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Functional Equations; Review of solution methods and applications in real and complex analysis.

The functional equation is the equation in which unknowns are functions (one or the several). For example,

$$f(x) + xf(x+1) = 1$$

$$f(x)+g(1-x)=f\left(g\left(\frac{2}{x+1}\right)\right)$$

Some functional equations $f(x) = f(-x)$, $f(-x) = -f(x)$, $f(x+T) = f(x)$ which set such properties of functions, as parity, oddness, periodicity.

The problem of the solution of the functional equations is one of the oldest in the mathematical analysis. They have appeared almost simultaneously with rudiments of the theory of functions. The first present blossoming of this discipline is connected with a problem of a parallelogramme of forces. In 1769 Dalamber has reduced a substantiation of the law of addition of forces to the solution of the functional equation

$$f(x+y)+f(x-y)=2\cdot f(x)\cdot f(y) \qquad (1)$$

The same equation and with the same purpose has been considered by Poisson in 1804 at some assumption of analyticity, meanwhile as in 1821 Cauchy (1789 – 1857) has found the common solutions

$$f(x)=\cos ax$$

$$f(x)=ch\ ax=\frac{e^{ax}+e^{-ax}}{2}$$

$$f(x)\equiv 0$$

This equation, assuming only a continuity f (x).

Even the known formula of non-euclidean geometry for a parallelism corner

$$f(x)=tg\frac{1}{2}\prod(x)=e^{-\frac{x}{k}}$$

has been received by N.I.Lobachevsky (1792 – 1856) from the functional equation

$$f^2(x)=f(x-y)\cdot f(x+y) \qquad (2)$$

Which he has solved a method similar to a method of Cauchy. This equation it is possible to lead to the equation

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$$

A number of the geometrical problems leading to the functional equations were considered by English mathematician C.Babbedzh (1792—1871). He studied, for example, the periodic curves of the second order defined by following property for any pair of points of a curve: if the absciss of the second point is equal to ordinate of the first then the ordinate of the second point is equal to an abscissa of the first. Let such curve be the graph of function $y = f(x)$; $(x, f(x))$ — its ordinary point. Then, according to a condition, a point with an abscissa $f(x)$ has an ordinate x. Hence,

$$f(f(x))=x \qquad (3)$$

In particular the following functions satisfy the functional equation (3) :

$$f(x)=\sqrt{a^2-x^2},\ x\in[0;|a|],\quad f(x)=\frac{a}{x},\ a\neq 0$$

One of the elementary functional equations are the equations of Cauchy

$$f(x+y) = f(x) + f(y), \quad (4)$$

$$f(x+y) = f(x) \cdot f(y), \quad (5)$$

$$f(xy) = f(x) + f(y), \quad (6)$$

$$f(xy) = f(x) \cdot f(y), \quad (7)$$

These equations Cauchy has in detail studied in his book (the Analysis Course) published in 1821. Continuous solutions of these four basic equations are

$$f(x) = ax, \quad a^x, \quad \log_a x, \quad x^a \ (x > 0)$$ respectively.

In the class of discontinuous functions there could be other solutions. The equation (4) was considered earlier by Legendre and Gauss at a conclusion of the basic theorem of projective geometry and at research of Gauss' law of distribution of probabilities.

The functional equation (4) has been again applied by Jean Gaston Darboux to a problem of a parallelogramme of forces and to the basic theorem of projective geometry; its main achievement - considerable easing of assumptions. We know that the functional equation of Cauchy (4) characterises in a continuous functions such as linear homogeneous function $f(x) = ax$. Darboux has shown that any solution is continuous at least in one point or limited from above (or from below) in any small interval, also should look like $f(x) = ax$. *The* further results on easing of assumptions followed quickly one after another (integration, measurability on set of a positive measure of the function). There is a question: is there any additive function (i.e. satisfying (4)), distinct from the linear homogeneous. To find such function really hardly! During work we will show that at rational x values of any additive function should coincide with values of some linear homogeneous function, i.e. $f(x) = ax$ for x Q∈. It would Seem that then $f(x) = ax$ for all valid x. If $f(x)$ - it is continuous, it is valid so if given assumption to reject - that is not present. The first example distinct from $f(x) = ax$ *the* explosive decision of the functional equation (4) was constructed in 1905 by the German mathematician G.Gamel by means of the basis of real numbers introduced by him.

Many functional equations do not define the given function, and set a wide class of functions, i.e. express the property characterising this or that class of functions. For example, the functional equation $f(x+1) = f(x)$ characterises a class of the functions having the period 1, and the equation $f(1+x) = f(1-x)$ - a class of the functions symmetric to the straight line $x = 1$, and etc.

In general, we know few about general methods of the solutions of the functional equations which are not reduced to differential or integrated. Further some receptions will be considered, allowing to solve the functional equations.

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Түйін

Бұл мақалада функциональдық теңдеулердің шешім әдістері мен қосымшалары қарастырылған.

Резюме

В данной статье рассматривается обзор методов решения и приложения функциональных уравнений реального и комплексного анализа.

Özet

Bu makalede fonksiyonel denklemlerin reel ve kompleks cözümlerinin ve metodlarının özeti incelenmiştir.

Қазақ тілі – қазақтың тағдыры.

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