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On existence of a solution of a class of second order differential equations

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ABSTRACT

The paper was devoted to the study of issues on the existence and compactness of a resolvent, as well as estimates of the distribution function of singular numbers (s-numbers) of a differential operator of mixed type in a noncompact domain. We used the embedding theorems for Sobolev-type weighted spaces, the method of a priori estimates, the uniform localization method. In this paper, besides the above methods, an approach is proposed that allows one to find two-sided estimates of singular numbers (approximation numbers, s-numbers) of differential operators containing a sign-changing parameter. As a result of the study, the following results were obtained for a class of mixed-type differential operators defined in a noncompact domain: a theorem on the existence of a bounded inverse operator was proved; a theorem on the compactness (discreteness of the spectrum) of the resolvent of a mixed-type operator was proved; a conditions to the two-sided estimate for the distribution function of singular numbers (s-numbers) of the resolvent of a mixed-type differential operator given in an unbounded domain are obtained.

АҢДАТПА

Бұл жұмыс компактілі емес облыста берілген аралас типті дифференциалдық оператордың резольвентасының бар болуы және компактылығы, және де сонымен қатар сингулярлық сандарының (s-сандардың) үлестірімділік функциялары бағалаулары мәселелерін зерттеуге Жұмыста Соболев типті салмақты кеңістіктердің енгізу теоремалары, априорлы бағалаулар әдісі, бірқалыпты локализациялау әдісі қолданылды. Жоғарыда аталған әдістерден бөлек, жұмыста таңбасы ауыспалы параметрі бар дифференциалдық операторлардың сингулярлы сандарының (аппроксимативті сандар, s-сандары) екіжақты бағалауларын табуға мүмкіндік беретін тәсілдер ұсынылған. Зерттеу нәтижесінде мынадайнәтижелер алынған: шенелген кері оператордың бар болуы туралы теорема дәлелденді;аралас типті оператор резольвентасының компактілігі (спектр дискреттілігі) туралы теорема дәлелденді;шенелмеген облыста берілген аралас типті дифференциалдық оператор резольвентасының сингулярлық сандарының (s-сандарының) үлестірім функцияларының екіжақты бағалауларыу алуға алғышарттар жасалған.

АННОТАЦИЯ

Работа была посвящена исследованию вопросов о существовании и компактности резольвенты, а также об оценках функции распределения сингулярных чисел (s-чисел) дифференциального оператора смешанного типа в некомпактной области. В работе были использованы теоремы вложения весовых пространств типа Соболева, метод априорных оценок, метод равномерной локализации. В работе, кроме вышеуказанных методов, предложен подход позволяющий найти двусторонние оценки сингулярных чисел (аппроксимационных чисел, s-чисел) дифференциальных операторов содержащих параметр меняющий знак. В результате исследования для одного класса дифференциальных операторов смешанного типа, заданных в некомпактной области, получены следующие результаты: доказана теорема о существовании ограниченного обратного оператора; доказана теорема о компактности (дискретности спектра) резольвенты оператора смешанного типа; получены условия для двусторонних оценок функции распределения сингулярных чисел (s-чисел) резольвенты дифференциального оператора смешанного типа заданного в неограниченной области.

CONTENT

INTRODUCTION

The modern theory of equations of mixed type is one of the most important sections in the theory of partial differential equations. This follows from the theoretical significance of the results obtained, and their presence in applications of gas dynamics, transonic flow, hydrodynamics, quantum mechanics and in other areas of the natural sciences.

Works are known in the case of a bounded domain, the solvability of boundary value problems and spectral questions are studied depending on the boundary conditions and the geometry of the domain.

However, in applications there are often differential operators of mixed type given in an unbounded domain and having increasing and oscillating coefficients. Also, as is known from the spectral theory of differential operators, that the spectra of differential operators defined in an unbounded domain can turn out to be not only discrete, but also continuous. Therefore, any function from the domain of the operator can not be expanded in terms of its eigenfunctions.

We note that equations of mixed type are called equations which in one part of the domain under consideration belong to the elliptic type, and in the other part hyperbolic. These parts are separated by a line (or surface) of the transition, on which the equation either degenerates into a parabolic equation or is not defined.

A very exhaustive bibliography on the investigation of equations of mixed type is contained, for example, in the works of M.A. Lavrent'ev [1], A.V. Bitsadze [2,3], A.M. Nakhushev [4,5], M.M. Smirnova [6,7], T.Sh. Kalmenova [8], E.I. Moiseeva [9,10], M.S. Salakhitdinov [11-12], T.Sh. Kalmenova, M.A. Sadibekov, N.B. Erzhanov [13], A.S. Berdyshev [14], S. Moravetz [15] and others. In these papers, in the case of a bounded domain, the solvability of boundary value problems and spectral questions are studied depending on the boundary conditions and the geometry of the domain.

However, in applications there are often differential operators of mixed type given in an unbounded domain and having increasing and oscillating coefficients. We note, as is known from the spectral theory of differential operators, that the spectra of differential operators defined in an unbounded domain can turn out to be not only discrete, but also continuous. Therefore, any function from the domain of the operator can not be expanded in terms of its eigenfunctions.

A review of the literature shows that for singular mixed differential operators given in an unbounded region, items such as:

1) the existence of a bounded inverse operator;

2) continuity and discreteness of the spectrum;

3) the asymptotics of the distribution function of eigenvalues have not been studied sufficiently.

A singular differential operator of mixed type in $L_2(R^2)$

$$Mu = s(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + a(x)u_t + c(x)u \ ,$$

is considered, where $u \in D(L)$. $s(x)$ is a piecewise-continuous and bounded function that changes sign in R, $a(x)$ is a locally summable real function. $c(x)$ is a nonnegative, locally summable function.

It is easy to verify that the spectral problems of the operator M can be reduced by the Fourier method to the study of the spectral characteristics of a one-dimensional Schrödinger operator with a sign changing parameter:

$$m_t u = -u''(x) + (-s(x)t^2 + ita(x) + c(x))u, \ u \in D(l_t), \tag{*}$$

where t is the parameter $-\infty < t < +\infty$.

As is well known, when $t = 0$ the spectral properties of the Schrödinger operator $M_0 \equiv -\frac{d^2}{dx^2} + c(x)$ strongly depend on the behavior $c(x)$ at infinity. In this case, the spectral characteristics of the Schrödinger operator are sufficiently well studied in the following papers: A.M. Molchanov [16], T. Cato [17], E.Ch. Titchmarsh [18], A.G. Kostyuchenko [19], V.G. Maz'ya [20], M.Sh. Birman, M.Z. Solomyak [21-22], K.Kh. Boimatov [23-24], M.G. Gasymov, B.M. Levitan [25], I.M. Glazman, V.A. Skachek [26], V.A. Ilyin [27], M.Otelbaev [28-29], Ya.T. Slutnayev [30], R.Richtmayer [31], C. Mizohato [32], M. Taylor [33], M.A. Naimark [34], M.Rid, B.Saimon [35], M.Rid, B.Saimon [36], V.B. Lidsky [37].

It is not difficult to see, for example, that in the operator m_t we get $-s(x)t^2 = -t^2 \to -\infty$ when $ks(x) \equiv 1$. It is clear from this that an entirely different situation arises, and in particular, the methods worked out for the Schrödinger operator at $t = 0$, turn out to be little adapted when studying the operator m_t for $t \neq 0$. We also note that the operator m_t is not semibounded.

From this and from the foregoing it follows that for the Schrodinger operator with the parameter changing the sign, the following questions are not sufficiently studied or practically left out:
- the existence of the resolvent for all t $(-\infty < t < +\infty)$;
- the compactness of the resolvent for all t $(-\infty < t < +\infty)$;
- estimates of eigenvalues and singular numbers (s-numbers).

Objective:
- study questions about the compactness and approximative properties of embedding of weighted functional spaces of Sobolev type, which are the domain of definition of differential operators under consideration;
- investigate and prove a theorem on the existence of the resolvent of a one-dimensional Schrödinger operator with a complex potential when the real part of the potential contains a parameter that changes sign (the case on the entire straight line);
- to study and obtain a representation of the resolvent of the one-dimensional Schrödinger operator with complex potential, when the real part of the potential

contains a parameter that changes sign (the case on the whole line).

Methods of research: embedding theorems for weighted spaces of Sobolev type, a priori estimates method, uniform localization method (developed mainly in the works of M. Otelbaev). In addition to the above methods, the paper proposes an approach that allows us to find two-sided estimates of the singular numbers (approximation numbers, s-numbers) of differential operators containing a sign changing parameter.

The results obtained and the novelty. As a result of the research, the following new results were obtained to a class of mixed differential operators defined in a noncompact region:

- the theorem on the existence of a bounded inverse operator is proved;
- a theorem on the compactness (discreteness of the spectrum) of the resolvent of an operator of mixed type is proved;
- two-sided estimates of the distribution function of singular numbers (s-numbers) of the resolvent of a differential operator of mixed type given in an unbounded domain are obtained.

The structure of the paper. The work consists of three sections. In the second section, the following results are obtained for the operator (*) on a finite interval (one-dimensional Schrödinger operator with potential when the real part contains a parameter changing sign):

- a priori and coercive estimates were obtained;
- the differential properties of the resolvent are studied;
- two-sided estimates of the distribution function of singular numbers (s-numbers) (case on a finite interval) are obtained.
- the theorem on the bounded invertibility of the operator (*) on the whole line is proved;
- the representation of the resolvent is obtained.
- the compactness of the resolvent of (*);
- estimates are obtained for the distribution function of the widths of the embedding.

In the third section, the spectral properties of a differential operator of mixed type are studied:

- the theorem on the bounded invertibility of a singular operator of mixed type is proved;
- the theorem on the discreteness of the spectrum (on compactness) is proved;
- two-sided estimates of the s-numbers of a mixed-type operator are obtained.

1 ON SPECTRAL PROPERTIES OF A MIXED TYPE OPERATOR

Consider a differential operator of mixed type

$$Lu = - s(y)u_{xx} - u_{yy} + p(y)u_x + q(y)u \tag{1.1.1}$$

initially defined on $C^{\infty}_{0,\pi}(\Omega)$, where s(y) is a piecewise-continuous, as well as bounded in R and s(0)=0, ys(y)>0, as y≠0, $\Omega = \{(x,y): -\pi < x < \pi, -\infty < y < \infty\}$

$C^{\infty}_{0,\pi}$ is a set consisting of infinitely differentiable functions and satisfying conditions u(-π,y)= u(π,y), $u_x(-\pi,y)= u_x(\pi,y)$ and compactly supported y.

It is known that operator (1.1.1) is called a mixed-type operator, which in the upper part of the considered domain belongs to the elliptic, and in the lower part, to the hyperbolic type; these parts are separated by a transition line, on which the operator degenerates into a parabolic.

Firstly, we are interested in how the solutions look, i.e. resolvent representations $R(\lambda,L)\equiv(L+\lambda E)^{-1}$ of the L as $\lambda\to\infty$.

Then, special attention is paid to:

1) questions about smoothness (coercivity, separability).

2) estimated solutions in different weights.

There are significant difficulties in estimating the derived functions from D(L).

And finally, the spectral properties of operators (1.1.1) are investigated, that is, the following questions:

3) discreteness of the spectrum of an operator of mixed type in a noncompact region;

4) Schmidt estimates of eigenvalues and their distribution functions;

5) estimates of eigenvalues and spectrum locations;

6) completeness of root vectors.

1.1 The existence of the resolvent and its compactness

The operator (1.1.1) admits a closure, which we denoted by L .

Suppose that the coefficients a (y), c (y) satisfy the condition

i) p(y), c(y) are the continuous functions: $q(y)\geq\delta>0$, $a(y)\geq 0$ or $a(y)\leq 0$;

Theorem 1.1.1. Let the condition i) be fulfilled. Then:

a) L + λE is continuously invertible for sufficiently large λ> 0:

b) the estimate is valid:

$$\|u\|_{2,1,\Omega}\leq c\|(L+\lambda+)u\|_{2,\Omega}. \tag{1.1.2}$$

for any u∈D (y),

Proof. We have:

$$|(L+\lambda E)u,u)|\geq\int_{\Omega}(|u'|^{2}+((c(y)+\lambda)|u|^{2})dxdy-\int_{\Omega}(|k(y)||u'|^{2})dxdy.$$

for $u\in C^{\infty}_{0,\pi}(\Omega)$.

Hence, on the basis of condition i) and Jung's inequality we have:

$$\|(L+\lambda E)u\|\geq c_1\int_{\Omega}(|u'|^{2}+|u|^{2})dxdy-\int_{\Omega}(|k(y)||u'|^{2})dxdy. \tag{1.1.3}$$

It is also easy to establish that

$$|<(Lu+\lambda E)u,u'\rangle|\geq\left|\int_{\Omega}(a(y)|u'|^{2}dxdy\right|.$$

From here we get that

$$\|(L+\lambda E)u\|_2^2\geq c_1(\delta)\|u'\|_2^2 \tag{1.1.4}$$

Combining (1.1.3) and (1.1.4):

$$\|(L+\lambda E)u\|_2^2\geq c\|u\|_2; \tag{1.1.5}$$

$$\|(L+\lambda E)u\|_2^2\geq c\|u\|_{2,1}, \tag{1.1.6}$$

where $c=c(c_1,c_2)$.

Due to (1.1.5), there is a continuous inverse operator of operator (1.1.1). How easy to see

$$(L+\lambda E)^{-1}f \doteq \sum_{n=-\infty}^{\infty}(l_n+\lambda E)^{-1}f_n e^{inx} \tag{1.1.7}$$

for all $f\in L_2(\Omega)$. The clause a) of Theorem 1.1.1 is proved. Proof of the clause b) follows from inequality (1.1.6). Theorem 1.1.1 is proved.

Theorem 1.1.2. Assume that i) is fulfilled. Then the resolvent of L is compact iff, then for all $\omega>0$

$$\lim_{|y|\to\infty}\int_{y}^{y+\omega}c(t)dt=\infty. \tag{1.1.8}$$

Proof. According to Theorem 5.3.1

$$\lim_{|n|\to\infty}\left\|(l_n+\lambda E)^{-1}\right\|=0.$$

From here and (1.1.7) it follows that the inverse operator $(L+\lambda E)^{-1}$ is completely continuous iff, then $(l_n+\lambda E)^{-1}$ is a completely continuous.

1.2 On the materiality of the condition on the coefficient with the lowest derivative on the solvability

Consider an operator

$$Lu = -s(y)u_{xx} - u_{yy} + p(y)u_x + q(y)u \tag{1.2.1}$$

in

$$\Omega = \{(x,y): -\pi < x < \pi,\ -1 < y < 1\}$$

with piecewise continuous and bounded coefficients, originally defined on $C^{\infty}_{0,\pi}(\Omega)$, where $C^{\infty}_{0,\pi}(\Omega)$ consist of infinitely differentiable elements that are compactly supported to y and satisfy the following conditions:

$$u(-\pi,y) = u(\pi,y),\ u_x(-\pi,y) = u_x(\pi,y) \tag{1.2.2}$$

$$u(x,-1) = u(x,1) = 0 \tag{1.2.3}$$

The operator L admits a closure and its we will denoted by L.

Let the coefficients of (1.2.1) satisfy the following conditions:

a) p(y), q(y), s(y) are piecewise continuous functions on the segment [-1,1]; $c(y) \ge \delta > 0$, $p(y) \ge 0$ or $p(y) \le 0$;

b) the following condition is satisfied:

$$\lim_{|t|\to\infty} \sup_{y\in[-1,1]} \frac{t^2}{[K_t^*(y)]^2} < c,$$

where c>0 is any fixed number and $K_t^*(y)$ is an averaging function defined by the following equality (introduced by M. Otelbaev)

$$K_t^*(y) = \inf_{d>0}\{d^{-1}; d^{-1} \ge \int_{y-\frac{d}{2}}^{y+\frac{d}{2}} K_t(\tau)d\tau\}$$

where the kernel $K_t(\tau) = t^2(m\,|\,p(\tau)\,|-|\,s(\tau)\,|) + q(\tau) > 0,\ for\ \forall\tau\in[-1,1],\ -\infty < t < \infty$

The main results of this section are formulated in the following Theorems.

Theorem 1.2.1. Assume that a)–b) be fulfilled. Then $L+\lambda E$ is a continuously invertible for large enough λ >0.

Theorem 1.2.2. Assume that a)–b) be fulfilled. Then there exists a sequence of positive eigenvalues of operators (1.2.1) and for them fair estimates

$$c^{-1}k^2 \le \lambda_k \le ck^2, \quad k = 1,2,....$$

where c is any constant number.

Before to the prove these theorems we have to consider several auxiliary lemmas.

Consider an operator

$$l_t u = -u''(y) + (t^2 s(y) + itp(y) + q(y))u(y)$$

or we may simply write

$$l_t u = -u'' + (t^2 s(y) + itp(y) + q(y))u$$

on the set $C_0^\infty(-1,1)$, где $-\infty < t < \infty$.

Lemma 1.2.1. Assume that a) be fulfilled. Then for all $u \in C_0^\infty(-1,1)$ for large enough λ>0 the inequality

$$(m+1)(t^2+1)\|(l_t + \lambda E)u\|_2^2 \ge \frac{1}{2}\int_{-1}^{1}[|u'|^2 + (t^2(m|p(y)| - |s(y) + q(y+1))|u|^2]dy;$$

holds, where m is any positive constant, $-\infty < t < \infty$.

Proof. Consider the scalar product

$$|<(l_t + \lambda E)u, -itu>| = \left| \int_{-1}^{1} t^2 p(y) | u|^2 \, dy - it\int_{-1}^{1}(t^2 s(y) + q(y) + \lambda \, |u|^2 \, dy + \right.$$

$$\left. + it\int_{-1}^{1} u''(y)\overline{u}(y)dy \right|$$

Using the method of Integrating by parts the last term and by virtue of the finiteness of u (y), we obtain

$$|<(l_t + \lambda E)u, -itu>| = \left| \int_{-1}^{1} t^2 p(y)|u|^2 \, dy - it\int_{-1}^{1}[|u'|^2 + (t^2 s(y) + q(y) + \lambda |u|^2]dy \right|.$$

Hence

$$|<(l_t + \lambda E)u, -itu>| \ge \left| \int_{-1}^{1} t^2 p(y)\,|u|^2 \, dy \right|,$$

Since the coefficient p(y) does not change the sign

$$|<(l_t+\lambda E)u,-itu>|\geq\int_{-1}^{1}t^2|p(y)||u|^2\,dy.$$

In view of the Cauchy-Bunyakovsky inequality, it follows that

$$\|(l_t+\lambda E)u\|_2\,\|-itu\|_2\geq\int_{-1}^{1}t^2\,|p(y)|\,|u|^2\,dy.$$

Based on this inequality and using the Cauchy inequality with ε> 0, we get

$$\frac{1}{2}(t^2+1)\|(l_t+\lambda E)u\|_2^2+\frac{1}{2}\frac{1}{t^2+1}\|itu\|_2^2\geq\int_{-1}^{1}t^2\,|p(y)|\,|u|^2\,dy, \tag{1.2.4}$$

for $\varepsilon=1/(t^2+1)$.

Now consider the following scalar product

$$|<l_tu+\lambda u,u>|=\left|-\int_{-1}^{1}u''\bar{u}dy+\int_{-1}^{1}(t^2s(y)+q(y+\lambda))\,|u|^2\,dy+it\int_{-1}^{1}p(y)\,|u|^2\,dy\right|$$

Using the method of integrating by parts to the first term and by virtue of the finiteness of u (y), we obtain

$$|<l_tu+\lambda u,\,u>|=\left|\int_{-1}^{1}[|u|^2+(t^2s(y)+q(y)+\lambda))\,|u|^2]dy+it\int_{-1}^{1}p(y)\,|u|^2\,dy\right|$$

Hence

$$|<l_tu+\lambda u,u>|\geq\left|\int_{-1}^{1}[|u'|^2+(t^2s(y)+q(y)+\lambda)\,|u|^2]dy\right|\geq$$

$$\geq\int_{-1}^{1}[|u'|^2+(q(y)+\lambda)\,|u|^2]dy-\int_{-1}^{1}t^2|s(y)|\,|u|^2\,dy$$

Similarly, as for the first scalar product using the Cauchy-Bunyakovsky inequality, and then Cauchy with ε >0 $(\varepsilon=1/(t^2+1))$, we have

$$\frac{(t^2+1)}{2}\|l_tu+\lambda u\|_2^{\,2}+\frac{1}{2}\left(\frac{1}{t^2+1}\right)\|u\|_2^2\geq$$

$$\geq\int_{-1}^{1}[|u'|^2+(q(y)+\lambda)\,|u|^2]dy-\int_{-1}^{1}t^2\,|s(y)|\,|u|^2\,dy$$

Considering condition a) from the last inequality and inequality (1.2.4) for some m> 0 and sufficiently large λ> 0, we finally get

$$(m+1)(t^2+1)\|(l_t+\lambda E)u\|_2^2 \geq \frac{1}{2}\int_{-1}^{1}[|u'|^2+(t^2(m|a(y)|-|s(y)|)+q(y))|u|^2]dy$$

Lemma is proved.

We introduce notation that will be useful in the future. Let d(y)=$[K_t^*(y)]^{-1}$, then $\Delta_{d(y)}(y)$ is the interval $(y-\frac{d}{2}, y+\frac{d}{2})$, $\Delta^{(k)}=\Delta_d(y_k)=(y_k-\frac{d}{2}, y_k+\frac{d}{2})$

From the above definition of Kt * (y) it is clear that it is positive. But besides this, Kt * (y) is a continuous function. The proof of this statement can be found in [27] and here it is omitted. We note only that the Lipschitz condition was used for the proof of continuity, which is sufficient condition to the continuity. In particular, in [27] an estimate was obtained

$$|[K_t^*(y_0)]^{-1}-[K_t^*(y)]^{-1}|\leq 2|y_0-y|,\ for\ \forall y\in\Delta_{\frac{d_0}{2}}(y_0)$$

where $\Delta_{\frac{d_0}{2}}(y_0)=(y_0-\frac{d_0}{4}, y_0+\frac{d_0}{4})\subset(-1,1)$

Lemma 1.2.2 For an interval (-1,1), there exists no more than a covering {Δ (k)} with nonintersecting intervals Δ (k) contained in the interval (-1, 1), up to a countable set.

The proof of this lemma can be found in the paper [27] and is given here for completeness.

Proof. Take any point $y_1\in(-1, 1)$ and put $\Delta^{(1)}=\Delta_{d(y)}$, $d(y_1)=[K_t^*(y_1)]^{-1}$. Assume that $\Delta^{(1)}$ does not cover the interval (y_1,b). Among the intervals $\{\Delta_{d(y)}(y): y_1\leq y<d,\ d(y)=[K_t^*(y)]^{-1}\}$ is like intersecting at intervals $\Delta^{(1)}$ (for example $\Delta^{(1)}$) so and not intersected. The last sentence follows the following property of $K_t^*(y)$: by definition $K_t^*(y)$ easy follows that $[K_t^*(y)]^{-1}\to 0$ as $y\to 1$, therefore the intervals $\Delta^{(1)}$ and $\Delta_{d(y)}$ do not intersect if u is close to 1. Consider the function $\psi(y)=y-\frac{[k_t^*(y)]^{-1}}{2}$ on $[y_1, 1]$. Field of values of this function $y_1-\varepsilon\leq\psi(y)\leq 1$. Therefore $y_1+\frac{d_1}{2}\in(y_1-\varepsilon, 1)$. Hence, by virtue of continuity of the function $\psi(y)$ there exist a function $y\in(y_1, 1)$, such that $\psi(y)=y_1+\frac{d_1}{2}$, i.e. left end of the interval $\Delta_{d(y)}$ coincides with the right end of the interval $\Delta^{(1)}$. The smallest environments of such y (it exists) we denote by y_2, $\Delta^{(2)}=\Delta_{d(y)}(y_2)$. If at any step (y_1, y) is contained in $\{\Delta^{(n)}\}_{n\leq k}$, then the process of completing the intervals to the right of $\Delta^{(1)}$ finished. Otherwise, it can be continued to infinity. Obtained intervals have to cover $(y_1, 1)$ because otherwise their centers y_k will convergent to the y<1, where $[K_t^*(y)]^{-1} = \lim_{k\to\infty}[K_t^*(y_k)]^{-1} = 0$, which contradicts the continuity and positivity of this function in (-1, 1). Similarly, the intervals are completed to the left of the interval $\Delta^{(1)}$.

Lemma 1.2.3. Assume that a) and b) be fulfilled. Then there exists a constant $c_0>0$, such that

$$\int_{\Delta_{d(y)}} [|u'|^2 + (t^2(m|a(\tau)|-|s(\tau)|)+q(\tau)|u|^2]d\tau \geq c_0^{-1}\left(\int_{\Delta_{d(y)}} |u'|^2\, d\tau + d^{-2}(y) \int_{\Delta_{d(y)}} |u|^2\, d\tau \right),$$

for all $u \in C_0^\infty(\Delta_{d(y)}(y))$, where $\Delta_{d(y)}(y) \subset (-1,1)$, where m>0 any constant.

Lemma 1.2.4. Assume that a) and b) be fulfilled. Then the following estimates

$$\| (l_t + \lambda E)u \|_2 \geq c \| u \|_2 \qquad (1.2.5)$$

holds for any $u \in D(l_t)$ as $\lambda>0$.

Proof. By lemmas 1.2.1–1.2.3 we have:

$$(t^2+1)(m+1)\| (l_t + \lambda E)u \|_2^2 \geq \frac{1}{2}\int_{-1}^{1}[|u'|^2 + (t^2(m|a(y)|-|s(y)|)+q(y)+\lambda)|u|^2]dy \geq$$

$$\geq \frac{1}{2}\sum_{\{k\}}\left(\int_{\Delta_{d_k}(y_k)} |u'|^2\, dy + d_k^{-2}(y_k) \int_{\Delta_{dk}(y_k)} |u|^2\, dy \right),$$

where the constant c_0 from Lemma 1.2.3 without detracting from commonality, they were taken as a unit in each interval $\Delta_d(y_k)$.

Hence

$$m(t^2+1)\| (l_t + \lambda E)u \|_2^2 \geq \frac{1}{2}\sum_{\{k\}} d_k^{-2}(y_k) \int_{\Delta_{d_k}(y_k)} |u|^2\, dy \geq \inf_{y\in(-1,1)} [K_t^*(y)]^2 \sum_{\{k\}} \int_{\Delta_{d_k}(y_k)} |u|^2\, dy =$$

$$= \inf_{y\in(-1,1)} [s_t^*(y)]^2 \int_{-1}^{1} |U|^2\, dy = \inf_{y\in(-1,1)} [s_t^*(y)]^2 \, \|U\|_2^2$$

From this inequality and considering b), we obtain

$$(m+1)(t^2+1)\| (l_t + \lambda E)u \|_2^2 \geq t^2 c \| u \|_2^2$$

Hence we obtain (1.2.5).

Lemma 1.2.5. Assume that a) and b) be fulfilled. Then $l_t+\lambda E$ continuously invertible as $\lambda>0$.

Proof. From Lemma 1.2.4 it follows that for the operator $l_t+\lambda E$ there is an inverse bounded operator on a set of values $R(l_t+\lambda E)$. Now, if the set of values of $l_t+\lambda E$ everywhere dense in $L_2(-1, 1)$, then the invers operator $(l_t+\lambda E)^{-1}$ will be continuous whole $L_2(-1, 1)$.

Suppose by contradiction that the set of values is not dense in L_2. Then there exists a function $\upsilon\in L_2$ $(\upsilon\neq 0)$ such that $\upsilon\perp R(l_t+\lambda E)$, i.e.

$$<l_t u+\lambda u, \upsilon>=0 \quad \text{for any} \quad u\in D(l_t)$$

By the Riesz theorem, it is clear that $\upsilon\in D(l_t^*)$ and mean $l_t^*\upsilon\in L_2(-1,1)$, where l_t^* is the operator adjoint to l_t, i.e. there is equality

$$<(l_t+\lambda E)u, \upsilon>=<u, (l_t^*+\lambda E)\upsilon>=0$$

and $(l_t^*+\lambda E)\upsilon=-\upsilon''+(t^2 s(y)-itp(y)+q(y)+\lambda)\upsilon=0$

Since $a(y)$, $c(y)$, $s(y)$ satisfied the above conditions, then $(t^2s(y)-itp(y)+q(y)++\lambda)\upsilon$ belong to $L_2(-1,1)$. From here $\upsilon''\in L_2(-1,1)$.

Now, if we show that $\upsilon=0$, then we get a contradiction and this Lemma will be proved. Indeed, consider the representation $<u, l_t^*\upsilon+\lambda\upsilon>=0$. For him the following calculations are true:

$$0=\langle u, l_t^*\upsilon+\lambda\upsilon\rangle=\int_{-1}^{1}u\left[-\bar{\upsilon}''+\overline{(t^2s(y)-itp(y)+q(y)+\lambda)\upsilon}\right]dy=$$

$$=-\int_{-1}^{1}u\bar{\upsilon}''dy+\int_{-1}^{1}(t^2s(y)+ita(y)+q(y)+\lambda)u\bar{\upsilon}dy=-\int_{-1}^{1}ud(\bar{\upsilon}')+$$

$$\int_{-1}^{1}(t^2s(y)+itp(y)+q(y)+\lambda)u\bar{\upsilon}dy=$$

$$=-u\bar{\upsilon}'\Big|_{-1}^{1}+\int_{-1}^{1}\bar{\upsilon}'du+\int_{-1}^{1}(t^2s(y)+itp(y)+q(y)+\lambda)u\bar{\upsilon}dy=\int_{-1}^{1}u'd\bar{\upsilon}+$$

$$+\int_{-1}^{1}(t^2s(y)+itp(y)+q(y)+\lambda)u\bar{\upsilon}dy=u'\bar{\upsilon}\Big|_{-1}^{1}-\int_{-1}^{1}u''\bar{\upsilon}\,dy+$$

$$+\int_{-1}^{1}(t^2s(y)+itp(y)+q(y)+\lambda)u\bar{\upsilon}dy=u'\bar{\upsilon}\Big|_{-1}^{1}+\int_{-1}^{1}(l_tu+\lambda u)\cdot\bar{\upsilon}dy=$$

$$=\langle(l_t+\lambda E)u,\upsilon\rangle+u'\bar{\upsilon}\Big|_{-1}^{1}=0$$

By assumption $<(l_t+\lambda E)u, \upsilon>=0$, which means $u'\bar{\upsilon}\,\Big|_{-1}^{1}=0$ or $u'(1)\bar{\upsilon}(1)=u'(-1)\bar{\upsilon}(-1)$.

Let $\bar{\upsilon}(1)=\alpha$, $\bar{\upsilon}(-1)=\beta$. Suppose that $\alpha\neq0$ and $\beta\neq0$ and by virtue of the arbitrariness of u, we get

$$u(y)=(y+1)^2(y-1), \quad (u(-1) = u(1)= 0).$$

Then

$$u'(y)=2(y^2-1) + (y+1)^2 \quad \text{и} \quad u'(-1)=0,\ u'(1)=4.$$

It follows that $\bar{\upsilon}(-1)=\bar{\upsilon}(1)=0$ or $\upsilon(-1)=\upsilon(1)=0$. Now, it is easy to verify that the $\upsilon(-1)=\upsilon(1)=0$, $\upsilon\in L_2(-1,1)$ the following inequality

$$\left\|(l_t^* + \lambda E)\upsilon\right\|_2 \geq \|\upsilon\|_2$$

holds, it follows from the above same calculations and arguments obtained for l_t.

From the last inequality, by virtue $l_t^*\upsilon+\lambda\upsilon = 0$ it follows that $\upsilon=0$.

Consider the operator

$$L'u = -s(y)u_{xx} - u_{yy} - p(y)u_x + q(y)u$$

initially defined on $C_{0,\pi}^{\infty}(\Omega)$.

It is easy to verify that the adjoint operator L' admits a closure, which we denote by L'.

Lemma 1.2.1. Assume that a)-c) be fulfilled. Then

$$D(\breve{L})\subseteq D(L'^*)$$

where the sets D(L) and D(L'*) is the domain definitions respectively of L and adjoint operator to L'.

Proof. According to the definition of a adjoint operator, the following equality

$$< L'u, v >=< u, L'^{*}v >,$$

holds for all $u(x,y)\in D(L'), v(x,y)\in D(L'^{*})$.

If we prove that the last equality holds for any v belong to D (L), then the statement of this lemma will be proved.

Let $u_n(x,y)\in C_{0,\pi}^{\infty}(\Omega)$ and $u_n\to u\in D(L')$, $v_n(x,y)\in C_{0,\pi}^{\infty}(\Omega)$ и $v_n\to v\in D(L)$. Then the following equality

$$< L'u_n, v_n >=< u_n, Lv_n > \qquad (1.2.7)$$

holds for all $u_n, v_n \in C_{0,\pi}^{\infty}(\Omega)$. Check this equality. We have (index n with u_n, v_n

omitted here to avoid notation complications):

$$L'u,v=\int_\Omega(-s(y)u_{xx}-u_{yy}-p(y)u_x+q(y)u)\bar{v}dxdy=\int_\Omega -s(y)u_{xx}\bar{v}dxdy-$$

$$-\int_\Omega u_{yy}\bar{v}dxdy-\int_\Omega p(y)u_x\bar{v}dxdy+\int_\Omega q(y)u\bar{v}dxdy=-\int_{-1}^{1}s(y)\left[\int_{-\pi}^{\pi}\bar{v}du_x\right]dy-\int_{-\pi}^{\pi}dx\int_{-1}^{1}\bar{v}du_y$$

$$-\int_{-1}^{1}p(y)\left[\int_{-\pi}^{\pi}\bar{v}du\right]dy+\int_\Omega q(y)u\bar{v}dxdy=-\int_{-1}^{1}s(y)\left[u_x\bar{v}\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}u_xd\bar{v}\right]dy-$$

$$-\int_{-\pi}^{\pi}\left[u_y\bar{v}\Big|_{-1}^{1}-\int_{-1}^{1}u_yd\bar{v}\right]dx-\int_{-1}^{1}p(y)\left[u\bar{v}\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}ud\bar{v}\right]dy+\int_\Omega q(y)u\bar{v}dxdy$$

for all $u,v\in C_{0,\pi}^{\pi}(\Omega)$.

By virtue of (1.2.2) - (1.2.3) for functions belong to $C_{0,\pi}^{\pi}(\Omega)$ we have:

$$\langle L'u,v\rangle=-\int_{-1}^{1}sk(y)\left[\int_{-\pi}^{\pi}\bar{v}_xdu\right]dy-\int_{-\pi}^{\pi}\left[\int_{-1}^{1}\bar{v}_ydu\right]dx-\int_{-1}^{1}p(y)\left[\int_{-\pi}^{\pi}\bar{v}_xudx\right]dy+$$

$$+\int_\Omega q(y)u\bar{v}dxdy=-\int_{-1}^{1}s(y)\left[u\bar{v}_x\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}u\bar{v}_{xx}dx\right]dy-\int_{-\pi}^{\pi}\left[u\bar{v}_y\Big|_{-1}^{1}-\int_{-1}^{1}u\bar{v}_{yy}dy\right]dx$$

$$-\int_{-1}^{1}\int_{-\pi}^{\pi}p(y)u\bar{v}_xdxdy+\int_\Omega q(y)u\bar{v}dxdy$$

Again, due to (1.2.2) - (1.2.3), we get

$$\langle L'u,v\rangle=\int_\Omega u(-s(y)\bar{v}_{xx}-\bar{v}_{yy}+p(y)\bar{v}_x+q(y)\bar{v})dxdy=\langle u,Lv\rangle$$

which proves equality (1.2.7).

Now moving to equality (1.2.7) to the limit, we have

$$\langle L'u,v\rangle=\langle u,Lv\rangle$$

Proof of Theorem 1.2.1.

Let $u(x,y)\in C_{0,\pi}^{\pi}(\Omega)$ и Lu=f. Then for operatora L+λE справедливо представление

$$(L+\lambda E)u_k=(L+\lambda E)\sum_{n=-k}^{k}u_n(y)e^{inx}=\sum_{n=-k}^{k}(l_n+\lambda E)u_ne^{inx}$$

where $u_k = \sum_{n=-k}^{k} u_n(y)e^{inx}$ and $u_k(x,y) \to u(x,y)$, as $k \to \infty$.

According to the Lemma 1.2.5 as $\lambda > 0$ we have

$$\| (L+\lambda E)u_k \|_2^2 = \sum_{n=-k}^{k} \| (l_n + \lambda E)u_n \|_2^2 \geq \sum_{n=-k}^{k} c^2 \| u_n \|_2^2 = c^2 \| u_k \|_2^2$$

Passing to the limit in the last inequality, we get

$$\|(L+\lambda E)u\|_2 \geq c\|u\|_2$$

and the last estimate is valid for all $u(x,y) \in C_{0,\pi}^{\infty}(\Omega)$

Since the considering operator admits closure, due to the continuity of the any norm, the above inequality is valid for all $u(x,y) \in D(L)$.

Now we show that the kernel of L is empty (consists only of the zero element, i.e. N(L)=Ker(L)={0}).

Repeating the calculations and reasoning used as proof of the reversibility of $l_n+\lambda E$, we get $l'_n+\lambda E$ has a continuous inverse operator $(l'_n+\lambda E)^{-1}$, here $l'_n+\lambda E = -u'' + (n^2 s(y) - inp(y) + q(y) + \lambda)u$.

It is known that for any $f \in L_2(\Omega)$ the inequality

$$f(x,y) = \sum_{n=-\infty}^{\infty} f_n(y)e^{inx}$$

holds. From here it is easy to show that

$$u_k(x,y) = \sum_{n=-k}^{k} (l'_n + \lambda E)^{-1} f_n(y)e^{inx}$$

is a solution of the following problem

$$(L'+\lambda E)u = f_k, \tag{1.2.1'}$$
$$u|_{-\pi} = u|_{\pi}, \quad u_x|_{-\pi} = u_x|_{\pi}, \tag{1.2.2'}$$
$$u(x,-1) = u(x,1) = 0, \tag{1.2.3'}$$

where

$$f_k(x,y) = \sum_{n=-k}^{k} f_n(y)e^{inx}$$

By Lemma 1.2.5 we get

$$\|f_k(x,y)\|_2^2=\|(L+\lambda E)u_k\|_2^2=\sum_{n=-k}^{k}\|(l'+\lambda E)u_n\|_2^2\geq$$

$$\geq c^2\sum_{n=-k}^{k}\|u_n\|_2^2=c^2\|u_k\|_2^2$$

as $\lambda>0$ or

$$\|f_k(x,y)\|_2\geq c\|u_k\|_2 \tag{1.2.8}$$

From here and from $f_k(x, y)\to f(x, y)$ it follows that the sequence $\{u_k(x,y)\}_{k=1}^{\infty}$ is a fundamental sequence (Cauchy sequence).

By virtue of the completeness of $L_2(\Omega)$ we obtain:

$$u_k(x, y)\to u(x, y)\in L_2 \tag{1.2.9}$$

Therefore, for all $f(x, y)\in L_2$ there exists so called a strong solution of (1.2.1′)-(1.2.3′).

It can be directly verified the definition of a strong solution of the problem is equivalent to the closure of $L'+\lambda E$, initially defined in $\overset{\circ}{W}{}^{2}_{2,\pi}(\Omega)$.

From the above it follows that the set $R(L'+\lambda E)$ it coincides with the whole $L_2(\Omega)$, i.e.

$$R(L'+\lambda E)=L_2(\Omega) \tag{1.2.10}$$

By virtue of the general theory of linear operators it is known that

$$L_2(\Omega)=R(L'+\lambda E)\perp N((L'+\lambda E)^*)$$

From here and (1.2.10) we obtain $N((L'+\lambda E)^*)=\{0\}$.

Therefore, from Lemma 1.2.6 we get

$$N(L+\lambda E)=Ker(L+\lambda E)\subseteq N((L'+\lambda E)^*)=Ker((L'+\lambda E)^*)=\{0\}$$

i.e. $N(L+\lambda E)=Ker(L+\lambda E)=\{0\}$.

Repeating all the calculations and arguments used to prove (1.2.8) and using the completeness of $L_2(\Omega)$,we get for any f there exists only one strong solution of (1.2.1)-(1.2.3), such that

$$c\|u\|_2\leq\|f\|_2$$

and for him there is the representation

$$u=(L+\lambda E)^{-1}f=\sum_{n=-\infty}^{\infty}(l_n+\lambda E)^{-1}f_n e^{inx}.$$

where c is any positive constant.

Proof of Theorem 1.2.2. Eigenfunctions of (1.2.1)-(1.2.3) we are looking for a $u(x,y)=\sum_{n=-\infty}^{\infty}u_n(y)e^{inx}$. Then with respect to the element u(y), we get the Sturm-Liouville spectral problem

$$-u''(y)+(n^2s(y)+inp(y)+q(y))u=\lambda u, \qquad (1.2.11)$$
$$u(-1)=u(1)=0 \qquad (1.2.12)$$

As condition a) in the case of n = 0, problem (1.2.11) - (1.2.12) takes the following form:

$$l_0u=-u''(y)+q(y)u=\lambda u,$$
$$u(-1)=u(1)=0.$$

It is easy to verify that, due to the boundedness of the function q(y), the domain of l_0 it coincides with $W_2^2(-1,1)$.

By M_0 we denote the unit sphere in $W_2^2(-1,1)$. For widths of the set M_0 the following two-sided evaluations take place:

$$c^{-1}\lambda^{-\frac{1}{2}}\le N_0(\lambda)\le c\lambda^{-\frac{1}{2}},$$

where $N_0(\lambda)$ is the number of widths $\overset{0}{d}_k$ of the set M_0 greater than $\lambda>0$, c is any positive constant. By the last inequalities and the properties of $N_0(\lambda)$ we get:

$$c^{-1}\frac{1}{k^2}\le \overset{0}{d}_k\le c\frac{1}{k^2}.$$

Further, given the self-adjointness of l_0 and equality $\lambda_{k+1}(l_0^{-1})=\overset{0}{d}_k$, for the eigenvalues of the operator l_0 finally get that

$$c^{-1}k^2\le\lambda_k(l_0)\le ck^2,\ \ k=1,2,...$$

Theorem is proved.

1.3 Separability of a operator and the completeness of its root vector system

Definition. Let's call the operator L is separable if for ny elements $u\in D(L)$ we have the following estimate

$$\|-s(y)u_{xx}-u_{yy}\|_2+\|p(y)u_x\|_2+\|cq(y)u\|_2 \le c(\|Lu\|_2+\|u\|_2)$$

here constant c does not depend on the function u (x,y).

Theorem 1.3.1. Assume that i)-iii) be fulfilled. Then L is separability.

Proof. From (1.1.7) and from the orthonormality of the system $\{e^{inx}\}_{n=-\infty}^{\infty}$ in L_2 $(-\pi,\pi)$

$$\|\rho(y)D_x^{\alpha}(L+\lambda E)^{-1}\|_{2\to 2}=\sup_{|n|}\||n|^{\alpha}\rho(y)(l_n+\lambda E)^{-1}\|_{2\to 2} \qquad (1.3.1)$$

n= ±1, ±2, . . ., where $D_x=\frac{\partial}{\partial x}$, $\alpha = 0,1$, ρ(y) is any continuous function.

Since for all $f\in L_2(\Omega)$

$$\|\rho(y)D_x^{\alpha}(L+\lambda E)^{-1}f\|_2^2=\sum_{n=-\infty}^{\infty}\|\rho(y)|n|^{\alpha}(l_n+\lambda E)^{-1}f_n\|_2^2,$$

$$f_n=(f,e^{inx}), \qquad (1.3.2)$$

then

$$\|\rho(y)|n|^{\alpha}(l_n+\lambda E)^{-1}f_n\|_2\le\|\rho(y)D_x^{\alpha}(L+\lambda E)^{-1}f\|_2,$$

i.e.

$$\|\rho(y)|n|^{\alpha}(l_n+\lambda E)^{-1}\|_2\le\|\rho(y)D_x^{\alpha}(L+\lambda E)^{-1}\|_2, \qquad (1.3.3)$$

n= ± 1, ± 2, . . .

On the other hand we get by (1.3.2):

$$\|\rho(y)D_x^{\alpha}(L+\lambda E)^{-1}f\|_2^2\le\sup_{|n|}\|\rho(y)|n|^{\alpha}(l_n+\lambda E)^{-1}\|\sum_{n=-\infty}^{\infty}\|f_n\|_2^2=$$

$$=\sup_{|n|}\|\rho(y)|n|^{\alpha}(l_n+\lambda E)^{-1}\|^2\|f\|_2^2$$

Hence

$$\|\rho(y)D_x^{\alpha}(L+\lambda E)^{-1}\|_2\le\sup_{|n|}\|\rho(y)|n|^{\alpha}(l_n+\lambda E)^{-1}\|. \qquad (1.3.4)$$

Inequalities (1.3.3) and (1.3.4) prove equality (1.3.1).

From equality (1.3.1), using Lemma 1.8, we obtain

$$\|\rho(y)D_x^{\alpha}(L+\lambda E)^{-1}\|_{2\to 2}\le c(\lambda)\sup_{|n|}\sup_{|j|}\|\rho(y)|n|^{\alpha}\varphi_j(l_{nj}+\lambda E)^{-1}\|_{2\to 2}. \qquad (1.3.5)$$

If we have $\rho(y) = p(y)$, $\rho(y) = q(y)$, then according to (1.3.5) we obtain

$$\left\|p(y)D_x(L+\lambda E)^{-1}\right\|_{2\to 2} \le \sup\sup \left\|p(y)n\varphi_n(l_n+\lambda E)^{-1}\right\|_{2\to 2}$$

$$\left\|q(y)(L+\lambda E)^{-1}\right\|_{2\to 2} \le \sup\sup \left\|q(y)\varphi_n(l_n+\lambda E)^{-1}\right\|_{2\to 2}$$

The boundedness of the last inequalities follows from Lemma 1.9, i.e.

$$\left\|pa(y)D_x(L+\lambda E)^{-1}\right\|_{2\to 2} < \infty, \quad \left\|q(y)(L+\lambda E)^{-1}\right\|_{2\to 2} < \infty \tag{1.3.6}$$

Using these inequalities, we obtain the inequality

$$\begin{gathered}\left\|-s(y)u_{xx}-u_{yy}\right\| = \left\|(L+\lambda E)u - p(y)u_x - q(y)u - \lambda u\right\| \le \\ \left\|(L+\lambda E)u\right\|_2 + \left\|p(y)u_x\right\|_2 + \left\|q(y)u\right\|_2 + \left\|\lambda u\right\|_2 \le \left\|(L_1+\lambda E)u\right\|_2 + \\ + \left\|p(y)D_x(L+\lambda E)^{-1}(L+\lambda E)u\right\|_2 + \left\|q(y)(L+\lambda E)^{-1}(L+\lambda E)u\right\|_2 + \\ + \left\|\lambda(L+\lambda E)^{-1}(L+\lambda E)u\right\|_2 \le \\ \le c\left\|(L+\lambda E)u\right\|_2, \qquad c = c(\mu_0, \mu, \lambda).\end{gathered}$$

It follows that

$$\left\|-s(y)u_{xx}-u_{yy}\right\|_2 + \left\|p(y)u_x\right\|_2 + \left\|q(y)u\right\|_2 \le \le c(\mu_0,\mu,\lambda)(\left\|Lu\right\|_2 + \left\|u\right\|_2).$$

Theorem 1.3.1 is proved.

Theorem 1.3.2. Assume that i)-iii) be fulfilled and $c^{-1}(y) \in L(R)$. Then root vectors of the resolvent operator (4.1.1) complete in $L_2(\Omega)$.

Proof. Eigenfunctions of the problem

$$Lu = -s(y)u_{xx} - u_{yy} + p(y)u_x + q(y)u = \lambda u, \tag{1.3.7}$$

$$u(-\pi, y) = u(\pi, y), \; u_x(-\pi, y) = u_x(\pi, y) \tag{1.3.8}$$

we are looking for a

$$u(x,y) = \sum_{n=-\infty}^{\infty} u_n(y)e^{inx}.$$

Then regarding function $u_n(y)$ we get the spectral problem

$$-u_n''(y) + (n^2 s(y) + inp(y) + q(y))u_n(y) = \lambda u_n(y). \tag{1.3.9}$$

From Theorem 5.2.2 we obtain that the root operator vectors (1.3.9) are complete in L_2 (R). Consequently, the root functions of the problem (1.3.7) - (1.3.8.) The corresponding eigenvalues λ_{nk} will be functions $u_{nk}(y)e^{inx}$. Obviously, the system $\{u_{nk}\}$ will be complete in L_2 (Ω).

Example. Assuming that $p(y)=e^{y}$, $q(y)=|y|^2+1$.

Then, according to Theorem 1.3.1 the operator L separable, i.e. the estimate

$$\|-s(y)u_{xx}-u_{yy}\|_2+\left\|e^{y}u_x\right\|_2+\left\|(|y|^2+1)u\right\|_2\le c(\|Lu\|_2+\|u\|_2),$$

holds, where the constant c does not depend on the element u(x,y).

Further, it is easy to verify that all the conditions of Theorem 1.3.2 are satisfied, therefore, the root vector system of operator $(L+\lambda E)^{-1}$ are complete in $L_2(\Omega)$.

1.4 On separation conditions for an operator of mixed type

In this section, we investigate the role of the coefficients for separability of a operator of mixed type in a non-compact region.

Consider a operator

$$Lu=-s(y)u_{xx}-u_{yy}+p(y)u_x+q(y)u \tag{1.4.1}$$

initially defined on $C^{\infty}_{0,\pi}(\Omega)$, where s(y) is piecewise continuous and bounded function in $R=(-\infty, +\infty)$ and $s(0)=0$, $ys(y)>0$ as $y\neq0$. Where $C^{\infty}_{0,\pi}(\Omega)$-is a set consisting of infinitely differentiable functions satisfying the conditions $u(-\pi,y)=u(\pi,y)$, $u_x(-\pi,y)=u_x(\pi,y)$ and finite variable y, where

$$\Omega=\{(x,y):-\pi<x<\pi,-\infty<y<\infty\}$$

It is easy to show that operator L admits a closure and its closure is also denoted by L. The domain of definition of an operator is denoted by D (L).

Definition. We say that operator L is separable if for all functions $u(x,y)\in D(L)$ the estimate

$$\|-s(y)u_{xx}-u_{yy}\|_2+\|p(y)u_x\|_2+\|q(y)u\|_2< c(\|Lu\|_2+\|u\|_2),$$

holds, where c>0 is a constant independent of u(x,y).

Here, and further by $\|\circ\|_2$ denotes the norm of L_2.

Theorem 1.4.1. Let the following conditions

i) p(y) and c(y) are piecewise continuous functions on any compact in R, $|p(y)|>\delta_0>0$;

ii) $q(y) \le c_0 p^2(y)$, for all $y\in R$, $c_0>0$ is a some fixed number;

iii) $\mu_1 = \sup\limits_{|y-t|\le 1} \frac{p(y)}{p(t)} < \infty$, $\mu_2 = \sup\limits_{|y-t|\le 1} \frac{q(y)}{q(t)} < \infty$

be fulfilled. Then value γ is finite, iff

$$\inf_{y\in R} q(y) = c > -\infty, \tag{*}$$

where

$$\gamma = \sup_{u\in D(L)} \frac{\| -s(y)u_{xx} - u_{yy} \|_2}{\| Lu \|_2 + \| u \|_2} \tag{1.4.2}$$

The necessity will be proved without the use of any auxiliary assertions. Then, as for the proof of sufficiency, we need a number of auxiliary lemmas.

Consider the operator l_n (n=0, + 1, +2, ...), defined by the equation

$$l_n u = -u''(y) + (n^2 s(y) + inp(y) + q(y)u(y) \tag{1.4.3}$$

initially defined in $C_0^\infty(R)$ is the set of functions infinitely differentiable and finite on $R=(-\infty, +\infty)$, where s(y) is a piecewise continuous on any compact in R and bounded in R, $ys(y)>0$ as $y\neq 0$ and $s(0)=0$.

The operator (1.4.3) allows closure in $L_2(R)$ and its closure will also be denoted by l_n and further, referring to expression (1.4.3), we will keep in mind this closure.

Denote by $D(l_n)$ the domain of the now-closed operator l_n, consisting of finite functions, which, together with their derivatives up to the second order, belong to $L_2(R)$.

For the operator l_n we have the following lemma.

Lemma 1.4.1. Let the conditions i) and (*) be fulfilled. Then the operator $l_n+\lambda E$ continuously invertible as $\lambda>0$.

Lemma 1.4.2. Let the conditions i)-iii) and (*) be fulfilled. Then the inequlity

$$\left\| \rho(y)|n|^\alpha (l_n + \lambda E)^{-1} \right\|_{2\to 2} \le c(\lambda) \sup_{j} \left\| \rho(y)|n|^\alpha \varphi_j (l_n + \lambda E)^{-1} \right\|_{2\to 2},$$

holds as $\lambda>0$, where the operator $l_{n,j}$ defined by equality

$$l_{n,j} u = -u''(y) + (n^2 s(y) + inp(y) + q(y))u$$

and by boundary conditions

$$u(\Delta_j^-)=u(\Delta_j^+)=0$$

where Δ_j^- и Δ_j^+ are the left and the right ends of intervals $\Delta_j=(j-1,j+1)$; ρ(y) is a continuous function in R; c(λ) depends on λ and $\alpha=0,1$.

Lemma 1.4.3. Let the conditions of Lemma 1.4.2 be satisfied. Then

$$\left\| a(y)|n|(l_n+\lambda E)^{-1}\right\|_{2\to 2}<\infty$$

$$\left\| c(y)(l_n+\lambda E)^{-1}\right\|_{2\to 2}<\infty$$

Proof of lemmas 1.4.1. - 1.4.3 and The proof of the sufficiency of the main theorem will be borrowed from [50] and for the completeness of the exposition is presented here in a more compact form.

Proof of Lemma 1.4.1. Consider a scalar product

$$\left|\langle l_n u+\lambda u,-inu\rangle\right|=\left|\int_{-\infty}^{\infty}p(y)\,|u|^2dy-in\int_{-\infty}^{\infty}[|u'|^2+(n^2s(y)+q(y)+\lambda)\,|u|^2]dy\right|,$$

where u(y) is an arbitrary function belonging to $C_0^\infty(R)$.

Hence, using the Cauchy-Bunyakovsky inequality it is easy to get

$$\|(l_n+\lambda E)u\|_2\geq|n|\delta_0\|u\|_2, \tag{1.4.4}$$

In the same time for n=0, i.e. for $l_0u=-u''(y)+c(y)u(y)$ the estimate

$$\|(l_0+\lambda E)u\|_2\geq\delta\|u\|_2, \tag{1.4.5}$$

holds, where $\delta=\tilde{c}+\lambda=\inf\limits_{y\in R}q(y)+\lambda>0$.

Indeed, having considered the scalar product

$$\left|\langle(l_0+\lambda E)u,u\rangle\right|=\left|\int_{-\infty}^{\infty}p(y)\,|u|^2dy-in\int_{-\infty}^{\infty}(q(y)+\lambda)\,|u|^2\,dy\right|$$

due to the Cauchy-Bunyakovsky inequality we get (1.4.5).

Now from (1.4.4) and (1.4.5), we get

$$\|(l_n+\lambda E)u\|_2\geq c_0(|n|+1)\|u\|_2, \tag{1.4.6}$$

where $c_0 = \min\{\frac{\delta_0}{2}, \frac{\delta}{2}\}$

Consider the operator $l_{n,j}$ from Lemma 1.4.2.

As fulfilling the conditions i) and (*) the operator $l_{n,j}+\lambda E$ has a continuous inverse defined whole $L_2(\Delta_j)$ for $\lambda>0$ and for the inverse operator the inequalities

$$\left\|(l_{n,j}+\lambda E)^{-1}u\right\|_{2\to 2} \geq \frac{c}{\lambda^{1/2}} \quad (1.4.7)$$

$$\left\|\frac{d}{dy}(l_{n,j}+\lambda E)^{-1}\right\|_{2\to 2} \geq \frac{c}{\lambda^{1/4}} \quad (1.4.8)$$

hold, where $c>0$ is a constant.

We prove these statements.

Integrating in parts, we get the inequality:

$$\left\|(l_n+\lambda E)u\right\|_2 \geq c_1\left\|u\right\|_2$$

for all $u(y)\in D(l_{n,j})$, where $c_1>0$ is a constant.

Now, if we show that the set of values of operators is everywhere dense in $L_2(\Delta_j)$, it will be clear that operator $l_{n,j}+\lambda E$ It has a continuous inverse for $\lambda>0$.

Assume the opposite, i.e. set of meanings $R(l_{n,j}+\lambda E)$ not everywhere dense in $L_2(\Delta_j)$. Then there exists an element $v\in L_2(v\neq 0)$ such that for all $u\in D(l_{n,j}+\lambda E)$

$$\left\langle u,(l_{n,j}+\lambda E)^* v\right\rangle = \left\langle (l_{n,j}+\lambda E)u, v\right\rangle = 0 \quad (1.4.9)$$

holds, where $(l_{n,j}+\lambda E)^*$ is the adjoint operator to $l_{n,j}+\lambda E$, which defined by the equation

$$(l_n+\lambda E)^* u = -v''(y) + (n^2 s(y) + inp(y) + q(y) + \lambda)v(y)$$

From (1.4.9) follows $(l_{n,j}+\lambda E)^* v=0$. Because s(y), p(y) and q(y) bounded and piecewise continuous functions on a segment Δ_j, then the function $(n^2s(y)-inp(y)+q(y)+\lambda)v\in L_2(\Delta_j)$ and therefore $v''\in L_2(\Delta_j)$

Now integrating the expression in parts

$$0=<u,(l_{n,j}+\lambda E)^{*}v>==\int_{j-1}^{j+1}u(y)[-\bar{v}''(y)+(\overline{n^2s(y)-inp(y)}+\overline{q(y)+\lambda})\bar{v}(y)]dy$$

and by virtue of arbitrariness $u\in D(l_{n,j}+\lambda E)$, easily make sure that $v(\Delta_j^-)=v(\Delta_j^+)=0$

Hence, by the method of integrating by parts, we obtain the inequality:

$$\left\|(l_{n,j}+\lambda E)^{*}v\right\|_2 > c_1\|v\|_2,$$

where $c_1>0$.

From this inequality to the force $(l_{n,j}+\lambda E)^{*}v=0$, it follows that $\square$ $v=0$. Thus, we come to a contradiction, which proves our statement.

It remains to prove inequalities (1.4.7) and (1.4.8)

For this we compose a scalar product $<(l_{n,j}+\lambda E)u, u>$, where $u\in D(l_{n,j}+\lambda E)$. Integrating in parts and using the boundary conditions, we get:

$|<(l_{n,j}+\lambda E)u, u>|=$

$$=\left|\int_{\Delta j}|u'|^2dy+\int_{\Delta j} n^2s(y)+inp(y)+((y)+\lambda)|u|^2]dy\right|. \quad (1.4.10)$$

Using the Cauchy-Bunyakovsky inequality and the condition i) we have:

$$c_0\|(l_{n,j}+\lambda E)u\|_2^2 > c_0|n|^2\min_{y\in\Delta_j}|p(y)^2\|u\|_2^2. \quad (1.4.11)$$

From (1.4.10), using Cauchy inequality with $\varepsilon>0$, we get

$$\frac{1}{2\varepsilon}\|(l_{n,j}+\lambda E)u\|_2^2 > \int_{\Delta j}|u'|^2+(q(y)+\lambda-\frac{\varepsilon}{2})|u|^2]dy-n^2\int_{\Delta j}s(y)|u|^2]dy$$

Combining this inequality with inequality (1.4.11) we get the inequality:

$$(\frac{1}{2\varepsilon}+c_0)\left\|(l_{n,j}+\lambda E)u\right\|_2^2\geq\int_{\Delta j}|u'|^2+(-(y)+\lambda-\frac{\varepsilon}{2})|u|^2]dy+$$

$$+\int_{\Delta j}n^2(c_0\min|p(y)|^2-s(y)|u^2|)dy.$$

From the last inequality, taking into account the condition (*), for sufficiently large $\lambda>0$, we get:

$$c(\varepsilon)\|(l_{n,j}+\lambda E)u\|_2^2 \geq \lambda\|u\|_2^2, \quad (1.4.12)$$

where $c(\varepsilon) = c_0+\frac{1}{2\varepsilon}$

Inequality (1.4.12) and proves inequality (1.4.7).

From (1.4.10), by virtue of (1.4.12), inequality (1.4.8) follows easily

Consider the operator K, defined by the equation

$$Kf=\sum_j \varphi_j(l_{n,j}+\lambda E)\varphi_j f, \qquad f\in L_2(R),$$

where $\{\varphi_j\}$ is a a set of non-negative functions belong to $C_0^\infty(R)$ such that $\sum_j \varphi_j \equiv 1$, $\operatorname{supp}\varphi_j\subset\Delta_j$ and $\bigcup_j \Delta_j = R$.

It is easy to show that $Kf\in D(l_n)$. Now acting on the Kf operator $(l_n+\lambda E)$ we have:

$$(l_n+\lambda E)Kf= f+\sum_j \varphi_j (l_{n,j}+\lambda E)^{-1}\varphi_j f+2\sum_j \varphi_j\frac{d}{dx}(l_{n,j}+\lambda E)^{-1}\varphi_j f.$$

Since f is arbitrary, it can be seen that equality is performed for any function $f\in L_2(R)$. From inequalities (1.4.7), (1.4.8) it is easy to get the following estimate:

$$\|(l_n+\lambda E)Kf\|_2 < \|f\|_2+\left[24c\sum_i\left(\frac{\|\varphi_i f\|^2}{\sqrt{\lambda}}+\frac{\|\varphi_i f\|^2}{\sqrt[4]{\lambda}}\right)\right]^{\frac{1}{2}}<\infty.$$

Based on the above reasoning and the inequality (1.4.6), we conclude that as sufficiently large $\lambda>0$, the operator $l_n+\lambda E$ has a continuous inverse in space $L_2(R)$. Lemma 1.4.1. is proved.

Proof of Lemma 1.4.2. Let the conditions i)-iii) be fulfilled. Then by Lemma 1.4.1 and continuous reversibility $l_{n,j}+\lambda E$ for any function $f\in C_0^\infty(R)$ the equality

$$(l_n+\lambda E)^{-1}f= (l_n+\lambda E)^{-1} B_{n,\lambda} f+ M_{n,\lambda}f, \quad (1.4.13)$$

holds, where $B_{n,\lambda}f= [E-\sum_j (l_{n,j}+\lambda E)\varphi_j(l_{n,j}+\lambda E)^{-1}\varphi_j]f$, $M_{n,\lambda} f=\sum_j \varphi_j(l_{n,j}+\lambda E)^{-1}\varphi_j f$

For $\lambda>0$, we get:

$$\|(l_{n,j}+\lambda E)^{-1}\|_{2\to 2}<\frac{2}{\lambda}, \quad \|\frac{d}{dx}(l_{n,j}+\lambda E)^{-1}\|_{2\to 2}<\frac{1}{\lambda}. \quad (1.4.14)$$

Now the representation of the operator $B_{n,\lambda}$ and $\sum \varphi_j^2 \equi 1$ и $f - \sum_j \varphi_j^2 f = 0$, for $B_{n,\lambda}$ we get:

$$B_{n,\lambda} f = \sum_j 2\varphi_j' \frac{d}{dy} (l_{n,j}+\lambda E)^{-1} \varphi_j^2 f + \sum_j \varphi_j'' (l_{n,j}+\lambda E)^{-1} \varphi_j f .$$

If we now turn to the estimation of the operator norm $B_{n,\lambda}$, then due to the fact that $\Delta_j = [j-1, j+1]$ only functions φ_{j-1}, φ_j, φ_{j+1} are not zero, we have:

$$\|B_{n,\lambda} f\|_2^2 < \sum_{j=-\lambda} \int_{j-1}^{j+1} \left| \sum_{j-1}^{j+1} [\varphi_j'' (l_{n,j} + \lambda E)^{-1} \varphi_j f + 2\varphi_j' \frac{d}{dy} (l_{n,j} + \lambda E)^{-1} \varphi_j f] \right|^2 dy$$

Hence, given that $(a+b+c)^2 < 3(a^2 + b^2 + c^2)$ and $(a+b)^2 < 2(a^2 + b^2)$ it follows that

$$\|B_{n,\lambda} f\|_2^2 < 24c \sum_j \left(\|(l_{n,j} + \lambda E)^{-1}\|_2^2 \|\varphi_j f\|_2^2 + \left\| \frac{d}{dy} (l_{n,j} + \lambda E)^{-1} \right\|_2^2 \|\varphi_j f\|_2^2 \right)$$

From the inequalities (1.4.14) and the last inequality, it is clear that if $\lambda > 0$ is large enough, then $\|B_{n,\lambda}\| < 1$. Then according to the well-known functional analysis theorem the operator $E - B_{n,\lambda}$ has a continuous inverse and for operator $(l_n+\lambda E)^{-1}$, by (1.4.13) the equality

$$(l_n+\lambda E)^{-1} = M_{n,\lambda} (E - B_{n,\lambda})^{-1}$$

holds. This shows that the operator $\rho(y)|n|^2(l_n+\lambda E)^{-1}$ is bounded (or unbounded) together with the operator $\rho(y)|n|^2 M_{n,\lambda}$, i.e. the inequality

$$\|\rho(y)|n|^2(l_n+\lambda E)^{-1}\|_2 < c(\lambda) \|\rho(y)|n|^2 M_{n,\lambda}\|_2 .$$

holds, where $\alpha = 0,1$; $\rho(y)$ is a continuous function.

From the last inequality and the definition of Mn, λ, we get the Proof of Lemma 1.4.2.

Proof of Lemma 1.4.3. For the operator $(l_{n,j}+\lambda E)^{-1}$ the estimate

$$\|(l_{n,j}+\lambda E)^{-1}\|_2 < \frac{2}{c(\tilde{y}_j)},$$

holds, where $c(\tilde{y}_j) = \min_{y \in \Delta_j} q(y)$.

According to this inequality and Lemma 1.4.2. we have:

$$\|q(y)(l_n+\lambda E)^{-1}\|_2 < c(\lambda)\ \sup_j \|q(y)\varphi_j(l_{n,j}+\lambda E)^{-1}\|_2 <$$

$$\le c(\lambda)\ \sup_{t,y\in\Delta} \frac{c(y)}{c(t)} < \infty$$

Similarly, we obtain the second inequality of Lemma 1.4.3 is proven.

P r o o f of the main theorem.

Necessity. Let $u(x,y)\in D(L)$. Since the system $\{e^{inx}\}_{n=-\infty}^{\infty}$ complete in $L_2(\Omega)$, then for $u(x,y)$ fair decomposition in the sense of L_2

$$u(x,y) = \sum_{n=-\infty}^{\infty} u_n(y)e^{inx}$$

It is not difficult to establish that

$$\gamma > \frac{\| -u''(y)\|^2}{\| -u''(y)+q(y)u(y)\|_2 + \| u(y)\|_2}, \qquad (1.4.15)$$

where $u(y)\in D(l_0)$, $l_0u = -u''(y) + q(y)u(y)$.

Indeed, it is clear from the representation of the function $u(x, y)$ as $n=0$. From it, in particular, it follows that $u(y)\in D(l_0)\subset D(L)$, from where by definition γ, follows (1.4.15).

Let the conditions i)-iii) hold, and the condition (*) not fulfilled and value γ is finite.

Take a sequence of non-overlapping intervals $\Delta_k=[-\frac{\pi}{2}+\pi k,\ \frac{\pi}{2}+\pi k]$, $\bigcup_{k} \Delta_k=R$, where $k=0, +1, +2, \dots$

Let $c(y) = -(k^2+1)$ as $y\in\Delta_k$. It is easy to check the condition iii) holds to the function $c(y)$.

Consider the following sequence of functions:

$$z_k(y) = \begin{cases} \sin k(y-y_k-\frac{\pi}{2}), \text{ при } y\in\Delta_k \\ 0, \text{ при } y\notin\Delta_k \end{cases};$$

where y_k is the average of the interval Δ_k .

Let

$$\alpha(y)=\begin{cases}1, & |y|\le \frac{\pi}{2}-\delta\\ 0, & |y|>\frac{\pi}{2}\end{cases} \quad \text{и} \quad \alpha(y)\in C_0^\infty\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

Now for the function $u(y)\in D(l_0)$ we will take a sequence of functions $\alpha(y-y_k)z_{s(y)}$ (y_k is the middle). Then

$$-(\alpha(y-y_k)z_{s(y)})''+q(y)\alpha(y-y_k)z_{s(y)}=-\alpha(y-y_k)\sin k(y-y_k-\frac{\pi}{2})-$$

$$-2k\alpha'(y-y_k)\cos k(y-y_k-\frac{\pi}{2})-\alpha''(y-y_k)\sin k(y-y_k-\frac{\pi}{2}) \qquad (1.4.16)$$

$$-(\alpha(y\text{-}y_k)z_{s(y)})''=k^2\alpha(y\text{-}y_k)\sin k(y\text{-}y_k\text{-}\frac{\pi}{2})-$$

$$-2k\alpha'(y\text{-}y_k)\cos k(y-y_k-\frac{\pi}{2})-\alpha''(y\text{-}y_k)\sin k(y\text{-}y_k-\frac{\pi}{2}) \qquad (1.4.17)$$

We estimate the norm:

$$\|-(\alpha(y-y_k)z_{s(y)})''+q(y)\alpha(y\text{-}y_k)z_{s(y)}\|_2< c_1+2kc_2+c_3$$
$$\|-(\alpha(y\text{-}y_k)z_{s(y)})''\|_2\ge k^2c_1-2kc_2-c_3,$$

where

$$c_1=\|\alpha(y\text{-}y_k)\sin k(y-y_k-\frac{\pi}{2})\|_2,$$

$$c_2=\|\alpha'(y\text{-}y_k)\cos k(y-y_k-\frac{\pi}{2})\|_2,$$

$$c_3=\|\alpha''(y-y_k)\sin k(y-y_k-\frac{\pi}{2})\|_2.$$

Here all norms are finite due to the fact that Δ_k functions vanish, i.e. are compactly supported.

Now by inequality (1.4.15) and the last inequality we get:

$$\gamma>\frac{k^2c_1-2kc_2-c_3}{c_1+2kc_2+c_3}$$

Letting k tend to infinity we see that $\gamma\to\infty$, which contradicts the finiteness of the magnitude γ.

Proof of sufficiency. Assume that i)-iii) and (*) be fulfilled. Prove that γ is finite.

Consider the scalar product $<(L+\lambda E)u, u>$ and $<(L+\lambda E)u, u_x>$, given the conditions i) and (*), we get

$$\|(L+\lambda E)u\|_2 > c\|u\|_2, \text{ for all } u\in D(L)$$

where c>0 is a constant.

Due to the last inequality and Lemma 1.4.1.we get $L+\lambda E$ is continuously invertible as $\lambda>0$ and:

$$(L+\lambda E)^{-1}f = \sum_{n=-\infty}^{\infty}(l_n+\lambda E)^{-1}f_n e^{inx}, \qquad (1.4.18)$$

holds, for any $f\in L_2(\Omega)$.

From (1.4.18) and orthonormality of the system $\{e^{inx}\}_{n=-\infty}^{\infty}$ in $L_2(-\pi, \pi)$ easy to follow

$$\|\rho(y)D_x^{\alpha}(L+\lambda E)^{-1}\|_{2\to 2} < \sup_{|n|}\|\rho(y)|n|^{\alpha}(l_n+\lambda E)^{-1}\|_{2\to 2}$$

where $D_x^{\alpha}=\frac{\partial^{\alpha}}{\partial x^{\alpha}}$, n=+ 1, +2 ...; $\rho(y)$ is a continuous in R.

From here and from Lemma 1.4.2 we get

$$\|\rho(y)D_x^{\alpha}(L+\lambda E)^{-1}\|_{2\to 2} < c(\lambda)\sup_{|n|}\sup_{|j|}\|\rho(y)|n|^{\alpha}\varphi_j\,(l_{n,j}+\lambda E)^{-1}\|_{2\to 2}$$

Now, if $\rho(y)=p(y)$, $\rho(y)=q(y)$ then according to the last inequality and Lemma 1.4.3 there is an inequality

$$\|p(y)D_x(L+\lambda E)^{-1}\|_{2\to 2} < \infty, \quad \|q(y)(L+\lambda E)^{-1}\|_{2\to 2} < \infty$$

Using the last inequality, we obtain the inequality

$$\begin{aligned}\|-s(y)u_{xx}-u_{yy}\|_2 &= \|(L+\lambda E)u - p(y)u_x - q(y)u - \lambda u\|_2 < \\ &< \|(L+\lambda E)u\|_2 + \|p(y)u_x\|_2 + \|q(y)u\|_2 + \|\lambda u\|_2 < \\ &< \|(L+\lambda E)u\|_2 + \|p(y)\,D_x(L+\lambda E)^{-1}(L+\lambda E)u\|_2 + \\ &+\|q(y)(L+\lambda E)^{-1}(L+\lambda E)u\|_2 + \|\lambda(L+\lambda E)^{-1}(L+\lambda E)u\|_2 \le \le c\|(L+\lambda E)u\|_2,\end{aligned}$$

where c>0 is a constant.

It follows that $\|-s(y)u_{xx}-u_{yy}\|_2 < c\|(L+\lambda E)u\|_2$ for any $u\in D(L)$. It means that $\gamma<\infty$, which was required to prove.
Theorem is proved.

1.5 Two-sides estimates of s-numbers (eigenvalues according to Schmidt) and eigenvalues of the resolvent of a mixed-type operator

This section is devoted to the study of the spectral properties of operators (1.1.1). Non-zero s-numbers (Schmidt eigenvalues) of the resolvent of operator

(1.1.1) will be numbered in the order of their decreasing taking into account their multiplicities.

We introduce the following function $N(\lambda)=\sum_{s_k>\lambda}1$ quantity of s_k greater than $\lambda>0$.

Theorem 1.5.1. Assume that i)-iii) be fulfilled. Then the inequality

$$c^{-1}\sum_{n=-\infty}^{\infty}\lambda^{-1/2}mes\left(y\in R::\left|n^2(s(y)+\varepsilon)+inp(y)+q(y)\right|\le c^{-1}\lambda^{-1/2}\right)\le$$

$$\le N(\lambda)\le c\sum_{n=-\infty}^{\infty}\lambda^{-1}mes\left(y\in R:\left|n^2s(y)+inp(y)+q(y)\right|\le c\lambda^{-1}\right), \qquad (1.5.1)$$

holds, where $c=c(\mu_1,\mu_0)$, $c_0=c_0(\mu_1,\mu_0,\varepsilon)$.

On the basis of this theorem, we can obtain some important properties of the eigenvalues of the operator (1.1.1), i.e. fair next theorem

Theorem 1.5.2. Let the conditions i) and (1.1.8) be fulfilled. Then:

a) there exists a sequence of positive eigenvalues of the operator's resolution (1.1.1);

b) Let in addition to the conditions i), (1.1.8) the condition iii) be fulfilled. Then for the distribution functions of this sequence, the following two-sided estimate is valid

$$c^{-1}\lambda^{1/2}mes(y\in R:q(y)\le c^{-1}\lambda^{1/2})\le N(\lambda)\le$$
$$\le c\lambda^{1/2}mes(y\in R:q(y)\le c\lambda^{1/2}), \qquad (1.5.2)$$

where $c=c(\mu)$;

c) if λ_1 is the upper bound of the spectrum of this sequence, then the estimate

$$c^{-1}\delta^{-2}\le\lambda_1\le c\delta^{-2}$$

holds with the constant $c=c(\mu)$.

Proof of Theorem 1.1. It is known that $\{\frac{1}{\sqrt{\pi}}e^{inx}\}_{n=-\infty}^{\infty}$ is the complete orthonormal eigenvector system of the next operator

$$Au=-u_{xx},$$
$$u(-\pi)=u(\pi), u_x(-\pi)=u_x(\pi)$$

Any element $f(x,y)\in L_2(\Omega)$ can be written as follows

$$f(x,y)=\sum_{n=-\infty}^{\infty}f_n(y)e^{inx}, \quad f_n(n)=(f(x,y),e^{inx})$$

From Lemma 5.3.1 it follows that

$$L^{-1}f = \sum_{n=-\infty}^{\infty} l_n^{-1} f_n e^{inx}$$

From this and the definition of the adjoint operatora we have:

$$(L^{-1})^* f = \sum_{n=-\infty}^{\infty} (l_n^{-1})^* f_n(y) e^{inx}$$

Now it is not difficult to determine the following action.

$$(L^{-1})^* L^{-1} f = \sum_{n=-\infty}^{\infty} (l_n^{-1})^* l_n^{-1} f_n(y) e^{inx} \tag{1.5.3}$$

From (1.5.3) it follows that if λ belongs to the spectrum of $(L^{-1})^* L^{-1}$, then λ is the point of the spectrum of one of the operators $(l_n^{-1})^* l_n^{-1}$ and vice versa if λ is the point of the spectrum of one of the operators $(l_n^{-1})^* l_n^{-1}$, then λ is the spectrum point of the operator $(L^{-1})^* L^{-1}$.

From here and from Theorem 5.3.1, the proof of Theorem 1.1 follows easily.

Proof of Theorem 1.5.2. The eigenfunctions of the operator (1.1.1) we are looking for a $u(x,y) = \sum_{n=-\infty}^{\infty} u_n(y) e^{inx}$. Then, with respect to the function u (y), we obtain the Sturm-Liouville spectral problem

$$-u_n''(y) + (n^2 s(y) + inp(y) + q(y)) u_n(y) = \lambda u_n(y) \tag{1.5.4}$$

It is known that as conditions i) and (1.1.8) are met in the case of n = 0, the operator (1.5.4) has positive eigenvalues. Further, repeating the arguments of the proofs of Theorem 1.5.1, we obtain the Proof of Theorem 1.5.2.

Corollary. Let the conditions of Theorem 1.5.2 be satisfied. Then fair inclusion $\{\lambda : |\lambda| > \frac{c}{\bar{\delta}}\} \subset \rho(L^{-1})$ is a resolvent set of operators L^{-1}, where $\bar{\delta} = \min\{\delta_0, \delta_1\}$, $c = c\{\mu, \delta_0, \delta_1\}$.

Remark. As far as we know, similar resolvents about estimates of singular numbers (eigenvalues by Schmidt) of the resolvent of a mixed-type operator have not previously appeared in other papers.

1.6 Spectrum estimates of one class of operators of mixed type in an unbounded domain

Consider the mixed type operator

$$Lu = -s(y)u_{xx} - u_{yy} + p(y)u_x + q(y)u \tag{1.6.1}$$

where s(y) is a piecewise continuous function on a segment [-1,1] and ys(y)>0 as $y \neq 0$. k(0)=0 (as y=0).

Initially define operator on $C_{0,\pi}^{\infty}(\Omega)$ is a set consisting of infinitely differentiable functions satisfying the conditions:

$$u(-\pi, y) = u(\pi, y), u_x(-\pi, y) = u_x(\pi, y)$$

and finite to variable y, where $\Omega = \{(x,y): -\pi < x < \pi, -1 < y < 1\}$

Note that operator L admits a closure, and we also denote it by L.

Let us introduce a number of necessary notation and definitions, which we will need in the future.

Let the function $u(x,y) \in L_2(\Omega)$. Then the following decomposition holds:

$$u(x,y) = \sum_{n=-\infty}^{\infty} u_n(y) e^{inx}.$$

Definition. Fractional derivative $D_x^{\alpha} u$ by x order α≥0 of function $u(x,y)$ let's call the expression [30]:

$$D_x^{\alpha} u = e^{\frac{i\pi\alpha}{2}} \sum_{n=0}^{\infty} n^{\alpha} u_n(y) e^{inx} + e^{-\frac{i\pi\alpha}{2}-1} \sum_{n=-\infty}^{-1} |n|^{\alpha} u_n(y) e^{inx}.$$

Here equality is understood in the metric in $L_2(\Omega)$.

Formulation of the main results

Theorem 1.6.1. Let the condition:

i) $|p(y)| \geq \delta_0 > 0, q(y) \geq \delta > 0$ is a continuous functions on the segment [-1,1]

be fulfilled. Then:

a) the operator $(L + \lambda E)$ is continuously invertible as λ>0;

b) the operators $r(y)D_x(L+\lambda E)^{-1}, r(y)D_y(L+\lambda E)^{-1}$ is bounded in $L_2(\Omega)$, here $D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y}$; r(y) is a continuous function on the segment [-1,1].

c) the operator $r(y)D_x^{\alpha}(L+\lambda E)^{-1}$ is a completely continuous if 0≤α<1.

Theorem 1.6.2. Let the conditions of Theorem 1.6.1 be satisfied. Then for eigenvalues according to Schmidt the following estimate holds

$$c^{-1}\frac{1}{k} \le s_k \le c\frac{1}{k^{1/2}},\ k = 1,2,\ldots,$$

where c>0 does not depend on k.

Theorem 1.6.3. Assume that i) is fulfilled. Then:

a) the spectrum $\sigma(L^{-1})$ is a discrete;

б) for any non-zero $\lambda \in \sigma(L^{-1})$ the following estimate holds:

$$|\lambda_k| \le c\frac{1}{k^{1/2}},\ k = 1,2,\ldots,$$

where the positive constant c>0 does not depend on k.

Recall that σp denotes the set of all completely continuous operators such that

$$\|A\|_{\sigma_p}^{p} = \sum_{k=1}^{\infty} s_k^p(A) < \infty$$

where $s_k(A)$ is a Schmidt eigenvalues of a completely continuous operator A.

In the following theorem, assertion is made about as the membership of the resolvent of the operator (1) to the class σ_p.

Theorem 1.6.4. Assume that i) is fulfilled. Then the resolvent of L belong to σ_p, if p>2.

Auxiliary lemmas and inequalities

Lemma 1.6.1. Assume that i) is fulfilled. Then L+λE is continuously invertible as λ≥0 and the estimate

$$(L+\lambda E)^{-1} f = \sum_{n=-\infty}^{\infty} (l_n + \lambda E)^{-1} f_n e^{inx} \qquad (1.6.2)$$

holds in $L_2(\Omega)$, where $(l_n+\lambda E)^{-1}$ - inverse operator to the closed operator $(l_n+\lambda E)$ determined by equality

$$(l_n + \lambda E)u = -u''(y) + (n^2 s(y)y) + inp(y) + q(y) + \lambda)u(y) \qquad (1.6.3)$$

initially on $C_0^\infty(-1,1)$.

A proof of this lemma can be found in the paper [12].

Lemma 1.6.2. Let $(l_n+\lambda E)$ defined by equality (1.6.3) on $C_0^\infty(-1,1)$ ($n = 0,\pm1,\pm2,\ldots$) and suppose that i) is fulfilled. Then the estimate

$$\left\|(l_n+\lambda E)^{-1}\right\|_{2\to 2} \le \frac{c}{\lambda^{1/2}}.$$

holds, where c>0 is a constant independent of n.

Proof. For any $u(y)\in C_0^\infty(-1,1)$

$$\langle (l_n+\lambda E)u,u\rangle = \int_{-1}^{1}\left[|u'|^2+(n^2 s(y)+inp(y)+q(y)+\lambda)|u|^2\right]dy \qquad (1.6.4)$$

holds. From here and considering the condition i) we find:

$$\left|\langle (l_n+\lambda E)u,u\rangle\right| \ge \left|\int_{-1}^{1} inp(y)|u|^2\,dy\right| \ge |n|\delta_0\|u\|^2.$$

Now using the Cauchy – Bunyakovsky inequality, we get

$$\|(l_n+\lambda E)u\|_2 \ge |n|\delta_0\|u\|_2. \qquad (1.6.5)$$

From (1.6.4) and the Cauchy inequality with ε = 1 it follows that

$$\frac{1}{2}\|(l_n+\lambda E)u\|_2^2 \ge \int_{-1}^{1}\left[|u'|^2+(q(y)+\lambda)|u|^2\right]dy-\int_{-1}^{1}n^2|s(y)||u|^2\,dy.$$

Using the condition i) and the fact that λ> 0, we obtain

$$\frac{1}{2}\|(l_n+\lambda E)u\|_2^2 \ge \frac{1}{2}\int_{-1}^{1}\left[|u'|^2+(q(y)+\lambda)|u|^2\right]dy-\int_{-1}^{1}n^2|s(y)||u|^2\,dy. \qquad (1.6.6)$$

Combining (1.6.5) and (1.6.6) finally we have

$$c^2\|(l_n+\lambda E)u\|_2^2 \ge \lambda\|u\|_2^2$$

The last inequality implies the assertion of Lemma 1.6.2.

Lemma 1.6.3. Let the conditions of Lemma 1.6.2 be satisfied. Then the estimate

$$\left\|(l_n+\lambda E)^{-1}\right\|_{2\to 2} \le \frac{1}{|n|\cdot\delta_0},\ n=\pm 1,\pm 2,\dots \qquad (1.6.7)$$

holds. The proof of Lemma 1.6.3 follows from inequality (1.6.5).

Lemma 1.6.4. Suppose that i) is fulfilled. Then the estimate

$$\left\| \frac{d}{dy}(l_n + \lambda E)^{-1} \right\|_{2\to 2} \le c,$$

holds, where c>0 any constant.

Proof. Due to condition i) and inequalities (1.6.5) and (1.6.6), we obtain that

$$c\|(l_n + \lambda E)u\|_2^2 \ge \|u'\|_2^2 + \|u\|_2^2,$$

where c>0 does not depend on u and n.

Hence

$$\left\| \frac{d}{dy}((l_n + \lambda E)^{-1} \right\|_{2\to 2} = \sup_{f\in \alpha_2(-1,1)} \frac{\left\| \frac{d}{dy}((l_n + \lambda E)^{-1} f \right\|_2}{\|f\|_2} =$$

$$= \sup_{u\in D(l_n+\lambda E)} \frac{\|u'\|_2}{\|(l_n + \lambda E)u\|_2} \le c < \infty .$$

Lemma is proved.

We now turn to the direct proof of the main theorems of this section..

Proof of Theorem 1.6.1.

Proof of a) the Theorem 1.6.1 immediately follows from the lemma 1.6.1.

We prove part b) of Theorem 1.6.1. By virtue of a) and Lemma 1.6.3 we have

$$\|r(y)D_x(L + \lambda E)^{-1} f\|_2^2 = \left\| r(y) \sum_{n=-\infty}^{\infty} in(l_n + \lambda E)^{-1} f_n e^{inx} \right\|_2^2 =$$

$$= \sum_{n=-\infty}^{\infty} \|r(y) in(l_n + \lambda E)^{-1} f_n e^{inx}\|_2^2 \le \max_{y\in[-1,1]} |r(y)| \sum_{n=-\infty}^{\infty} n^2 \|(l_n + \lambda E)^{-1}\|_2^2 \cdot$$

$$\cdot \|f_n\|_2^2 \le c_0 \sup_{\{n\}} |n|^2 \|(l_n + \lambda E)^{-1}\|_2^2 \sum_{n=-\infty}^{\infty} \|f_n\|_2^2 \le \frac{c_0}{\delta_0^2} \|f\|_2^2 .$$

Hence

$$\|r(y)D_x(L + \lambda E)^{-1}\|_{2\to 2} \le \frac{c_0}{\delta_0} < \infty .$$

Next, calculate the norm:

$$\left\|r(x)D_x(L+\lambda E)^{-1}f\right\|_2^2 = \sum_{n=-\infty}^{\infty}\left\|r(x)\frac{d}{dx}(l_n+\lambda E)^{-1}f_n(x)\right\|_2^2$$

$$\le \max_{x\in[-1,1]}|r(x)|\sum_{n=-\infty}^{\infty}\left\|\frac{d}{dx}(l_n+\lambda E)^{-1}f_n\right\|_2^2 \le c_0\sum_{n=-\infty}^{\infty}\left\|\frac{d}{dx}(l_n+\lambda E)^{-1}\right\|_{2\to2}^2\left\|f_n\right\|_2^2.$$

From here, by Lemma 1.6.4, we get that

$$\left\|r(x)D_x(L+\lambda E)^{-1}\right\|_{2\to2} \le c<\infty.$$

Item b) of Theorem 1.6.1 is proved..

Using the operator view and the definition of the fractional derivative, we obtain that

$$r(x)D_x^{\alpha}(L+\lambda E)^{-1}f = r(x)e^{\frac{i\pi\alpha}{2}}\sum_{n=0}^{\infty}n^{\alpha}(l_n+\lambda E)^{-1}f_n(x)e^{inx}+$$

$$+r(x)^{\frac{-i\pi\alpha}{2}}\sum_{n=-\infty}^{1}|n|^{\alpha}(l_n+\lambda E)^{-1}f_n(x)e^{inx}$$

From Lemma 1.6.1 it follows that $(l_n+\lambda E)$ has a continuous reverse operator $(l_n+\lambda E)^{-1}$, and from Lemma 1.6.4 it is clear that the domain of $(l_n+\lambda E)^{-1}$ belong to $W_2^1(-1,1)$ for any n. Then, from the well-known theorems of the Sobolev space (see namer [21]), it follows that operator $r(x)(l_n+\lambda E)^{-1}$ is completely continuous for each n and as this holds the inequality

$$\left\|r(x)|n|^{\alpha}(l_n+\lambda E)^{-1}f\right\|_{2\to2} \le \frac{|n|^{\alpha}}{|n|\cdot\delta_0}, 0\le\alpha<1 \qquad (1.6.8)$$

The last inequality follows from Lemma 1.6.3.

Since for each n the operator $r(x)|n|^{\alpha}(l_n+\lambda E)^{-1}$ completely continuous from L_2 to L_2, then from the well-known theorems for completely continuous operators (see [1]), it follows that operator $r(x)D_x^{\alpha}(L+\lambda E)^{-1}$ is completely continuous if

$$\mu=\lim_{n\to\infty}\mu=\lim_{n\to\infty}\left\|r(x)|n|^{\alpha}(l_n+\lambda E)^{-1}\right\|_{2\to2}=0$$

From (1.6.8) it can be seen that $\mu\to0$, as $n\to\infty$. Completely continuity of the operator $r(x)D^{\alpha}(L+\lambda E)^{-1}$ is proved. Theorem 1.6.1 is proved.

For further discussion, we need some important estimates and inclusions.

We introduce the following sets:

$$M = \left\{u \in L_2(\Omega) : \|Lu\|_2^2 + \|u\|_2^2 \le 1\right\}$$

$$\widetilde{M}_c = \left\{u \in L_2(\Omega) : \|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2 \le c\right\}$$

$$\widetilde{\widetilde{M}}_{c^{-1}} = \left\{u \in L_2(\Omega) : \|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2 \le c^{-1}\right\}$$

Then the following Lemma takes place.

Lemma 1.6.5. Let condition i) is fulfilled. Then the following inclusions are valid

$$\widetilde{\widetilde{M}}_{c^{-1}} \subseteq M \subseteq \widetilde{M}_c,$$

where c>0 is a constant independent of u(x, y).

Proof. Let $u(x,y) \in \widetilde{\widetilde{M}}_{c^{-1}}$. Then

$$\|Lu\|_2^2 + \|u\|_2^2 = \|-s(y)u_{xx} - u_{yy} + p(y)u_x + q(y)u\|_2^2 + \|u\|_2^2 \le$$

$$\le \|-s(y)u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|p(y)u_x\|_2^2 + \|q(y)u\|_2^2 + \|u\|_2^2 \le$$

$$\le c(\|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|u_x\|_2^2 + \|u\|_2^2) \le c(\|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + \|u_x\|_2^2 +$$

$$+\|u_y\|_2^2 + \|u\|_2^2) \le c \cdot c^{-1} \le 1,$$

where $c = \max\limits_{y \in [-1,1]} \{|k(y)|, |a(y)|, |c(y)|\}$.

It follows that

$$\widetilde{\widetilde{M}}_{c^{-1}} \subseteq M .$$

Let $u \in M$. Then by virtue of point b) of Theorem 1.6.1 we have:

$$\|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2) \le c(\|Lu\|_2^2 + \|u\|_2^2) \le c,$$

i.e.

$$M \subseteq \widetilde{M}_c$$

Lemma is proved.

Lemma 1.6.1. Let condition i) is fulfilled. Then the estimate

$$c^{-1}\widetilde{\widetilde{d}}_k \le s_{k+1} \le c\widetilde{d}_k, k = 1,2,...,$$

holds. where c>0 any constant. s_{k+1} are singular numbers of operators L^{-1}: $\tilde{d}_k, \tilde{\tilde{d}}_k$ are widths of corresponding sets $\tilde{M}, \tilde{\tilde{M}}$.

Proof. It follows from Lemma 1.6.5 and the properties of the diameters that

$$c^{-1}\tilde{\tilde{d}}_k \leq d_k \leq c\tilde{d}_k$$

Hence, given the equality $s_{k+1} = d_k$ (second definition of s-numbers) we obtain the proof of Lemma 1.6.1.

We introduce the following function $N(\lambda) = \sum_{d_k > \lambda} 1$ is a quantity of d_k greater than $\lambda>0$.

Lemma 1.6.7. Let the condition of Lemma 1.6.5 be satisfied.Then the estimate

$$\tilde{\tilde{N}}(c\lambda) \leq N(\lambda) \leq \tilde{N}(c^{-1}\lambda) \tag{1.6.9}$$

holds, where $N(\lambda) = \sum_{s_{k+1} > \lambda} 1$, $\tilde{N}(\lambda) = \sum_{\tilde{d}_k > \lambda} 1$, $\tilde{\tilde{N}}(\lambda) = \sum_{\tilde{\tilde{d}}_k > \lambda} 1$.

Proof. Using Lemma 1.6.6 we find:

$$N(\lambda) = \sum_{s_{k+1} > \lambda} 1 \leq \sum_{c\tilde{d}_k > \lambda} 1 = \sum_{\tilde{d}_k > c^{-1}\lambda} 1 = \tilde{N}(c^{-1}\lambda)$$

Similarly

$$\tilde{\tilde{N}}(c\lambda) = \sum_{\tilde{\tilde{d}}_k > c\lambda} 1 = \sum_{c^{-1}\tilde{\tilde{d}}_k > \lambda} 1 \leq \sum_{s_{k+1} > \lambda} 1 = N(\lambda)$$

Where we finally come to inequality (1.6.9). Lemma proven.

We now turn to the proof of Theorem 1.6.2 and 1.6.3.

Proof of Theorem 1.6.2.

For the function $\tilde{N}(\lambda) = \sum_{\tilde{d}_k > \lambda} 1$, $\tilde{\tilde{N}}(\lambda) = \sum_{\tilde{\tilde{d}}_k > \lambda} 1$ the estimates

$$c^{-1}\lambda^{-2} \leq \tilde{N}(\lambda) \leq c\lambda^{-2}, \tag{1.6.10}$$

$$c^{-1}\lambda^{-1} \leq \tilde{\tilde{N}}(\lambda) \leq c\lambda^{-1}, \tag{1.6.11}$$

hold, where c does not depend on $\lambda>0$.

Let $\lambda = \tilde{d}_k$, Then $\tilde{N}(\tilde{d}_k) = k$ and from (1.6.10) it follows that

$$c^{-1}d_k^{-2} \leq k \leq cd_k^{-2}.$$

Hence

$$c^{-1}\frac{1}{k^{1/2}} \le \tilde{d}_k \le c\frac{1}{k^{1/2}}.$$

Same way we have:

$$c^{-1}\frac{1}{k} \le \tilde{\tilde{d}}_k \le c\frac{1}{k}.$$

Now, using Lemma 1.6.6, we obtain that

$$c^{-1}\frac{1}{k} \le s_k \le c\frac{1}{k^{1/2}}, k = 1,2,\dots . \qquad (1.6.12)$$

Theorem 1.6.2 is proved.

Proof of Theorem 1.6.3.

For completely uninterrupted operators, Weil's inequality holds true [9]:

$$\prod_{j=1}^{k}|\lambda_j(A)| \le \prod_{j=1}^{k} s_j(A), \quad k = 1,2,\dots, \qquad (1.6.13)$$

where A is a completely continuous operator, $\lambda_j(A)$ are the eigenvalues numbered in order of non-increasing absolute values, $s_j(A)$ are the non-increasing singular numbers.

From (1.6.12) and (1.6.13) it follows that

$$|\lambda_k|^k \le \prod_{j=1}^{k}|\lambda_j| \le \prod_{j=1}^{k} s_j \le c^k (k!)^{-\frac{1}{2}}$$

Next, using the inequality $e^k k! > k^k (k = 1,2,\dots)$ we get

$$|\lambda_k|^k \le c^k (k!)^{-\frac{1}{2}} \le c^k e^{\frac{k}{2}} k^{-\frac{k}{2}}$$

Hence finally we have:

$$|\lambda_k| \le ck^{-\frac{1}{2}}, \quad k = 1,2,\dots$$

Theorem 1.6.3 is proved.

Proof теоремы 1.6.4. The proof of the assertion of Theorem 1.6.4 immediately follows from Theorems 1.6.1-1.6.2.

2 ON THE EXISTENCE OF RESOLVENTS OF A SECOND ORDER OF A SECOND ORDER WITH A COMPLEX POTENTIAL WHEN THE CURRENT PART OF CAPACITY CONTAINS A PARAMETER A CHANGING SIGN

2.1 A priori and coercive estimates for a second-order differential operator with complex coefficient (case on a finite interval)

Consider the operator

$$\left(m_{t,i,\gamma}+\lambda I\right)u \equiv -u''+\left(-s(y)t^2+it\left(a(y)+\gamma\right)+c(y)+\lambda\right)u \tag{2.1}$$

defined on the set $C_0^2(\overline{\Delta}_i)$ of twice continuously differentiable functions satisfying the conditions: $u(i-1)=u(i+1)=0$, where $i-1$ and $i+1$ are respectively the right and left endpoints of the interval $\Delta_i=(i-1,\ i+1)$, $i\in Z$, the sign of the real number γ coincides with the sign of the function $a(y)$, t is the parameter $(-\infty<t<\infty)$, $s(y)$ is a piecewise continuous and bounded function that changes sign in R $(R=(-\infty,\infty))$, $i^2=-1$, $a(y)$, $c(y)$ are continuous functions in R, $\lambda\ge 0$. In what follows we denote Δ_i^-, Δ_i^+ respectively the right and left endpoints of the interval Δ_i.

It is not difficult to verify that the operator $m_{t,i,\gamma}+\lambda I$ admits a closure in $L_2(\Delta_i)$. We also denote the closure of this operator by $m_{t,i,\gamma}+\lambda I$.

Remark. The real number γ plays an auxiliary character.

In what follows we assume that the coefficients $a(y)$ and $c(y)$ $(y\in R)$ satisfy the condition:

a) $|a(y)|\ge\delta_0>0$, $c(y)\ge\delta>0$ are the continuous functions in R $(R=(-\infty,\infty))$.

In this subsection the following assertions are proved:

Lemma 2.1 (a priori estimate). Suppose that condition *a)* holds. Then the estimate in $L_2(\Delta_i)$

$$\left\|\left(m_{t,i,\gamma}+\lambda I\right)u\right\|_2\ge c\|u\|_2 \tag{2.2}$$

holds for all $u\in D\left(L_{t,i,\gamma}\right)$ as $\mu\ge 0$, where $c(\delta,\delta_0)>0$ is a constant number.

Lemma 2.2. Suppose that condition a) holds. Then the operator $l_{t,i,\gamma}+\lambda I$ is continuously invertible in $L_2(\Delta_i)$ as $\lambda\ge 0$.

Lemma 2.3. (coercive estimates). Suppose that condition *a)* holds. Then the following estimates

$$c\cdot\left\|\left(m_{t,i,\gamma}+\lambda I\right)u\right\|_2\ge\sqrt{(\delta+\lambda)}\|u\|_2, \tag{2.3}$$

$$c\cdot\left\|\left(m_{t,i,\gamma}+\lambda I\right)u\right\|_2\ge(\delta+\lambda)^{1/4}\|u'\|_2, \tag{2.4}$$

$$c \cdot \left\| (m_{t,i,\gamma} + \lambda I) u \right\|_2 \geq c \left(\|u'\|_2 + \left\| \sqrt{c(y) + \lambda}\, u \right\|_2 + \left\| \sqrt{|t(a(y) + \gamma)|}\, u \right\|_2 \right). \qquad (2.5)$$

hold for all $u \in D(l_{t,i,\gamma})$, where $c > 0$ is a constant number (different in different places).

The proof of these assertions can be found in [1-3].

2.2 Spectral and differential properties of the resolvent of a second-order local differential operator with complex coefficient, when the real part of the coefficient contains a parameter that changes sign (the case on a finite interval)

In this subsection we have been studying the spectral and differential properties of the following operator:

Operator

$$(m_{t,i,\gamma} + \lambda I) u \equiv -u'' + \left(-s(y) t^2 + it(a(y) + \gamma) + c(y) + \lambda\right) u \qquad (2.6)$$

,

defined on the set $C_0^2(\bar{\Delta}_i)$ of twice continuously differentiable functions satisfying the conditions: $u(i-1) = u(i+1) = 0$, where, $i-1, i+1$ the right and left ends of the interval $\Delta_i = (i-1, i+1)$, $i \in Z$, respectively, t is a parameter $(-\infty < t < \infty)$, $s(y)$ is a piecewise-continuous and bounded function that changes the sign in R, $a(y)$, $c(y)$ are continuous functions in R, $i^2 = -1$, $\lambda \geq 0$.

When $t = 0$ the operator $l_{0,i,\gamma} + \mu I$ is a Sturm-Liouville operator

$$(m_{0,i} + \lambda I) y \equiv -y'' + (c(y) + \lambda) y, \quad y \in D(l_{0,i}).$$

As is known, in this case the elements from the domain of the Sturm-Liouville operator belong to the space $W_2^2(\Delta_i)$ (closed operator), where $W_2^2(\Delta_i)$ is the Sobolev space. The operator itself is semibounded and the spectrum is discrete, nonnegative.

For the operator (2.6), for example, in the case $s(y) \equiv 1$, $-t^2 \to -\infty$ as $|t| \to \infty$.

Hence we see that the operator is not semibounded for $t \neq 0$. Consequently, a comparison of the properties of a given operator for $t \neq 0$, with similar results, related to the Sturm-Liouville operator for $t = 0$ and its various generalizations show that the operator $m_{t,i,\gamma} + \lambda I$ differs significantly from the Sturm-Liouville operator (for $t = 0$).

Its main distinguish is non-semiboundness.

Lemma 2.4. (differential properties). Let condition *a)* and $\lambda \geq 0$. Then for all $u \in D(m_{t,i,\gamma})$ we have the estimate

$$c\cdot\left\|(m_{t,i,\gamma}+\lambda I)u\right\|_2\geq\|u'\|_2+\|u\|_2$$

Evidence. Lemma 2.4 is proved in exactly the same way as Lemmas 2.3-2.4.

Further, we established bilateral estimates for the s-numbers (singular numbers) for the resolvent of the operator $m_{t,i,\gamma}+\lambda I$. As is known, the properties of s-numbers play an important role in the construction of a general theory of symmetrically-normed ideals of completely continuous operators, and also in the investigation of asymptotic properties of spectra in other problems.

Definition 2.2. [38] Let A be a completely continuous operator. Then the eigenvalues of the operator $(A^*A)^{1/2}$ are called the s-numbers of the operator A (Schmidt eigenvalues).

We shall s-number the non-zero numbers in the order of their decrease, taking into account their multiplicity so that

$$s_k(A)=\lambda_k\left((A^*A)^{1/2}\right)\quad k=1,2,\ldots.$$

We give another equivalent definition of s-numbers. But first we give a definition of the notion of k-widths of Kolmogorov and their properties.

Let be M a centrally symmetric subset of a (H is a hilbert space), i.e. $M=-M$.

The value

$$d_k=\inf_{\{y_k\}}\sup_{u\in M}\inf_{v\in y_k}\|u-v\|_2,\quad k=0,1,2,\ldots$$

are called k-widths of Kolmogorov of the set M, where y_k is a subspace of dimension k.

Widths have the following properties:

1) $d_0\geq d_1\geq d_2\geq\ldots$;

2) $d_k(\tilde{M})\leq d_k(M)$, $\tilde{M}\subset M$, $k=1,2,\ldots$;

3) $d_k(nM)=nd_k(M)$, $n>0$, $nM=\{x'=nx, x\in M\}$.

The following statement allows us to give the second equivalent definition of s-numbers.

Proposition 2.2. [38]. Let A be some completely continuous operator. Then $s_{k+1}(A)$ $(k=1,2,\ldots)$ coincide with the k-widths of Kolmogorov of the set $M=AS$, onto which the operator A maps the $S=\{x\in H:\|x\|\leq 1\}$.

In many cases this definition is more convenient than the original one.

Main results of this section.

Lemma 2.5. (at $t\in R$). Suppose that condition *a)* holds. Then we have the estimate

$$c^{-1}\lambda^{-1/2}mes\left(y\in\Delta_i:Q_t(y)\leq c^{-1}\lambda^{-1}\right)\leq N(\lambda)\leq c\cdot\lambda^{-1}mes\left(y\in\Delta_i:K_t^{1/2}(y)\leq c\lambda^{-1}\right)$$

Where $Q_t(y)=\left|t^2+ita(y)+c(y)\right|$, $K_t=\left(|ta(y)|+c(y)\right)$, $c>0$ is a constant, $N(\lambda)=\sum_{s_k>\lambda}1$ is the number of s_k greater than $\lambda>0$ large, s_k is the singular numbers of the operator $m_{t,i,y}^{-1}$.

Lemma 2.6. Let condition *a)* be satisfied and $\mu \geq 0$. Then:

1) the spectrum $\sigma\left(m_{t,i,y}^{-1}\right)$ is a discrete set, where $\sigma\left(m_{t,i,y}^{-1}\right)$ is the spectrum of the operator $l_{t,i,y}^{-1}$;

2) for any $0 \neq \lambda \in \sigma\left(m_{t,i,y}^{-1}\right)$ the following estimate holds

$$|\lambda_k| \leq c\frac{1}{k}, \quad k = 1,2,\ldots,$$

where $c > 0$ is a constant independent of k.

Lemmas 2.5 and 2.6 are proved in exactly the same way as Lemmas 4 and 5 of [39].

2.3 Weighted spaces and their approximative properties

Here we studied the spectral properties of a second-order two-term differential operator with a complex coefficient (potential) when the real part of the coefficient contains a parameter that changes sign:

$$(m_t + \lambda I)u = -u''(y) + (-s(y)t^2 + ita(y) + c(y) + \lambda)u(y), \tag{2.7}$$

with domain $D(m_t)$, where $D(m_t)$ is the set of infinitely differentiable and finite functions, $s(y)$ is a piecewise-continuous bounded function in R $(R = -\infty,+\infty))$, $\lambda \geq 0$, $-\infty < t < +\infty$.

As is well known, the spectral properties of the operator (the Sturm-Liouville operator) $m_0 u = -u''(y) + c(x)u(y)$, $u \in D(L_0)$ strongly depend on the behavior of $c(x)$ at infinity for $t = 0$. In this case, the spectral characteristics of the above operator are sufficiently well studied in [16-37].

This section is devoted to the study of embeddings of spaces with weight: $L_2^2(R, Q_t(y))$, $L_2^1(R, K_t(y))$, which are obtained by completion with respect to norms:

$$\|u\|_{L_2^2(R,Q_t(y))} = \left(\int_{-\infty}^{\infty}(|u''|^2 + Q_t(y)|u|^2)dy\right)^{\frac{1}{2}},$$

$$\|u\|_{L_2^1(R,K_t(y))} = \left(\int_{-\infty}^{\infty}(|u'|^2 + K_t(y)|u|^2)dy\right)^{\frac{1}{2}}$$

where $Q_t(y) = |t^2 + ita(y) + c(y)|$, $K_t(y) = |ta(y)| + c(y)$ are called weight functions.

Here we give some results on compactness and on estimates of the widths of the embedding operators of spaces:

$$\overset{2}{L}_2(R,Q_t(y)) \subset_{\to} L_2(R);$$

$$\overset{1}{L}_2(R,K_t(y)) \subset_{\to} L_2(R)$$

Definition 2.3. Let B_1 and B_2 be are Banach spaces. We say that it is B_1 invested in B_2, if

1) B_1 is contained in B_2 as a linear subset;

2) there exists a constant $c > 0$ such that

$\|u\|_{B_2} \le c\|u\|_{B_1}$ for anyone $u \in B_1$.

To emphasize this, write $B_1 \subset_{\to} B_2$.

If $B_1 \subset_{\to} B_2$, then the identity operator E acting from B_1 to B_2 is called the imbedding operator.

They say that B_1 it is embedded (embedded) in B_2 bounded or completely continuous if the corresponding imbedding operator is bounded or completely continuous.

We define functions Q_t^*, K_t^* as in [28-29]:

$$Q_t^*(y) = \inf_{d>0}\{d^{-1} : d^{-1} \ge \int_{y-\frac{d}{2}}^{y+\frac{d}{2}} Q(\tau)d\tau\},$$

$$K_t^*(y) = \inf_{d>0}\{d^{-1} : d^{-1} \ge \int_{y-\frac{d}{2}}^{y+\frac{d}{2}} K(\tau)d\tau\},$$

Theorem 2.1. Suppose that condition *i)* is satisfied. Then the embedding operator $\overset{1}{L}_2(R,K_t(y)) \subset_{\to} L_2(R)$ is compact if and only if

$$K_t^*(y) \to \infty \text{ as } |y| \to \infty.$$

Theorem 2.2. Suppose that condition *i)* is satisfied. Then the embedding operator $\overset{2}{L}_2(R,Q_t(y)) \subset_{\to} L_2(R)$ is compact if and only if

$$Q_t^*(y) \to \infty \text{ as } |y| \to \infty.$$

Proofs of Theorems 2.1 and 2.2. Repeating the calculations and arguments used in [28-29], we obtain the proofs of the above theorems.

The theorems given here can be formulated in terms of coefficients. To this end, we prove the following lemma.

Suppose, without loss of generality, that the coefficients $a(y)$, $c(y)$ satisfy the following conditions:

i) $|a(y)| \ge 1$, $c(y) \ge 1$ are continuous functions in R;

ii) $c_0 = \sup\limits_{|y-\tau| \le 1} \dfrac{c(y)}{c(\tau)} < \infty$, $c_1 = \sup\limits_{|y-\tau| \le 1} \dfrac{a(y)}{a(\tau)} < \infty$.

Lemma 2.7. Suppose that conditions *i) -ii)* are satisfied. Then

$$c^{-\frac{1}{2}} K_t^{\frac{1}{2}}(y) \le K_t^{*}(y) \le c^{\frac{1}{2}} K_t^{\frac{1}{2}}(y), \tag{2.8}$$

where $c > 0$ is a constant number.

Evidence. Let $d_y = K_t^{*-1}(y)$, then the following equality holds:

$$d_y^{-1} = \int\limits_{y-\frac{d}{2}}^{y+\frac{d}{2}} K_t(\tau) d\tau, \tag{2.9}$$

The last equality follows from the definition of the function $K_t^*(y)$. Further, according to condition *i)* $K_t(y) \ge 1$, therefore, taking into account the last inequality, we have:

$$d_y^{-1} = \int\limits_{y-\frac{d}{2}}^{y+\frac{d}{2}} K_t(\tau) d\tau \ge d_y.$$

From this it is not difficult to verify that $d_y \le 1$. Consequently, by virtue of condition *i) -ii)*, on the interval $d_y \le 1$ the following inequality holds:

$$\sup_{|y-\tau| \le 1} \frac{K_t(y)}{K_t(\tau)} = \sup_{|y-\tau| \le 1} \frac{|ta(y)| + c(y)}{|ta(\tau)| + c(\tau)} \le \sup_{|y-\tau| \le 1} \frac{|ta(y)|}{|ta(\tau)| + c(\tau)} + \sup_{|y-\tau| \le 1} \frac{c(y)}{|ta(\tau)| + c(\tau)} \le$$

$$\le \sup_{|y-\tau| \le 1} \frac{|ta(y)|}{|ta(\tau)|} + \sup_{|y-\tau| \le 1} \frac{c(y)}{c(\tau)} \le \sup_{|y-\tau| \le 1} \frac{|a(y)|}{|a(\tau)|} + \sup_{|y-\tau| \le 1} \frac{c(y)}{c(\tau)}. \tag{2.10}$$

In proving inequality (2.10), condition *i)* was taken into account.

Now taking into account conditions *i) -ii)* from (2.10), we find that

$$\sup_{|y-\tau| \le 1} \frac{K_t(y)}{K_t(\tau)} \le \sup_{|y-\tau| \le 1} \frac{|a(y)|}{|a(\tau)|} + \sup_{|y-\tau| \le 1} \frac{c(y)}{c(\tau)} \le$$

$$\leq \sup_{|y-\tau|\leq 1} \frac{a(y)}{a(\tau)} + \sup_{|y-\tau|\leq 1} \frac{c(y)}{c(\tau)} \leq c_0 + c_1 < c < \infty \tag{2.11}$$

Here we also took into account, according to condition *ii)*, that the function $a(y)$ does not change sign in.

From inequality (2.11) we obtain that

$$c^{-1} \leq \frac{K_t(y)}{K_t(\tau)} \leq c \text{ при } |y-\tau| \leq 1 \tag{2.12}$$

Further, from (2.9), by virtue of inequality (2.12), we find that

$$d_y^{-1} \geq c^{-1} \int_{y-\frac{d}{2}}^{y+\frac{d}{2}} K_t(y)d\tau = c^{-1} K_t(y) d_y$$

Hence we obtain that

$$d_y^{-2} \geq c^{-1} K_t(y).$$

From the last inequality we have the following inequality

$$d_y^{-1} \geq c^{-\frac{1}{2}} K_t^{\frac{1}{2}}(y).$$

Thus, we obtain the left-hand side of (2.8).

To obtain the right-hand side of (2.8), we proceed analogously. Indeed, using (2.9) and using (2.12), we obtain that

$$d_y^{-1} \leq c \int_{y-\frac{d}{2}}^{y+\frac{d}{2}} K_t(y)d\tau \leq c K_t(y) d_y$$

Hence we have

$$d_y^{-2} \leq c K_t(y)$$

or

$$d_y^{-1} \leq c^{\frac{1}{2}} K_t^{\frac{1}{2}}(y).$$

Thus, the right-hand side of (2.8) is proved. Lemma 2.7 is completely proved.

Lemma 2.8. Suppose that conditions *i) -ii)* are satisfied. Then

$$c^{-1} Q_t^{\frac{1}{2}}(y) \leq Q_t^{*}(y) \leq c Q_t^{\frac{1}{2}}(y), \tag{2.13}$$

where $c > 0$ is a constant number.

Theorem 2.3. Suppose that conditions *i) -ii)* are satisfied.. Then the embedding operator $\overset{1}{L}_2(R, K_t(y)) \subset_{\to} L_2(R)$ is compact if and only if

$$K_t(y) \to \infty \text{ as } |y| \to \infty, \ (-\infty < t < +\infty).$$

Theorem 2.4. Suppose that conditions *i) -ii)* are satisfied. Then the embedding operator $\overset{2}{L}_2(R, Q_t(y)) \subset_{\to} L_2(R)$ is compact if and only if

$$Q_t(y) \to \infty \text{ as } |y| \to \infty, \ (-\infty < t < +\infty).$$

The proofs of Theorems 2.3 and 2.4 follow from Lemmas 2.7 and 2.8, and also from Theorems 2.1 and 2.2.

Definition 2.4. Let M be a centrally symmetric subset of the Hilbert space H; M = -M. Then the quantities

$$d_k = \inf_{\{y_k\}} \sup_{u \in M} \inf_{v \in y_k} \|u - v\|, \ k = 0,1,2,...$$

are called k-width of Kolmogorov widths of the set M, where y_k is a subspace of dimension k.

Definition 2.5. k-width of Kolmogorov of the embedding $B_1 \subset_{\to} B_2$ is called k-width of Kolmogorov of the unit sphere of B_1 in B_2.

We introduce the following function: $N(\lambda) = \sum_{d_k > \lambda} 1$ is the number of d_k, k-width of the embedding, greater than $\lambda > 0$ which is called the distribution function (counting function) of the widths of the embedding $B_1 \subset_{\to} B_2$.

k-width are restored by their "counting" function by means of formula

$$d_k = \inf\{\lambda > 0 : N(\lambda) \le k\}, \ \forall k > 0$$

It follows from the last formula that instead of estimating d_k it suffices to deal with the estimation of their distribution function.

Let $N(\lambda)$ be the distribution function of the widths of the embedding

$$\overset{1}{L}_2(R, K_t(y)) \subset_{\to} L_2(R).$$

Then the following theorem holds.

Theorem 2.5. Suppose that conditions *i) -ii)* are satisfied. Then the estimates

$$c^{-1}\lambda^{-1} \cdot mes(y \in R : K_t^{\frac{1}{2}}(y) \le c^{-1}\lambda^{-1}) \le N(\lambda) \le c\lambda^{-1} \cdot mes(y \in R : K_t^{\frac{1}{2}}(y) \le c\lambda^{-1}),$$

hold, where $c > 0$ is a constant number.

Evidence. The proof of Theorem 2.5 follows from Lemmas 2.7 and 2.8, and Theorem 2.3.

Proceeding in a completely analogous way, one can obtain an estimate for the distribution function of the widths of the embedding

$$\overset{2}{L}_2(R, Q_t(y)) \subset_{\to} L_2(R).$$

Theorem 2.6. Suppose that conditions *i) -ii)* are satisfied. Then the estimates

$$c^{-1}\lambda^{-\frac{1}{2}}\cdot mes(y \in R : Q_t(y) \leq c^{-1}\lambda^{-1}) \leq N(\lambda) \leq c\lambda^{-\frac{1}{2}}\cdot mes(y \in R : Q_t(y) \leq c\lambda^{-1}),$$

hold, where $c > 0$ is a constant number.

3 SPECTRAL PROPERTIES OF ONE CLASS OF SINGULAR DIFFERENTIAL OPERATORS OF A MIXED TYPE

3.1 On the existence of a bounded inverse operator of a singular operator of mixed type and the discreteness of the spectrum

Singular differential operators, for example, operators defined in an unbounded domain can, in general, not only have a discrete but also a continuous spectrum. In this connection, the expansion of an arbitrary function into a series in terms of their eigenfunctions becomes impossible in the general case. Therefore, for this reason, the most important question of spectral theory in the study of the spectrum, depending on the behavior of the coefficients, in the case of an unbounded region, is the sign of the discreteness of the spectrum.

The spectral characteristics of singular differential operators of elliptic type have been fairly well studied and the typical difficulties encountered in connection with the poor behavior of the coefficients are elucidated. An extensive literature is devoted to their study, we mention papers [16-37].

A review of the literature shows that such questions as: 1) the existence and compactness of the resolvent; 2) the discreteness of the spectrum of differential operators of mixed type given in an unbounded domain is not sufficiently studied.

We note that a systematic study of spectral questions, depending on the boundary conditions and the geometry of the domain in the case of a bounded domain for operators of mixed type, has been started [8-10].

We consider a differential operator of mixed type in $L_2(\Omega)$

$$M_0 u = s(y)u_{xx} - u_{yy} + a(y)u_x + c(y)u$$

with a domain $C_{0,\pi}^{\infty}(\overline{\Omega})$, which consist of infinitely differentiable functions that satisfy the conditions $u(-\pi;y)=u(\pi;y)$, $u_x(-\pi;y)=u_x(\pi;y)$ and are finite belong to y, where $\overline{\Omega}=\{(x,y): -\pi \le x \le \pi,\ -\infty < y < \infty\}$, $s(y)$ is a piecewise-continuous and bounded function in $R=(-\infty,+\infty)$ и $s(0)=0$, $ys(y)>0$ for $y \ne 0$.

In what follows we assume that the coefficients satisfy condition

i) $|a(y)| \ge \delta_0 > 0$, $c(y) \ge \delta > 0$ are continuous functions in $R=(-\infty;\infty)$.

It is not difficult to verify that the operator M_0 in $L_2(\Omega)$ admits a closure, which we denote by M.

The main results of this paper are the following theorems.

Theorem 3.1. Let the condition *i)* is satisfied. Then the operator $M+\lambda E$ is continuously invertible for $\lambda \ge 0$.

Theorem 3.2. Suppose that condition *i)* is satisfied. Then the resolvent of the operator M is compact if and only if

$$\lim_{y \to \infty} \int_y^{y+w} c(t)dt = \infty. \qquad (**)$$

for any $w>0$. The last theorem shows that condition (**) is a necessary and sufficient condition for the discreteness of the spectrum of the operator.

The question of the existence of a resolvent and the discreteness of the spectrum in an unbounded domain with increasing and oscillating coefficients was previously investigated only in the case of elliptic and pseudo-differential operators [16-37].

Suppose that the coefficients of the operator M, in addition to the condition *i)*, satisfy condition

$$ii)\ \mu_0 = \sup_{|y-t|\le 1} \frac{c(y)}{c(t)} < \infty,\ \mu = \sup_{|y-t|\le 1} \frac{a(y)}{a(t)} < \infty.$$

Then from Theorem 3.2 easily implies the following theorem:

Theorem 3.3. Let the conditions $i)-ii)$ are satisfied. Then the resolvent of the operator M is compact if and only if

$$\lim_{|y|\to\infty} c(y) = \infty.$$

3.2 Auxiliary lemmas and inequalities

Lemma 3.1. Suppose that the condition *i)* is satisfied and $\mu \ge 0$. Then the inequality

$$\|(M+\lambda I)u\|_2 \ge c\|u\|_2 \quad , \qquad (\|\cdot\|_2 \text{ is the norm in } L_2(\Omega)),\ (3.1)$$

holds, for all $u \in D(M)$, where $c = c(\delta_0, \delta)$.

Proof. Let $C_{0,\pi}^{\infty}(\overline{\Omega})$. Integrating by parts of the expressions $<(M+\lambda I)u, u>$ and $<(M+\lambda I)u, u_x>$, we obtain the following inequalities

$$\frac{1}{2\varepsilon}\|(M+\lambda I)u\|_2^2 \ge \int_\Omega \left[|u_y|^2 + (\delta+\lambda-\frac{\varepsilon}{2})|u|^2\right]dxdy - \int_\Omega |s(y)||u_x|^2\,dxdy, \tag{3.2}$$

$$\|(M+\lambda I)u\|_2^2 \ge \delta_0^2\|u_x\|_2^2, \tag{3.3}$$

where $<\cdot,\cdot>$ is the scalar product in $L_2(\Omega)$. Here we have used the Cauchy inequality with «$\varepsilon > 0$». We can choose $\varepsilon = \frac{\delta}{2}$. Now we multiply both sides of inequality (3.3) by $c_0 > 0$ and combining this inequality with (3.2) we have

$$\frac{1}{\delta}\|(M+\lambda I)u\|_2^2 + c_0\|(M+\lambda I)u\|_2^2 \ge c_1(\delta)\int_\Omega (|u_y|^2 + |u|^2)dxdy + \int_\Omega (c_0\delta_0^2 - |s(y)|)|u_x|^2\,dxdy$$

Choosing c_0 so that $(c_0\delta_0^2 - |s(y)|) \ge 0$ from the last inequality we get that

$$c\|(M+\lambda I)u\|_2^2 \ge \|u\|_2^2,$$

where $c = c(\delta, c_1(\delta))$. By virtue of the closedness of the operator $M+\lambda I$, it remains valid for $u \in D(M+\lambda I)$.

Let $\Delta_j = (\Delta_j, \Delta_j') := (j-1, j+1)$ $(j \in Z)$, γ such a constant that $\gamma a(y) > 0$. We denote by $m_{n,j,\gamma}+\lambda I$ the closure in $L_2(\Delta_j)$ the differential expression $(m_{n,j,\gamma}+\lambda I)u=-u''+\left[-s(y)n^2+in(a(y)+\gamma)+c(y)+\lambda\right]u$, $(n=0,\pm1,\pm2,\ldots)$, defined on the set $C_0^2(\overline{\Delta}_j)$ of twice continuously differentiable functions in $\overline{\Delta}_j$ satisfying the conditions $u(\Delta^-_j)=u(\Delta^+_j)=0$.

Lemma 3.2. Suppose that the condition *i)* is satisfied and $\lambda\ge 0$. Then the following inequalities hold:

a) $\left\|(m_{n,\gamma,j}+\lambda I)u\right\|_2\ge c_1\left(\|u'\|_2+\left\|\sqrt{c(y)+\lambda}\,u\right\|_2+\left\|\sqrt{|n|(|a(y)|+|\gamma|)}\,u\right\|_2\right)$, $u\in D(m_{n,\gamma,j}+\lambda I)$;

b) $\left\|(m_{n,j,\gamma}+\lambda I)^{-1}\right\|_{2\to2}\le\dfrac{c_0}{(\delta+\lambda)^{1/2}}$;

c) $\left\|\dfrac{d}{dy}(m_{n,j,\gamma}+\lambda I)^{-1}\right\|_{2\to2}\le\dfrac{c_2}{(\delta+\lambda)^{1/4}}$,

where $c_0=c_0(\delta)$, $c_1=c_1(\delta_0,\delta)$, $c_2=c_2(\delta)$.

Proof. Let $u\in C_0^2(\overline{\Delta}_j)$. We have

$$\left|<(m_{n,j,\gamma}+\lambda I)u,u>\right|\ge\left|\|u'\|_2^2+\int_{\Delta_j}(c(y)+\lambda)|u|^2dy\right|-\left|\int_{\Delta_j}n^2s(y)|u|^2dy\right|,\tag{3.4}$$

from which the inequalities follow

$$\left\|(m_{n,j,\gamma}+\lambda I)u\right\|_2\cdot\|u\|_2\ge\int_{\Delta_j}|u'|^2dy-\int_{\Delta_j}|n|^2|s(y)||u|^2dy,\tag{3.5}$$

and

$$\frac{1}{\delta}\left\|(m_{n,j,\gamma}+\lambda I)u\right\|_2^2\ge\frac{1}{2}\int_{\Delta_j}(c(y)+\lambda)|u|^2dy-\int_{\Delta_j}|n|^2|s(y)||u|^2dy.\tag{3.6}$$

Here we again used the Cauchy inequality with «$\varepsilon>0$», where $\varepsilon=\frac{\delta}{2}$.

On the other hand, by transforming the expressions $<(m_{n,j,\gamma}+\lambda I)u,u>$ and $<(m_{n,j,\gamma}+\lambda I)u,-inu>$, $u\in C_0^2(\overline{\Delta}_j)$, we have

$$c(\delta_0)\left\|(m_{n,j,\gamma}+\lambda I)u\right\|_2\ge\left\|\sqrt{|n|(|a(y)|+|\gamma|)}\,u\right\|_2,\tag{3.7}$$

and

$$\left\|(m_{n,j,\gamma}+\lambda I)u\right\|_2^2\ge(\delta_0+|\gamma|)^2\cdot|n|^2\,\|u\|_2^2.\tag{3.8}$$

Combining inequalities (3.6), (3.8) and choosing γ so as $(\delta_0+|\gamma|)^2-|k(y)|\ge0$ we obtain

$$c_0(\delta)\left\|(m_{n,j,\gamma}+\lambda I)u\right\|_2\ge\left\|\sqrt{c(y)+\lambda}\,u\right\|_2,\tag{3.9}$$

where $c_0(\delta) = 2(\frac{1}{\delta}+1)$.

Hence, according to condition *i)*, we have

$$\frac{c_0(\delta)}{\sqrt{\delta+\lambda}}\|(m_{n,j,\gamma}+\lambda I)u\|_2 \geq \|u\|_2. \tag{3.10}$$

From the inequalities (3.5), (3.8), and (3.10) we obtain the estimate

$$\frac{c_0(\delta)+1}{\sqrt{\delta+\lambda}}\|(m_{n,j,\gamma}+\lambda I)u\|_2^2 \geq \int_{\Delta_j}|u'|^2 dy + |n|^2\int_{\Delta_j}\left(\frac{(\delta_0+|\gamma|)^2}{\sqrt{\delta+\lambda}} - |s(y)|\right)|u|^2 dy. \tag{3.11}$$

Choosing γ so that $\frac{(\delta_0+|\gamma|)^2}{\sqrt{\delta+\lambda}} - |s(y)| \geq 0$ from the inequalities (3.7), (3.9) and (3.11) we obtain the estimate *a)*. The estimate *b)* follows from (3.10). The inequality (3.11) implies the estimate $\frac{c_2(\delta)}{\sqrt{\delta+\lambda}}\|(m_{n,j,\gamma}+\lambda I)u\|_2^2 \geq \|u'\|_2^2$, $c_2(\delta) = c_0(\delta)+1$. The estimate *c)* is proved.

Lemma 3.2 is proved.

Lemma 3.3. The operator $m_{n,j,\gamma}+\lambda I$ has a continuous inverse $(m_{n,j,\gamma}+\lambda I)^{-1}$ as $\lambda \geq 0$, defined on all $L_2(\Delta_j)$, $j \in Z$.

Proof. By virtue of estimate b) in Lemma 3.2, it suffices to show that the range $R(m_{n,\gamma,j}+\lambda I)$ of the operator $m_{n,\gamma,j}+\lambda I$ coincides with $L_2(\Delta_j)$. Let us assume the contrary. Then there is an element $\upsilon \in L_2(\Delta_j)$, $\upsilon \neq 0$,, which satisfies equation

$$(m_{n,\gamma,j}+\lambda I)^*\upsilon = -\upsilon'' + \left[-s(y)n^2 - in(a(y)+\gamma) + c(y) + \lambda\right]\upsilon = 0 \tag{3.12}$$

in the sense of the theory of distributions. It follows from (3.12) $\upsilon'' \in L_2(\Delta_j)$. Integrating by parts $\langle (m_{n,j,\gamma}+\lambda I)u, \upsilon\rangle_{\Delta_j}$, we have $u'(j+1)\bar{\upsilon}(j+1) - u'(j-1)\bar{\upsilon}(j-1) = 0$ where u is an arbitrary function of $C_0^2(\bar{\Delta}_j)$. Then $\upsilon(j+1) = \upsilon(j-1) = 0$. Using these conditions, similarly to the derivation of (2.10), we obtain the inequality

$$\|(m_{n,\gamma,j}+\lambda I)^*\upsilon\|_2^2 \geq c\|\upsilon\|_2^2, \tag{3.13}$$

this implies $\upsilon = 0$. The lemma is proved.

We denote by $m_{n,\gamma}+\lambda I$ $(n = 0,\pm 1,\pm 2,\ldots)$ the closure in $L_2(R)$ a differential expression $(m_{n,\gamma}+\lambda I)u = -u'' + (-s(y)n^2 + in(a(y)+\gamma) + c(y) + \lambda)u$, defined on a set $C_0^\infty(R)$ of arbitrarily differentiable finite functions.

Lemma 3.4. Suppose that the condition *i)* is satisfied and $\mu \geq 0$. Then the inequalities

$$\|(m_{0,\gamma}+\lambda I)u\|_2 \geq \sqrt{\delta+\lambda}\|u\|_2;\ \|(m_{n,\gamma}+\lambda I)u\|_2 \geq |n|(\delta_0+|\gamma|)\|u\|_2, \quad n \neq 0$$

hold for $u \in D(m_{n,\gamma} + \lambda I)$.

The lemma is proved by the transformation of the expression $< (m_{n,\gamma} + \lambda I)u, -inu >$, where $u \in C_0^\infty(R)$.

Let $\{\varphi_j(x)\}_{j=-\infty}^{+\infty} \subset C_0^\infty(R)$ be the set of such functions such that $\varphi_j \geq 0$, $\operatorname{supp}\varphi_j \subseteq \Delta_j (j \in Z)$, $\sum_{j=-\infty} \varphi_j^2(x) = 1$. Suppose $K_{\lambda,\gamma} f = \sum_{j=-\infty}^{+\infty} \varphi_j (m_{n,j,\gamma} + \lambda I)^{-1} \varphi_j f$,

$$B_{\lambda,\gamma} f = \sum_{j=-\infty}^{+\infty} \varphi_j'' (m_{n,j,\gamma} + \lambda I)^{-1} \varphi_j f + 2\sum_{j=-\infty}^{+\infty} \varphi_j' \frac{d}{dx}(m_{n,j,\gamma} + \lambda I)^{-1} \varphi_j f,$$

$f \in C_0^\infty(R)$, $\lambda \geq 0$. Clearly

$$\left(m_{n,\gamma} + \lambda I\right) K_{\lambda,\gamma} f = f - B_{\lambda,\gamma} f \tag{3.14}$$

Lemma 2.5. Suppose that condition *i)* is satisfied. Then there is a number $\lambda_0 > 0$ such $\|B_{\lambda,\gamma}\|_{2\to 2} < 1$ that for all $\lambda \geq \lambda_0$.

Proof. Let $f \in C_0^\infty(R)$. On the interval $\overline{\Delta}_j (j \in Z)$, only the functions $\varphi_{j-1}, \varphi_j, \varphi_{j-1}$ differ from zero, hence

$$\|B_{\lambda,\gamma} f\|_{L_2(R)}^2$$

$$= \int_{-\infty}^{\infty} \left| \sum_{j=-\infty}^{\infty} \varphi_j'' (m_{n,j,\gamma} + \lambda I)^{-1} \varphi_j f + 2\sum_{j=-\infty}^{\infty} \varphi_j' \frac{d}{dx}(m_{n,j,\gamma} + \lambda I)^{-1} \varphi_j f \right|^2 dx \leq$$

$$\leq \sum_{j=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| \sum_{k=j-1}^{j+1} \left[\varphi_k'' (m_{n,k,\gamma} + \lambda I)^{-1} \varphi_k f + 2\varphi_k' \frac{d}{dx}(m_{n,k,\gamma} + \lambda I)^{-1} \varphi_k f \right] \right|^2 dx.$$

Hence, using the obvious inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ and estimates b) and c) in Lemma 2.2, we have $\|B_{\lambda,\gamma} f\|_2^2 \leq c\left(\frac{c_0}{(\delta+\lambda)^{1/2}} + \frac{c_2}{(\delta+\lambda)^{1/4}}\right)\|f\|_2^2$ where $c = 24\max\{|\varphi_j''|, |\varphi_j'|\}$ and the constants c_0, c_2 from Lemma 2.2. Hence it follows that it is not difficult to find a number $\lambda_0 > 0$ such that $\|B_{\lambda,\gamma}\|_{2\to 2} < 1$ for $\lambda \geq \lambda_0$.

The lemma is proved.

Lemma 3.6. Suppose that condition *i)* is satisfied. Then the operator $m_{n,\gamma} + \lambda I$ is boundedly invertible as $\lambda \geq \lambda_0 > 0$ and the inverse operator $\left(m_{n,\gamma} + \lambda I\right)^{-1}$ the equality

$$\left(m_{n,\gamma} + \lambda I\right)^{-1} = K_{\lambda,\gamma}(E - B_{\lambda,\gamma})^{-1}. \tag{3.15}$$

holds.

The lemma follows from the representation (3.14), taking Lemmas 3.5 and 3.4 into account.

Lemma 3.7. Suppose that the condition *i)* is satisfied and $\lambda \geq \lambda_0$, $\alpha = 0,1$, $\rho(x)$ is a function defined on a continuous function. Then we have the estimate

$$\left\|\rho(y)n^{\alpha}(m_{n,\gamma}+\lambda I)^{-1}\right\|^2_{2\to 2}\le c_4(\lambda)\sup_{j\in Z}\left\|\rho(y)n^{\alpha}\varphi_j(m_{n,j,\gamma}+\lambda I)^{-1}\right\|^2_{2\to 2}. \quad (3.16)$$

Proof. Let $f\in C_0^\infty(R)$. From the representation (3.15), taking into account the properties of the functions $\{\varphi_j\}(j\in Z)$, we have

$$\left\|\rho(y)n^{\alpha}(m_{n,\gamma}+\lambda I)^{-1}f\right\|_2^2\le\sum_{j=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\sum_{k=j-1}^{j+1}\left[\rho(y)|n|^{\alpha}\varphi_k(m_{n,k,\gamma}+\lambda I)^{-1}\varphi_k(E-B_{\lambda,\gamma})^{-1}f\right]\right|^2 dy$$

Hence, applying the inequality $(a_0+b_0+c_0)^2\le 3(a_0^2+b_0^2+c_0^2)$ again, and using Lemma 3.6, we obtain (3.16). Lemma 3.7 is proved.

The following lemma follows from Lemma 3.2, taking into account inequality (3.16).

Lemma 3.8. Let condition *i)* and $\lambda\ge\lambda_0$. Then the following inequalities hold:

a) $\left\|\sqrt{c(y)+\lambda}(m_{n,\gamma}+\lambda I)^{-1}\right\|_{2\to 2}<\infty$ $(n=0,\pm1,\pm2,\dots)$;

b) $\left\|in(m_{n,\gamma}+\lambda I)^{-1}\right\|_{2\to 2}<\infty$;

c) $\left\|\frac{d}{dy}(m_{n,\gamma}+\lambda I)^{-1}\right\|_{2\to 2}<\infty$ $(n=0,\pm1,\pm2,\dots)$.

Consider the equation

$$(m_n+\lambda I)u\equiv -u''+(-n^2+ina(y)+c(y)+\lambda)u=f, \quad (3.17)$$

where $f\in L_2(R)$. By a solution of equation (3.17) we mean a function $u\in L_2(R)$ for which there exists a sequence $\{u_n\}_{n=1}^{\infty}\subset C_0^\infty(R)$ such that $\|u_n-u\|_{L_2(R)}\to 0$, $\|(m_n+\lambda I)u_n-f\|_{L_2(R)}\to 0$ as $n\to\infty$.

Lemma 3.9. The operator $m_n+\lambda I$ $(n=0,\pm1,\pm2,\dots)$ is boundedly invertible for $\mu\ge\mu_0$ and for the inverse operator $(m_n+\lambda I)^{-1}$ the equality

$$(m_n+\lambda I)^{-1}f=(m_{n,\gamma}+\lambda I)^{-1}(\lambda-A_{\lambda,\gamma})^{-1}f,\ f\in L_2(R) \quad (3.18)$$

where $\|A_{\lambda,\gamma}\|_{2\to 2}<1$.

Proof. Let $n\ne 0$. We rewrite the equation $(m_n+\lambda I)u=f\in L_2(R)$ in the form, $\upsilon-A_{\lambda,\gamma}\upsilon=f$, where $\upsilon=(m_{n,\gamma}+\lambda I)u$, $A_{\mu,\gamma}=in\gamma(m_{n,\gamma}+\lambda I)^{-1}$. It follows from Lemma 3.4 that $\|A_{\mu,\gamma}\upsilon\|_{2\to 2}\le\frac{|n|\cdot|\gamma|}{|n|(\delta_0+|\gamma|)}\|v\|_2$ or $\|A_{\lambda,\gamma}\|_{2\to 2}<1$. Hence $u=(m_n+\lambda I)^{-1}f=(m_{n,\gamma}+\lambda I)^{-1}(I-A_{\lambda,\gamma})^{-1}f$. When $n=0$ the operator $m_0+\lambda I$ is essentially self-adjoint [36, p.208], and the estimate $\|(m_0+\lambda I)u\|_2\ge(\delta+\lambda)\|u\|_2$ is valid for all $u\in D(m_0+\lambda I)$. It follows that the operator $m_0+\lambda I$ has a bounded inverse $(m_0+\lambda I)^{-1}$, defined on whole $L_2(R)$. Lemma 3.9 is proved.

From Lemma 3.8 and the representation (3.18) it follows that

Lemma 3.10. The estimates

a) $\left\|\sqrt{c(y)+\lambda}\,(m_n+\lambda I)^{-1}\right\|_{2\to 2}<\infty \;(n=0,\pm 1,\pm 2,\dots)$;

b) $\left\|in(m_n+\lambda I)^{-1}\right\|_{2\to 2}<\infty$;

c) $\left\|\frac{d}{dy}(m_n+\lambda I)^{-1}\right\|_{2\to 2}<\infty \;(n=0,\pm 1,\pm 2,\dots)$.

hold for $\lambda\geq\lambda_0$.

The following lemma is known [40, p. 350].

Lemma 3.12. Let the operator $M+\lambda_0 I$ $(\lambda_0>0)$ is boundedly invertible in $L_2(R)$ and the estimate $\|(M+\lambda I)u\|_2\geq C\|u\|_2$, $u\in D(M+\lambda I)$ is satisfied for $\lambda\in[0,\lambda_0)$. Then the operator $M: L_2(R)\to L_2(R)$ is also boundedly invertible.

Lemma 3.12. Let condition *i)* and $\lambda\geq 0$. Then for all $n\,(n=\pm 1,\pm 2,\dots)$ we have the inequality

$$\left\|(\mathrm{m}_n+\lambda I)^{-1}\right\|_{2\to 2}\leq\frac{1}{|n|\cdot\delta_0}. \tag{3.19}$$

Proof. Let $u\in C_0^\infty(R)$. Integrating by parts the expression $<(m_n+\lambda I)u,u>$, we obtain the inequality

$$\left|<(m_n+\lambda I)u,u>\right|=\left|\int_{-\infty}^{\infty}[|u'|^2+(-s(y)n^2+c(y))|u|^2]dy+\int_{-\infty}^{\infty}ina(y)|u|^2dy\right|.$$

Hence, taking into account condition *i)* and using the property of complex numbers, we have

$$\|(m_n+\lambda I)u\|_2\geq|n|\cdot\delta_0\|u\|_2. \tag{3.20}$$

Here it was taken into account that it does not change sign. From the inequality (3.20) follows the proof of Lemma 3.12.

Lemma 3.13. Let condition *i)* and $\lambda\geq 0$. Then the operator $(m_n+\lambda I)^{-1}$ is completely continuous for all $n\,(n=0,\pm 1,\pm 2,\dots)$ if and only if for any $\omega>0$

$$\lim_{y\to\infty}\int_y^{y+\omega}c(t)dt=\infty. \tag{*}$$

Proof. It follows from Lemma 3.12 that it is sufficient to prove Lemma 3.13 for every finite $n\neq 0$.

Let $n=0$. Then the operator $l_0+\mu E$ is a Sturm-Liouville operator with a potential $c(y)$, i.e.

$$(m_0+\lambda I)u=-u''+c(y)u,\; u\in D(m_0).$$

In this case, repeating all the calculations and the arguments used in [16, 17], we obtain the proof of Lemma 3.13, that is, Equality (*) is a necessary and sufficient condition for the compactness of the resolvent of the operator $m_0+\lambda I$.

Let us consider the case $n\neq 0$. Let

$$\lim_{y \to \infty} \int_{y}^{y+n} (|na(t)| + c(t))dt = \infty. \qquad (3.21)$$

Necessity. Suppose that condition (3.21) does not hold. Then there exists a sequence of intervals $Q_d(y_j) \subset R$ such that

$$\sup_{|j|} \int_{Q_d(y_j)} (|na(t)| + c(t))dt < \infty \text{ for each finite } n. \qquad (3.22)$$

where $d > 0$. i.e. the interval $Q_d(y_j)$ preserving the length, it goes to infinity. Let $w(x) \in C_0^\infty(Q_d(0))$ and consider the set of functions such that $u_j(y) = w(y - y_j)$. It is not difficult to establish for each finite $n \neq 0$ inequality the following:

$$\left\| -u_j'' + (-s(y)n^2 + ina(y) + c(y) + \lambda)u_j(y) \right\|_2^2 =$$

$$= \int_{-\infty}^{\infty} \left| -u_j''(y) + (-s(y)n^2 + ina(y) + c(y) + \lambda)u_j(y) \right|^2 dy \le$$

$$\le 8(\int_{Q_d(y_j)} (\left| -u_j''(y) \right|^2 + (|s(y)|^2 n^4 + n^2 a^2(y) + c^2(y) + \lambda^2)|u_j(y)|^2 dy \qquad (3.23)$$

From inequality (3.23), taking into account inequality (3.22) and the properties of the function $u_j(y) = \omega(y - y_j)$, for each finite $n \neq 0$, we obtain that

$$\left\| -u_j'' + (-s(y)n^2 + ina(y) + c(y) + \lambda)u_j(y) \right\|_2^2 < c < \infty,$$

where $c > 0$ it does not depend on j.

We set

$$F_j(y) = -u_j''(y) + (-s(y)n^2 + ina(y) + c(y) + \lambda)u_j(y), \quad \sup pF_j(y) \subseteq Q_d(y_j).$$

Now we show that $F_j(y)$ the weakly convergent to zero in $L_2(R)$:

$$< F_j(y), v(y) >= \left| \int_{-\infty}^{\infty} F_j(y)v(y)dy \right| \le \left(\int_{Q_d(y_j)} F_j^2(y)dy \right)^{\frac{1}{2}} \left(\int_{Q_d(y_j)} v^2(y)dy \right)^{\frac{1}{2}} \qquad (3.24)$$

It is obvious that $\int_{Q_d(y_j)} v^2(y)dy \to 0$ as $j \to \infty$, since $v \in L_2(R)$. Taking into account this from (3.24), we find that the sequence $\{F_j\}$ converges weakly to zero.

It can be directly seen that

$$\left\| u_j(y) \right\|_2 = \alpha > 0 \ . \qquad (3.25)$$

Since, if the operator $(l_n + \mu E)^{-1}$ is compact, then $\{u_j(y)\}$ it must converge to zero in the norm of $L_2(R)$. It is impossible by virtue of (3.25); we came to a contradiction.

Thus, we have proved that in the case $n \neq 0$ of a necessary condition for the compactness of the resolvent of the operator $m_n + \lambda I$, condition (3.21).

It follows from (*) and (3.21) that the equality (*) is a necessary condition for the compactness of the resolvent of the operator $m_n + \lambda I$ for all $n = 0, \pm 1, \pm 2, \ldots$

Sufficiently. It follows from Lemma 3.10 that $E(m_n + \lambda I)^{-1} \subset L_2^1(R, c(y))$, where $E(m_n + \lambda I)^{-1}$ is the range of the operator $(m_n + \lambda I)^{-1}$, $L_2^1(R, c(y))$ is the space obtained by the completion $C_0^\infty(R)$ with respect to the norm

$$\left\| u : L_2^1(R, c(y)) \right\| = \left(\int_{-\infty}^{+\infty} \left(|u'|^2 + (c(y) + \lambda)|u|^2 \right) dy \right)^{\frac{1}{2}}.$$

To complete the proof it remains to show that the embedding operator from $L_2^1(R, c(y))$ to $L_2(R)$ is a compact. The answer for this question follows from the results of Theorems 3.1-3.2. In this paper it is proved that any bounded set of space $L_2^1(R, c(y))$ is compact in $L_2(R)$ if and only if

$$\lim_{y \to \infty} c^*(y) = \infty, \tag{3.26}$$

where the function $c^*(y)$ represents a special averaging of the function $c(y)$ [28]. Furthermore, it was proved in [28-29] that the conditions (*) and (3.26) are equivalent. The sufficiency of Lemma 3.13 is proved.

Proof of Theorems 3.1-3.3. It follows from Lemma 3.9 that

$$u_k(x, y) = \sum_{n=-k}^{k} (m_n + \lambda I)^{-1} f_n(y) \cdot e^{inx} \tag{3.27}$$

is a solution of the problem

$$(M + \lambda I) u_k(x, y) = f_k(x, y),$$

$$u_k(-\pi, y) = u_k(\pi, y), \quad u_{kx}(-\pi, y) = u_{kx}(\pi, y),$$

where $f_k(x, y) \xrightarrow{L_2} f(x, y)$, $f_k(x, y) = \sum_{n=-k}^{k} f_n(y) \cdot e^{inx}$ $(i^2 = -1)$, $(l_n + \mu E)^{-1}$ is the inverse operator to the operator $(l_n + \mu E)$.

Using inequality (3.1), we have:

$$\| u_k(x, y) \|_2 \le c \| f_k(x, y) \|_2, \tag{3.28}$$

where $c > 0$ is a constant independent of k.

Since $f_k \xrightarrow{L_2} f$, we then find from (3.28) that

$$\| u_k - u_m \|_2 \le c \| f_k - f_m \|_2 \to 0 \text{ as } k, m \to \infty.$$

Hence, in view of the completeness of the space $L_2(\Omega)$, it follows that there exists a unique function $u \in L_2(\Omega)$ such that

$$u_k \to u \text{ as } k \to \infty. \tag{3.29}$$

It follows from (3.29) that for any $f \in L_2(\Omega)$

$$u(x, y) = (M + \lambda I)^{-1} f(x, y) = \sum_{n=-\infty}^{\infty} (m_n + \lambda I)^{-1} f_n(y) \cdot e^{inx} \tag{3.30}$$

is a strong solution to the problem:

$$(M + \lambda I) u = f \tag{3.31}$$

$$u(-\pi, y) = u(\pi, y),\ u_y(-\pi, y) = u_y(\pi, y) \tag{3.32}$$

We recall the definition of a strong solution.

We call a function $u \in L_2(\Omega)$ a strong solution of the problem (3.31) - (3.32) if there is a sequence $\{u_k\}_{k=1}^{\infty} \subset D(L_0)$ such that $\|u_k - u\|_2 \to 0, \|(M + \lambda I)u_k - f\|_2 \to 0$ as $k \to \infty$.

Now, it is not hard to see that (3.30) is the inverse of a closed operator $L + \mu E$. It follows from Lemma 3.11 that the last sentence is valid for all $\mu \geq 0$.

Theorem 3.1 is proved.

Proof of Theorem 3.2. Using Lemma 3.12, it is easy to see that

$$\lim_{n \to \infty} \left\| (m_n + \lambda I)^{-1} \right\|_{2 \to 2} = 0.$$

In view of this, and using the ε-set from (3.30), we have that the operator $(M + \lambda I)^{-1}$ is compact if and only if $(m_n + \lambda I)^{-1}$ is completely continuous. Now the theorem follows from Lemma 3.13.

Proof of Theorem 3.3. Without loss of generality, we assume that $w \leq 1$, according to condition *ii)*, we have

$$\mu_0^{-1} \cdot w \cdot c(y) \leq \int_y^{y+w} c(t)dt \leq \lambda_0 \cdot w \cdot c(y)$$

This inequality and Theorem 3.2 imply the proof of Theorem 3.3.

3.3 Two-sided estimates of the distribution function of singular numbers of a singular operator of mixed type

With the solution of the problem of discreteness of the spectrum, problems such as estimates of singular numbers and the distribution functions are related.

We shall number the nonzero s-numbers of the operator $(L + \mu E)^{-1}$ in the order of their decrease taking into account their multiplicity so that

$$s_k((M + \lambda I)^{-1}) = \lambda_k \sqrt{((M + \lambda I)^{-1})^{*} (M + \lambda I)^{-1}}\ ,\ k = 1,2,...$$

We introduce the function $N(\lambda) = \sum_{s_k > 1} 1$ is the number of s_k greater than $\lambda > 0$.

Theorem 3.4 Suppose that conditions *i) -ii)* are satisfied. Then the estimate

$$c^{-1} \sum_{n=-r}^{r} \lambda^{-\frac{1}{2}} mes(y \in R : Q_n(y) \leq c^{-1}\lambda^{-1}) \leq N(x) \leq$$

$$c \sum_{n=-r}^{r} \lambda^{-1} mes(y \in R : K_n^{\frac{1}{2}}(y) \leq c\lambda^{-1}),$$

where $Q_n(y) = \left| n^2 + in \cdot a(y) + c(y) + \mu \right|$, $K_n(y) = \left| n \cdot a(y) \right| + c(y) + \mu$, a constant $c > 0$ does not depend on $Q_n(y), K_n(y), \mu$.

Proof. Any element can be written as follows:

$$f(x, y) = \sum f_n(y) \cdot e^{inx},$$

where $\Omega = \{(x, y): -\pi < x < \pi, -\infty < y < \infty\}$, $i^2 = -1$, $f_n(y)$ are the Fourier coefficients.

From the representation (3.30) it follows that

$$(M + \lambda I)^{-1} f = \sum_{n=-\infty}^{\infty} (m_n + \lambda I)^{-1} f_n \cdot e^{inx}$$

From this and the definition of the adjoint operator we have:

$$\left[(M + \lambda I)^{-1}\right]^{*} f = \sum_{n=-\infty}^{\infty} \left((m_n + \lambda I)^{-1}\right)^{*} f_n(y) \cdot e^{inx}$$

Now it is not difficult to define the following action:

$$\left[(M + \lambda I)^{-1}\right]^{*} (M + \lambda I)^{-1} = \sum_{n=-\infty}^{\infty} \left((m_n + \lambda I)^{-1}\right)^{*} (m_n + \lambda I)^{-1} f_n(y) \cdot e^{inx} \qquad (3.33)$$

It follows from (3.33) that if λ is the point of the spectrum of the operator $\left[(M + \lambda I)^{-1}\right]^{*} (M + \lambda I)^{-1}$, then λ is a point of the spectrum of one of the operators $\left[(m_n + \lambda I)^{-1}\right]^{*} (m_n + \lambda I)^{-1}$ and vice versa, if λ is the point of the spectrum of one of the operators $\left[(m_n + \lambda I)^{-1}\right]^{*} (m_n + \lambda I)^{-1}$, then λ is a point of the spectrum of the operator $\left[(M + \lambda I)^{-1}\right]^{*} (M + \lambda I)^{-1}$. From this and Theorems 2.5 and 2.6, the proof of Theorem 3.4 follows easily. Here we have taken into account that the $s_{k+1} = d_k$ (d_k-widths of the embedding) [38].

CONCLUSION

The paper is devoted to the study of the existence and compactness of the resolvent, as well as estimates of the distribution function of singular numbers (s-numbers) of a differential operator of mixed type in a noncompact domain. This research allows us to formulate the following new scientific results:

- a theorem on the existence of the resolvent of a one-dimensional Schrödinger operator with a complex potential is proved, when the real part of the potential contains a parameter that changes sign (the case on the whole line);
- a representation of the resolvent of a one-dimensional Schrödinger operator with a complex potential is obtained, when the real part of the potential contains a parameter that changes sign (the case on the whole line).

On the basis of these results, the following new results are obtained for a class of mixed-type differential operators defined in a noncompact domain:

- the theorem on the existence of a bounded inverse operator is proved;
- a theorem on the compactness (discreteness of the spectrum) of the resolvent of an operator of mixed type is proved;
- two-sided estimates of the distribution function of singular numbers (s-numbers) of the resolvent of a differential operator of mixed type given in an unbounded domain are obtained.

The obtained results are of theoretical interest and can find application in the spectral theory of differential operators of mixed type and in other sections of the theory of boundary value problems.

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