v+(u(t_0(x_0(y_0z_0))))v =\\ &= (u(t_0(z_0(x_0y_0))))(z_1(x_1y_1))-(u(t_0(z_0(x_0y_0))))(x_1(y_1z_1))\\ &\quad -(u(t_0(z_0(x_0y_0))))(z_1(x_1y_1))+(u(t_0(x_0(y_0z_0))))(z_1(x_1y_1))\\ &= -(u(t_0(z_0(x_0y_0))))(x_1(y_1z_1))+(u(t_0(x_0(y_0z_0))))(z_1(x_1y_1))\\ &= -(u(t_0(x_0(y_0z_0))))(x_1(y_1z_1))+(u(t_0(x_0(y_0z_0))))(x_1(y_1z_1))\\ &\equiv 0 \mod (S^0,\overline{g})\end{aligned}$$

25. $h_1 = uh_0$

$$t_1(z_1(x_1y_1)) = u(t_0(z_0(x_0y_0)))$$

$$t_1 = u$$

$$z_1 = t_0$$

$$x_1 = z_0$$

$$y_1 = x_0y_0$$

$$\begin{aligned}(g,h_0)_{\overline{g}} = t_1(z_1(x_1y_1)) - t_1(x_1(y_1z_1)) - u(t_0(z_0(x_0y_0))) + u(t_0(x_0(y_0z_0)))\\ = u(t_0(z_0(x_0y_0))) - u(z_0((x_0y_0)t_0))\\ -u(t_0(z_0(x_0y_0))) + u(t_0(x_0(y_0z_0)))\\ = -u(z_0((x_0y_0)t_0)) + u(t_0(x_0(y_0z_0)))\\ = -u(z_0((x_0y_0)t_0)) + u(t_0(x_0(y_0z_0)))\\ = -u((x_0y_0)(t_0z_0)) + u(x_0((y_0z_0)t_0))\\ = -u(x_0(y_0(t_0z_0))) + u(x_0(y_0(z_0t_0)))\\ \equiv 0 \mod (S^0, \overline{g})\end{aligned}$$

26. $h_1 = h_0v$

$$t_1(z_1(x_1y_1)) = (t_0(z_0(x_0y_0)))v$$

$$t_1 = t_0(z_0(x_0y_0))$$

$$z_1(x_1y_1) = v$$

$$\begin{aligned}(g,h_0)_{\overline{g}} = t_1(z_1(x_1y_1)) - t_1(x_1(y_1z_1)) - (t_0(z_0(x_0y_0)))v + (t_0(x_0(y_0z_0)))v\\ = (t_0(z_0(x_0y_0)))(z_1(x_1y_1)) - (t_0(z_0(x_0y_0)))(x_1(y_1z_1))\\ -(t_0(z_0(x_0y_0)))(z_1(x_1y_1)) + (t_0(x_0(y_0z_0)))(z_1(x_1y_1))\\ = -(t_0(z_0(x_0y_0)))(x_1(y_1z_1)) + (t_0(x_0(y_0z_0)))(z_1(x_1y_1))\\ = -(t_0(x_0(y_0z_0)))(x_1(y_1z_1)) + (t_0(x_0(y_0z_0)))(x_1(y_1z_1))\\ \equiv 0 \mod (S^0, \overline{g})\end{aligned}$$

27. $\overline{h_1} = u(\overline{g}v)$

$$t_1(z_1(x_1y_1)) = u((a(bc))v)$$

$$t_1 = u$$

$$z_1 = a(bc)$$

$$x_1y_1 = v$$

$$\begin{aligned}(h_1,g)_{\overline{h_1}} = t_1(z_1(x_1y_1)) - t_1(x_1(y_1z_1)) - u((a(bc))v) + u((a(cb))v)\\ = u((a(bc))(x_1y_1)) - u(x_1(y_1(a(bc)))) - u((a(bc))(x_1y_1)) + u((a(cb))(x_1y_1))\\ = -u(x_1(y_1(a(bc)))) + u((a(cb))(x_1y_1))\\ = -u(x_1(y_1(a(cb)))) + u(x_1(y_1(a(cb))))\\ \equiv 0 \mod (S^0, \overline{h_1})\end{aligned}$$

28. $\overline{h_1} = (u\overline{g})v$

$$t_1(z_1(x_1y_1)) = (u(a(bc)))v$$

$$t_1 = u(a(bc))$$

$$z_1(x_1y_1) = v$$

$$\begin{aligned}
(h_1, g)_{\overline{h_1}} &= t_1(z_1(x_1y_1)) - t_1(x_1(y_1z_1)) - u((a(bc))v) + u((a(cb))v) \\
&= (u(a(bc)))(z_1(x_1y_1)) - (u(a(bc)))(x_1(y_1z_1)) \\
&\quad -u((a(bc))(z_1(x_1y_1))) + u((a(cb))(z_1(x_1y_1))) \\
&= -(u(a(bc)))(x_1(y_1z_1)) + u((a(cb))(z_1(x_1y_1))) \\
&= -(u(a(bc)))(x_1(y_1z_1)) + u((a(cb))(z_1(x_1y_1))) \\
&= -(u(a(cb)))(x_1(y_1z_1)) + u((a(cb))(x_1(y_1z_1))) \\
&\equiv 0 \mod (S^0, \overline{h_1})
\end{aligned}$$

29. $\overline{h_1} = u\overline{g}$

$$t_1(z_1(x_1y_1)) = u(a(bc))$$

$$\therefore t_1 = u$$

$$z_1 = a$$

$$x_1 = b$$

$$y_1 = c$$

$$\begin{aligned}
(h_1, g)_{\overline{h_1}} &= t_1(z_1(x_1y_1)) - t_1(x_1(y_1z_1)) - u(a(bc)) + u(a(cb)) \\
&= u(a(bc)) - u(b(ca)) - u(a(bc)) + u(a(cb)) \\
&= -u(b(ca)) + u(a(cb)) \\
&= -u(c(ab)) + u(a(cb)) \\
&\equiv 0 \mod (S^0, \overline{h_1})
\end{aligned}$$

30. $\overline{h_1} = \overline{g}v$

$$t_1(z_1(x_1y_1)) = (a(bc))v$$

$$t_1 = a(bc)$$

$$z_1(x_1y_1) = v$$

$$\begin{aligned}
(h_1, g)_{\overline{h_1}} &= t_1(z_1(x_1y_1)) - t_1(x_1(y_1z_1)) - (a(bc))v + (a(cb))v \\
&= (a(bc))(z_1(x_1y_1)) - (a(bc))(x_1(y_1z_1)) \\
&\quad -(a(bc))(z_1(x_1y_1)) + (a(cb))(z_1(x_1y_1)) \\
&= -(a(cb))(x_1(y_1z_1)) + (a(cb))(x_1(y_1z_1)) \\
&\equiv 0 \mod (S^0, \overline{h_1})
\end{aligned}$$

31. $\overline{f} = \overline{h_0}$

$$(xy)z = t_0(z_0(x_0y_0))$$

$$xy = t_0$$
$$z = z_0(x_0y_0)$$

$$\begin{aligned}(f, h_0)_{\overline{f}} &= (xy)z - x(yz) - t_0(z_0(x_0y_0)) + t_0(x_0(y_0z_0)) \\ &= (xy)(z_0(x_0y_0)) - x(y(z_0(x_0y_0))) - (xy)(z_0(x_0y_0)) + (xy)(x_0(y_0z_0)) \\ &= -x(y(z_0(x_0y_0))) + (xy)(x_0(y_0z_0)) \\ &= -x(y(x_0(y_0z_0))) + x(y(x_0(y_0z_0))) \\ &\equiv 0 \mod (S^0, \overline{f})\end{aligned}$$

32. $\overline{f} = (u\overline{h_0})v$

$$(xy)z = (u(t_0(z_0(x_0y_0))))v$$
$$x = u$$
$$y = t_0(z_0(x_0y_0))$$
$$z = v$$

$$\begin{aligned}(f, h_0)_{\overline{f}} &= (xy)z - x(yz) - (u(t_0(z_0(x_0y_0))))v + (u(t_0(x_0(y_0z_0))))v \\ &= (u(t_0(z_0(x_0y_0))))v - u((t_0(z_0(x_0y_0)))v) \\ &\quad -(u(t_0(z_0(x_0y_0))))v + (u(t_0(x_0(y_0z_0))))v \\ &= -u((t_0(z_0(x_0y_0)))v) + (u(t_0(x_0(y_0z_0))))v \\ &= -u((t_0(x_0(y_0z_0)))v) + u((t_0(x_0(y_0z_0)))v) \\ &\equiv 0 \mod (S^0, \overline{f})\end{aligned}$$

33. $\overline{f} = u(\overline{h_0}v)$

$$(xy)z = u((t_0(z_0(x_0y_0)))v)$$
$$xy = u$$
$$z = (t_0(z_0(x_0y_0)))v$$

$$\begin{aligned}(f, h_0)_{\overline{f}} &= (xy)z - x(yz) - u((t_0(z_0(x_0y_0)))v) + u((t_0(x_0(y_0z_0)))v) \\ &= (xy)((t_0(z_0(x_0y_0)))v) - x(y((t_0(z_0(x_0y_0)))v)) \\ &\quad -(xy)((t_0(z_0(x_0y_0)))v) + (xy)((t_0(x_0(y_0z_0)))v) \\ &= -x(y((t_0(x_0(y_0z_0)))v)) + x(y((t_0(x_0(y_0z_0)))v)) \\ &\equiv 0 \mod (S^0, \overline{f})\end{aligned}$$

34. $\overline{f} = \overline{h_0}v$

$$(xy)z = (t_0(z_0(x_0y_0)))v$$
$$x = t_0$$
$$y = z_0(x_0y_0)$$

$$z = v$$

$$\begin{aligned}(f,h_0)_{\overline{f}} &= (xy)z - x(yz) - (t_0(z_0(x_0y_0)))v + (t_0(x_0(y_0z_0)))v \\ &= (t_0(z_0(x_0y_0)))v - t_0((z_0(x_0y_0))v) - (t_0(z_0(x_0y_0)))v + (t_0(x_0(y_0z_0)))v \\ &= -t_0((z_0(x_0y_0))v) + (t_0(x_0(y_0z_0)))v \\ &= -t_0(z_0(x_0y_0)v)) + (t_0(x_0(y_0z_0v))) \\ &= -t_0((x_0y_0)(vz_0)) + (t_0(x_0(y_0z_0v))) \\ &= -t_0(x_0(y_0(vz_0))) + (t_0(x_0(y_0z_0v))) \\ &\equiv 0 \mod (S^0, \overline{f})\end{aligned}$$

35. $\overline{f} = u\overline{h_0}$

$$(xy)z = u(t_0(z_0(x_0y_0)))$$

$$xy = u$$

$$z = t_0(z_0(x_0y_0))$$

$$\begin{aligned}(f,h_0)_{\overline{f}} &= (xy)z - x(yz) - u(t_0(z_0(x_0y_0))) + u(t_0(x_0(y_0z_0))) \\ &= (xy)(t_0(z_0(x_0y_0))) - x(y(t_0(z_0(x_0y_0)))) \\ &\quad -(xy)(t_0(z_0(x_0y_0))) + (xy)(t_0(x_0(y_0z_0))) \\ &= -x(y(t_0(z_0(x_0y_0)))) + (xy)(t_0(x_0(y_0z_0))) \\ &= -x(y(t_0(x_0(y_0z_0)))) + x(y(t_0(x_0(y_0z_0)))) \\ &\equiv 0 \mod (S^0, \overline{f})\end{aligned}$$

36. $\overline{h_0} = u(\overline{f}v)$

$$t_0(z_0(x_0y_0)) = u(((xy)z)v)$$

$$t_0 = u$$

$$z_0 = (xy)z$$

$$x_0y_0 = v$$

$$\begin{aligned}(h_0,f)_{\overline{h_0}} &= t_0(z_0(x_0y_0)) - t_0(x_0(y_0z_0)) - u(((xy)z)v) + u((x(yz))v) \\ &= u(((xy)z)(x_0y_0)) - u(x_0(y_0((xy)z))) \\ &\quad -u(((xy)z)(x_0y_0)) + u((x(yz))(x_0y_0)) \\ &= -u(x_0(y_0((xy)z))) + u((x(yz))(x_0y_0)) \\ &= -u(x_0(y_0(x(yz)))) + u(x_0(y_0(x(yz)))) \\ &\equiv 0 \mod (S^0, \overline{h_0})\end{aligned}$$

37. $\overline{h_0} = (u\overline{f})v$

$$t_0(z_0(x_0y_0)) = (u((xy)z))v$$

$$t_0 = u((xy)z)$$

$$z_0(x_0y_0) = v$$

$$\begin{aligned}(h_0, f)_{\overline{h_0}} &= t_0(z_0(x_0y_0)) - t_0(x_0(y_0z_0)) - (u(((xy)z))v + (u((x(yz)))v \\ &= (u((xy)z))(z_0(x_0y_0)) - (u((xy)z))(x_0(y_0z_0)) \\ &\quad -(u(((xy)z))(z_0(x_0y_0)) + (u(x(yz)))(z_0(x_0y_0)) \\ &= -(u((xy)z))(x_0(y_0z_0)) + (u(x(yz)))(z_0(x_0y_0)) \\ &= -(u(x(yz)))(x_0(y_0z_0)) + (u(x(yz)))(x_0(y_0z_0)) \\ &\equiv 0 \mod (S^0, \overline{h_0})\end{aligned}$$

38. $\overline{h_0} = \overline{f}v$

$$t_0(z_0(x_0y_0)) = ((xy)z)v$$

$$t_0 = (xy)z$$

$$z_0(x_0y_0) = v$$

$$\begin{aligned}(h_0, f)_{\overline{h_0}} &= t_0(z_0(x_0y_0)) - t_0(x_0(y_0z_0)) - ((xy)z)v + (x(yz))v \\ &= ((xy)z)(z_0(x_0y_0)) - ((xy)z)(x_0(y_0z_0)) \\ &\quad -((xy)z)(z_0(x_0y_0)) + (x(yz))(z_0(x_0y_0)) \\ &= -((xy)z)(x_0(y_0z_0)) + (x(yz))(z_0(x_0y_0)) \\ &= -(x(yz))(x_0(y_0z_0)) + (x(yz))(x_0(y_0z_0)) \\ &\equiv 0 \mod (S^0, \overline{h_0})\end{aligned}$$

39. $\overline{h_0} = u\overline{f}$

$$t_0(z_0(x_0y_0)) = u((xy)z)$$

$$t_0 = u$$

$$z_0 = xy$$

$$x_0y_0 = z$$

$$\begin{aligned}(h_0, f)_{\overline{h_0}} &= t_0(z_0(x_0y_0)) - t_0(x_0(y_0z_0)) - u((xy)z) + u((x(yz))) \\ &= u((xy)(x_0y_0)) - u(x_0(y_0(xy))) - u((xy)(x_0y_0)) + u((x(y(x_0y_0)))) \\ &= -u(x_0(y_0(xy))) + u(x(y(x_0y_0)))\end{aligned}$$

Consider the following cases

(a) If $y > y_0$ then $x_0 > x$ because $x_0y_0 > z_0 = xy$

$$\begin{aligned}
(h_0, f)_{\overline{h_0}} &= -u(x_0(y_0(xy))) + u(x(y(x_0y_0))) \\
&= -u(x_0(x(yy_0))) + u(x(x_0(y_0y))) \\
&= -u((x((yy_0)x_0))) + u(x(x_0(y_0y))) \\
&= -u((x(x_0(y_0y)))) + u(x(x_0(y_0y))) \\
&\equiv 0 \mod (S^0, \overline{h_0})
\end{aligned}$$

(b) If $y < y_0$ and $x > x_0$

$$\begin{aligned}
(h_0, f)_{\overline{h_0}} &= -u(x_0(y_0(xy))) + u(x(y(x_0y_0))) \\
&= -u(x_0(x(yy_0))) + u(x(x_0(y_0y))) \\
&= -u(x_0(x(yy_0))) + u(x_0((y_0y)x)) \\
&= -u(x_0(x(yy_0))) + u(x_0(x(yy_0))) \\
&\equiv 0 \mod (S^0, \overline{h_0})
\end{aligned}$$

(c) If $y < y_0$ and $x < x_0$

$$\begin{aligned}
(h_0, f)_{\overline{h_0}} &= -u(x_0(y_0(xy))) + u(x(y(x_0y_0))) \\
&= -u(x_0(x(yy_0))) + u(x(x_0(y_0y))) \\
&= -u(x((yy_0)x_0)) + u(x(x_0(yy_0))) \\
&= -u(x(x_0(yy_0))) + u(x(x_0(yy_0))) \\
&\equiv 0 \mod (S^0, \overline{h_0})
\end{aligned}$$

All cases leads composition cases leads composition to 0

We have reviewed all the compositions in S^0. We have reviewed all the compositions . All compositions are trivial. Then S^0- Gröbner-Shirshov basis. $Irr(S^0)$-linear basis Perm algebras.

Chapter 5

Gröbner–Shirshov bases theory for Zinbiel algebras

5.1 Composition-Diamond lemma for Zinbiel algebras

Let $Z(X)$ linear basis of Zinbiel algebras generated by X.By theorem 4.3 $Z(X)$ consists of $[x_1...x_n]$. Then any element of $f \in Zin(X)$ $\exists! w_1, ..., w_n \in Z(X), w_1 > w_2 > ...w_n, \lambda_1, ...\lambda_n \in k\backslash\{0\}$:

$$f = \lambda_1 w_1 + ... + \lambda_n w_n$$

Denote by $\overline{f}$ the leading word w_1 of f
An element f is called monic if the cöfficient of $\overline{f}$ is 1
Let S be a monic subset of $Zin(X)$. Denote by $Id(S)$ the ideal of $Zin(X)$ generated by S.
Definition 5.1. For every $s \in S$, we call s an $S - polynomial$ and for every $S - polynomial$ h, for every μ in β, both μh and $h\mu$ are called $S - polynomials$.
Definition.5.2 For each $s \in S$,$n \geq 0$, $x_1, ...x_n \in X$ the $[x_1, ..., x_n, s]_R$ polynomial called normal S- polynomial (nSp)
For every $f \in Zin(X)$ and $\omega \in Z(X)$, the polynomial f is sa to be trivial modulo S with respect to ω, denoted by

$$h \equiv 0 \mod (S, \omega)$$

if f can be written as a linear combination of normal S-polynomials such that their leading monomials are $< \omega$
Definition 5.3. For all $f, g \in S$, we define compositions of S as follows:

1. If $\overline{f} = [b_1, ..., b_m, \overline{g}]$ for some $b_1, ..., b_m \in X$ then

$$(f, g)_{\overline{f}} = f - [b_1, b_2, ..., b_m, g]$$

is called **inclusion composition**

2. For each $\mu \in \beta$, if $fa \neq 0$ Then we call $f \cdot \mu$ a **right mulitiplication composition.**

The set S called a Gröbner-Shirshov basis in $Zin(X)$ if every right mulitiplication composition fa trivial modulo $(S, \overline{f}a)$ and every inclusion composition $(f, g)_{\overline{f}}$ trivial modulo $(S, \overline{f})$

Lemma 1. Let S - Gröbner-Shirshov basis. Then $\forall \mu \in Z(X)$ and $\forall h - nSp$.The expression $= \mu h + h\mu$-S can be written as a linear combination of normal S-polynomials whose leading monomials are $\leq \max\{\overline{\mu h}, \overline{h\mu}\}$.

Proof.$h - nSp, h = [a_1, a_2, , , , a_k, s]$ for n_s function is defined as $n_s(h) = k$. We perform induction on $|\mu| + n_s(h) = 1$.Then $|\mu| = 1$ & $n_s(h) = 0$ or $|\mu| = 0$ & $n_s(p) = 1$. It is enough to indicate one of two cases.For $|\mu| = 1, n_s(h) = 0$

$$\mu h + h\mu = \mu s + s\mu$$

the first connector is nSp and the second is right nultiplication. There, the conclusion of the Lemma is appropriate for them. For $|\mu| + n_s(h) < k$, the lemma is executed . We will show that the lemma also holds for $|\mu| + n_s(h) = k$. The following situations may occur.

1. $|\mu| = 1 \Rightarrow \mu \in X$ & $n_s(h) \geq 1 \Rightarrow h = ah_1$

   $$\mu h + h\mu = \mu h + (ah_1)\mu = \mu h + a(h_1\mu) + a(\mu h_1) = \mu h + a(h_1\mu + \mu h_1)$$

   the first connector is nSp and the second by induction , the conclusion of the lemma is fulfilled.

2. $|\mu| > 1$ & $n_s(h) = 0 \Rightarrow h = s \in S$ & $\mu = \mu_1\mu_2$ & $\mu_1 \in X$

   $$\mu h + h\mu = (\mu_1\mu_2)s + s\mu = \mu_1(\mu_2 s + s\mu_2) + s\mu$$

   by induction , the conclusion of the lemma is fulfilled.

3. $|\mu| > 1$ & $n_s(h) > 0, \Rightarrow h = ah_1$ & $\mu = \mu_1\mu_2$ & $a, \mu_1 \in X$

   $$\mu h + h\mu = (\mu_1\mu_2)h + (ah_1)\mu = \mu_1(\mu_2 h + h\mu_2) + a(h_1\mu + \mu h_1)$$

   by induction , the conclusion of the lemma is fulfilled.

Corollary 1. Let S - Gröbner-Shirshov basis. Then for $\forall \mu \in Z(X)$ and $\forall h - nSp$, μh can be written as a linear combination of normal S-polynomials whose leading monomials are $\leq \overline{\mu h}$.

Corollary 2. Let S - Gröbner-Shirshov basis. Then for $\forall \mu \in Z(X)$ and $\forall h - nSp$, $h\mu$ can be written as a linear combination of normal S-polynomials whose leading monomials are $\leq \overline{h\mu}$.

Corollary 3. Let S - Gröbner-Shirshov basis. Then $\forall l$- S-polynomial can be written as a linear combination of normal S-polynomials such that their leading monomials are $\leq \overline{l}$

Lemma 2. Let S - Gröbner-Shirshov basis. If $f, g - nSp$ & $\overline{f} = \overline{g}$, then

$$f - g \equiv 0 \mod (S, \overline{f})$$

Proof. Let $f = [a_1, a_2, ..., a_n, s_1]$, $g = [b_1, b_2, ..., b_n, s_2]$. Then $\overline{f} = \overline{g} \Rightarrow$

$$[a_1, a_2, ...a_n, \overline{s_1}] = [b_1, b_2, ...b_m, \overline{s_2}]$$

$$\Rightarrow a_1 = b_1, ..., a_n = b_n, \overline{s_1} = [b_{n+1}, ...b_m, \overline{s_2}]$$

$$f-g = [a_1, a_2, ...a_n, \overline{s_1}] - [b_1, b_2, ...b_m, \overline{s_2}] = [a_1, a_2, ...a_n, (s_1, s_2)_{\overline{s_1}}] \equiv 0 \mod (S, \overline{f})$$

Definition 5.4.

$$Irr(S) := \{x \in Z(X) : x \neq \overline{h} \quad \forall h - nSp\}$$

Lemma 3. For every $f \in Zin(X)$ we have

$$f = \sum_{u_i \leq \overline{f}} \lambda_i u_i + \sum_{\overline{h} \leq \overline{f}} \alpha_i h_i$$

where $\lambda_i, \alpha_i \in k,\ u_i \in Irr(S), h_i - nSp$.
Proof. $f = \sum_i \gamma_i f_i, \gamma_i \neq 0, f_1 > f_2 > f_3 > \dots$. If $f_1 \in Irr(S)$,then we take $g = f - \gamma_1 f_1$, if $f_1 \notin Irr(S) \Rightarrow \exists h - nSp[f_1 = \overline{h}] \Rightarrow$ then we take $g = f - \gamma_1 h$. Both case $\overline{g} < \overline{f}$. then we will do the induction by $\overline{f}$.
Theorem 5.1 (Composition-Diamond lemma for Zinbiel algebras). Let S monic subset of $Zin(X)$. Then the followings are equivalent:

1. S- Gröbner-Shirshov basis
2. If $f \in Id(S)$ and $f \neq 0$ then $\overline{f} = \overline{h}$ for some normal S-polynomial h
3. The set $Irr(S)$ is linear basis of the algebra $Zin(X|S) = Zin(X)/Id(S)$

Proof.1) $\Rightarrow$ 2) $f \in Id(S)$ Then

$$f = \sum_{i=1}^{n} \alpha_i h_i$$

where $\overline{h_1} = \overline{h_2} = \dots = \overline{h_l} > \overline{h_{l+1}} \geq \dots$.
If $\sum_{i=1}^{l} \neq 0$, the conclusion is correct. Let $\sum_{i=1}^{l} = 0$.

$$f = \sum_{i=1}^{n} \alpha_i h_i = \sum_{i=1}^{l} \alpha_i h_i + \sum_{i=l+1}^{n} \alpha_i h_i = \sum_{i=1}^{l} \alpha_i h_1 - \sum_{i=2}^{l} \alpha_i (h_1 - h_i) + \sum_{i=l+1}^{n} \alpha_i h_i =$$

$$= \sum_{i=1}^{l} \gamma_i h_i' + \sum_{i=l+1}^{n} \alpha_i h_i.$$

$\overline{h_i'} < \overline{h_1}$. Claim 2) follows by induction hypothesis

$2) \Rightarrow 3)$ Let in algebras $Zin(X|S)$ $\sum\limits_i \alpha_i\omega_i = 0$. Then $\sum\limits_i \alpha_i\omega_i \in Id(S) \Rightarrow \forall i[\alpha_i =$ $0]$. Because, if $\exists i[\alpha_i \neq 0]$ then $\overline{\sum\limits_i \alpha_i\omega_i} = \omega_i$. This is contrary to 2) .
By Lemma 3 $Irr(S)$ linear shell $Zin(X|S)$ algebras. Therefore, $Irr(S)$ linear basis $Zin(X|S)$ algebras.
$3) \Rightarrow 1)$ Let $f \in S$ .By Lemma 3 and 3) $f = \sum\limits_i \alpha_i h_i$. All composiotions trivial then we obtain 1).

Chapter 6

Conclusions and future work

6.1 Conclusions

The paper finds a linear basis for the Zinbiel algebra, Leibniz algebra, Associative algebra, and Perm algebra using the Gröbner-Shirshov theory for magmatic algebras. The following theorem for Zinbiel algebras is obtained Let S monic subset of $Zin(X)$. Then the followings are equivalent:

1. S- Gröbner-Shirshov basis
2. If $f \in Id(S)$ and $f \neq 0$ then $\overline{f} = \overline{h}$ for some normal S-polynomial h
3. The set $Irr(S)$ is linear basis of the algebra $Zin(X|S) = Zin(X)/Id(S)$

This theorem is Composition-Diamond lemma for Zinbiel algebras. It is theory of basis of Gröbner-Shirshov for Zinbiel algebras.

6.2 Future work

In [13], the Zinbiel superalgebras were defined. It is necessary to construct a theory of the Gröbner Shirshov basis for Zinbiel superalgebra

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