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SULEYMAN DEMIREL UNIVERSITY

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Categories of dialgebras

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Аңдатпа

Бұл идея xy көбейтіндісіне және yx көбейтіндісіне арналған екі түрлі операциядан басталды және ол қисықсыз – симметриялық болуы міндетті емес. Шын мәнінде, біз 3 аксиомаларды қанағаттандыратын $\dashv$ және $\vdash$ сәйкесінше сол және оң көбейтінді деп аталатын екі ассоциативті операциямен жабдықталған векторлық кеңістік ретінде диалгебраны анықтаймыз. Бұл аксиомаларды қолдану арқылы біз кейбір диалгебраларды қарастырдық.

Гельфанд - Дорфман биалгебра (GD-алгебралары) – екі түрлі (ассоциативті емес) көбейтіндімен, Гамильтонианның меншік сипатын вариация формальды есептемесінде көрсетеді. Бұл жұмыста $GD-$алгебралардың дифференциалды Пуассон алгебраларына қатысы барлығының зерттеуіне арналған. Көбейтінді ретінде, $a \succ b = d(a)b$ және $a \prec b = ad(b)$ операцияларына арналған сәйкестіктердің жалпы сипаттамасын аламыз, мұндағы d – алгебраның дифференциалды туындысы (ассоциативті емес) болып табылады.

Аннотация

Идея состоит в том, чтобы начать с двух различных операций для умножения xy и умножения yx, так что скобка больше не обязательно является кососимметричной. Мы определяем диалгебру как векторное пространство, снабженное двумя ассоциативными операциями $\dashv$ и $\vdash$, называемыми соответственно левым и правым произведением, удовлетворяющими также 3 аксиомам. Мы рассмотрели некоторые разновидности алгебр с помощью этих аксиом.

Биалгебры Гельфанда - Дорфмана (GD-алгебры) являются неассоциативными системами с двумя билинейными операциями, удовлетворяющими ряду тождеств которые выражают гамильтоново свойство оператора в формальном вариационном исчислении. Работа посвящена исследованию GD-алгебр, связанных с дифференциальными алгебрами Пуассона. Как побочный продукт, мы получаем общее описание тождеств, которые выполняются для операций $a \succ b = d(a)b$ и $a \prec b = ad(b)$ на (неассоциативной) дифференциальной алгебре с дифференцированием d.

Acknowledgements

This work would not have been possible without the financial support and state grant.

I am grateful to my supervisor with whom I have had the pleasure to work during this and other related works. Each of the members of my Dissertation Committee has provided me extensive personal and professional guidance and taught me a great deal about both scientific research and life in general. I would especially like to thanks Dr. Kolesnikov P. S and Dr. Dzhumadil'dayev A. S.. As my teachers and mentors, they have taught me more than I could ever give them credit for here. They have shown me, by his example, what a good scientist and person should be. Nobody has been more important to me in the pursuit of this project than the members of my family. I would like to thank my parents and wife, whose love and guidance are with me in whatever I pursue. They are the ultimate role models.

Contents

1. Introduction

1.1 Motivation

A category (sometimes called an abstract category to distinguish it from a concrete category) is a set of "objects" that are connected by "arrows". A category has two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object.

Definition 1. A category is:

1). A set $ob(C)$ of objects.

2).A set $hom(C)$ of morphisms, or arrows, or maps, between the objects. All morphism f has a source object a and a target object b where a and b are in $ob(C)$. We denote $f : a \to b$, and we say "f is a linear map from a to b" and f is a morphism. We write $hom(a, b)$ (or $hom_c(a, b)$ to set the hom-class of all morphisms from a to b.

3). For every three objects a, b and c, a binary operation $hom(a, b) \times hom(b, c) \to hom(a, c)$ called composition of morphisms; the composition of $f : a \to b$ and $g : b \to c$ is written as $g \circ f$ or gf.

We consider category of algebras with two different multiplication and construct homomorphism between them.

In chapter 4 we consider algebras with two binary operations $(a, b) \mapsto a \vdash b$ and $(a, b) \mapsto a \dashv b$. J.L. Loday has introduced so called 0-dialgebras [2]. They satisfy the following 0-dialgebra identities:

$$a \dashv (b \dashv c) = a \dashv (b \vdash c) \tag{1.1}$$

$$(a \vdash b) \vdash c = (a \dashv b) \vdash c \tag{1.2}$$

There are three natural constructions for associativity identities in 0-dialgebras:

$$a \vdash (b \vdash c) = (a \vdash b) \vdash c \quad \text{(left-associative)} \tag{1.3}$$

$$a \vdash (b \dashv c) = (a \vdash b) \dashv c \quad \text{(central-associative)} \tag{1.4}$$

$$a \dashv (b \dashv c) = (a \dashv b) \dashv c \quad \text{(right-associative)} \tag{1.5}$$

A 0-dialgebra is said *left-associative* (shortly lass) if it satisfies left-associative identity. Similarly, 0-dalgebra is *central-associative* (cass), *right-associative* (rass), if it satisfies identities (1.3), (1.4), (1.5) correspondingly.

Left-central-right-associative dialgebras was firstly considered by Loday and called associative dialgebras [1],[2]. Cental-associative dialgebras are also called L-associative [4]. Free bases for associative dialgebras was constructed in [2]. Free bases for L-associative dialgebras was constructed in [3] and [4].

In our work we construct free bases of 0-dialgebras and their associative versions. Our constructions are based on binary trees. We consider trees whose inner vertices are colored by two colors: black and white. We say that tree is black (white) if all vertices are black (white).

Motivation of chapter 5 is series of non-associative structures related with the formal calculus of variations appeared in [13]. These structures are nowadays known as Novikov algebras (they were independently introduced in [6] as a tool for classification of linear Poisson brackets of hydrodynamic type), Novikov–Poisson algebras, and Gelfand–Dorfman bialgebras. The latter class of systems was initially invented in [13] as a source of Hamiltonian operators: the structure constants of a Gelfand–Dorfman bialgebra may be used as coefficients of a differential operator.

By definition, a linear space A equipped with one bilinear operation $\circ : A \times A \to A$ is said to be a *Novikov algebra* (the term was proposed in [21]) if it satisfies the following identities:

$$(a \circ b) \circ c - a \circ (b \circ c) = (b \circ a) \circ c - b \circ (a \circ c), \tag{1.6}$$

$$(a \circ b) \circ c = (a \circ c) \circ b. \tag{1.7}$$

A *Gelfand–Dorfmann bialgebra* [25] is a system $(A, \circ, [\cdot,\cdot])$ with two bilinear operations such that $(A, \circ)$ is a Novikov algebra, $(A, [\cdot,\cdot])$ is a Lie algebra, and the following additional identity holds:

$$[a, b \circ c] - [c, b \circ a] + [b, a] \circ c - [b, c] \circ a - b \circ [a, c] = 0. \tag{1.8}$$

In order to avoid a collision with the well-known notion of a bialgebra as an algebra equipped with a coproduct, we will use the term *GD-algebra* for a Gelfand–Dorfman bialgebra.

1.2 Aims and Objectives

Our aim is consider some classical problems for algebras with two multiplications. One of them is construction of base and give a formula of dimension for multilinear part and formula for degree n with $k_1 - x_1 th, k_2 - x_2 th, \ldots, k_m - x_m th$ elements, where $k_1 + \ldots + k_m = n$.

1.3 Thesis Outline

Chameleon trees is a appropriate type of trees for explain bases of $0 - dialgebras$. Firstly we considered magmatic dialgebras and for each $0 - dialgebra$ we define special kind of chameleon trees.

We found homomorphism from $GD-$algebra to $Poisson$ algebra and special identities. Also there constructed base of Koszul-dual algebra and identities for any algebra with derivation.

2. Preliminaries

We consider a binary trees with black and white vertices.

Colored tree is called *chameleon* if for any vertex v

- its left subtree is one vertex tree or black tree if v is black, and,
- its right subtree is one vertex tree or white tree if v is white vertex.

A chameleon tree is called *left S-chameleon* (left stairs chameleon), if left subtree of any black vertex is a tree with one vertex or black stairs tree.

We say that a chameleon tree is *right S-chameleon* (right stairs chameleon), if right subtree of any white vertex is a tree with one vertex or white stairs tree.

A two sided S-chameleon tree, i.e. left and right S-chameleon tree is called *S-chameleon*.

We say that a S-chameleon tree is

- *C-chameleon* (central chameleon) if right son of any black vertex is black.
- *L-chameleon* (left chameleon) is right son of any black vertex is white.
- *R-chameleon* (right chameleon) if left son of any white vertex is black.

3. Algebras with two multiplications and basic notions of algebra

Definition 2. A vector space R is called an *algebra* if is equipped with a binary operation $*$ *(i.e. a mapping* $* : (R, R) \longrightarrow R)$, called multiplication, such that for any $a, b, c \in R$ and any $\alpha \in K$

$$(a + b) * c = a * c + b * c$$

$$a * (b + c) = a * b + a * c$$

$$\alpha(a * b) = (\alpha a) * b = a(\alpha * b)$$

.

If the algebra R has a basis $\{e_i | i \in I\}$, then in order to define the multiplication in R it is sufficient to know the multiplication between the basis elements:

$$e_i * e_j = \sum_{k \in I} \alpha_{ij}^k e_k, \alpha_{ij}^k \in K$$

where for fixed i and j only a finite number of α_{ij}^k are different from 0. Oppositely, for any given basis $\{e_i | i \in I\}$ of the vector space R and a given system of elements $\alpha_{ij}^k \in K$ with the property that for fixed i, j only a finite number of $\alpha_{i,j}^k$ are not 0, we can define the multiplication in R by

$$(\sum_{i \in I} \xi_i e_i) * (\sum_{j \in I} \eta_j e_j) = \sum_{i,j \in I} \xi_i \eta_j (e_i * e_j), e_i * e_j = \sum_{k \in I} \alpha_{i,j}^k e_k.$$

The subset S of the algebra R is called a *subalgebra* if it is closed with respect to the multiplication, i.e. $s_1, s_2 \in S$ implies $s_1 * s_2 \in S$. The subalgebra I of R is called a *left ideal* of R if $RI \subseteq I$ (i.e. $r*i \in I$ for all $r \in R, i \in I$). Similarly one defines a *right ideal* and a *two-sided ideal* (or simply ideal), (=*left+right ideal* in the same time, notation $I \triangleleft R$).

The vector space homomorphism $\phi : R_1 \to R_2$ of the algebras R_1, R_2 is an (algebra) *homomorphism* if

$$\phi(a * b) = \phi(a) * \phi(b), a, b \in R.$$

Similarly one introduces the notion of *isomorphism*, *automorphism* (=isomorphism of R and R), *endomorphism*(=homomorphism from R to R).

Let R be an algebra over K. The basic identities in algebras is

(i) R is *associative* if $(a * b) * c = a * (b * c)$ for every $a, b, c \in R$;

(ii) R is *commutative* if $a * b = b * a$ for every $a, b \in R$;

(iii) R is *Lie algebra* if for every $a, b, c \in R$,

$a * a = 0,$ *the anticommutative law*,

$(a * b) * c + (b * c) * a + (c * a) * b = 0,$ *the jacoby law*;

(iv) *Ris Jordan algebra* if for every $a, b \in R$,

$a * b = b * a,$ *the commutative law*,

$(a^2 b)a = a^2(ba)$;

Let K be a field of characteristic 0 and $\mathcal{K} = K\langle t_1, t_2, \ldots\rangle$ be a free magma, i.e., a space of non-associative non-commutative polynomials with generators $t_1, t_2, \ldots$. An ideal I of $\mathcal{K}$ is called *T-ideal* if for any $f(t_1, \ldots, t_k) \in I$ and for any endomorphism ϕ of $\mathcal{K}$,

$$f(\phi(t_1), \ldots, \phi(t_k)) \in I.$$

For non-associative non-commutative polynomials $f_1, \ldots, f_l \in \mathcal{K}$ denote by

$$J(f_1, \ldots, f_l)$$

a T-ideal of $\mathcal{K}$ generated by these elements.

As the same can be defined algebra with two multiplications is a set X equipped with two maps called respectively *left product* and *right product*:

$$\vdash : X \times X \to X,$$

$$\dashv : X \times X \to X.$$

Dialgebras satisfying the following axioms:

$$x \dashv (y \dashv z) = (x \dashv y) \dashv z = x \dashv (y \vdash z)$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z)$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z) = (x \vdash y) \vdash z$$

for all x, y and $z \in X$ is called diassociative algebras. This identities firstly introduced by Loday [2].

In the assumption $x \dashv y$, $x \vdash y$, the element x is said to be on the pointer side and the element y is said to be on the bar side.

Observe that the given relations are the *associativity* of the products $\dashv$ and $\vdash$.

A morphism of dialgebras is a map $f : X \to Y$ between two dialgebras X and Y which are satisfies to a following conditions: $f(x \dashv y) = f(x) \dashv f(y)$ and $f(x \vdash y) = f(x) \vdash f(y)$ for any $x, y \in X$.

We can see that there can be defined di-object in any monoidal category. One does not need the monoidal category to be symmetric because all elements stay in the same order.

4. 0-dialgebras

4.1 Colored trees

Colored tree is called *chameleon* if for any vertex v

- its left subtree is one vertex tree or black tree if v is black, and,
- its right subtree is one vertex tree or white tree if v is white vertex.

A chameleon tree is called *left S-chameleon* (left stairs chameleon), if left subtree of any black vertex is a tree with one vertex or black stairs tree.

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We say that a S-chameleon tree is

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- *L-chameleon* (left chameleon) is right son of any black vertex is white.
- *R-chameleon* (right chameleon) if left son of any white vertex is black.

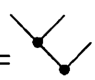
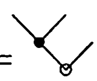
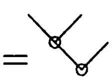
In our paper we prove following results:

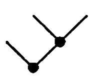
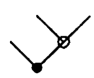
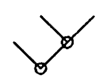
- chameleon trees generate free base of 0-dialgebras.
- For $W = C, L, R$ any W-associative 0-dialgebra is stairs algebra.
- If $W = S, C, L, R$, then W-chameleon trees generate free base of W-dialgebras.

For chameleon trees with n leaves we establish the following results.

- number of chameleon trees is $\binom{2n-2}{n-1}$.
- number of left (right) S-chameleon trees is $3^{n-2} + \frac{1}{2}\binom{2n-2}{n-1}$, if $n > 1$ and 1, if $n = 1$
- number of S-chameleon trees is $2 \cdot 3^{n-2}$, if $n > 1$, and 1, if $n = 1$
- number of C-chameleon trees is $2^{n-3}(n+2)$ if $n > 1$ and 1, if $n = 1$.
- number of L-chameleon (R-chameleon) trees is F_{2n-1}, where F_i are Fibonacci numbers with $F_1 = 1, F_2 = 1$.

Example. Let $n = 3$. Then set of chameleon trees with leave-degree 3 consists of the following 6 trees

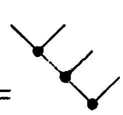
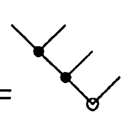
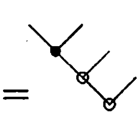
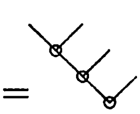
$t_1 =$  $t_2 =$  $t_3 =$ 

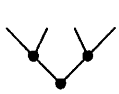
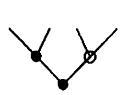
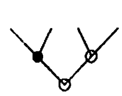
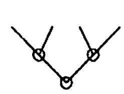
$t_4 =$  $t_5 =$  $t_6 =$ 

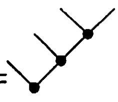
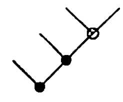
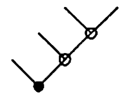
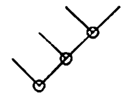
and all of these trees,

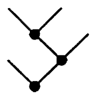
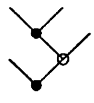
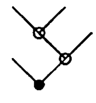
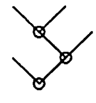
- are left (right) and two-sided S-chameleon
- except t_5 are C-chameleon
- except t_4 are L-chameleon
- except t_3 are R-chameleon

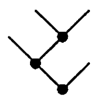
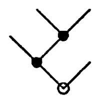
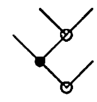
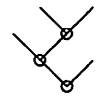
Example. $n = 4$. Then set of chameleon trees with leave-degree 4 consists of the following 20 trees

$t_1 =$  $t_2 =$  $t_3 =$  $t_4 =$ 

$t_5 =$  $t_6 =$  $t_7 =$  $t_8 =$ 

$t_9 =$  $t_{10} =$  $t_{11} =$  $t_{12} =$ 

$t_{13} =$  $t_{14} =$  $t_{15} =$  $t_{16} =$ 

$t_{17} =$  $t_{18} =$  $t_{19} =$  $t_{20} =$ 

and, if number of leaves $n = 4$, all of these trees,

- except t_{17}, are left S-chameleon (19 left C-chameleon trees)
- except t_{16}, are right S-chameleon (19 right S-chameleon trees)
- excet t_{16} and t_{17}, are S-chameleon (18 S-chameleon trees)
- except $t_6, t_{10}, t_{11}, t_{14}, t_{15}, t_{16}, t_{17}, t_{19}$, are C-chameleon (12 C-chameleon trees)
- except $t_5, t_9, t_{10}, t_{13}, t_{16}, t_{17}, t_{18}$ are L-chameleon (13 L-chameleon trees)
- except $t_3, t_4, t_8, t_{15}, t_{16}, t_{17}, t_{20}$ are R-chameleon (13 R-chameleon trees)

4.2 Binary trees and magmatic algebras

Tree is a graph without cycle. A tree is called *rooted*, if one vertex is selected as a root. We endow a tree by orientation directed from root to leaf. For a vertex of tree number of its in-edges and out-edges is called *in-degree* and *out-degree.* Rooted tree is called *binary,* if each vertex has out-degree 0 or 2. A vertex with out-degree 0 is a *leaf* a vertex with in-degree 0 is root. Non-leaf vertex is called *inner.* For example, root of tree is inner vertex.

Let $V(t)$ be a set of vertices and $E(t)$ set of edges of tree t. We denote by $root(t)$ root of tree t. Let $V_{leaf}(t)$ be a set of leaves of t and $V_{inner}(t)$ set of inner vertices of t.

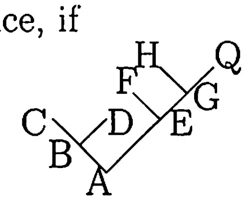
If $n = |V_{leaf}(t)|$ is number of leaves of t, say that t has *leave degree* $n = |t|_l$. Let *degree* of tree, denoted $deg(t)$, be number of all vertices. Then

$$|V_{inner}(t)| = |t|_l - 1,$$

$$deg(t) = 2|t|_l - 1.$$

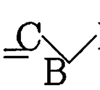
Each inner vertex $v \in V(t)$ generates binary subtree t_v of t with root v. Any vertex $u \in V(t_v)$ is a vertex of t connected with v by positive oriented edges. The root v of subtree t_v has *left* son denoted $l(v)$ and *right* son denoted $r(v)$. Any of these sons generate subtrees. Left subtree $l_v(t) = t_{l(v)}$ is generated by left son $l(v)$ served as a root, all vertices of these subtree are connected by $l(v)$ and all edges are directed from $l(v)$. Similarly, right son $r(v)$ serves as a root of right subtree $r_v(t) = t_{r(v)}$ and all vertices of $r_v(t)$ are connected by $r(v)$ and edges are directed from $r(v)$.

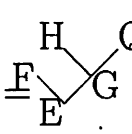
For instance, if

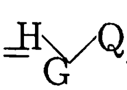
$t =$ 

then

$$V_{leaf}(t) = \{C, D, F, H, Q\}, \quad V_{inner}(t) = \{A, B, E, G\}, \quad root(t) = A,$$

$t_B = l_A(t) =$ 

$, t_E = r_A(t) =$ 

$, l_E(t) = F; r_E(t) =$ 

.

Let $\mathbb{T}$ be set of binary trees and $\mathbb{T}_n$ set of binary trees with n leafes. It is known that

$$|\mathbb{T}_n| = \frac{1}{n}\binom{2n-2}{n-1}.$$

(n-th Catalan number). For example, $\mathbb{T}_2$ consists of one tree and $\mathbb{T}_3$ consists of two trees.

We endow set of trees by multiplication (bouquet product)

$$\mathbb{T} \times \mathbb{T} \to \mathbb{T}, \quad (t', t'') \mapsto t' \vee t''$$

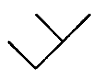
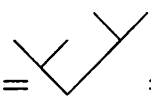
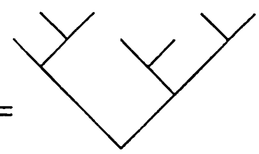
by the following rule

$$t = t' \vee t'', \quad \text{if } l_{root(t)}(t) = t' \text{ and } r_{root(t)}(t) = t''.$$

We see that

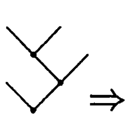
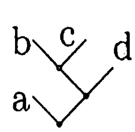
$$|t' \vee t''| = |t'| + |t''| + 1.$$

For example,

$t' =$  , $t'' =$  $\Rightarrow t' \vee t'' =$ 

Let $\mathcal{T}$ be a linear span of $\mathbb{T}$. This means that we consider vector space $\mathcal{T}$ whose base vectors are binary trees. We prolong by linearity bouquet product on to linear span. So, we obtain binary trees algebra $(\mathcal{T}, \vee)$.

A *magmatic* algebra generated by $X = \{a, b, c, \ldots\}$ is absolute free algebra generated by X. Sometimes it is called non-commutative non-associative algebra. It has base generated by words, i.e. by sequence of elements of X with bracketing type. The language of binary trees is very convenient to construct magmatic words. Just one corresponds to leaves elements of X and to each inner vertex correspond multiplication in magma. In other words, label leaves by elements of X and color inner vertices by binary operation in magma. If binary tree t has q leaves, then for any word $u_i = u_i(a, b, c, \ldots), i = 1, 2, \ldots, q$, by $t(u_1, \ldots, u_q)$ we denote a word constructed by tree t whose labels of leaves (from left to right) are $u_1, \ldots u_q$ and to each inner vertex v with subtree t_v whose leave labels are $u_{i_1}, \ldots u_{i_k}$, corresponds a word constructed by subtree $t_v(u_{i_1}, \ldots, u_{i_k})$. For example,

$t =$  $\Rightarrow \quad t(a, b, c, d) =$ 

$= a \cdot ((b \cdot c) \cdot d)$

The following fact is well known

Theorem 1. *Let $X = \{a_1, \ldots, a_n\}$ and $F(X)$ be magmatic algebra generated by X. Then set of words $t(x_1, \ldots, x_q)$, where t runs set binary trees with leave degree q and $x_1, \ldots, x_q \in X$, forms base of degree q part of magmatic algebra $F(X)$. Dimension of subspace of $F(X)$ generated by l_1 elements a_1, l_2 elements a_2, etc, l_n elements a_n, is equal to*

$$d_{l_1,\ldots,l_n} = \frac{(2\sum_{i=1}^n l_i - 2)!}{(\sum_{i=1}^n l_i - 1)!\prod_{i=1}^n l_i}.$$

In partucular, dimension of multilinear part of $F(X)$ is

$$dim\, F^{multi}(X) = 2^{n-1}(2n-3)!!$$

One generator magmatic algebra is isomorphic to trees algebra $(\mathcal{T}, \vee)$. In particular, dimension of homogeneous part of degree n of magmatic algebra generated by one element is equal to Catalan number

$$dim\, F(a_1) = \frac{1}{n}\binom{2n-2}{n-1}.$$

4.3 Colored trees and magmatic dialgebras

Let $X = \{a, b, c, \ldots\}$ be some alphabet. Recall that an algebra is a vector space with one binary operation. Magmatic algebra $F(X)$ with generators on X has base constructed by words with components on X. In terms of trees its base elements have a form $t(x_1, \ldots x_q)$.

A dialgebra is a vector space with two binary operations. After J.L. Loday we denote these operations as $\vdash$ and $\dashv$. In this section we construct di-analog of words and magmatic algebras. Any di-word is a word constructed by two operations on generators. A magmatic dialgebra as a vector space is a linear span of di-words. Di-operations on words are di-concatination operations. They are extended by linearity to di-operations on di-algebra. For example, if $u = (a \dashv b) \vdash c$ and $v = d \vdash e$, then

$$u \dashv v = ((a \dashv b) \vdash c) \dashv (d \vdash e), \quad u \vdash v = ((a \dashv b) \vdash c) \vdash (d \vdash e),$$

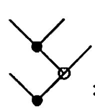
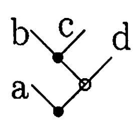
$$v \vdash u = (d \vdash e) \vdash ((a \dashv b) \vdash c), \quad v \dashv u = (d \vdash e) \dashv ((a \dashv b) \vdash c).$$

Now we give these constructions in terms of trees. To do that we color inner vertices of binary trees by two colors: black and white. Such a tree is called *colored* tree. So, any colored tree is defined by *frame* $t \in \mathbb{T}$ and coloring $\alpha : V_{inner}(t) \to \{\bullet, \circ\}$. We denote colored tree t whose all vertices are black by $b(t)$. Similarly, by $w(t)$ we denote colored tree t whose all vertices are white.

Let *color* $(\mathbb{T})$ be set of colored trees. We can prolong bouquet product $\vee : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ to bouquet product of colored trees in two ways. For $t', t'' \in color(\mathbb{T})$ denote $t' \vee_\bullet t''$ a tree $t' \vee t''$ whose new root is colored by black color and colors of inner vertices of t' and t'' are not changed. Similarly, $t' \vee_\circ t''$ is a tree whose new root is white and other vertices are as before.

Let C be linear span of base vectors indexed by colored trees. Under two products constructed above we obtain dialgebra $(C, \vee_\bullet, \vee_\circ)$. It gives us di-analog of magmatic algebra.

We imagine that leaves of colored trees are labelled by elements of X and colored inner vertices correspond to di-operations. Namely, $\bullet$ corresponds to $\vdash$ and $\circ$ to $\dashv$. For instance,

$t =$  $\Rightarrow \quad t(a, b, c, d) =$  $= a \vdash ((b \vdash c) \dashv d)$

The following is a di-version of Theorem 1.

Theorem 2. *Let $X = \{a_1, \ldots, a_n\}$ and $F_{di}(X)$ be magmatic di-algebra generated by X. Then set of words $t(x_1, \ldots, x_q)$, where t runs set colored binary trees with leave degree q and $x_1, \ldots, x_q \in X$, forms base of degree q part of magmatic dialgebra $F_{di}(X)$. Dimension of subspace of $F_{di}(X)$ generated by l_1 elements a_1, l_2 elements a_2, etc, l_n elements a_n, is equal to*

$$d_{l_1, \ldots, l_n} = 2^{l_1 + \cdots + l_n - 1} \frac{(2 \sum_{i=1}^n l_i - 2)!}{(\sum_{i=1}^n l_i - 1)! \prod_{i=1}^n l_i}.$$

In particular, dimension of multilinear part of $F_{di}(X)$ is

$$dim\, F_{di}^{multi}(X) = 4^{n-1}(2n - 3)!!$$

Magmatic dialgebra with one generator is isomorphic to colored trees dialgebra $\mathcal{C}$. In particular, dimension of homogeneous part of degree n of magmatic dialgebra generated by one element is equal to

$$dim\, F_{di}(a_1) = \frac{4^{n-1}(2n-3)!!}{n!}.$$

4.4 Chameleon trees and magmatic 0-dialgebras

4.4.1 Chameleon map

A dialgebra $(A, \vdash, \dashv)$ is called 0-*dialgebra* if it satisfies the following identities

$$a \dashv (b \dashv c) = a \dashv (b \vdash c) \tag{4.1}$$

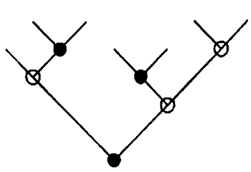
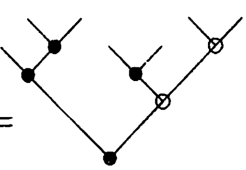
$$(a \vdash b) \vdash c = (a \dashv b) \vdash c \tag{4.2}$$

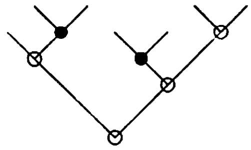
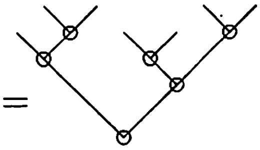
In the notation $a \dashv b, b \vdash a$, the element a is said to be on the *pointer side* and the element b is said to be on the *bar side*. So, identities (4.1) and (4.2) can be summarized as "on the bar side, does not matter which product". To formulate this rule in terms of trees we define a map $cham : color\ (\mathbb{T}) \to color\ (\mathbb{T})$ that we call *chameleon* map.

If $|t| = 1$, we set $cham(t) = t$. Suppose that for any colored tree t' with $|t'| < |t|$ a chameleon colored tree $cham(t')$ is constructed. Let t be a colored tree with left subtree $t' = l_{root(t)}(t)$ and right subtree $t'' = r_{root(t)}(t)$. We set

$$cham(t) = \begin{cases} b(t') \vee cham(t'') & \text{if } root(t) \text{ is black} \\ cham(t') \vee w(t'') & \text{if } root(t) \text{ is white} \end{cases}$$

So, for any $t \in color(\mathbb{T})$ and $v \in V_{inner}(t)$ color of left or right subtrees of v is uniquely defined by color of v: if v is black, then $t_{l(v)}$ is black, and, if v white, then $t_{r(v)}$ is white. For example,

$t =$  $\Rightarrow$ $cham(t) =$ 

$t =$  $\Rightarrow$ $cham(t) =$ 

We say that colored tree t is *chameleon* if $cham(t) = t$.

Example. Any black (white) tree is chameleon.

Let

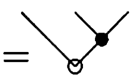
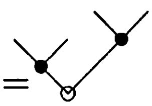
$$\mathbb{C}_0 = \{t \in color(\mathbb{T}) | cham(t) = t\}$$

be set of chameleon trees. Let $\mathcal{C}_0$ be a linear span of $\mathbb{C}_0$. Now we endow set of colored trees $\mathcal{C}$ by structure of dialgebra. For chameleon trees t' and t'' we set

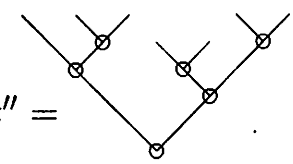
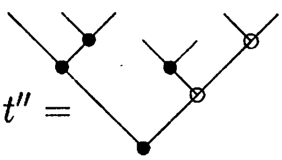
$$t' \vee_{\bullet,0} t'' = cham(t' \vee_{\bullet} t''),$$

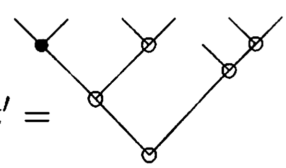
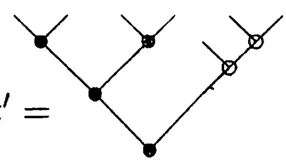
$$t' \vee_{\circ,0} t'' = cham(t' \vee_{\circ} t'').$$

For example, if

$t' =$  $t'' =$ 

then

$t' \vee_{\circ,0} t'' =$  $t' \vee_{\bullet,0} t'' =$ 

$t'' \vee_{\circ,0} t' =$  $t'' \vee_{\bullet,0} t' =$ 

In this section we prove the following result.

Theorem 3. *Let $X = \{a_1, \dots, a_n\}$ and $F_{di,0}(X)$ be magmatic 0-dialgebra generated by X. Then set of words $t(x_1, \dots, x_q)$, where t runs set of chameleon trees with leave degree q and $x_1, \dots, x_q \in X$, forms base of degree q part of magmatic 0-dialgebra $F_{di,0}(X)$. Dimension of subspace of $F_{di,0}(X)$ generated by l_1 elements a_1, l_2 elements a_2, etc, l_n elements a_n, is equal to*

$$d_{l_1,\dots,l_n} = \frac{(2\sum_{i=1}^{n} l_i - 2)!}{(\sum_{i=1}^{n} l_i - 1)!\prod_{i=1}^{n} l_i}\frac{Di_{l_1+\dots+l_n}}{C_{l_1+\dots+l_n}},$$

where $C_i = \frac{1}{i}\binom{2i-2}{i-1}$ is i-th Catalan number and Di_i is dimension of dialgebra generated by one generator.

In particular, dimension of multilinear part of $F_{di,0}(X)$ is

$$dim\, F_{di,0}^{multi}(X) = n!\binom{2n-2}{n-1}.$$

Magmatic dialgebra with one generator is isomorphic to chameleon trees dialgebra C_0. In particular, dimension of homogeneous part of degree n of magmatic dialgebra generated by one element is equal to

$$dim\, F_{di}(a_1) = \binom{2n-2}{n-1}.$$

4.4.2 Labeling of chameleon trees

In this section we give labeling of vertices of chameleon trees. This means that we construct bijective map $N : V(t) \to \{1, 2, \dots, |t|\}$. Let T be chameleon tree with degree $|t|$. If $|t| = 1$ then t coincides with one vertex. Its label by definition is 1.

Let $t' = l_{root(t)}(t), t'' = r_{root(t)}(t)$. Suppose that labels of vertices of t' and t'' are known. Then vertices of t' are numerated by $1, 2, \dots, k = |t'|$. Let us label root of t by $k+1$ and to all labels of vertices t'' add $k+1$.

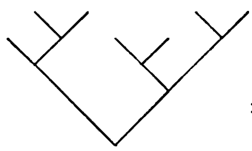
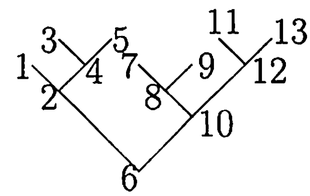
So, labeling algoritheorem of a tree with n leafes is the following

- label all leaves from left to right by consequative odd numbers, $1, 3, \dots, 2n-1$.

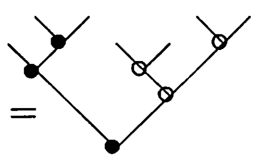
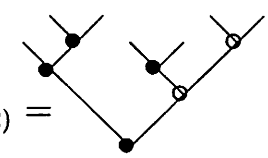
- if v is inner vertex with left subtree $l_v(t)$ and u is leftmost leave of $l_v(t)$, then set $N(v) = N(u) + |l_v(t)|$.

To construct chameleon tree with frame t and weight $w(t) = (a, b)$, denoted $t_{(a,b)}$ we color all first a inner vertices with increasing levels by black color.

Example.

$t =$  $\Rightarrow$ $N(t) =$ 

We see that labeling sequence of inner vertices is $(2, 4, 6, 8, 10, 12)$. Therefore, for chameleon tree with weight $(3, 3)$ black color have vertices with labels $2, 4, 6$ and white color have vertices with labels $8, 10, 12$. As far as chameleon tree with weight $(4, 2)$ labels of black vertices are $2, 4, 6, 8$ and labels of white vertices are $10, 12$. So,

$t_{(3,3)} =$  $t_{(4,2)} =$ 

4.4.3 A chameleon equivalence relation

Let t' and t'' are colored trees with the same leave degree q. For any di-words $u_1, \ldots, u_q$ we can construct elements of 0-dialgebra $t'(u_1, \ldots, u_q)$ and $t''(u_1, \ldots, u_q)$. Say that colored trees t' and t'' are *equivalent*, and denote $t' \sim t''$, if as elements of 0-dialgebra

$$t'(u_1, \ldots, u_q) = t''(u_1, \ldots, u_q),$$

for any di-words $u_1, \ldots, u_q$. It is easy to see that the raltion $\sim$ is really equivalent relation on a set of colored trees.

Lemma 1. *For any trees t', t'' and $\alpha = \bullet$ or $\circ$*

$$cham(t' \vee_\alpha t'') = cham(t') \vee_\alpha cham(t''),$$

Proof. We have to prove that for any $a_1, \ldots, a_{k+m} \in A$ where A is 0-dialgebra, $|t'|_l = k$, $|t''| = m$,

$$cham(t' \vee_\bullet t'')(a_1, \ldots, a_{k+m}) = (b(t') \vee_\bullet cham(t''))(a_1, \ldots, a_{k+m}),$$

$$cham(t' \vee_\circ t'')(a_1, \ldots, a_{k+m}) = (cham(t') \vee_\circ w(t''))(a_1, \ldots, a_{k+m}).$$

Suppose that

We label vertices of $t = t' \vee_\bullet t''$ as in section 4.4.2.

Let $\alpha = \bullet$. Then $2k$-th vertex is black. Suppose that for some $l < 2k$ l-th vertex is inner and its color is white and $l+1$-th vertex is black. Then by 0-dialgebra principle "on the bar side, does not matter which product" for any $l' \leq l$ we can change colors of l'-th vertices to black. Repeat this procedure for all $l \leq 2k+1$. We obtain that

$$cham(t' \vee_\bullet t'')(a_1, \ldots, a_{k+m}) = (b(t')(a_1, \ldots, a_k) \bullet cham(t'')(a_{k+1}, \ldots, a_{k+m}).$$

In a similar way one consider the case $\alpha = \circ$. In this case we also use 0-dialgebra principle "on the bar side, does not matter which product" and we recolor black inner vertices with labels $2k+1 \leq l \leq 2(k+m)-1$ by white. Then we obtain

$$cham(t' \vee_\circ t'')(a_1, \ldots, a_{k+m}) = (cham(t')(a_1, \ldots, a_k) \circ w(t'')(a_{k+1}, \ldots, a_{k+m}).$$

So, for $\alpha = \bullet$ or $\circ$

$$cham(t' \vee_\alpha t'') = cham(t') \vee_\alpha cham(t''),$$

□

Lemma 2. *Any colored tree t is equivalent to chameleon tree $cham(t)$.*

Proof. Since for any colored tree frames of $cham(t)$ and t are coincide,

$$(t' \vee_{\bullet,0} t'')(u_1, \ldots, u_{n+m}) = (b(t'))(u_1, \ldots, u_n) \vee_\bullet (cham(t''))(u_{n+1}, \ldots, u_{n+m}),$$

$$(t' \vee_{\circ,0} t'')(u_1,\ldots,u_{n+m}) = (cham(t'))(u_1,\ldots,u_n) \vee_\circ (w(t''))(u_{n+1},\ldots,u_{n+m}).$$

Therefore, by Lemma 1

$$t' \vee_\bullet t'' \sim b(t') \vee_\bullet cham(t'') = cham(t' \vee_\bullet t'')$$

$$t' \vee_\circ t'' \sim cham(t') \vee_\circ w(t'') = cham(t' \vee_\circ t'')$$

□

Lemma 3. *If t is chameleon tree with root vertex r and $|t|_l > 1$. Then its left subtree is black if r is black and right subtree is white if r white.*

Proof. Let r is black. Since

$$cham(t) = b(t') \vee_\bullet t'',$$

the condition $cham(t) = t$ implies that t' is black,

$$t' = b(t').$$

In a similar way, one consider the case when r white. In this case

$$cham(t) = t' \vee_\circ w(t''), cham(t) = t \Rightarrow t'' = w(t'').$$

Lemma 4. *The map $cham : color(\mathbb{T}) \to color(\mathbb{T})$ is idempotent map,*

$$cham^2 = cham.$$

Proof. We have to prove that for any $t \in color(\mathbb{T})$

$$cham^2(t) = cham(t).$$

We use induction on leave degree $|t|_l$. If $|t|_l = 1$, our statement is evident. Suppose that

$$cham^2(t') = cham(t'),$$

for any colored tree t' with $|t'|_l < |t|_l$. Let t be chameleon tree with root r, left and right subtrees $t' = t_{l(v)}$ and $t'' = t_{r(v)}$ with $|t'| < |t|_l, |t''| < |t|_l$.

Now we consider the case $|t|_l > 1$. Let t be chameleon tree with root r, left and right subtrees $t' = t_{l(v)}$ and $t'' = t_{r(v)}$.

Suppose that r is black. Then by Lemma 3 $t' = b(t')$. Therefore,

$$cham(t) = b(t') \vee_\bullet cham(t'') = t' \vee_\bullet cham(t'').$$

Since by inductive suggestion $cham^2(t'') = cham(t'')$, this means that

$$cham^2(t) = b(t') \vee_\bullet cham(cham(t'')) = t' \vee_{bullet} cham(t'') = cham(t).$$

Similarly, if r white, then by Lemma 3 $t'' = w(t''$, and

$$cham^2(t) = cham(t' \vee_\circ w(t'')) = cham(t') \vee_\circ w(t'') = cham(t') \vee_\circ t''.$$

By inductive suggestion $cham^2(t') = cham(t')$. Hence

$$cham^2(t) = cham(t') \vee_\bullet w(t'') = cham(t).$$

Lemma 5. *Let t' and t'' chameleon trees with leave degrees $|t'|_l = n, |t''|_l = m$. Then $t' \vee_{\bullet,0} t''$ and $t' \vee_{\circ,0} t''$ are also chameleon trees with leave degree $n+m$.*

and for any di-words $u_1, \ldots, u_{n+m}$,

$$(t' \vee_{\bullet,0} t'')(u_1, \ldots, u_{n+m}) = (b(t'))(u_1, \ldots, u_n) \vee_\bullet (cham(t''))(u_{n+1}, \ldots, u_{n+m}),$$

$$(t' \vee_{\circ,0} t'')(u_1, \ldots, u_{n+m}) = (cham(t'))(u_1, \ldots, u_n) \vee_\circ (w(t''))(u_{n+1}, \ldots, u_{n+m}).$$

Proof. Since

$$cham(t') = t', \quad cham(t'') = t'',$$

by Lemma 3

$$cham(t' \vee_\bullet t'') = b(t') \vee_\bullet cham(t'') = t' \vee_\bullet t'',$$

$$cham(t' \vee_\circ t'') = cham(t') \vee_\circ w(t'') = t' \vee_\circ t''.$$

Hence by Lemma 4

$$cham(t' \vee_{\bullet,0} t'') = cham^2(t' \vee_{\bullet} t'') = cham(t' \vee_{\bullet} t'') = t' \vee_{\bullet} t'',$$

$$cham(t' \vee_{\circ,0} t'') = cham^2(t' \vee_{\circ} t'') = cham(t' \vee_{\circ} t'') = t' \vee_{\circ} t''.$$

So, $t' \vee_{\bullet,0} t''$ and $t' \vee_{\circ,0} t''$ are chameleon trees. It is evident that

$$|t' \vee_{\bullet,0} t''|_l = |t' \vee_{\circ,0} t''|_l = |t'|_l + |t''|_l = n + m.$$

Lemma 6. $(C_0, \vee_{\bullet,0}, \vee_{\circ,0})$ *is* 0*-dialgebra.*

Proof. For any chameleon trees t', t'', t''' with leave degree n, m, s,

$$(t' \vee_{\circ,0} t'') \vee_{\bullet,0} t'' = b(t' \vee_{\circ,0} t'') \vee_{\bullet} cham(t''') = (b(t') \vee_{\bullet} b(t'')) \vee_{\bullet} cham(t'''),$$

$$(t' \vee_{\bullet,0} t'') \vee_{\bullet,0} t'' = b(t' \vee_{\bullet,0} t'') \vee_{\bullet} cham(t''') = (b(t') \vee_{\bullet} b(t'')) \vee_{\bullet} cham(t''').$$

Hence

$$(t' \vee_{\bullet,0} t'') \vee_{\bullet,0} t''' = (t' \vee_{\circ,0} t'') \vee_{\bullet,0} t'''.$$

Similarly,

$$t' \vee_{\circ,0} (t'' \vee_{\bullet,0} t''') = cham(t') \vee_{\circ} w(t'' \vee_{\bullet} t''') = cham(t') \vee_{\circ} (w(t'') \vee_{\circ} w(t''')),$$

$$t' \vee_{\circ,0} (t'' \vee_{\circ,0} t''') = cham(t') \vee_{\circ} w(t'' \vee_{\circ,0} t''') = cham(t') \vee_{\circ} (w(t'') \vee_{\circ} w(t''')),$$

and

$$t' \vee_{\circ,0} (t'' \vee_{\bullet,0} t''') = t' \vee_{\circ.0} (t'' \vee_{\circ,0} t''').$$

4.4.4 Proof of Theorem 3.

By Lemmas 2 and by 6 elements $t(x_1, \ldots, x_q)$, where $t \in cham(\mathbb{T})$, $|t|_l = q$, and $x_1, \ldots, x_q \in X$, forms base of magmatic algebra $F_{di,0}(X)$.

By section 4.2 for any tree with n leaves there can be constructed n *chameleon trees*

Number of trees is Catalan number i.e.

$$dim\, F(a_1) = \frac{1}{n}\binom{2n-2}{n-1}.$$

So number of

$$dim\, F_{di}(a_1) = \binom{2n-2}{n-1}.$$

Notice that identities (4.1) and (4.2) change only multiplications and there elements remain at the same place. So there can be only equivalent trees with generators which is stay at the same place. So dimensions of all colored trees are equal to each other. By multiplication of tree with one generator and divided to Catalan number there can be obtained dimension of any associative $0-dialgebra$. There known that number of this trees is $dim\, F_{di}(a_1)$. So

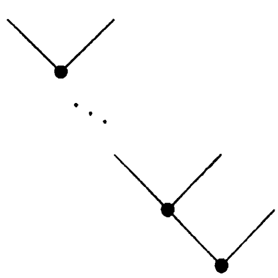
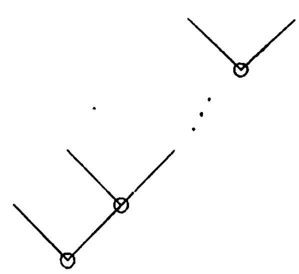
$$d_{l_1,\dots,l_n} = \frac{(2\sum_{i=1}^{n} l_i - 2)!}{(\sum_{i=1}^{n} l_i - 1)!\prod_{i=1}^{n} l_i}\frac{Di_{l_1+\dots+l_n}}{C_{l_1+\dots+l_n}}.$$

4.5 Strong chameleon trees and associative versions of dialgebras

4.5.1 Special kind of chameleon trees.

Let n is degree of tree. Recall that degree of tree is odd number and $|t| = 2|t|_l - 1$, where $|t|_l$ is leave-degree of t, i.e. a number of leafes.

We define special kinds of trees so called left and right stairs tree and denoted ls_n and rs_n respectively. Let ls_1 and rs_1 denote a tree with one vertex. For $n > 1$ *left stairs tree* ls_n and *right stars tree* rs_n are trees of the form

$ls_n =$  $rs_n =$ 

So, by definition left stairs tree is black and right stairs tree is white if $n > 1$.

Definition. S-chameleon is the subset of chameleon trees if left subtree of any black vertex is ls_n and right subtree of any white vertex is rs_n.

C-chameleon is the subset of S-chameleon trees if right son of any black vertex is black. L-chameleon is the subset of S-chameleon trees if right son of any black vertex is white. R-chameleon is the subset of S-chameleon trees if left son of any white vertex is black.

Theorem 4. *Bases of free dialgebras can be constructed by chameleon trees by the following ways.*

- *Set of chameleon trees generates base of free* 0*-dialgebra.*
- *Set of left S-chameleon trees generates base of free left stairs* 0*-dialgebra. Set of right S-chameleon trees generates base of free right stairs* 0*-dialgebra. Set of S-chamelon trees generates base of free stairs* 0*-dialgebra.*
- *Set of C-chameleon trees generates base of free central associative* 0*-dialgebra.*
- *Set of L-chameleon trees generates base of free left associative* 0*-dialgebra.*
- *Set of R-chameleon trees generates base of free right associative* 0*-dialgebra.*

Theorem 5. *Dimensions of given* $0-$ *dialgebras generated by l_1 x_1's ,etc., l_n x_n's is*

$$d_{l_1,\dots,l_n} = \frac{(2\sum_{i=1}^{n} l_i - 2)!}{(\sum_{i=1}^{n} l_i - 1)!\prod_{i=1}^{n} l_i} \frac{Di_{l_1+\dots+l_n}}{C_{l_1+\dots+l_n}},$$

where $C_i = \frac{1}{i}\binom{2i-2}{i-1}$ is i-th Catalan number and Di_i is dimension of dialgebra generated by one generator.

Corollary 1. *Let n is number of leafes.*

Dimension of 0*-dialgebra generated by one generator of degree n is $\binom{2n-2}{n-1}$.*

Dimension of cass generated by one generator is $2^{n-3}(n+2)$ if $n > 1$ and dimension for degree 1 is 1.

Dimension of lass and rass generated by one generator is F_{2n-3} where F_i is Fibonacci numbers and $n > 1$. For degree 1 dimension is 1.

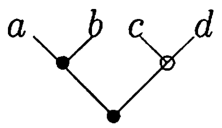
Corollary 2. *Let n is number of leafes.*

Multilinear parts of 0*-dialgebra of degree n is $\binom{2n-2}{n-1}n!$.*

Multilinear parts of cass is $2^{n-3}(n+2)n!$.

Multilinear parts of lass and rass is $F_{2n-3}n!$ *where* F_i *is Fibonacci numbers and* $n > 1$. *For degree 1 dimension is 1.*

Now we interpret colored trees as elements of dialgebras. Leaves of trees correspond to elements of dialgebras and black vertex to multiplication $\vdash$ and white vertex to the multiplication $\dashv$. For example,



$$= (a \vdash b) \vdash (c \dashv d)$$

So, any colored tree we interpret as an element of 0-dialgebra and the linear map $f : color(\mathbb{T}_n) \to A_n$ is epimorphism. There A_n any 0-dialgebra of degree n.

There can be defined $f^{-1} : A_n \to color(\mathbb{T}_n)$. Let call that element a of $0-$ *dialgebra* satisfy to *chameleon rule* if $f^{-1}(a) \in cham(\mathbb{T})$. Below we will prove that if a is element of any $0-$ *dialgebra*, then $f^{-1}(a) \in cham(\mathbb{T})$.

4.6 Chameleon trees

Lemma 7. *Any element of* 0- *dialgebra satisfy to the chameleon rule.*

Proof: Firstly consider case when $(a * b) \vdash c$ where a, b, c some elements and $*$ can be any multiplication by (4.2) then we must show that $a * b$ contains only $\vdash$ type of multiplication. We can say that if there will be proved that b contains only $\vdash$ then for a it can be proved by the same way. Let $b = b' * b''$ and there can be any multiplication because

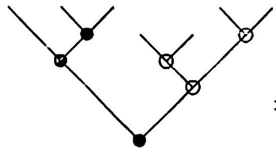
$$(a * b) \vdash c = (a \dashv (b' * b'')) \vdash c$$

and by (4.1) it is true. If b' is not leave then make the same operation i.e. $b' = d' * d''$ and continue it to leave. By this way we can change any multiplication to $\vdash$ by using (4.1) and (4.2.)

Analogically can be proved for case $a \dashv (b * c)$. $\square$

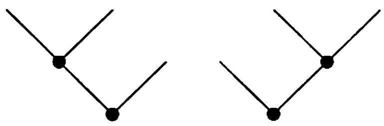
There we can say that *lass*, *rass*, *cass* also satisfy to the *chameleon rule*.

For $t \in cham(\mathbb{T})$ let us define its *weight* $w(t)$ as a pair of integers (a, b), where a is a number of black vertices and b is a number of white vertices. For example,

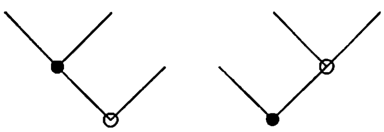
$t =$  $\Rightarrow \quad w(t) = (3,3).$

Example. Let us construct all chameleon trees for degree 3 by weights. There are three kinds of weights: $(2,0),(1,1),(0,2)$.

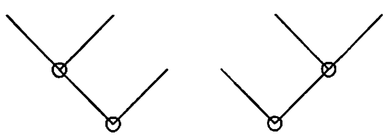
Trees with weight $(2,0)$:



Trees with weight $(1,1)$:



Trees with weight $(0,2)$:



Notice that in this example for $A_3 = cass$ two trees in weight $(1,1)$ are equal by (1.4). Analogically for *lass* two trees in weight $(2,0)$ are equal by (1.3). For *rass* two trees in weight $(0,2)$ are equal by (1.5). So number of independent trees for all this 0-dialgebras of degree 3 equal to 5.

By (7) we can consider only trees of the set $cham(\mathbb{T})$. We write every weight in the form (a,b) for degree n. There $a+b+1=n$ and we have $n+1$ different weights $\cup_{i=0}^{n}(n-i,i)$.

Let call associativity identity for $0-dialgebra$ is some identity of the form:

$$(a*b)*c = a*(b*c) \tag{4.3}$$

where $* = \vdash$ *or* $\dashv$.

Let call that element of 0-dialgebra satisfy to the left and right stairs if $f^{-1}(a) \in S - chameleon.$

Lemma 8. *Any associative 0-dialgebra satisfy to the left and right stairs.*

Proof: Let we have some associative $0-dialgebra$ with identity (4.3). By (7) we know that any $0-dialgeba$ satisfy to *chameleon rule.* We prove that

$$a \vdash b = ((\dots(a_1 \vdash a_2) \vdash \dots) \vdash a_n) \vdash b.$$

By using (4.1) and (4.2) in a there can be obtained any type of multiplication without changing of brackets. So there can be anytime used (4.3) and changed any right normed element to the left normed i.e.

$$a \vdash b = (a_1 * (a_2 * a_3)) \vdash b = ((a_1 * a_2) * a_3) \vdash b = ((a_1 \vdash a_2) \vdash a_3) \vdash b.$$

Analogically can be proved *right stairs.*□

Lemma 9. *For any tree t with $a+b+1$ leaves and weight (a,b) there exist bijection to labeling sequence of inner vertices.*

Proof: There easy to see that in sequences which is give in **section 2.3** all integer are even numbers i.e. all vertices labeled by even numbers and leaves by odd numbers. Let prove that all trees which correspond to given sequences is chameleon trees. By induction suppose that this labeling is true for all trees of degree t where $deg(t) \leq a+b$. Then we must prove that multiplication of two trees which is their degree less than $a+b+1$ that their labeling also chameleon tree.

Let take any trees a, b where $deg(a) = m, deg(b) = n$ s.t $m, n < a+b+1$. Then we consider the tree $a \vee b$ of weight $(0, m+n-1)$. Tree of the weight $(1, m+n-2)$ is equal to $a_{(1,m-2)} \vee_\circ b_{(0,n-1)}$. For weight $(2, m+n-3)$ is equal to $a_{(2,m-3)} \vee_\circ b_{(0,n-1)}$. So for weight $(i, m+n-1-i)$ where $i \leq m-1$ tree is equal to

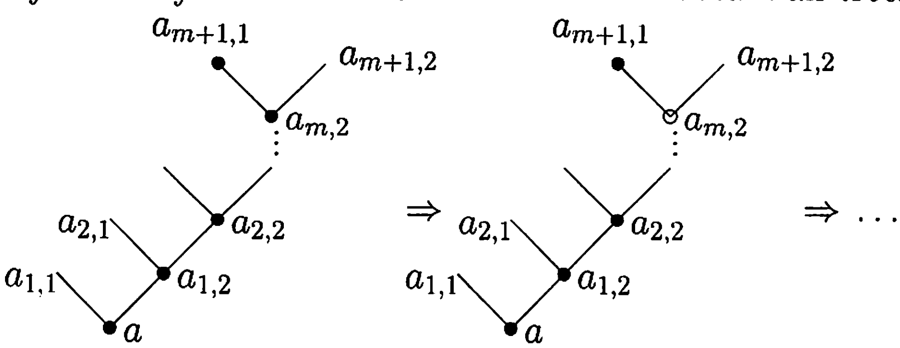
$$a_{(i,m-1-i)} \vee_\circ b_{(0,n-1)}.$$

For $(m, n-1)$ tree is $a_{(m-1,0)} \vee_\bullet b_{(0,n-1)}$. For $(i, m+n-1-i)$ where $m < i \leq m+n-1$ tree is

$$a_{(m-1,0)} \vee_\bullet b_{(i,n-1-i)}.$$

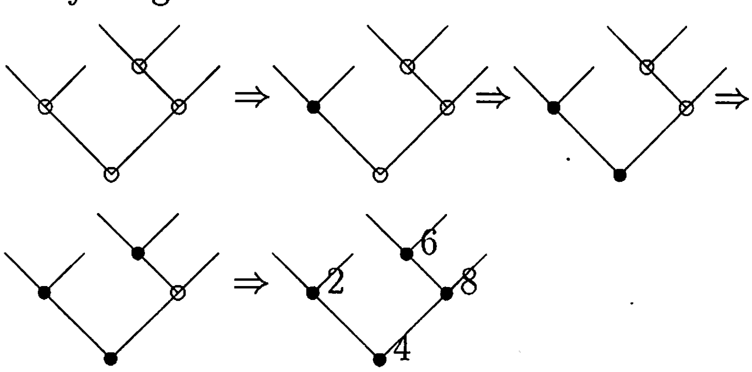
We can not write it by the other way because it will contradict to definition of chameleon trees.In particular all chameleon trees of degree $a+b+1$ satisfy to this labeling sequences. So induction is proved. It is easy to see that for this tree $a \vee b$ bijection between every weight and labeling sequence is satisfied. □

For any tree of degree $a+b+1$ we will give the way of construction of vertices for all weights. It is equivalent to find inner vertex which is not change color of other vertices by chameleon rule. Let we have tree $a = a_{1,1} \vee_{\bullet} a_{1,2}; a_{1,2} = a_{2,1} \vee_{\bullet} a_{2,2}; \dots ; a_{m,2} = a_{m+1,1} \vee_{\bullet} a_{m+1,2}$ and let $a_{m+1,2}$ is leaf, then we start from $a_{m,2}$ and change it's color to white and this vertex do not change color of other vertices. Let numerate this vertex by $2(a+b)$. This new tree will be element of weight $(a+b-1, 1)$. Then we need to obtain element of weight $(a+b-2, 2)$. We make the same thing for $a_{m+1,1}$ like for a i.e. change color of vertex which is stay at right side and it has only leaf at the right side i.e. this vertex also not change color of other vertices. Numerate this vertex by $2(a+b) - 2$. Continue this operation. By this way we numerate all vertices and obtain all trees for each weight i.e.



We can't change color of vertices by other way because it will contradict to definition of $cham(\mathbb{T})$. □

Example. Let choose tree of the form $\mathbb{T}_2 \vee (\mathbb{T}_2 \vee \mathbb{T}_1)$ and write it's form for every weight.



Corollary 3. *Let fix any tree t with $(a+b+1)$ leaves then for any weight it defined uniquely.*

4.7 Free 0-dialgebras on trees.

Let construct some ***di*** dialgebra with following multiplication: if a, b some trees then

$$a \vee_{\bullet} b = cham(a \vee_{\bullet} b)$$

$$a \vee_{\circ} b = cham(a \vee_{\circ} b)$$

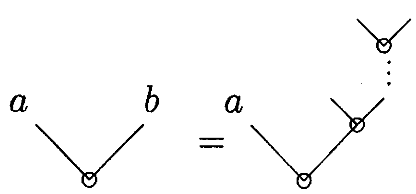
By (7) this algebra $di \in Var(0 - dialgebra)$ where $Var(X)$ is variety of X algebra.

Let prove that this $0-$dialgebra is free. Let take set of generator $\mathbb{Y}$ which is leaves of trees. Let take any algebra $A \in Var(0 - dialgebra)$. Then make some map $\tau : \mathbb{Y} \to A$. This map can be canonically extended to homomorphism $\eta : \Phi(\mathbb{Y}) \to A$ where $\Phi(\mathbb{Y})$ is free magma dialgebra i.e. dialgebra without any identities. But ideal $I(\Phi(\mathbb{Y}))$ which is constructed by (4.1) and (4.2) lies in any kernel of homomorphism of $\Phi(\mathbb{Y})$ to any dialgebra A. In particular it lies in kernel of η. It means that there exist homomorphism $\varphi : \Phi(\mathbb{Y})/I(\Phi(\mathbb{Y})) \to A$ which is make this diagram commutative

$$\begin{tikzcd} \Phi(\mathbb{Y}) \arrow[r, "\sigma"] \arrow[dr, "\eta"'] & \Phi(\mathbb{Y})/I(\Phi(\mathbb{Y})) \arrow[d, "\varphi"] \\ & A \end{tikzcd}$$

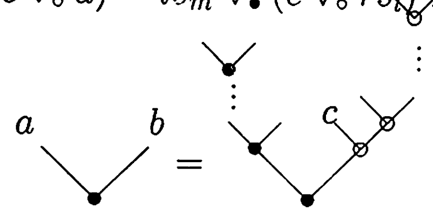
So φ is needed homomorphism. Uniqueness is obvious. This way is work for any dialgebra.

Now we construct ***dil*** dialgebra by the following multiplication: if a, b some trees of degree m, n then $a \vee_{\circ} b = a \vee_{\circ} rs_n$ i.e.

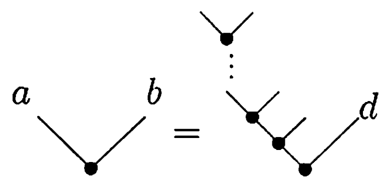


For $a \vee_{\bullet} b$ we consider 2 cases: $b = c \vee_{\circ} d$ or $c \vee_{\bullet} d$ where c, d some trees of degree k, l. At first case

$a \vee_\bullet (c \vee_\circ d) = ls_m \vee_\bullet (c \vee_\circ rs_l)$ i.e.

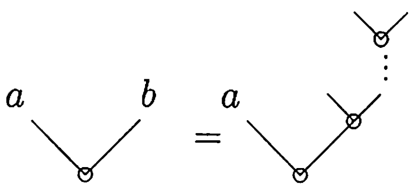


At the second case $a \vee_\bullet (c \vee_\bullet d) = ls_{m+k} \vee_\bullet d$ i.e.

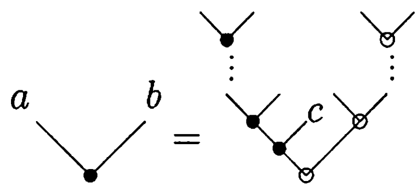


It can be easily checked that $dil \in Var(lass)$. Analogically can be constructed for $rass$ dir dialgebra.

Let construct dialgebra ***dic*** for some trees a, b of degree m, n with multiplication $a \vee_\circ b = a \vee_\circ rs_n$ i.e.



For $a \vee_\bullet b$ we consider 2 cases: $b = c \vee_\circ d$ or $c \vee_\bullet d$ where c, d some trees of degree k, l. At first case $a \vee_\bullet (c \vee_\circ d) = (ls_m \vee_\bullet c) \vee_\circ rs_l$.



For second case $a\vee_\bullet(c\vee_\bullet d) = 0$. It can be easily checked that $dic \in Var(cass)$.

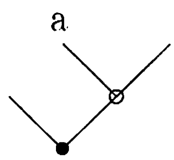
4.8 Proof of (4),(5).

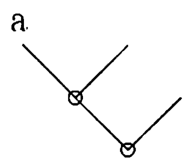
Set of chameleon tree with subtree of a form $t = b\vee_\bullet(a\vee_\circ c)$ is called $\bar{C}$-*chameleon,* where a is chameleon tree,$b = bs_n$ and $c = ws_m$ or it can be leaves.

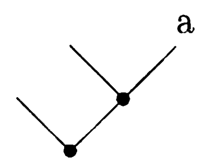
Set of chameleon tree with subtree of a form $t = b \vee_{\bullet} (c \vee_{\bullet} a)$ is called $\bar{L}$-*chameleon.*, where a is chameleon tree, $b = bs_n$ and $c = bs_m$ or it can be leaves.

Set of chameleon tree with subtree of a form $(a \vee_{\circ} b) \vee_{\circ} c$ is called $\bar{R}$-*chameleon,* where a is chameleon tree, $b = ws_n$ and $c = ws_m$ or it can be leaves.

By the other words it means that there set of trees with subtrees of the following forms correspondingly:

$\bar{C} =$ 

$\bar{R} =$ 

$\bar{L} =$ 

Let define set of trees for any dialgebra A such that:

$$Ann_{l,b}(A) = \{x \in A | x \vee_{\bullet} a = 0, where\ \forall a \in A\}.$$

$$Ann_{r,w'}(A) = \{x \in A | a \vee_{\bullet} x = 0, where\ \forall a \in A\}.$$

Definition: There can be defined $left$ and $right\ stairs$ for $cham(\mathbb{T})$ if

$$a' \vee_{\bullet} a'' = bs_{|a'|} \vee_{\bullet} a''$$

$$b' \vee_{\circ} b'' = b' \vee_{\circ} ws_{|b''|}.$$

For $left$ and $right\ stairs$ $Ann_{l,b}(A) = \{t - bs_{|t|} | \forall t \in 0-dialgebra\}$, $Ann_{r,w}(A) = \{t - ws_{|t|} | \forall t \in 0 - dialgebra\}$.

Set of trees which is satisfy to $left$ and $right\ stairs$ called $stairs$ *chameleon trees*. There we define subsets of trees

$$C-, L-, R - chameleon \in cham(\mathbb{T})$$

which are satisfies to $left$ and $right\ stairs$ and

$$R - chameleon = cham(\mathbb{T}) \backslash \bar{R},$$

$$L - chameleon = cham(\mathbb{T}) \backslash \bar{L},$$

$$C - chameleon = cham(\mathbb{T}) \backslash \bar{C}.$$

Let $\mathbb{B}_n$ be set of all black trees with n leafs, i.e. trees whose all vertices are black and similarly denote by $\mathbb{W}_n$ set of white trees and

$$\mathbb{K}_n \vee_\alpha \mathbb{K}_m = \{t' \vee_\alpha t'' | t' \in \mathbb{K}_n, t'' \in \mathbb{K}_m\}$$

where $\mathbb{K} = \mathbb{B}$ or $\mathbb{W}$.

$$\mathbb{B}_n \vee_\alpha \mathbb{W}_m = \{t' \vee_\alpha t'' | t' \in \mathbb{B}_n, t'' \in \mathbb{W}_m\}.$$

$$\mathbb{W}_n \vee_\alpha \mathbb{B}_m = \{t' \vee_\alpha t'' | t' \in \mathbb{W}_n, t'' \in \mathbb{B}_m\}.$$

$$\mathbb{W}_n \vee_\alpha \mathbb{W}_m = \{t' \vee_\alpha t'' | t' \in \mathbb{W}_n, t'' \in \mathbb{W}_m\}.$$

For convenience let write basis trees of central associative 0-dialgebra by other way of degree m:

$$C - chameleon = \{\bigcup (\ldots(\mathbb{B}_a \vee_\circ \mathbb{W}_{k_1}) \ldots \vee_\circ \mathbb{W}_{k_n})\}$$

where $a + k_1 + \ldots + k_n = m$,$a \neq 1$ and $\mathbb{B}_i, \mathbb{W}_j$ satisfies to $left$ and $right\ stairs$.

For $lass$, $rass$ equations (1.3) and (1.5) are means associativity for a one multiplication. So $\mathbb{B}_n$ and $\mathbb{W}_n$ in this algebras we consider in the forms of one tree i.e.

$$f(\mathbb{B}_n) = (\ldots((a_1 \vdash a_2) \vdash a_3) \ldots \vdash a_{n-1}) \vdash a_n$$

$$f(\mathbb{W}_n) = a_n \dashv (a_{n-1} \dashv \ldots (a_3 \dashv (a_2 \dashv a_1)) \ldots)$$

correspondingly.

$\bar{L}$ set means that subset of trees $cham(\mathbb{T})$ such that right son of any black vertex is black i.e. contains all trees with subtrees of the form

$$\ldots(\mathbb{B}_k \vee_\bullet (\mathbb{B}_l \vee_\bullet (\ldots))).$$

Analogically $\bar{R}$ set means that subset of trees $cham(\mathbb{T})$ such that left son of any white vertex is white i.e. contain all trees with subtrees of the form

$$\ldots(((\ldots) \vee_\circ \mathbb{W}_k) \vee_\circ \mathbb{W}_l).$$

$\bar{C}$ set is the subset of $cham(\mathbb{T})$ such that right son of any black vertex is white i.e. contains all trees with subtrees of the form

$$\ldots(\mathbb{B}_k \vee_\bullet ((\ldots) \vee_\circ \mathbb{W}_l)).$$

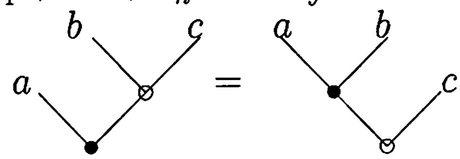
Example. Let construct base of *cass* for degree 3 and 4. By (4) for degree 3, 4 is constructed above i.e. $\mathbb{B}_3$, $\mathbb{B}_2 \vee_\circ \mathbb{W}_1$,$\mathbb{W}_3$.

For degree 4 is $\mathbb{B}_4$,$\mathbb{B}_3 \vee_\circ \mathbb{W}_1$,$(\mathbb{B}_2 \vee_\circ \mathbb{W}_1) \vee_\circ \mathbb{W}_1$,$\mathbb{B}_2 \vee_\circ \mathbb{W}_2$,$\mathbb{W}_4$.

Firstly let prove it for free $0-dialgebra$. In **section 4** we constructed *di* free $0-dialgebra$ and base of this dialgebra is $cham(\mathbb{T})$. In (7) there shown that we can vanish all trees which is not satisfy chameleon trees. So base of *di* and 0-dialgebra coincides. So basis of $0-dialgebra$ is $cham(\mathbb{T})$. By using linear map $f(cham(\mathbb{T}))$ there can be obtained basis of $0-dialgebra$. □

By (8) we know that cass, lass, rass satisfies to the $left$ and $right\ stairs$.

Let firstly consider *cenral associative* $0-dialgebra$. Then by construction which is given in (9) any tree of weight (a, b) defined uniquely and by using (1.4) we can change any element of weight (a, b) to $(\ldots(\mathbb{B}_{a+1} \vee_\circ \mathbb{W}_{k_1})\ldots\vee_\circ \mathbb{W}_{k_n})$ where $k_1 + \ldots + k_n = b$. By the other words (1.4) in terminology of trees means that:



where tree c is contain only white vertices and $c = \mathbb{W}_{k_n}$. If tree b contain white vertices then like above make the same operation by using (1.4) and there can be obtained the form $(\ldots((\mathbb{B}_{a+1}) \vee_\circ \mathbb{W}_{k_1})\ldots \vee_\circ \mathbb{W}_{k_n})$.

In **section 4** given *dic* free central associative 0−dialgebra and base of *dic* and free central associative 0−dialgebra are coincides. By using f we obtain base of $cass = f(cham(\mathrm{T})/\bar{C})$.

Let consider $left\ associative$ $0-dialgebra$. In this algebra all elements which are satisfy to $\bar{L}$ set can be written in this form $\mathbb{B}_k \vee_\bullet (\mathbb{B}_l \vee_\bullet (\ldots))$. By (1.3) we can write

$$\mathbb{B}_k \vee_\bullet (\mathbb{B}_l \vee_\bullet (\ldots)) = (\mathbb{B}_k \vee_\bullet \mathbb{B}_l) \vee_\bullet (\ldots) = \mathbb{B}_{k+l} \vee_\bullet (\ldots)$$

By this way we can vanish all elements which are satisfy to $\bar{L}$ set.

We constructed *dil* free left associative 0−dialgebra and and base of *dil* and free left associative 0−dialgebra are coincides. by using f we obtain base of $lass = f(cham(\mathtt{T})/\bar{L})$. Analogically it can be wrote for $rass$.$\square$

For prove (5) there known that dimension of free magma algebra

$$d_{l_1,\dots,l_n} = \frac{(2\sum_{i=1}^{n} l_i - 2)!}{(\sum_{i=1}^{n} l_i - 1)!\prod_{i=1}^{n} l_i}.$$

For free magma algebra any tree of degree n has the same dimension with the other tree of the same degree. We know that given identities of dialgebras do not change place of generators. There changed only brackets and multiplication. So dimensions of all colored trees are equal to each other. By multiplication of tree with one generator and divided to Catalan number there can be obtained dimension of any associative $0-dialgebra$.

In particular for $0-dialgebra$ dimension is

$$d_{l_1,\dots,l_n} = \frac{(2\sum_{i=1}^{n} l_i - 2)!}{(\sum_{i=1}^{n} l_i - 1)!\prod_{i=1}^{n} l_i}(l_1 + \ldots + l_n).\square$$

5. GD-algebras

5.1 Some notions of GD-algebras

It turned out [25] (see also [15]) that GD-algebras are closely related with conformal Lie algebras appeared in the theory of vertex operators [17]. Conformal algebras and their generalizations (pseudo-algebras) also turn to be useful for the classification of Poisson brackets of hydrodynamic type [5]. It worth mentioning that Poisson algebras are also closely related with representations of Lie conformal algebras [18].

The following statement provides a series of examples of GD-algebras. Recall that a Poisson algebra is a linear space V equipped with two bilinear operations, i.e., a system $(V, \cdot, \{\cdot,\cdot\})$, where $(V, \cdot)$ is an associative and commutative algebra, $(V, \{\cdot,\cdot\})$ is a Lie algebra, and the following *Leibniz identity* holds:

$$\{x, yz\} = \{x, y\}z + y\{x, z\}.$$

Theorem 6. *Let V be a Poisson algebra with a derivation $d \in \operatorname{Der} V$ relative to the both products. Define a new operation $\circ$ on V in the following way:*

$$x \circ y = xd(y). \tag{5.1}$$

Then $(V, \circ, \{\cdot,\cdot\})$ is a GD-algebra.

The proof is straightforward. It is well-known [6, 13] that $x\circ y = xd(y)$ satisfies (1.6) and (1.7). It remains to show that (1.8) holds. Indeed, evaluate the left-hand side of (1.8) in a differential Poisson algebra: $[u, w\circ v]-[v, w\circ u]+[w, u]\circ v-[w, v]\circ u-w\circ[u, v] = \{u, wd(v)\}-\{v, wd(u)\}+\{w, u\}d(v)-\{w, v\}d(u)-wd(\{u, v\}) = \{u, w\}d(v) + \{u, d(v)\}w - \{v, w\}d(u) - \{v, d(u)\}w + \{w, u\}d(v) - \{w, v\}d(u) - w\{d(u), v\} - w\{u, d(v)\} = 0.$ □

Let us call a GD-algebra A *special* if it can be embedded into a differential Poisson algebra with operations $\{\cdot,\cdot\}$ and $\circ$ given by (5.1). The class of all homomorphic images of all special GD-algebras is a variety (i.e., a class of algebras which is closed with respect to homomorphic images, subalgebras, and Cartesian products). Let us denote this variety by SGD. The main purpose of this paper is to describe a way for finding the defining identities of SGD.

In order to solve that problem, we need to study differential algebras and answer the following question which is of independent interest. Suppose A is a (non-associative) algebra with a derivation d. Introduce the following new binary operations on A:

$$a \succ b = d(a)b, \quad a \prec b = ad(b), \quad a, b \in A. \tag{5.2}$$

Assuming A satisfies a family of polynomial identities, what can be said about identities on $A^{(d)} = (A, \succ, \prec)$?

For example, if A is associative and commutative then $a \succ b = b \prec a$ and the operation $\prec$ satisfies (1.6) and (1.7). It was shown in [11] that there are no more independent identities that hold on $A^{(d)}$ for all associative commutative algebras A and for all their derivations d. Moreover, it was proved in [8] that every Novikov algebra may be embedded into a differential associative commutative algebra.

We find an explicit answer to the question about identities of $A^{(d)}$. It turns out that, given a variety Var defined by multilinear identities (multilinear variety, for short), the systems $A^{(d)}$, $A \in \mathrm{Var}$, $d \in \mathrm{Der}(A)$, generate a variety governed by the operad $\mathrm{Var} \circ \mathrm{Nov}$, where Nov is the operad of Novikov algebras and $\circ$ stands for the Manin white product of operads [14, 20]. Hereinafter we identify notations for multilinear varieties and the corresponding operads.

If Var is a variety of algebras then the free algebra in Var generated by a set X is denoted $\mathrm{Var}\langle X \rangle$. By Var Der we denote the class of all differential Var-algebras with one derivation (this is a variety of algebraic systems with one additional unary operation d in the language). The free algebra in Var Der generated by a set X is denoted $\mathrm{Var\,Der}\langle X, d \rangle$. We will often need to know the structure of the free differential Var-algebra. It can be easily derived in general (c.f. [12] for associative algebra case) that the following statement holds.

Lemma 10. *For a multilinear variety* Var*, the free differential* Var*-algebra gen-*

erated by a set X with one derivation d coincides with $\mathrm{Var}\langle d^\omega X\rangle$*, where* $d^\omega X = \{d^s(x) \mid s \geq 0, x \in X\}$.

Indeed, consider the free nonassociative (magmatic) algebra $M\langle d^\omega X\rangle$ and define a linear map $d : F(X;d) \to F(X;d)$ in such a way that $d(d^s(x)) = d^{s+1}(x)$, $d(uv) = d(u)v + ud(v)$. Denote by $T_{\mathrm{Var}}(X)$ the T-ideal of all identities in a set of variables X that hold on Var.

Since Var is defined by multilinear identities, the T-ideal $T_{\mathrm{Var}}(d^\omega X)$ of $M\langle d^\omega X\rangle$ is d-invariant:

$$d(f(d^{s_1}(x_1),\dots,d^{s_n}(x_n))=\sum_{i=1}^{n} f(d^{s_1}(x_1),\dots,d^{s_i+1}(x_i),\dots,d^{s_n}(x_n)).$$

Hence, $U = F(X;d)/T_{\mathrm{Var}}(d^\omega X)$ is a Var-algebra with a derivation d. Since all relations from $T_{\mathrm{Var}}(d^\omega X)$ hold in every differential Var-algebra, U is isomorphic to the free differential Var-algebra. □

Remark 1. In Lemma 10, a derivation d may be replaced with a *generalized derivation*, i.e., a linear map D such that

$$D(xy) = D(x)y + xD(y) + \lambda xy,$$

where λ is a fixed scalar in $\Bbbk$. Indeed, it is enough to note that $d = D - \lambda\mathrm{id}$ is an ordinary derivation.

In order to study the relations between GD-algebras and differential Poisson algebras we need more general settings. Namely, given an algebra A with a derivation d, consider the system $A^{(d*)} = (A, \succ, \prec, \cdot)$, where $\succ$, $\prec$ are defined by (5.2), and $\cdot$ is the initial multiplication on A. How to describe the identities that hold on $A^{(d*)}$ provided that we know the identities of A? It turns out that the answer involves the operad GD of GD-algebras. Namely, for a multilinear variety Var, the systems $A^{(d*)}$, $A \in \mathrm{Var}$, $d \in \mathrm{Der}(A)$, generate a variety governed by $\mathrm{Var} \circ \mathrm{GD}^!$, where $\mathrm{GD}^!$ is the Koszul-dual of the operad GD.

The paper is organized as follows. In Section 5.2, we calculate the operad $\mathrm{GD}^!$ and describe the free $\mathrm{GD}^!$-algebra. In fact, the variety of $\mathrm{GD}^!$-algebras is a proper sub-variety of the class of Novikov–Poisson algebras. The defining identities of $\mathrm{GD}^!$ were earlier obtained in [8], but it was assumed in [8] that a Novikov–Poisson algebra has an identity element. We can not use this assumption for our purpose, so we describe the free $\mathrm{GD}^!$-algebra explicitly and prove that it can be embedded into the free differential associative and commutative algebra.

In Section 5.3, we define (generalized) derived identities of a multilinear variety Var of algebras with binary operations as those identities that hold on all $A^{(d)}$ (resp., $A^{(d*)}$) for all $A \in \mathrm{Var}$ and for all $d \in \mathrm{Der}(A)$. Denote the operads defined by (generalized) derived identities by $D\,\mathrm{Var}$ (resp., $D^*\,\mathrm{Var}$). Then we prove

$$D\,\mathrm{Var} = \mathrm{Var} \circ \mathrm{Nov}, \quad D^*\,\mathrm{Var} = \mathrm{Var} \circ \mathrm{GD}^!.$$

In Section 5.4, we describe the linear basis of the free special GD-algebra $\mathrm{SGD}\langle X \rangle$ in terms of the enveloping differential Poisson algebra. This result generalizes what was done in [11]: here we have to add one more operation $[\cdot,\cdot]$ into consideration.

Section 5.5 is devoted to the description of defining relations of the operad SGD governing the variety generated by all special GD-algebras. It is easy to see that the operad SGD is a sub-operad of $D^*\mathrm{Pois}$ generated by a proper subspace of binary operations. We find the entire family of defining relations for $D^*\mathrm{Pois}$, so all defining identities of SGD occur among their corollaries. In particular, we are interested in *special identities* of GD-algebras, i.e., those identities that hold on all SGD-algebras but do not hold on the class of all GD-algebras. There are no special identities of degree 3, but we can find two independent special identities of degree 4. The complete list of special identities of GD-algebras remains unknown.

5.2 Dual Gelfand—Dorfmann algebras

Suppose Var is a multilinear variety of algebras. Fix a countable set of variables $X = \{x_1, x_2, \dots\}$ and denote

$$\mathrm{Var}(n) = M_n(X)/M_n(X) \cap T_{\mathrm{Var}}(X),$$

where $M_n(X)$ is the space of all (non-associative) multilinear polynomials of degree n in $x_1, \dots, x_n$, $T_{\mathrm{Var}}(X)$ is the T-ideal of all identities that hold in Var.

The collection of spaces $(\mathrm{Var}(n))_{n\geq 1}$ forms a symmetric operad relative to the natural composition rule and symmetric group action (see, e.g., [20]). We will denote this operad by the same symbol Var. The operad Var is a *binary* one, i.e., it is generated (as a symmetric operad) by the elements of $\mathrm{Var}(2)$. In the generic case of algebras with one binary operation of multiplication, $\mathrm{Var}(2)$ is spanned by x_1x_2 and x_2x_1.

The variety of GD-algebras is defined by identities of degree 2 or 3. Therefore, the corresponding operad is a *quadratic* one [14]. Let us compute the Koszul dual operad $\mathrm{GD}^!$.

Denote two operations of a GD-algebra by μ and ν, namely, $\mu = [x_1, x_2]$, $\nu = x_1 \circ x_2$. Then $\mathrm{GD}(2) = \mathrm{Span}\{\mu, \nu, \nu^{(12)}\}$, where $\nu_2^{(12)} = x_2 \circ x_1$.

As in [14], the S_3-space of multilinear terms of degree 3 is identified with

$$O_3 = \Bbbk S_3 \otimes_{\Bbbk S_2} (\mathrm{GD}(2) \otimes \mathrm{GD}(2)),$$

where the action of $(12) \in S_2$ on $\mathrm{GD}(2)^{\otimes 2}$ is given by $\mathrm{id} \otimes (12)$.

We may choose a basis $a_1, \dots, a_{27}$ of O_3 as follows:

$$\begin{aligned}
a_1 &= 1 \otimes_{\Bbbk S_2} (\mu \otimes \mu) = [[x_1, x_2], x_3], & a_2 &= 1 \otimes_{\Bbbk S_2} (\mu \otimes \nu) = [x_1 \circ x_2, x_3], \\
a_3 &= 1 \otimes_{\Bbbk S_2} \left(\mu \otimes \nu^{(12)}\right) = [x_2 \circ x_1, x_3], & a_4 &= 1 \otimes_{\Bbbk S_2} (\nu \otimes \mu) = [x_1, x_2] \circ x_3, \\
a_5 &= 1 \otimes_{\Bbbk S_2} (\nu \otimes \nu) = (x_1 \circ x_2) \circ x_3, & a_6 &= 1 \otimes_{\Bbbk S_2} \left(\nu \otimes \nu^{(12)}\right) = (x_2 \circ x_1) \circ x_3, \\
a_7 &= 1 \otimes_{\Bbbk S_2} \left(\nu^{(12)} \otimes \mu\right) = x_3 \circ [x_1, x_2], & a_8 &= 1 \otimes_{\Bbbk S_2} \left(\nu^{(12)} \otimes \nu\right) = x_3 \circ (x_1 \circ x_2), \\
a_9 &= 1 \otimes_{\Bbbk S_2} \left(\nu^{(12)} \otimes \nu^{(12)}\right) = x_3 \circ (x_2 \circ x_1), & &
\end{aligned}$$

and $a_{9+i} = (13)a_i$, $a_{18+i} = (23)a_i$ for $i = 1, \dots, 9$.

Relative to the basis $a_1, \dots, a_{27}$, the vectors representing the defining relations of GD may be presented by the rows of the following matrix:

$$\begin{array}{ccccccccccccccccccccccccccc}
1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&1&-1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&1&-1&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&1&-1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&-1&0&-1&0&0&0&0&0\\
0&0&1&-1&0&0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&-1&1&0&0&0&0&0
\end{array}$$

Compute the sign-twisted orthogonal complement $U^\perp \subset O_3^\vee$ [14] for the S_3-module U spanned by these vectors to obtain the defining relations of the Koszul dual operad $\mathrm{GD}^!$. It turns out that a $\mathrm{GD}^!$-algebra has two operations $\mu^\vee(x_1, x_2) =$

$x_1 * x_2$ and $\nu^\vee(x_1, x_2) = x_1 \star x_2$, where

$$x * y = y * x, \quad x * (y * z) = (x * y) * z, \tag{5.3}$$

i.e., $*$ is associative and commutative,

$$(x \star y) \star z - x \star (y \star z) = (x \star z) \star y - x \star (z \star y), \tag{5.4}$$

$$x \star (y \star z) = y \star (x \star z), \tag{5.5}$$

i.e., $\star$ is a right Novikov product, and

$$x \star (y * z) = (x \star y) * z, \tag{5.6}$$

$$x \star (y * z) + y \star (x * z) = (x * y) \star z. \tag{5.7}$$

Remark 2. Relations (5.3)–(5.7) up to a change of notations

$$xy = x * y, \quad x \circ y = y \star x$$

coincide with the identities defining a *differential GDN-Poisson algebra* [8]. This is a proper sub-variety of the variety of Novikov—Poisson algebras introduced in [24]. However, all algebras in [8] are assumed to be unital, i.e., with an identity e relative to the associative and commutative operation $*$. This is an essential restriction since it is unclear in general how to join a unit to a $\mathrm{GD}^!$-algebra. This is why in this section we have to study the free $\mathrm{GD}^!$-algebra without an identity assumption.

Let us find a normal form of elements in the free $\mathrm{GD}^!$-algebra, i.e., a linear basis of $\mathrm{GD}^!\langle X\rangle$.

Denote by BiCom the variety of algebras with two associative and commutative operations $*$ and $\odot$ such that

$$(x \odot y) * z = x \odot (y * z). \tag{5.8}$$

It is easy to see that the following identities hold in BiCom:

$$(x \odot y) * z = (x * y) \odot z, \quad x * (y \odot z) = y * (x \odot z). \tag{5.9}$$

Indeed, apply commutativity of $*$ and (5.8) to obtain

$$(x \odot y) * z = (y \odot x) * z = y \odot (x * z) = y \odot (z * x) = (y \odot z) * x$$
$$= (z \odot y) * x = z \odot (y * x) = (x * y) \odot z.$$

Similarly,

$$x * (y \odot z) = (z \odot y) * x = z \odot (y * x) = z \odot (x * y) = (z \odot x) * y = y * (x \odot z).$$

Theorem 7. *A linear basis of the free* BiCom*-algebra* BiCom$\langle X \rangle$ *generated by a linearly ordered set X consists of the words*

$$x_1 * \cdots * x_t * (y_1 \odot \cdots \odot y_k), \quad x_1 \leq \cdots \leq x_t \leq y_1 \leq \cdots \leq y_k, \tag{5.10}$$

$x_i, y_j \in X$.

Proof. It can be easily seen that every element of a BiCom algebra generated by a set X can be transformed to a linear combination of words that have the following form:

$$x_{i_1} * x_{i_2} * \ldots * x_{i_t} * (y_{j_1} \odot \ldots \odot y_{j_k}), \quad x_i, y_j \in X.$$

Commutativity of $*$ and $\odot$ allows us to assume $x_{i_1} \leq \cdots \leq x_{i_t}$ and $y_{j_1} \leq \cdots \leq y_{j_k}$. If $x_{i_t} > y_{j_1}$ then we may interchange them by means of the second relation in (5.9). Hence, every element of BiCom can be transformed to a linear combination of monomials (5.10).

To prove linear independence of (5.10), consider a linear space B' spanned by (5.10) and define multiplications $*$ and $\odot$ on B' in the following way. If $u = x_1 * \ldots * x_n * (y_1 \odot \ldots \odot y_m)$, $v = z_1 * \ldots * z_p * (t_1 \odot \ldots \odot t_q)$ then

$$u * v = l_1 * \ldots * l_{n+p+1} * (l_{n+p+2} \odot \ldots \odot l_{n+p+m+q})$$

$$u \odot v = l_1 * \ldots * l_{n+p} * (l_{n+p+1} \odot \ldots \odot l_{n+p+m+q})$$

where the sequence $(l_1, \ldots, l_{n+p+m+q})$ is just the ordering of the sequence $(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_p, t_1, \ldots, t_q)$. It is easy to check that B' is a BiCom algebra. Hence, $B' \simeq$ BiCom$\langle X \rangle$. $\square$

Let $\mathrm{BiComDer}\langle X, d\rangle$ stand for the free differential BiCom-algebra generated by a set X with a derivation d (i.e., d is a derivation relative to the both operations $*$ and $\odot$).

Theorem 8. *Let A be a* BiCom*-algebra with a derivation d. Then A is a* $\mathrm{GD}^!$*-algebra relative to the operations $*$ and $\star$ defined by*

$$x \star y = d(x) \odot y.$$

Proof. The operation $x \star y = d(x) \odot y$ is known to be right Novikov. It remains to check the identities (5.6), (5.7):

$$x \star (y * z) - (x \star y) * z = d(x) \odot (y * z) - (d(x) \odot y) * z = 0,$$

$$\begin{aligned} x\star(y*z)+y\star(x*z) &= d(x)\odot(y*z)+d(y)\odot(x*z) = (d(x)\odot y)*z+(d(y)\odot x)*z \\ &= (d(x) * y) \odot z + (d(y) * x) \odot z = d(x * y) \odot z = (x * y) \star z \end{aligned}$$

by (5.9). $\square$

Let us say that a $\mathrm{GD}^!$-algebra B is *special* if it can be embedded into an appropriate differential BiCom-algebra. Denote by $\mathrm{SGD}^!$ the variety generated by special $\mathrm{GD}^!$-algebras (all homomorphic images of special $\mathrm{GD}^!$-algebras). Obviously, the free $\mathrm{SGD}^!$-algebra $\mathrm{SGD}^!\langle X\rangle$ generated by a set X is isomorphic to the subalgebra of $\mathrm{BiComDer}\langle X, d\rangle$ generated by X relative to the operations $\star$ and $*$.

For every monomial $u \in \mathrm{BiComDer}\langle X, d\rangle$ define the weight of u by the following induction rule:

$$\begin{aligned} \mathrm{wt}(x) &= -1, \quad x \in X; \\ \mathrm{wt}(d(u)) &= \mathrm{wt}(u) + 1; \\ \mathrm{wt}(u * v) &= \mathrm{wt}(u) + \mathrm{wt}(v) + 1; \\ \mathrm{wt}(u \odot v) &= \mathrm{wt}(u) + \mathrm{wt}(v). \end{aligned}$$

The space of differential BiCom-monomials of weight -1 has the following basis:

$$\begin{gathered}
x_1 * \cdots * x_t * (y_1 \odot \cdots \odot y_i \odot d^{r_{i+1}}(y_{i+1}) \odot \cdots \odot d^{r_k}(y_k)), \\
x_1 \leq \cdots \leq x_t \leq y_1 \leq \cdots \leq y_i, \\
r_{i+1}, \ldots, r_k > 0, \quad r_{i+1} + \cdots + r_k - k = -1, \\
(r_{i+1}, y_{i+1}) \leq \cdots \leq (r_k, y_k) \text{ lexicographically,}
\end{gathered} \tag{5.11}$$

Theorem 9. *An element $f \in \mathrm{BiComDer}\langle X, d\rangle$ belongs to $\mathrm{SGD}^!\langle X\rangle$ if and only if all monomials that appear in f have weight -1.*

Proof. The "only if" part can be easily proved by induction. For the converse ("if") part, it is enough to consider the case when f is a monomial of the form (5.11). Note that if $\mathrm{wt}(f) = -1$ then $f = x_1 * \cdots * x_t * w$, where $w = y_1 \odot \cdots \odot y_i \odot d^{r_{i+1}}(y_{i+1}) \odot \cdots \odot d^{r_k}(y_k)$, $\mathrm{wt}(w) = -1$. As shown in [11], w can be expressed via $\star$. Then $f \in \mathrm{SGD}^!\langle X\rangle$. $\square$

Theorem 10. *For a linearly ordered set X, $\mathrm{SGD}^!\langle X\rangle$ is isomorphic to $\mathrm{GD}^!\langle X\rangle$.*

Proof. First, let us prove the following technical statement.

Lemma 11. *The following elements span the free $\mathrm{GD}^!$-algebra $\mathrm{GD}^!\langle X\rangle$ generated by a linearly ordered set X:*

$$x_1 * \cdots * x_t * w, \quad x_1 \leq \cdots \leq x_t, \; x_i \in X,$$

where w is a monomial from $\mathrm{RNov}\langle X\rangle$ relative to the operation $\star$.

Here RNov denotes the variety of right Novikov algebras defined by the identities (5.4) and (5.5).

Proof. Let f be a monomial in $\mathrm{GD}^!\langle X\rangle$. Proceed by induction on $n = \deg f$. If $n = 1, 2$ then the statement is clear. Suppose $n \geq 3$. Then $f = u * v$ or $f = u \star v$, $\deg u, \deg v < n$. By the inductive assumption,

$$u = x_1 * \ldots x_r * w_1, \quad v = y_1 * \ldots y_s * w_2,$$

where $w_1, w_2 \in \mathrm{RNov}\langle X\rangle$.

Case 1: $f = u * v$. Then $f = x_1 * \cdots * x_r * y_1 * \cdots * y_s * (w_1 * w_2)$. If $\deg w_1 = 1$ or $\deg w_2 = 1$ then we are done. If $w_i = (w_{i1} \star w_{i2})$, $i = 1, 2$, then the inductive assumption implies

$$(w_{11} \star w_{12}) * (w_{21} \star w_{22}) = w_{11} \star (w_{12} * (w_{21} \star w_{22})) = w_{11} \star (z * w') = z * (w_{11} \star w'),$$

where $z \in X$, $w' \in \mathrm{RNov}\langle X\rangle$. Hence, $f = x_1 * \cdots * x_r * y_1 * \cdots * y_s * z * (w_{11} \star w')$.

Case 2: $f = u \star v$. Then $f = (x_1 * \cdots * x_r * w_1) \star (y_1 * \cdots * y_s * w_2)$. If $\deg u = 1$ then the statement follows from (5.6) and commutativity of $*$:

$$f = x_1 \star (y_1 * \cdots * y_s * w_2) = y_1 * \cdots * y_r * (x_1 \star w_2).$$

For $\deg u > 1$, we have to consider the following sub-cases: $r = 0$ and $r > 0$. The first one is similar to what is done for $\deg u = 1$, in the second we have $u = x_1 * u'$, so by (5.7)

$$f = (x_1 * u') \star (y_1 * \cdots * y_s * w_2) = ((x_1 \star u') + (u' \star x_1)) * (y_1 * \cdots * y_s * w_2).$$

The latter expression is of the type considered in Case 1. $\square$

Suppose X is linearly ordered. Consider the epimorphism

$$\varphi : \mathrm{GD}^!\langle X\rangle \to \mathrm{SGD}^!\langle X\rangle \subset \mathrm{BiComDer}\langle X, d\rangle$$

given by $\varphi(x) = x$ for $x \in X$, $\varphi(u \star v) = d(\varphi(u)) \odot \varphi(v)$, $\varphi(u * v) = \varphi(u) * \varphi(v)$.

Recall that elements from $\mathrm{RNov}\langle X\rangle$ may be presented as differential polynomials from the free differential commutative algebra $\mathrm{ComDer}\langle X, d\rangle$ (relative to binary operation $\odot$ and derivation d) by means of $x \star y = d(x) \odot y$. Such a presentation is known to be unique [11]. Therefore, by Lemma 11 $\mathrm{GD}^!\langle X\rangle$ is spanned by elements

$$u = x_1 * \cdots * x_t * (d^{s_1}(y_1) \odot \cdots \odot d^{s_k}(y_k)), \quad x_i \in X,\ y_j \in X, \tag{5.12}$$

where $x_1 \leq \cdots \leq x_t$, $s_1 + \cdots + s_k - k = -1$, and

$$(s_1, y_1) \leq \cdots \leq (s_k, y_k) \text{ lexicographically.}$$

Note that if u is of the form (5.12) then $\varphi(u) = u \in \mathrm{BiComDer}\langle X, d\rangle$. Moreover, $s_1 = 0$.

Relation (5.9) implies that we may add one more condition on the words (5.12): $x_t \leq y_1$.

Therefore, we have found a set of linear generators in $\mathrm{GD}^!\langle X\rangle$ whose images in $\mathrm{SGD}^!\langle X\rangle$ are linearly independent. Hence, $\mathrm{GD}^!\langle X\rangle \simeq \mathrm{SGD}^!\langle X\rangle$. In particular, the set of monomials (5.11) is a linear basis of $\mathrm{GD}^!\langle X\rangle$. □

Consider the class SNP of all algebras with two operations $*$ and $\star$ obtained from differential associative commutative algebras via

$$a * b = ab, \quad a \star b = d(a)b. \tag{5.13}$$

Corollary 4. *The class of all homomorphic images of algebras from* SNP *coincides with the variety of* $\mathrm{GD}^!$*-algebras.*

In other words, the defining identities of $\mathrm{GD}^!$ and their corollaries exhaust all identities that hold for operations (5.13) on commutative differential algebras.

Proof. Consider the homomorphism of $\mathrm{GD}^!$-algebras

$$\psi : \mathrm{GD}^!\langle X\rangle \to \mathrm{ComDer}\langle X, d\rangle$$

defined as follows: $\psi(x) = x$ for $x \in X$, $\psi(f * g) = \psi(f)\psi(g)$, $\psi(f \star g) = d(\psi(f))\psi(g)$. We are interested in the kernel of ψ.

By Theorem 10, $\mathrm{GD}^!\langle X\rangle \subset \mathrm{BiComDer}\langle X, d\rangle$. Therefore, ψ is the restriction of the following homomorphism of differential BiCom-algebras:

$$\varphi : \mathrm{BiComDer}\langle X, d\rangle \to \mathrm{ComDer}\langle X, d\rangle,$$

$\varphi(x) = x$ for $x \in X$, $\varphi(f * g) = \varphi(f \odot g) = \varphi(f)\varphi(g)$.

Suppose X is equipped with a linear order and w is a monomial in $\mathrm{BiComDer}\langle X, d\rangle$ of the form (5.11). Then

$$\varphi(w) = x_1 \dots x_t y_1 \dots y_i d^{r_{i+1}}(y_{i+1}) \dots d^{r_k}(y_k),$$

where $x_1 \leq \dots \leq x_t \leq y_1 \leq \dots \leq y_i$, $r_{i+1} > 0$, $(r_{i+1}, y_{i+1}) \leq \dots \leq (r_k, y_k)$ lexicographically. Therefore, the images of the basic monomials of $\mathrm{GD}^!\langle X\rangle$ are linearly independent, so

$$\operatorname{Ker}\psi = GD^!\langle X\rangle \cap \operatorname{Ker}\varphi = \{0\}.$$

□

In particular, we may find a simple form for a basis of the free algebra $\mathrm{GD}^!\langle X\rangle$.

Let w be a monomial in $\mathrm{ComDer}\langle X, d\rangle$. Define the weight of w by induction as follows: $\mathrm{wt}(x) = -1$, $x \in X$; $\mathrm{wt}(uv) = \mathrm{wt}(u) + \mathrm{wt}(v)$; $\mathrm{wt}(d(u)) = \mathrm{wt}(u) + 1$.

Corollary 5. *A linear basis of* $\mathrm{GD}^!\langle X\rangle$ *consists of all monomials in* $\mathrm{ComDer}\langle X, d\rangle$ *of weight* ≤ -1.

Remark 3. For unital $\mathrm{GD}^!$-algebras, a more precise result was obtained in [8]: if V is a $\mathrm{GD}^!$-algebra with an identity e relative to the operation $*$ then V coincides with an associative commutative algebra A with a derivation $d \in \mathrm{Der}(A)$ equipped with the operations $x * y = xy$, $x \star y = d(x)y$.

5.3 Derived identities

Let $\mathcal{P}$ and $\mathcal{Q}$ be two operads. Then the family of spaces $(\mathcal{P}(n) \otimes \mathcal{Q}(n))_{n\geq 1}$ is an operad relative to the componentwise composition and symmetric group action. This operad, denoted $\mathcal{P} \otimes \mathcal{Q}$, is known as the *Hadamard product* of $\mathcal{P}$ and $\mathcal{Q}$. If $\mathcal{P}$ and $\mathcal{Q}$ are binary operads then $\mathcal{P} \otimes \mathcal{Q}$ may be non-binary. The sub-operad of $\mathcal{P} \otimes \mathcal{Q}$ generated by $\mathcal{P}(2) \otimes \mathcal{Q}(2)$ is known as the *Manin white product* of $\mathcal{P}$ and $\mathcal{Q}$, it is denoted $\mathcal{P} \circ \mathcal{Q}$ [14].

Example 1. The operad Lie $\circ$ Nov is isomorphic to the operad governing the class of all algebras (*magmatic* operad).

Indeed, both Lie and Nov are quadratic operads, and so is Lie $\circ$ Nov [14]. Let $[x_1, x_2]$ and $x_1 \circ x_2$ be the generators of Lie and Nov. It is enough to find the defining identities of Lie $\circ$ Nov that are quadratic with respect to the generators.

Identify $[x_1, x_2] \otimes (x_1 \circ x_2)$ with $[x_1 \prec x_2]$, then $[x_1, x_2] \otimes (x_2 \circ x_1) = ([x_2, x_1] \otimes (x_1 \circ x_2))^{(12)} = -[x_2 \prec x_1]$. Hence, (Lie $\circ$ Nov)(n) is an image of the space $M_n(X)$ of all multilinear non-associative polynomials of degree n in

$X = \{x_1, x_2, \dots\}$ relative to the operation $[\cdot \prec \cdot]$. Calculating the compositions in the componentwise way, we obtain

$$m_1 = [[x_1 \prec x_2] \prec x_3] = [[x_1x_2]x_3] \otimes (x_1 \circ x_2) \circ x_3,$$
$$m_2 = [x_1 \prec [x_2 \prec x_3]] = [x_1[x_2x_3]] \otimes x_1 \circ (x_2 \circ x_3),$$

It remains to find the intersection of the S_3-submodule generated by m_1 and m_2 in $M_3(X) \otimes M_3(X)$ with the kernel of the projection $M_3(X) \otimes M_3(X) \to \mathrm{Lie}(3)\otimes\mathrm{Nov}(3)$. Straightforward calculation shows the intersection is zero. Hence, the operation $[\cdot \prec \cdot]$ satisfies no identities.

Example 2. The operad As $\circ$Nov is generated by 4-dimensional space (As $\circ$Nov)(2) spanned by

$$(x_1 \prec x_2) = x_1x_2 \otimes x_1 \circ x_2, \quad (x_2 \prec x_1) \quad = x_2x_1 \otimes x_2 \circ x_1,$$
$$(x_1 \succ x_2), = x_1x_2 \otimes x_2 \circ x_1, \quad (x_2 \succ x_1), \quad = x_2x_1 \otimes x_1 \circ x_2,$$

relative to the following identities:

$$(x_1 \succ x_2) \prec x_3 - x_1 \succ (x_2 \prec x_3) = 0, \tag{5.14}$$
$$(x_1 \prec x_2) \prec x_3 - x_1 \prec (x_2 \succ x_3) + (x_1 \prec x_2) \succ x_3 - x_1 \succ (x_2 \succ x_3) = 0. \tag{5.15}$$

This result may be checked with a straightforward computation. Namely, one has to find the intersection of the S_3-submodule in $M_3(X) \otimes M_3(X)$ generated by

$$(x_1 \prec x_2) \prec x_3 = (x_1x_2)x_3 \otimes (x_1x_2)x_3, \quad x_1 \succ (x_2 \succ x_3) = x_1(x_2x_3) \otimes (x_3x_2)x_1,$$
$$(x_1 \succ x_2) \prec x_3 = (x_1x_2)x_3 \otimes (x_2x_1)x_3, \quad x_1 \succ (x_2 \prec x_3) = x_1(x_2x_3) \otimes (x_2x_3)x_1,$$
$$(x_1 \prec x_2) \succ x_3 = (x_1x_2)x_3 \otimes x_3(x_1x_2), \quad x_1 \prec (x_2 \succ x_3) = x_1(x_2x_3) \otimes x_1(x_3x_2),$$
$$x_1 \prec (x_2 \prec x_3) = x_1(x_2x_3) \otimes x_1(x_2x_3), \quad (x_1 \succ x_2) \succ x_3 = (x_1x_2)x_3 \otimes x_3(x_2x_1)$$

with the kernel of the projection $M_3(X) \otimes M_3(X) \to \mathrm{As}(3) \otimes \mathrm{Nov}(3)$.

Given a (non-associative) algebra A with a derivation d, denote by $A^{(d)}$ the same linear space A considered as a system with two binary linear operations of multiplication $\prec$ and $\succ$ defined by

$$x \prec y = xd(y), \quad x \succ y = d(x)y, \quad x, y \in A.$$

In the same settings, denote by A^{d*} the system $(A, \prec, \succ, \cdot)$, where $\cdot$ is the initial multiplication on A.

Definition 3. Let Var be a multilinear variety of algebras. A non-associative polynomial $f(x_1, \ldots, x_n)$ in two operations of multiplication $\prec$ and $\succ$ is called a *derived identity* of Var if for every $A \in \mathrm{Var}$ and for every derivation $d \in \mathrm{Der}(A)$ the algebra $A^{(d)}$ satisfies the identity $f(x_1, \ldots, x_n) = 0$. A *generalized derived identity* is a polynomial in three operations $\prec$, $\succ$, and $\cdot$ that turns into an identity on A^{d*} for every $A \in \mathrm{Var}$ and for every $d \in \mathrm{Der}(A)$.

Obviously, the set of all (generalized) derived identities is a T-ideal of the algebra of non-associative polynomials in two (three) operations.

For example, $(x \succ y) = (y \prec x)$ is a derived identity of Com. Moreover, the operation $(\cdot \prec \cdot)$ satisfies the axioms of Novikov algebras (1.6) and (1.7). It was actually shown in [11] that the entire T-ideal of derived identities of Com is generated by these identities. Similarly, Corollary 4 implies that the identities of a $\mathrm{GD}^!$-algebra (5.3)–(5.7) present the complete list of generalized derived identities of Com relative to the following change of notations:

$$x * y = xy, \quad x \star y = x \succ y = y \prec x.$$

Remark 4. For the variety of associative algebras, it was mentioned in [19] that (5.14) and (5.15) are derived identities of As.

Our aim in this section is to show that (generalized) derived identities of Var are exactly the same as defining relations of $\mathrm{Var} \circ \mathrm{Nov}$ ($\mathrm{Var} \circ \mathrm{GD}^!$). We will focus on the proof of the description of generalized derived identities.

Let $\mathcal{N}_{\mathrm{Var}}$ be the class of all differential Var-algebras with one locally nilpotent derivation, i.e., for every $A \in \mathcal{N}_{\mathrm{Var}}$ and for every $a \in A$ there exists $n \geq 1$ such that $d^n(a) = 0$.

Note that $\mathcal{N}_{\mathrm{Var}}$ and the entire class Var Der satisfy the same (differential) identities: we prove that within the following statement.

Lemma 12. *Suppose $f = f(x_1, \dots, x_n)$ is a multilinear identity that holds on $A^{(d*)}$ for all $A \in \mathcal{N}_{\mathrm{Var}}$. Then f is a generalized derived identity of* Var.

Proof. Consider the free differential Var-algebra U_n generated by the set $X = \{x_1, x_2, \dots\}$ with one derivation d modulo defining relations $d^n(x) = 0$, $x \in X$.

Then $U_n \in \mathcal{N}_{\mathrm{Var}}$. As a differential Var-algebra, U_n is a homomorphic image of the free magmatic algebra $M\langle d^\omega X\rangle$. Denote by I_n the kernel of the homomorphism $M\langle d^\omega X\rangle \to U_n$. As an ideal of $M\langle d^\omega X\rangle$, I_n is a sum of two ideals: $I_n = T_{\mathrm{Var}}(d^\omega X) + N_n$, where N_n is generated by $d^{n+t}(x)$, $x \in X$, $t \geq 0$. Note that the last relations form a d-invariant subset of $M\langle d^\omega X\rangle$, so the ideal N_n is d-invariant.

If a polynomial $F \in M\langle d^\omega X\rangle$ belongs to I_n and its degree in d is less than n then F belongs to $T_{\mathrm{Var}}(d^\omega X)$. If $f = f(x_1, \dots, x_n)$ is a multilinear identity in $U_n^{(d)}$ then the image F of f in $M\langle d^\omega X\rangle$ has degree $n-1$ in d, so $F \in T_{\mathrm{Var}}(d^\omega)$. Hence, f is a derived identity of Var. □

Theorem 11. *If a multilinear identity f holds on the variety governed by the operad* $\mathrm{Var} \circ \mathrm{GD}^!$ *then f is a generalized derived identity of* Var.

Proof. For every $A \in \mathcal{N}_{\mathrm{Var}}$ we may construct an algebra in the variety $\mathrm{Var} \circ \mathrm{GD}^!$ as follows. Consider the linear space H spanned by elements $x^{(n)}$, $n \geq 0$, with multiplication

$$x^{(n)} \star x^{(m)} = \binom{n+m-1}{m} x^{(n+m-1)}, \quad x^{(n)} * x^{(m)} = \binom{n+m}{m} x^{(n+m)}.$$

This is a $\mathrm{GD}^!$-algebra (in characteristic 0, this is just the ordinary polynomial algebra with $x^{(n)} = x^n/n!$). Denote $\hat{A} = A \otimes H$ and define operations $\prec$, $\succ$, and $\cdot$ on $\hat{A}$ by

$$(a \otimes f) \succ (b \otimes g) = ab \otimes (f \star g), \tag{5.16}$$

$$(a \otimes f) \prec (b \otimes g) = ab \otimes (g \star f), \tag{5.17}$$

$$(a \otimes f)(b \otimes g) = ab \otimes (f * g), \tag{5.18}$$

The algebra $\hat{A}$ obtained belongs to the variety governed by the operad $\mathrm{Var} \circ \mathrm{GD}^!$.

There is an injective map

$$\begin{aligned}\Phi : A &\to \hat{A},\\ a &\mapsto \sum_{s\geq 0} d^s(a)\otimes x^{(s)},\end{aligned} \tag{5.19}$$

which preserves operations $\prec$, $\succ$, and $\cdot$. Indeed, let us check that Φ preserves $\prec$. Apply (5.19) to $a \prec b$, $a, b \in A$:

$$(a \prec b) = ad(b) \mapsto \sum_{s\geq 0} d^s(ad(b)) \otimes x^{(s)} = \sum_{s,t\geq 0} \binom{s+t}{s} d^s(a)d^{t+1}(b)\otimes x^{(s+t)}$$

$$= \sum_{s,t\geq 0} d^s(a)d^{t+1}(b)\otimes x^{(s)}\cdot x^{(t+1)} = \left(\sum_{s\geq 0} d^s(a)\otimes x^{(s)}\right) \prec \left(\sum_{t\geq 0} d^t(a)\otimes x^{(t)}\right).$$

Since Φ is injective, $A^{(d*)}$ is isomorphic to a subalgebra of a $(\mathrm{Var}\circ \mathrm{GD}^!)$-algebra, so all defining relations of $\mathrm{Var}\circ\mathrm{GD}^!$ hold on $A^{(d)}$. Lemma 12 completes the proof. □

It is easy to see that if we forget about the operation $\cdot$ on $\hat{A}$ then we obtain a proof of the following

Theorem 12. *If a multilinear identity f holds on the variety governed by the operad* $\mathrm{Var}\circ\mathrm{Nov}$ *then f is a derived identity of* Var. □

It remains to prove the converse: why all generalized derived identities of Var hold on all $(\mathrm{Var}\circ\mathrm{GD}^!)$-algebras.

Theorem 13. *Let f be a multilinear generalized derived identity of* Var. *Then f holds on all* $(\mathrm{Var}\circ\mathrm{GD}^!)$*-algebras.*

Proof. Let $X = \{x_1, x_2, \dots\}$, and let $f(x_1,\dots,x_n)$ be a non-associative multilinear polynomial in three operations of multiplication $\cdot$, $\prec$, and $\succ$. By the construction of the Manin white product, if an identity holds on $\mathrm{Var}\langle X\rangle \otimes \mathrm{GD}^!\langle X\rangle$ equipped with operations (5.16)–(5.18) then it holds on all $(\mathrm{Var}\circ\mathrm{GD}^!)$-algebras.

By Corollary 5, $\mathrm{GD}^!\langle X\rangle$ is a subalgebra of $C^{(d*)}$ for the free differential commutative algebra $C = \mathrm{ComDer}\langle X, d\rangle$. Therefore, $\mathrm{Var}\langle X\rangle\otimes\mathrm{GD}^!\langle X\rangle$ is a subalgebra of $(\mathrm{Var}\langle X\rangle\otimes C)^{(D*)}$, where $D(u\otimes f) = u\otimes d(f)$ for $u\in\mathrm{Var}\langle X\rangle$, $f\in C$. Since $A = \mathrm{Var}\langle X\rangle\otimes C\in\mathrm{Var}$, the identity $f(x_1,\dots,x_n) = 0$ holds on $A^{(D*)}$, so it holds on $\mathrm{Var}\langle X\rangle\otimes\mathrm{GD}^!\langle X\rangle$. □

Similarly, we have

Theorem 14. *Let f be a multilinear derived identity of* Var. *Then f holds on all* (Var $\circ$Nov)*-algebras.*

Remark 5. In this section, we have considered algebras with one binary operation, so the operad Var has only one generator. However, Theorems 11–14 are straightforward to generalize for algebras with multiple binary operations, i.e., for an arbitrary binary operad Var. In the next sections, we apply these results to Poisson algebras.

5.4 Free special GD-algebra

Denote by Pois the variety of all Poisson algebras, and let PoisDer stand for the variety of differential Poisson algebras. Theorem 6 determines a functor from PoisDer to GD corresponding to the following morphism of operads:

$$\begin{aligned} \delta : \mathrm{GD} &\to \mathrm{PoisDer}, \\ x_1 \circ x_2 &\mapsto x_1 d(x_2), \\ [x_1, x_2] &\mapsto \{x_1, x_2\} \end{aligned}$$

Let us say that a GD-algebra A is *special* if there exists $V \in$ Pois such that A is a subalgebra of $V^{(\delta)}$. Denote by SGD the variety generated by special GD-algebras, i.e., the class of all homomorphic images of all special GD-algebras. The main purpose of this section is to describe the free SGD-algebra.

Let X be a linearly ordered set of generators. Denote by PoisDer$\langle X, d\rangle$ the free Poisson differential algebra with a derivation d. Consider the set $d^\omega X = \{d^n(x) \mid n \geq 0, x \in X\}$. Let us consider the elements $d^n(x) \in d^\omega X$ as pairs (n, x) and compare them lexicographically. Then PoisDer$\langle X, d\rangle$ is isomorphic to Pois$\langle d^\omega X\rangle$ as a Poisson algebra.

As a commutative algebra, PoisDer$\langle X, d\rangle$ is isomorphic to the symmetric algebra over the space Lie$\langle d^\omega X\rangle$. Recall that a linear basis of the free Lie algebra Lie$\langle Y\rangle$ generated by a linearly ordered set Y may be chosen as follows. An associative word $U \in Y^*$ is said to be a *Lyndon–Shirshov word* if for every presentation $U = U_1U_2$, $U_i \in Y^*$, we have $U > U_2U_1$ lexicographically. This definition goes back to [9] and [23] (see also [7]). For every Lyndon–Shirshov word U there is a unique (*canonical*) bracketing $[U]$ such that $[U] \in \mathrm{Lie}\langle Y\rangle$ and the set of all Lyndon–Shirshov words with canonical bracketing forms a linear basis of $\mathrm{Lie}\langle Y\rangle$.

Definition 4. Let u be a monomial in $\mathrm{PoisDer}\langle X, d\rangle$. Define the *weight* function $\mathrm{wt}(u) \in \mathbb{Z}$ by induction in the following way:

$$\begin{aligned} \mathrm{wt}(x) &= -1, \quad x \in X; \\ \mathrm{wt}(d(u)) &= \mathrm{wt}(u) + 1; \\ \mathrm{wt}(\{u, v\}) &= \mathrm{wt}(u) + \mathrm{wt}(v) + 1; \\ \mathrm{wt}(uv) &= \mathrm{wt}(u) + \mathrm{wt}(v). \end{aligned}$$

Since all defining identities of Poisson algebras are homogeneous relative to $\{\cdot, \cdot\}$ and $\cdot$, the function wt is well-defined.

By the general algebra arguments (see, e.g., [16]) $\mathrm{SGD}\langle X\rangle$ is isomorphic to the subalgebra of $\mathrm{PoisDer}\langle X, d\rangle$ generated by X relative to the operations $[u, v] = \{u, v\}$ and $u \circ v = ud(v)$.

Consider a linear map

$$\begin{gathered} \phi : \mathrm{GD}\langle X\rangle \to \mathrm{SGD}\langle X\rangle \subset \mathrm{PoisDer}\langle X, d\rangle, \\ \phi(x) = x, \quad x \in X, \\ \phi([u, v]) = \{\phi(u), \phi(v)\}, \quad \phi(u \circ v) = \phi(u)d(\phi(v)). \end{gathered}$$

By Theorem 6, ϕ is an epimorphism of GD-algebras.

For a monomial $u \in \mathrm{GD}\langle X\rangle$, denote by $m(u)$ the number of Novikov multiplications in u or, which is the same, the number of commutative multiplications in $\phi(u)$. For a monomial $a \in \mathrm{PoisDer}\langle X, d\rangle$, denote $D(a) = n$ the total number of derivations d in a. It is easy to see that $D(\phi(u)) = m(u)$.

Lemma 13. *Let* $a = x_1 \dots x_t[U] \in \mathrm{PoisDer}\langle X, d\rangle$, *where* $x_i \in X$, U *is a Lyndon–Shirshov word in* $d^\omega X$, $D([U]) = t$. *Then there exists* $f \in \mathrm{GD}\langle X\rangle$ *such that* $\phi(f) = a$.

Proof. Let us prove the statement by induction on $D([U])$ and $|U|$, where $|U|$ is the length of U as of a word in $d^\omega X$.

If $D([U]) = t = 0$ then the desired pre-image may be obtained just by replacing $\{\cdot, \cdot\}$ with $[\cdot, \cdot]$. We do not distinguish notations for $[U] \in \mathrm{Lie}\langle X\rangle \subset \mathrm{PoisDer}\langle X, d\rangle$ and its pre-image in $\mathrm{GD}\langle X\rangle$.

Suppose $D(U) = 1$. If $|U| = 1$ then the result is obvious. Let $|U| = 2$. Then

$$x_1\{d(y_1), y_2\} = \{x_1 d(y_1), y_2)\} - \{x_1, y_2\}d(y_1),$$

so $[x_1 \circ y_1, y_2] - [x_1, y_2] \circ y_1$ is a desired pre-image.

If $|U| = l > 2$ then $[U] = \{[d(y_1)u], [v]\}$ for some words u, $|u| = i < l-1$, and v, $|v| = j < l$, by the definition of a Lyndon–Shirshov word ($d(y_1)$ is the greatest letter in U). Assume there exists

$$F_{1,j}(x_1, [W]) \in \phi^{-1}(x_1[W])$$

for all Lyndon–Shirshov words W such that $D(W) = 1$, $|W| = j < l$. Then

$$x_1[U] = x_1\{[d(y_1)u], [v]\} = \{x_1[d(y_1)u], [v]\} - [d(y_1)u]\{x_1, [v]\}.$$

Therefore,

$$F_{1,l}(x_1, [U]) = [F_{1,i+1}(x_1, [d(y_1)u]), [v]] - F_{1,i+1}([x_1, [v]], [d(y_1)u])$$

is a desired pre-image of $x_1[U]$.

Suppose $D(U) = t > 1$ and $|U| = 1$, i.e., $U = d^t(y)$, $y \in X$. Then there exists a pre-image of $x_1 \dots x_t d^t(y)$,

$$F_{t,1}(x_1, \dots, x_t, d^t(y)) \in \mathrm{Nov}\langle X\rangle \subset \mathrm{GD}\langle X\rangle,$$

as shown in [11].

Assume $D(U) = t > 1$, $|U| = l > 1$, and

$$F_{i,j}(x_1, \dots, x_i, [W]) \in \phi^{-1}(x_1 \dots x_i[W])$$

exists for all Lyndon–Shirshov words W such that either $i = D(W) < t$ or $D(W) = t$, $j = |W| < l$. Since

$$[U] = \{[d^s(y)u], [v]\}, \quad 0 < s \le t, \; y \in X,$$

where

$$1 \leq D(d^s(y)u) = i \leq t, \quad D(v) = t - i < t, \quad |d^s(y)u| = j < l, \quad |v| = q < l,$$

we have

$$x_1 \dots x_t[U] = x_{i+1} \dots x_t\{x_1 \dots x_i[d^s(y)u], [v]\} - \sum_{k=1}^{i} x_1 \dots \hat{x}_k \dots x_t[d^s(y)u]\{x_k, [v]\},$$

where $\hat{x}_k$ means that the factor x_k is omitted. By the inductive assumption, there exist

$$P \in \phi^{-1}(x_1 \dots x_i[d^s(y)u]) = F_{i,j}(x_1, \dots, x_i, [d^s(y)u])$$

and

$$P_k \in \phi^{-1}(x_{i+1} \dots x_t\overline{\{x_k, [v]\}}),$$

where $\overline{f}$ stands for the linear combination of Lyndon–Shirshov words with canonical bracketing representing a Lie polynomial $f \in \mathrm{Lie}\langle d^\omega X\rangle$. Obviously, we may choose

$$\phi^{-1}(x_1 \dots x_t[U]) = F_{t-i,q+1}(x_{i+1}, \dots, x_t, \overline{\{z, [v]\}})|_{z=P} - \sum_{k=1}^{i} F_{t-i,q+1}(z, x_1, \dots, \hat{x}_k, \dots, x_i, [d^s(y)u])|_{z=P_k} \quad (5.20)$$

Here we consider the variable z in the right-hand side of (5.20) as a new one, and assume $z < x$ for all $x \in X$. □

Theorem 15. *Differential Poisson monomials of weight* -1 *span the special GD-algebra* $\mathrm{SGD}\langle X\rangle \subset \mathrm{PoisDer}\langle X, d\rangle$.

Proof. Obviously, $\phi(f)$ is a linear combination of differential Poisson monomials of weight -1 for every $f \in \mathrm{GD}\langle X\rangle$.

Conversely, let a be a monomial in $\mathrm{PoisDer}\langle X, d\rangle$ such that $\mathrm{wt}(a) = -1$. We have to show that there exists $\phi^{-1}(a) \in \mathrm{GD}\langle X\rangle$. It is enough to prove it for basic elements of $\mathrm{PoisDer}\langle X, d\rangle$, i.e., elements of the form

$$a = [X_1][X_2] \cdots [X_n], \tag{5.21}$$

where X_i are Lyndon–Shirshov words in $d^\omega X$, $[\cdot]$ denotes the canonical bracketing, $X_1 \le \cdots \le X_n$.

Let us proceed by induction on $D(a)$. If $D([X_i]) = k_i$ then $\mathrm{wt}([X_i]) = k_i - 1$, so $\mathrm{wt}([X_i]) \ge -1$. Note that $\mathrm{wt}(a) = \mathrm{wt}([X_1]) + \cdots + \mathrm{wt}([X_n]) = D(a) - n$. Hence, if $D(a) = 0$ then $n = 1$ and $a = [X_1]$ is a pure Lie element in X which belongs to $\mathrm{GD}\langle X \rangle$.

Assume that for some $m > 0$ all monomials $b \in \mathrm{PoisDer}\langle X, d\rangle$ such that $\mathrm{wt}(b) = -1$, $D(b) < m$ have pre-images in $\mathrm{GD}\langle X \rangle$. Suppose $D(a) = m$ then, without loss of generality, we may assume $D([X_1]) = k > 0$, $k \le m$. Then there must exist k differential-free factors, say $[X_2], \dots, [X_{k+1}]$ of weight -1. Lemma 13 allows us to replace $[X_1]\cdots[X_{k+1}]$ with a new variable y, $\mathrm{wt}(y) = -1$, find a pre-image $f(y)$ of $y[X_{k+2}]\cdots[X_n]) \in \mathrm{GD}\langle X \cup \{y\}\rangle$, and find the desired pre-image of a as a substitute $f(y)|_{y\in\phi^{-1}([X_1]\cdots[X_{k+1}])}$. □

In order to calculate $\dim \mathrm{SGD}(n)$, we need to count Poisson differential monomials in $x_1, \dots, x_n$ of weight -1. Let us introduce the following function on $n, k \in \mathbb{Z}$, $n, k \ge 1$:

$$H(n,1) = 1,$$
$$H(n,k) = \sum_{1\le d_1<\cdots<d_{k-1}\le n-1} \frac{1}{d_1 \dots d_{k-1}}, \qquad k \ge 2.$$

In particular, $H(n, 2)$ is the nth partial sum of the harmonic series.

Corollary 6. *For the operad* SGD, *we have*

$$\dim \mathrm{SGD}(n) = \sum_{k=1}^{n} \frac{(n+k-2)!}{(k-1)!} H(n,k).$$

Proof. Consider the set $X = \{x_1, x_2, \dots\}$ which is linearly ordered as above: $x_1 < x_2 < \dots$. Let us first calculate the number $L(n,k)$ of homogeneous Poisson monomials $u_1 \dots u_k \in \mathrm{Pois}(n)$ that split into k Lyndon–Shirshov factors $u_i \in \mathrm{Lie}\langle X \rangle$, $u_1 \le \cdots \le u_k$ lexicographically. If $k = 1$, $L(n,1) = (n-1)!$. If $k > 1$ then u_k starts with the greatest letter x_n. Suppose the length of u_k is d, $1 \le d \le n-1$. The number of such Lyndon–Shirshov words is $(d-1)!\binom{n-1}{d-1}$. The remaining $n-d$ letters form a product of $k-1$ Lyndon–Shirshov words

$u_1, \ldots, u_{k-1}$. Hence,

$$L(n,k) = \sum_{d=1}^{n-k+1} (d-1)! \binom{n-1}{d-1} L(n-d, k-1)$$

Therefore,

$$\begin{aligned} &L(n,k) \\ &= \sum_{d_1,\ldots,d_{k-1}} \frac{(n-1)!(n-d_1-1)!(n-d_1-d_2-1)!\ldots(n-d_1-\cdots-d_{k-1}-1)!}{(n-d_1)!(n-d_1-d_2)!\ldots(n-d_1-\cdots-d_{k-1})!} \\ &= (n-1)! \sum_{d_1,\ldots,d_{k-1}} \frac{1}{(n-d_1)(n-d_1-d_2)\ldots(n-d_1-\cdots-d_{k-1})}, \end{aligned}$$

where the sum ranges over all $(d_1, \ldots, d_{k-1})$ such that $d_i \geq 1$ and $d_1 + \cdots + d_{k-1} < n$. It is easy to see that

$$L(n,k) = (n-1)! H(n,k)$$

for $n, k \geq 2$.

In order to get a differential Poisson monomial of weight -1, we have to apply $k-1$ derivations d to the letters of a Poisson monomial $u_1 \ldots u_k$. There are n letters $x_1, \ldots, x_n$, and we may differentiate every letter more than once. Hence, the number of all possible combinations is

$$\binom{n+k-2}{k-1} L(n,k) = \frac{(n+k-2)!}{(k-1)!} H(n,k).$$

□

In particular, for $n = 1, \ldots, 7$ we have the following values:

n	1	2	3	4	5	6	7
$\dim \mathrm{SGD}(n)$	1	3	17	130	1219	13391	167656

5.5 Defining identities of special GD-algebras

In this section, we describe the defining relations of the operad SGD and derive examples of *special identities*, i.e., those identities that distinguish GD and SGD.

By definition, the operad SGD is the sub-operad of $\mathrm{Pois} \circ \mathrm{GD}^!$ generated by $x_1x_2 \otimes x_2 \star x_1$ and $\{x_1, x_2\} \otimes x_1 * x_2$.

Proposition 1. *The variety of* $(\mathrm{Pois} \circ \mathrm{GD}^!)$*-algebras consists of linear spaces A equipped with four bilinear operations $*$, $\circ$, $[\cdot,\cdot]$, and $[\cdot \succ \cdot]$ such that $(A, *, [\cdot,\cdot])$ is a Poisson algebra, $(A, \circ)$ is a Novikov algebra, and the following identities hold:*

$$[[x \succ z], y] = [[z \succ x], y] - [[x, z] \succ y], \tag{5.22}$$

$$[[y \succ x], z] = [[y \succ z], x] + [y \succ [x, z]], \tag{5.23}$$

$$(x * y) \circ z = x * (y \circ z), \tag{5.24}$$

$$x \circ (y * z) = (z * x) \circ y + (y * x) \circ z, \tag{5.25}$$

$$[(x * z) \succ y]) = [(x \circ z), y] + [(z \circ x), y], \tag{5.26}$$

$$[x, z] \circ y = [y \succ x] * z - [(z \circ y), x], \tag{5.27}$$

$$y \circ [x, z] = [x \succ z] * y - [z \succ x] * y, \tag{5.28}$$

$$[y \succ (x * z)] = [(x \circ y), z] + [(z \circ y), x], \tag{5.29}$$

$$[y \succ z] \circ x + [x \succ z] \circ y = [x \succ (z \circ y)] + [y \succ (z \circ x)], \tag{5.30}$$

$$\begin{aligned}[(y \circ z) \succ x] + [(x \circ z) \succ y] &= x \circ [z \succ y] + [z \succ y] \circ x \\ &+ [x \succ (y \circ z)] + y \circ [z \succ x] + [z \succ x] \circ y + [y \succ (x \circ z)].\end{aligned} \tag{5.31}$$

Proof. Apply the computation process similar to that in Examples 1, 2 relative to the following notations for the generators of $\mathrm{Pois} \circ \mathrm{GD}^!$ in $\mathrm{Pois} \otimes \mathrm{GD}^!$:

$$x_1 * x_2 = x_1x_2 \otimes x_1 * x_2, \quad x_1 \circ x_2 = x_1x_2 \otimes x_2 \star x_1,$$

$$[x_1, x_2] = \{x_1, x_2\} \otimes x_1 * x_2, \quad [x_1 \succ x_2] = \{x_1, x_2\} \otimes x_1 \star x_2.$$

Straightforward solution of the corresponding linear algebra problem leads to the required identities. □

Corollary 7. *The T-ideal $T_{\mathrm{SGD}}(X)$ coincides with the intersection of $T_{\mathrm{Pois}\circ\mathrm{GD}^!}(X)$ with the space of nonassociative polynomials on operations $\circ$ and $[\cdot,\cdot]$. In other words, to get the defining relations of the operad SGD it is enough to find all those corollaries of the identities stated in Proposition 1 that contain only the operations $\circ$ and $[\cdot,\cdot]$.*

According to Corollary 7, defining identities of SGD may be derived from the identities of Pois $\circ$ GD$^!$ given by Proposition 1. For example, we may use (5.27) to eliminate the operation $\succ$ in (5.28). As a result, we obtain (1.8).

Finding the complete list of defining relations for the operad SGD seems to be a hard problem. We will deduce some of these relations below.

Consider the monomial $[y \succ x] * (z \circ u)$. Apply (5.24) and then (5.27) to get

$$[y \succ x] * (z \circ u) = ([y \succ x] * z) \circ u = ([x, z] \circ y + [(z \circ y), x]) \circ u.$$

On the other hand, apply (5.27) to get

$$[y \succ x] * (z \circ u) = [x, (z \circ u)] \circ y + [((z \circ u) \circ y), x].$$

Therefore, the following identity holds on SGD-algebras:

$$[x_1, x_3 \circ x_2] \circ x_4 + ([x_3, x_1] \circ x_2) \circ x_4 = [x_1, (x_3 \circ x_4) \circ x_2] + [x_3 \circ x_4, x_1] \circ x_2. \quad (5.32)$$

(It is also easy to check in a straightforward way that the corresponding identity holds on differential Poisson algebras.)

Proposition 2. *The identity* (5.32) *does not hold on the class of all* GD-*algebras.*

Proof. Without loss of generality we may assume the base field $\Bbbk$ is algebraically closed. We will find an example of a GD-algebra which does not satisfy (5.32) by means of algebraic geometry arguments using the approach of [22].

Let us fix a structure of a Lie algebra $\mathfrak{g}$ on a n-dimensional linear space spanned by $x_1, \dots, x_n$. Namely, we fix a multiplication table of this Lie algebra

$$[x_i, x_j] = \sum_{k=1}^{n} c_k^{ij} x_k, \quad c_k^{ij} \in \Bbbk.$$

Consider the class of all Novikov products $\circ$ on the space $\mathfrak{g}$ satisfying (1.8) as a Zariski closed set $\mathcal{V}(\mathfrak{g})$ in the n^3-dimensional affine space assuming

$$x_i \circ x_j = \sum_{k=1}^{n} a_k^{ij} x_k, \quad a_k^{ij} \in \Bbbk.$$

A point $(a_k^{ij})_{i,j,k=1,\dots,n} \in \Bbbk^{n^3}$ belongs to $\mathcal{V}(\mathfrak{g})$ if and only if

$$\sum_{k=1}^{n} \left(a_k^{ij} a_l^{kp} - a_k^{ip} a_l^{kj}\right) = 0, \tag{5.33}$$

$$\sum_{k=1}^{n} \left(a_k^{ij} a_l^{kp} - a_l^{ik} a_k^{jp} - a_k^{ji} a_l^{kp} + a_l^{jk} a_k^{ip}\right) = 0, \tag{5.34}$$

$$\sum_{k=1}^{n} \left(c_l^{ik} a_k^{jp} - c_l^{pk} a_k^{ji} + c_k^{ji} a_l^{kp} - c_k^{jp} a_l^{ki} - c_k^{ip} a_l^{jk}\right) = 0 \tag{5.35}$$

for all $i, j, p, l = 1, \dots, n$. These relations represent the defining identities of GD-algebras.

Denote by $I(\mathfrak{g})$ the ideal in $\Bbbk[a_k^{ij} \mid i, j, k = 1, \dots, n]$ generated by the left-hand sides of (5.33)–(5.35). If we assume that (5.32) holds for all GD-algebras in the class $\mathcal{V}(\mathfrak{g})$ then the corresponding polynomials

$$\sum_{k,l=1}^{n} \left(c_l^{ik} a_k^{jp} a_s^{lq} - c_l^{ij} a_k^{lp} a_s^{lq} - c_s^{il} a_k^{jq} a_l^{kp} + c_k^{il} a_l^{jq} a_s^{kp}\right), \quad i, j, p, q, s = 1, \dots, n, \tag{5.36}$$

belong to the radical of $I(\mathfrak{g})$.

For dimensions $n = 2, 3$ it is possible to compute the radical of $I(\mathfrak{g})$ in a straightforward way by means of the computer algebra system Singular [10]. These computations show that for every 2-dimensional Lie algebra $\mathfrak{g}$ all polynomials (5.36) belong to $\sqrt{I(\mathfrak{g})}$.

However, even for the 3-dimensional Heisenberg algebra $\mathfrak{h}_3$ spanned by x_1, x_2, x_3, $[x_1, x_2] = x_3$, x_3 is central, there exist polynomials of the form (5.36) that do not belong to $\sqrt{I(\mathfrak{h}_3)}$. □

Let us find one more (independent) special identity for GD-algebras. Consider the expression

$$([x \succ (z \circ y)] + [y \succ (z \circ x)]) * u.$$

Apply (5.30) first, then (5.24) and commutativity of $*$:

$$\begin{aligned}
([x \succ (z \circ y)] + [y \succ (z \circ x)]) * u &= ([x \succ z] \circ y + [y \succ z] \circ x) * u \\
&= ([x \succ z] * u) \circ y + ([y \succ z] * u) \circ x.
\end{aligned}$$

Relation (5.27) and right commutativity of $\circ$ imply

$$([x \succ (z \circ y)] + [y \succ (z \circ x)]) * u = 2([x, u] \circ x) \circ y + [(u \circ x), z] \circ y + [(u \circ y), z] \circ x.$$

On the other hand, it follows from (5.27) that

$$\begin{aligned}([x \succ (z \circ y)] &+ [y \succ (z \circ x)]) * u \\ &= [(z \circ y), u] \circ x + [(u \circ x), (z \circ y)] + [(z \circ x), u] \circ y + [(u \circ y), (z \circ x)].\end{aligned}$$

Finally, using (5.32) we obtain

$$\begin{aligned}2[x_4, (x_3 \circ x_2) \circ x_1] &= [x_4 \circ x_1, x_3 \circ x_2] + [x_4 \circ x_2, x_3 \circ x_1] \\ &+ [x_4, x_3 \circ x_2] \circ x_1 + [x_4, x_3 \circ x_1] \circ x_2 + [x_3, x_4 \circ x_1] \circ x_2 + [x_3, x_4 \circ x_2] \circ x_1.\end{aligned} \tag{5.37}$$

Proposition 3. *The identity* (5.37) *does not hold on the class of all* GD-*algebras satisfying* (5.32).

Proof. Let us consider the class of algebras $(V, \circ, [\cdot, \cdot])$ satisfying the following identities:

$$\begin{aligned} x_1 \circ (x_2 \circ x_3) = (x_1 \circ x_2) \circ x_3 = 0, \\ [x_1, x_2] \circ x_3 = 0, \\ [x_1, x_2 \circ x_3] - [x_3, x_2 \circ x_1] = x_2 \circ [x_1, x_3], \\ [x_1, x_2] = -[x_2, x_1], \\ [x_1, [x_2, x_3]] = 0. \end{aligned} \tag{5.38}$$

Obviously, all these algebras belong to GD and (5.32) holds. If we assume that (5.37) holds on all these algebras then

$$[x_1 \circ x_2, x_3 \circ x_4] + [x_1 \circ x_4, x_3 \circ x_2] = 0 \tag{5.39}$$

has to be a corollary of (5.38). If $\operatorname{char} \Bbbk \neq 2$, relation (5.39) may be transformed to

$$x_1 \circ [x_2 \circ x_3, x_4] = 0 \tag{5.40}$$

modulo (5.38). However, if we write down all those corollaries of (5.38) that contain two operations $\circ$ and one operation $[\cdot, \cdot]$ then we see that (5.40) does not belong to this list. $\square$

Remark 6. We do not suppose that (5.32) and (5.37) exhaust all special identities of GD-algebras. Finding the complete list of special identities (or at least determining whether it is finite) is an interesting problem.

Remark 7. If $(A, \circ)$ is a Novikov algebra then $(A, \circ, [\cdot, \cdot])$ with

$$[a, b] = a \circ b - b \circ a$$

is a GD-algebra [13]. It is easy to check that (5.32) and (5.37) hold for such a GD-algebra. We conjecture that all GD-algebras obtained in this way are special.

6. Conclusion

We considered new type of trees and called them Chameleon trees. In this work is given base of free $0-\mathit{dialgebras}$ and method of construction them by using Chameleon trees. Also there wrote dimension for each class of such algebras. Prove of freeness for algebra with one multiplication can be easily extended for algebras with two multiplications. It will be interesting to find connection between $centralassociative-0-dialgebra$ and $Novikov$ algebra. Main reason is equality of dimensions of this two algebras.

Koszul-dual algebra for $GD-$algebra is a $BiCom$ algebra. We constructed free base for $BiCom$ algebra and find all identities of free special $BiCom$ algebra. There is not found all identities of a free special $GD-$ algebra. Problem to find all special identities is actual for this problem. Also there is given a way to find all identities for free algebra with derivation. There is no the same equality for Rota Baxter operator. It is interesting problem to find for what type of algebras this equality holds.

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