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Fractal dimension of exceptional sets in semi-regular continued fraction

THESIS

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Declaration

I confirm that I did this work myself, and I have clearly and fully mentioned all the sources I used.

Symbat Duisen

June 2025

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I would like to thank my supervisor,PhD Shirali Kadyrov,for his help and support during my master's project.His advice and guidance were very important for my work.I also thank SDU university for giving me the change to study and do my master project.

Dedication

I would like to dedicate this diploma work to my dear family and teachers. Your support, encouragement, and belief in me helped me reach this important moment. Thank you for always being there for me.

Abstract

This thesis investigates the interplay between Diophantine approximation, continued fraction representations, and fractal geometry. We begin by exploring the classical notion of *badly approximable numbers*-real numbers whose continued fraction expansions have bounded partial quotients. These numbers, while forming a set of zero Lebesgue measure, exhibit full Hausdorff dimension, highlighting their rich geometric structure.

Building on this foundation, we introduce and analyze a generalization known as *semi-regular continued fractions*, wherein a fixed sequence of signs modifies the classical expansion. For such expansions, we define the class of σ-badly approximable numbers and study their distribution and fractal properties. We demonstrate that these generalized expansions preserve many of the geometric complexities of their classical counterparts, while offering new degrees of arithmetic freedom.

In the second part of the thesis, we shift our focus to Lehner expansions of real numbers and examine how the statistical behavior of the associated digit sequence (b_n) influences the fractal geometry of the corresponding number sets. Specifically, we investigate the impact of the average value of b_n on the *box dimension*-a quantitative measure of geometric complexity. Employing the box-counting method, we perform numerical experiments to estimate the box dimension and uncover how variations in the digit sequence relate to the irregularity and structure of the expansion.

By synthesizing the analytical and numerical approaches, this thesis provides a comprehensive view of how modifications to continued fraction representations influence the fractal characteristics of real number sets, contributing to the broader understanding of number-theoretic and geometric interrelations.

Аңдатпа

Бұл диссертациялық жұмыс Диофанттық жуықтау, тізбекті бөлшек түріндегі өрнектеулер және фрактал геометрия арасындағы байланысты зерттейді. Алдымен, классикалық *нашар жуықталатын сандар* ұғымы қарастырылады-яғни, тізбекті бөлшек түріндегі өрнектеулерінде бөліктері шектеулі болатын нақты сандар. Мұндай сандар жиыны Лебег өлшемі бойынша нөлге тең болса да, Хаусдорф өлшемі толық, бұл олардың геометриялық құрылымының күрделілігін көрсетеді.

Осы негізде біз *жартылай-реттелген тізбекті бөлшектер* деп аталатын жалпыламаны енгіземіз және зерттейміз. Мұнда классикалық өрнектеуді белгіленген таңбалар тізбегі өзгертеді. Мұндай өрнектеулер үшін біз σ-нашар жуықталатын сандар класын анықтап, олардың таралуын және фракталдық қасиеттерін қарастырамыз. Біз бұл жалпыланған өрнектеулер классикалық түріне тән геометриялық күрделілікті сақтап қана қоймай, арифметикалық тұрғыдан жаңа еркіндіктер беретінін көрсетеміз.

Жұмыстың екінші бөлімінде біз нақты сандардың Лёнер өрнектеулеріне назар аударамыз және осы өрнектеуге сәйкес (b_n) сандарының статистикалық мінез-құлқы фрактал геометриясына қалай әсер ететінін зерттейміз. Атап айтқанда, b_n сандарының орташа мәні *қорапша өлшемі* деп аталатын геометриялық күрделіліктің сандық өлшеміне қалай әсер ететінін қарастырамыз. Қорапша санау әдісін қолдана отырып, біз қорапша өлшемін сандық түрде бағалап, цифрлар тізбегіндегі өзгерістер өрнектеудің құрылымы мен ретсіздігіне қалай әсер ететінін анықтаймыз.

Теориялық және сандық тәсілдерді біріктіре отырып, бұл жұмыс тізбекті бөлшек өрнектеулеріндегі өзгерістер нақты сандар жиынының фракталдық сипаттамаларына қалай әсер ететінін жан-жақты қарастырады және сандар теориясы мен геометрия арасындағы байланыстарды тереңірек түсінуге үлес қосады.

Аннотация

В данной диссертационной работе исследуется взаимосвязь между диофантовыми приближениями, представлениями чисел в виде непрерывных дробей и фрактальной геометрией. Мы начинаем с изучения классического понятия *плохо приближаемых чисел* — действительных чисел, разложение которых в непрерывную дробь имеет ограниченные неполные частные. Несмотря на то что множество таких чисел имеет нулевую меру Лебега, оно обладает полной размерностью Хаусдорфа, что подчёркивает их богатую геометрическую структуру.

Основываясь на этой теории, мы вводим и анализируем обобщение, известное как *полурегулярные непрерывные дроби*, где фиксированная последовательность знаков модифицирует классическое разложение. Для таких разложений мы определяем класс σ-плохо приближаемых чисел и изучаем их распределение и фрактальные свойства. Мы показываем, что такие обобщённые разложения сохраняют многие геометрические особенности классических, при этом предоставляя новые степени арифметической свободы.

Во второй части работы мы сосредотачиваемся на разложениях Лёнера действительных чисел и исследуем, как статистическое поведение соответствующей последовательности цифр (b_n) влияет на фрактальную геометрию соответствующих множеств чисел. В частности, мы анализируем влияние среднего значения b_n на *бокс-гаусдорфову размерность* — количественную меру геометрической сложности. Используя метод подсчёта ячеек, мы проводим численные эксперименты для оценки размерности и устанавливаем, как изменения в последовательности цифр связаны с нерегулярностью и структурой разложения.

Объединяя аналитические и численные подходы, данная работа предоставляет всесторонний взгляд на то, как модификации представлений в виде непрерывных дробей влияют на фрактальные характеристики множеств действительных чисел, способствуя более глубокому пониманию взаимосвязей между теорией чисел и геометрией.

Content

1. Introduction

The degree to which rational numbers can approximate real numbers is a crucial question in number theory. Dirichlet's theorem is a fundamental finding that states that for any real number α and any positive integer N, there exist integers p and q such that

$$1 \leq q \leq N, \quad \left|\alpha - \frac{p}{q}\right| < \frac{1}{qN}.$$

This means every real number can be closely approximated by rational numbers. But some numbers can be approximated better than others. The concept of badly *approximable numbers* results from this.

We say a number α is *badly approximable* if there is some constant $c > 0$ such that, no matter which integers p and $q > 0$ you choose, the difference between α and the fraction $\frac{p}{q}$ is always at least $\frac{c}{q^2}$,

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2}.$$

The set of badly approximable numbers, written as BA, is small in terms of Lebesgue measure (it has measure zero), but it is large in terms of fractal[1] geometry it has Hausdorff dimension 1 [2].

A tool to study approximation is the *continued fraction expansion.*Every irrational number x can be written in one special way as a continued fraction, and this form is always unique:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}, \quad a_0 \in \mathbb{Z}, \quad a_n \in \mathbb{N}.$$

A number is badly approximable if and only if the numbers $a_1, a_2, \ldots$ are bounded.

We now look at a more general type of expansion called the *semi-regular continued fraction (SRCF)*. Fix a sequence $\sigma = (\sigma_n)_{n=1}^{\infty}$ where each $\sigma_n = \pm 1$. Then we write:

$$x = a_0 + \cfrac{\sigma_1}{a_1 + \cfrac{\sigma_2}{a_2 + \cfrac{\sigma_3}{a_3 + \ddots}}}.$$

This expansion is valid if the two conditions listed below are true:

$$a_n + \sigma_{n+1} \geq 1 \text{ for all } n,$$

$$a_n + \sigma_{n+1} \geq 2 \text{ for infinitely many } n.$$

These conditions guarantee that the expansion makes sense and converges. The regular continued fraction is a special case when all $\sigma_n = 1$.

The set of σ-badly approximable numbers, written as BA^σ, includes all numbers that have a constant $M > 0$ such that every term $a_n(x)$ in their continued fraction expansion is less than or equal to M.

Next, we get the following outcome:

Theorem 1.0.1. *For any $\sigma \in \{-1, 1\}^{\mathbb{N}}$, the set of σ-badly approximable numbers BA^σ contains the set of badly approximable numbers BA. In particular,*

$$\dim_H BA^\sigma = 1.$$

So the new set BA^σ always contains BA, and it also has full Hausdorff dimension.

In some cases, BA^σ is strictly larger than BA. One example is the case when $\sigma_n = -1$ for all n. This gives the *backward continued fraction*. We write the set of such numbers as BA^-.

Theorem 1.0.2. *The set of numbers that belong to BA^- but not to BA is a not empty null set. Moreover, it has Hausdorff dimension*

$$\dim_H(BA^- \cap BA^c) \geq \frac{1}{2}.$$

So BA^σ can be strictly bigger than BA depending on the choice of σ.

We then investigate the *Lehner continued fraction*, a particular kind of semi-regular continued fraction. It is defined for numbers in the interval $(1, 2)$. Every irrational number $x \in (1, 2)$ can be expressed as:

$$x = [b_0; \sigma_1/b_1, \sigma_2/b_2, \dots],$$

where each pair (b_i, σ_{i+1}) is either $(1, 1)$ or $(2, -1)$.

This expansion is generated by the map

$$Lx = \begin{cases} \frac{1}{2-x}, & \text{if } 1 \leq x < \frac{3}{2}, \\ \frac{1}{x-1}, & \text{if } \frac{3}{2} \leq x < 2. \end{cases}$$

For almost all $x \in (1, 2)$, the average of the b_n values approaches 2:

$$\lim_{n\to\infty} \frac{b_1 + \cdots + b_n}{n} = 2.$$

We define the set

$$S(\epsilon, c) = \left\{ x \in (1, 2) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} b_i \in (c - \epsilon, c + \epsilon) \right\}.$$

This set has measure zero, but it may have positive box dimension. Box dimension is a way to measure how "dense" a set is. It is defined as:

$$\dim_B(S) = \lim_{\delta\to 0} \frac{\log N(\delta)}{-\log \delta},$$

where $N(\delta)$ is the number of δ intervals required to cover the set S.

In this thesis, we study and compare the sets BA, BA^σ, BA^-, and $S(\epsilon, c)$ using tools from continued fractions and fractal geometry. We give both theoretical results and numerical estimates of dimension.

2. Background

2.1 Regular continued fraction(RCF)

Definition 2.1.1. *A regular continued fraction (RCF) is a special way to write a number using a pattern of nested fractions. It looks like this:*

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}, \tag{2.1.1}$$

We often write this more compactly as:

$$x = [a_0; a_1, a_2, a_3, \ldots],$$

where a_0 is any integer ($a_0 \in \mathbb{Z}$), and each a_i for $i \geq 1$ is a natural number ($a_i \in \mathbb{N}$).

This type of fraction is useful because it gives some of the best possible approximations of real numbers using simple fractions. The fraction you get by stopping at the n-th step is called the n-th convergent *of x:*

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n],$$

where the top numbers p_n and the bottom numbers q_n follow these rules:

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2},$$

with the starting values: $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, and $q_0 = 1$.

Example 2.1.1. *Consider the rational number $\frac{13}{5}$. The steps to represent it as a finite continued fraction are as follows:*

$$\frac{13}{5} = [2; 1, 1, 2]$$

This means:

$$\frac{13}{5} = 2 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{2}}}$$

Example 2.1.2. *The continued fraction representation of π is:*

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 2, \dots]$$

This can be expressed as:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}$$

The continued fraction for π begins with the integer part 3,and the remaining terms 7,15,1,292.. are the partial quotients.The continued fraction provides increasingly accurate approximations of π

Example 2.1.3. *Let us look at the square root of 13:*

$$\sqrt{13} = [3; \overline{1, 1, 1, 1, 6}]$$

This means the continued fraction for $\sqrt{13}$ becomes a repeating pattern after the first number. The full expression is:

$$\sqrt{13} = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cdots}}}}}}$$

Because 13 is not a perfect square, $\sqrt{13}$ is an irrational number. A well-known result in number theory says that if n is a positive number that is not a perfect square, then the continued fraction of $\sqrt{n}$ always repeats.

The bar above the numbers, $\overline{1, 1, 1, 1, 6}$, shows that this part of the sequence *goes on forever in the same way.*

2.2 Transforming

Definition 2.2.1. *Let $x \in \mathbb{R}$ be a real number. Its semi-regular continued fraction (SRCF) expansion can be written as:*

$$x = [b_0; \tfrac{e_1}{b_1}, \tfrac{e_2}{b_2}, \tfrac{e_3}{b_3}, \dots] = b_0 + \cfrac{e_1}{b_1 + \cfrac{e_2}{b_2 + \cfrac{e_3}{b_3 + \cdots}}},$$

where each $b_i \geq 2$ is an integer and each e_i is either $+1$ or -1 for $i \geq 1$.

Now suppose that for some index $n \geq 0$, we have:

$$b_{n+1} > 1 \quad \textit{and} \quad e_{n+1} = 1.$$

Then we can apply a process called an insertion at position n, using a transformation called τ_n, which changes the SRCF:

$$x = [b_0; \tfrac{e_1}{b_1}, \dots, \tfrac{e_n}{b_n}, \tfrac{1}{b_{n+1}}, \dots]$$

into a new SRCF:

$$\tau_n(x) = [b_0; \tfrac{e_1}{b_1}, \dots, \tfrac{e_n}{b_n+1}, \tfrac{-1}{1}, \tfrac{1}{b_{n+1}-1}, \dots].$$

This change is correct because of the id

Example 2.2.1.

$$x = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{6}}}}}} \qquad x = \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{-1}{5 + \cfrac{1}{5 + \cfrac{1}{6}}}}}$$

2.3 Semi-regular continued fraction(SRCF

Definition 2.3.1. *An expression written in the following way is called a semi-regular continued fraction (SRCF):*

$$x = b_0 + \cfrac{e_1}{b_1 + \cfrac{e_2}{b_2 + \cfrac{e_3}{b_3 + \cdots}}} = [b_0; \tfrac{e_1}{b_1}, \tfrac{e_2}{b_2}, \tfrac{e_3}{b_3}, \ldots], \tag{2.3.1}$$

where $b_0 \in \mathbb{Z}$, $b_n \in \mathbb{N}$ for $n \geq 1$, and $e_n \in \{-1, +1\}$ is a fixed sequence of signs. The sequence $\sigma = (e_n)$ is called the sign sequence of the expansion.

Example 2.3.1.

$$x = \cfrac{1}{2 + \cfrac{-1}{4 + \cfrac{1}{2 + \cfrac{-1}{2 + \cfrac{-1}{4 + \cfrac{1}{3 + \cfrac{-1}{8 + \cfrac{1}{9 + \cfrac{-1}{2 + \cfrac{-1}{12 + \cfrac{1}{2 + \cfrac{1}{3 + \ldots}}}}}}}}}}}}$$

2.4 Backward expansion

Suppose $x \in [0, 1)$ has a regular continued fraction (RCF) expansion like the one shown in Equation 2.1.1, and that $a_0 = 0$. We can find the backward continued fraction expansion of x by following these steps:

(I) If $a_1 = 1$, we apply a process called *singularization* to the first term. This gives a new semi-regular continued fraction (SRCF) expansion:

$$x = [1; \tfrac{-1}{a_2+1}, \tfrac{1}{a_3}, \tfrac{-1}{a_4+1}, \tfrac{1}{a_5}, \tfrac{-1}{a_6+1}, \tfrac{1}{a_7}, \ldots]$$

If instead $a_1 > 1$, we insert the value $\frac{-1}{1}$ a total of $a_1 - 1$ times before a_1, resulting in:

$$x = [1; \underbrace{\tfrac{-1}{2}, \ldots, \tfrac{-1}{2}}_{a_1-2 \text{ times}}, \tfrac{-1}{1}, \tfrac{1}{1}, \tfrac{1}{a_2}, \ldots]$$

This new expression is again considered an SRCF expansion of x. Then, we apply the singularization process to the $\frac{1}{1}$ term, which appears at the a_1-th position. In both cases, we arrive at the same structure:

$$x = [1; \underbrace{\tfrac{-1}{2}, \ldots, \tfrac{-1}{2}}_{a_1-1 \text{ times}}, \tfrac{-1}{a_2+1}, \tfrac{1}{a_3}, \ldots]$$

Here, the notation $(-1/2)^{a_1-1}$ stands for writing $\frac{-1}{2}$ exactly $a_1 - 1$ times in a row.

(II) Let $m \geq 1$ be the first position in this new SRCF expansion where the sign $e_m = 1$. Then, repeat the same steps from part (I), starting at this point and using the updated expansion.

Following the pattern of insertions and singularizations, we eventually find that x has the following backward continued fraction expansion:

$$x = [1; (-1/2)^{a_1-1}, \tfrac{-1}{a_2+2}, (-1/2)^{a_3-1}, \tfrac{-1}{a_4+2}, \ldots] \tag{2.4.1}$$

Example 2.4.1.

$$\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6+\ldots}}} \qquad 1+\cfrac{-1}{1+\cfrac{1}{5+\cfrac{1}{6+\ldots}}} \qquad 1+\cfrac{-1}{2+\cfrac{-1}{1+\cfrac{1}{4+\cfrac{1}{6+\ldots}}}}$$

$$1+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{1+\cfrac{1}{3+\cfrac{1}{6+\ldots}}}}} \qquad 1+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{1+\cfrac{1}{2+\cfrac{1}{6+\ldots}}}}}}$$

$$1+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{1+\cfrac{1}{1+\cfrac{1}{6+\dots}}}}}}} \qquad 1+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{7+\cfrac{1}{6+\dots}}}}}}}$$

3. Related Work

The study of Diophantine approximation and continued fractions has a long history, beginning with results in classical number theory. Over time, this topic has developed into a field connecting approximation theory, dynamical systems, and fractal geometry. We divide the literature into two parts: historical background and recent developments.

3.1 Historical Background

One of the earliest and most important results in number theory is known as Dirichlet's approximation theorem. It states that for any real number α and any positive integer N, there always exist integers p and q such that $1 \leq q \leq N$ and

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN}.$$

This result led to deeper investigations into how closely real numbers can be approximated by rational numbers. A real number θ is called *badly approximable* [3] if there is a constant $k > 0$ such that for every integer p and every positive integer q,

$$\left|\theta - \frac{p}{q}\right| > \frac{k}{q^2}.$$

In the 1920s, Jarník [2] proved that the set of badly approximable numbers has zero Lebesgue measure, but its Hausdorff dimension is exactly 1. He also showed that a number is badly approximable if and only if the terms in its continued fraction expansion remain bounded.

Later on, Schmidt [4] introduced a mathematical framework known as *Schmidt's game*, which is used to study badly approximable numbers through the lens of dynamical systems. He showed that the set of badly approximable numbers is a winning set in this game. This implies that although the set has measure zero, it is still large in a topological sense—it is dense and has full Hausdorff dimension.

The regular continued fraction expansion of an irrational number,

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}},$$

gives the best possible rational approximations to x, and it plays a central role in understanding the Diophantine properties of real numbers.

3.2 Recent Work

Recent research has looked at new ways to extend the idea of continued fractions and explore their geometric and measurement-related properties. One such extension is called the *semi-regular continued fraction (SRCF)*. This version is based on a sequence of signs $\sigma \in \{-1, 1\}^{\mathbb{N}}$, and it includes the regular continued fraction as a special case.

Nakajima and Takahasi [5] explored the Hausdorff dimension of sets of numbers whose SRCF partial quotients stay within a fixed limit. They found that for any sign sequence σ, the set of σ-badly approximable numbers, written as BA^{σ}, always has Hausdorff dimension 1. They also showed that this set includes the standard set of badly approximable numbers, BA.

S.Kadyrov, A.Kazin, and Mashurov[6] considered SRCFs where the partial quotients grow rapidly. They estimated the Hausdorff dimension of the corresponding sets and studied the structure of overlaps with the classical sets.

Another direction studies *Lehner continued fractions*[7], a specific type of SRCF for numbers in $(1, 2)$. These expansions use only the digits 1 and 2 in a fixed pattern, generated by the map

$$Lx = \begin{cases} \frac{1}{2-x}, & \text{if } 1 \leq x < \frac{3}{2}, \\ \frac{1}{x-1}, & \text{if } \frac{3}{2} \leq x < 2. \end{cases}$$

Fang et al. showed that for almost all $x \in (1, 2)$,

$$\lim_{n \to \infty} \frac{b_1 + \cdots + b_n}{n} = 2,$$

where b_n are the partial quotients in the Lehner expansion. The exceptional set, where this limit is not 2, has Lebesgue measure zero but nontrivial fractal structure.

Kazin and Kadyrov estimated the box dimension of such sets using symbolic coding and methods from multifractal analysis. These ideas build on the work of Barreira and Schmeling, who used binary word representations to study complex patterns in dynamical systems.

In summary, modern research explores new forms of continued fractions, new classes of badly approximable numbers, and new methods to estimate fractal dimensions. These results connect number theory, dynamics, and geometry.

4. Fractal Geometry

4.1 Fractal geometry and fractal dimension

Fractal geometry is a field of mathematics that studies irregular sets with fine structure at every scale. These sets often cannot be described by classical geometry, which focuses on smooth curves and surfaces. Fractals are typically defined by recursive rules and may have properties such as self-similarity and non-integer dimension.

Fractal geometry is important because many natural and mathematical structures are not smooth or regular. Classical tools like calculus do not apply well to such objects. Fractals appear in physics, biology, computer graphics, and number theory. In number theory, fractal ideas help describe sets with complex structure, such as the set of badly approximable numbers.

Self-similar sets are a common type of fractal. A set is called *self-similar* if it can be written as a union of smaller versions of itself, each one scaled by a fixed ratio. A famous example is the middle-third Cantor set. It is built by starting with the interval $[0, 1]$ and repeatedly removing the open middle third from each remaining part (see Figure 4.1).

The final set includes only those numbers in $[0, 1]$ whose base-3 expansions use only the digits 0 and 2. This set has zero total length, is self-similar, and is uncountable.

Fractals like the Cantor set can be analyzed using tools such as Hausdorff dimension, which measures their complexity. Fractal dimension allows us to distinguish between sets that are too irregular for classical dimensions like length, area, or volume.

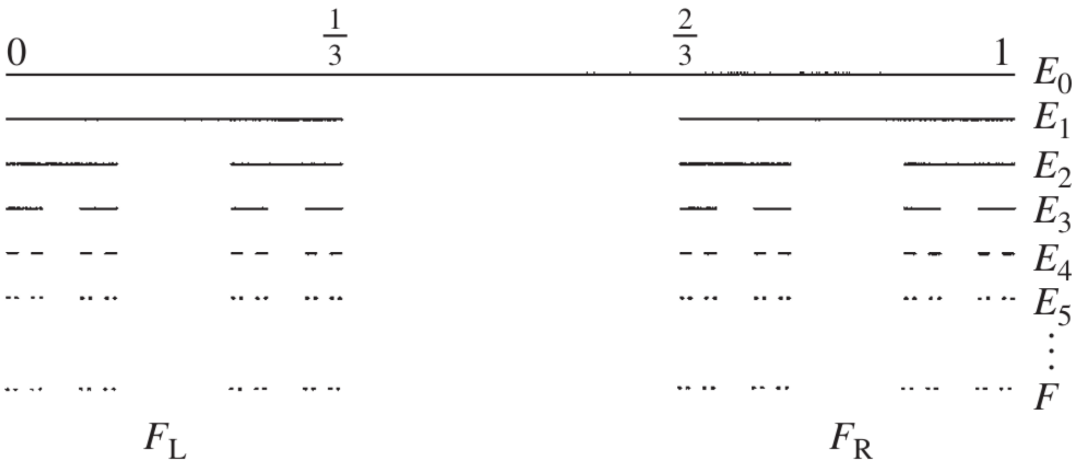


Figure 4.1: By repeatedly removing the middle third of intervals, the middle third of Cantor set F is constructed.The left and right portions of F, F_L and F_R, are copies of F scaled by a factor $\frac{1}{3}$.

The von Koch curve is another well-known example of a fractal (see Figure 4.2). We start with a straight line segment of length 1, called E_0. To form the next step, E_1, we remove the middle third of the segment and replace it with two sides of an equilateral triangle (leaving out the base). This process creates four line segments.

We apply the same procedure to every segment of E_1 to create E_2. This procedure is repeated: for E_k, we substitute the two triangle sides for the middle third of each line in E_{k-1}.

As we keep going, the shapes E_k get more detailed and look closer to a final shape, called the von Koch curve F. This curve has a pattern similar to the Cantor set. It has four parts, each like the whole, just smaller by a factor of $\frac{1}{3}$.

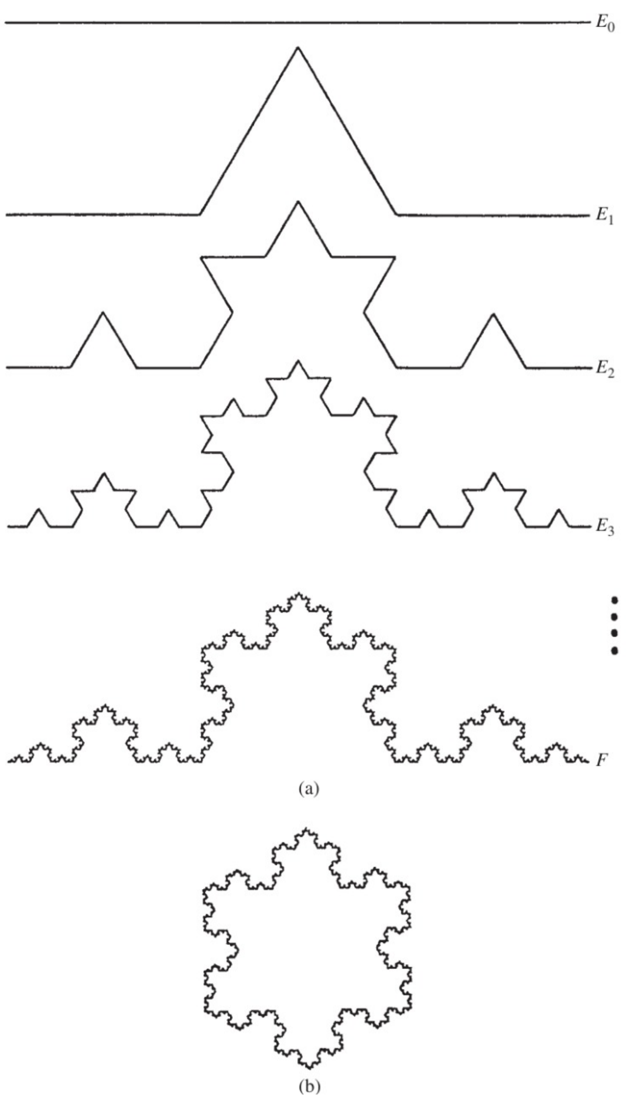


Figure 4.2: (a)In order to construct the von Koch cure F, we substitute two sides of an equilaternal triangle for the middle third of each line segment at each stage.(b)A snowflake-shaped cure is created when three von Koch curves are connected.

The curve looks very detailed and uneven at all scales. It is too jagged to have tangents in the usual sense. If we calculate the length of E_k, it is $\left(\frac{4}{3}\right)^k$, which grows without limit as k increases. So the final curve F has infinite length. But even though it's infinitely long, it covers zero area on the plane. So, normal ideas like length or area don't help describe how big the von Koch curve is.

4.2 Box dimension and box dimension of Cantor set

The box dimension, also called the Minkowski dimension, is a way to measure how "complicated" or "detailed" a shape or set is—especially if it has a fractal structure. Unlike regular dimensions (like a line being 1D or a square being 2D), the box dimension can be a decimal, which helps describe sets that have fine details at every scale.

It works by covering the set with small boxes and checking how the number of boxes changes as the boxes get smaller. This idea is useful in many areas of math, especially when regular tools don't work well.
Let us take a set G in the plane. For any small number $\varepsilon > 0$, we try to cover the set G using shapes (like boxes or balls) that have a diameter no bigger than ε. The smallest number of such shapes needed is called $M_\varepsilon(G)$. This tells us how many small pieces of size about ε are needed to cover G.
The box dimension of G describes how $M_\varepsilon(G)$ changes as ε becomes very small. If the number of pieces follows a pattern like

$$M_\varepsilon(G) \approx c\varepsilon^{-d}$$

for some positive constants c and d, then we say the box dimension of G is d. (We'll explain the name "box dimension" shortly.)

To find the value of d, we take the logarithm of both sides:

$$\log M_\varepsilon(G) \approx \log c - d \log \varepsilon.$$

So, we can estimate the dimension d using the formula:

$$d \approx \frac{\log M_\varepsilon(G)}{-\log \varepsilon} + \frac{\log c}{\log \varepsilon}.$$

As ε becomes very small, the second term $\frac{\log c}{\log \varepsilon}$ becomes less significant and tends to zero. Therefore, we can define the box dimension d as the following limit:

$$d = \lim_{\varepsilon \to 0} \frac{\log M_\varepsilon(G)}{-\log \varepsilon}.$$

Below is the formal definition of the box-counting dimension. Let V be a non-empty set in n-dimensional space $\mathbb{R}^n$. The *diameter* of V is defined as the greatest distance between any two points in the set. In other words,

$$|V| = \sup\{|x - y| : x, y \in V\}.$$

Now, imagine we have a shape G, and we want to cover it using small sets $\{V_i\}$, each of which has diameter at most ε. If

$$G \subset \bigcup_{i=1}^{\infty} V_i \quad \text{and} \quad |V_i| \leq \varepsilon,$$

then we say this collection $\{V_i\}$ is a ε-*cover* of G.

Let $M_\varepsilon(G)$ be the smallest number of such sets needed to completely cover the set G. Then, the *lower* and *upper box-counting dimensions* of G are defined as:

$$\underline{\dim}_{\mathrm{B}} G = \lim_{\varepsilon \to 0} \frac{\log M_\varepsilon(G)}{-\log \varepsilon} \tag{4.2.1}$$

$$\overline{\dim}_{\mathrm{B}} G = \lim_{\varepsilon \to 0} \frac{\log M_\varepsilon(G)}{-\log \varepsilon} \tag{4.2.2}$$

Of course,

$$\underline{\dim}_{\mathrm{B}} G \leq \overline{\dim}_{\mathrm{B}} G.$$

If both are equal, we just call this the *box-counting dimension* or *box dimension* of G:

$$\dim_{\mathrm{B}} G = \lim_{\varepsilon \to 0} \frac{\log M_\varepsilon(G)}{-\log \varepsilon} \tag{4.2.3}$$

Definition 4.2.1 (Equivalent definitions of box dimension)**.** *The lower and upper box-counting dimensions of a subset G of $\mathbb{R}^n$ are given by*

$$\underline{\dim}_{B} G = \lim_{\varepsilon \to 0} \frac{\log M_\varepsilon(G)}{-\log \varepsilon} \tag{4.2.4}$$

$$\overline{\dim}_{B} G = \lim_{\varepsilon \to 0} \frac{\log M_\varepsilon(G)}{-\log \varepsilon} \tag{4.2.5}$$

and the box-counting dimension of G by

$$\dim_B G = \lim_{\varepsilon \to 0} \frac{\log M_\varepsilon(G)}{-\log \varepsilon} \tag{4.2.6}$$

(if this limit exists), where $M_\varepsilon(G)$ can be understood in different ways, for example:

- *(i) the smallest number of sets, each with diameter at most ε, needed to cover the set G;*
- *(ii) the smallest number of closed balls of radius ε required to fully cover G;*
- *(iii) the smallest number of cubes with side length ε that are enough to cover G;*
- *(iv) the number of cubes in a grid of size ε that intersect or touch the set G;*
- *(v) the largest number of non-overlapping balls of radius ε that can fit inside G, with their centers lying in G.*

Example 4.2.1. *Let us look at the middle third Cantor set G (see Figure 4.3). Its box-counting dimension is given by:*

$$\dim_B G = \frac{\log 2}{\log 3}$$

We start with the upper bound.

We consider the level-k intervals from the Cantor set construction, each of length 3^{-k}. There are 2^k such intervals that cover G. So for $3^{-k} < \varepsilon \leq 3^{-k+1}$, we can write:

$$M_\varepsilon(G) \leq 2^k$$

This gives an upper bound for the box-counting dimension:

$$\overline{\dim}_B G \leq \lim_{k \to \infty} \frac{\log 2^k}{-\log 3^{-k+1}} = \lim_{k \to \infty} \frac{k \log 2}{(k-1)\log 3} = \frac{\log 2}{\log 3}$$

Next, we look at the lower bound.

Suppose $3^{-k-1} \leq \varepsilon < 3^{-k}$. In this case, any interval of length ε can overlap with at most one of the level-k intervals, since the gaps between those intervals are at least 3^{-k}. This means we need at least 2^k intervals of length ε to fully cover the set G. In other words,

$$M_\varepsilon(G) \geq 2^k$$

This gives the lower bound:

$$\underline{\dim}_B G \geq \lim_{k\to\infty} \frac{\log 2^k}{-\log 3^{-k-1}} = \lim_{k\to\infty} \frac{k \log 2}{(k+1)\log 3} = \frac{\log 2}{\log 3}$$

Finally, since the lower and upper bounds are equal, we find the box-counting dimension:

$$\dim_B G = \frac{\log 2}{\log 3}$$

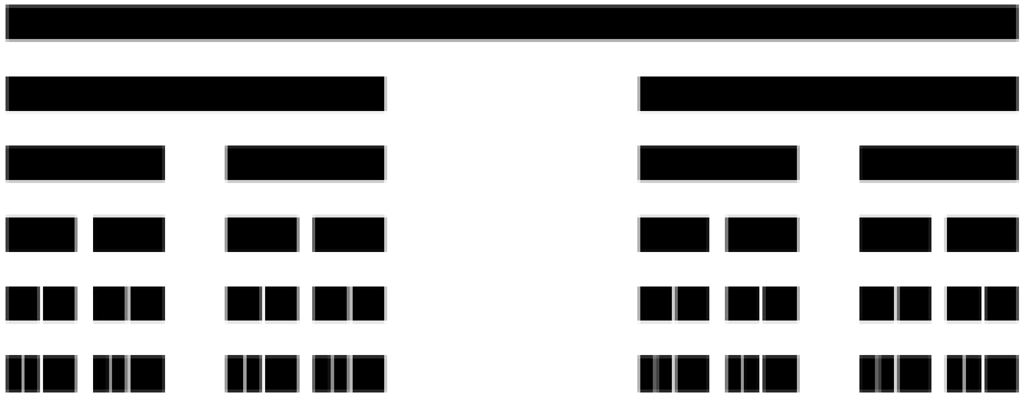


Figure 4.3: Cantor set

Example 4.2.2. *Let G be the Sierpiński triangle with side length 1 (see Figure 4.4). Then, its box-counting dimension is:*

$$\dim_B G = \frac{\log 3}{\log 2}$$

Construction idea. *At step k of building the triangle, we create 3^k smaller triangles, each with side length 2^{-k}. So if the box size ε is between 2^{-k} and 2^{-k+1}, then:*

$$M_\varepsilon(G) \leq 3^k$$

and this gives:

$$\dim_B G = \lim_{\varepsilon\to 0} \frac{\log M_\varepsilon(G)}{-\log \varepsilon} \leq \lim_{k\to\infty} \frac{\log 3^k}{-\log 2^{-k+1}} = \frac{\log 3}{\log 2}$$

On the other hand, if ε lies between 2^{-k-1} and 2^{-k}, then we need at least 3^{k-1} boxes of size ε to cover the triangle:

$$M_\varepsilon(G) \geq 3^{k-1}$$

which gives:

$$\dim_B G = \lim_{\varepsilon \to 0} \frac{\log M_\varepsilon(G)}{-\log \varepsilon} \geq \lim_{k \to \infty} \frac{\log 3^{k-1}}{-\log 2^{-k-1}} = \frac{\log 3}{\log 2}$$

So finally since both bounds are equal, we have:

$$\dim_B G = \frac{\log 3}{\log 2}$$

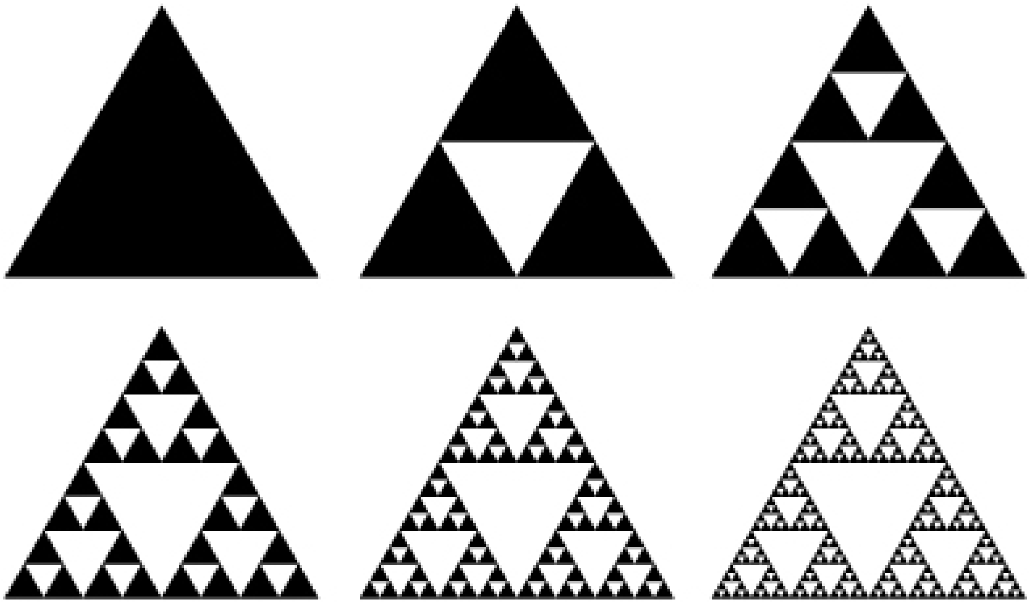


Figure 4.4: Sierpinski triangle

4.3 Hausdorff Measure and Dimension

Definition 4.3.1 (Hausdorff Measure). *Let $F \subset \mathbb{R}^n$ and fix a non-negative real number s. Given a precision parameter $\delta > 0$, we say that a collection of subsets $\{U_i\}_{i=1}^{\infty}$ forms a δ-cover of F if:*

- *Each $U_i \subset \mathbb{R}^n$,*
- *The union $\bigcup_{i=1}^{\infty} U_i$ contains F,*
- *And each U_i has diameter at most δ, i.e., $diam(U_i) \leq \delta$.*

We then define the s-dimensional pre-Hausdorff content of F at scale δ as:

$$\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} (diam(U_i))^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

Taking the limit as δ tends to zero, we obtain the Hausdorff measure:

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

From this construction, it is clear that as the dimension parameter s increases, the value of $\mathcal{H}^s_\delta(F)$ cannot increase (for fixed $\delta < 1$). This monotonic behavior extends to the full Hausdorff measure $\mathcal{H}^s(F)$, which is thus non-increasing with respect to s.

To understand the decay more concretely, suppose $t > s$, and consider any δ-cover $\{U_i\}$ of F. Then:

$$\sum_i (\text{diam}(U_i))^t = \sum_i (\text{diam}(U_i))^{t-s} (\text{diam}(U_i))^s \leq \delta^{t-s} \sum_i (\text{diam}(U_i))^s.$$

Taking the infimum over all such covers yields:

$$\mathcal{H}^t_\delta(F) \leq \delta^{t-s} \mathcal{H}^s_\delta(F).$$

Letting $\delta \to 0$, this inequality implies that if $\mathcal{H}^s(F) < \infty$, then for any $t > s$, the measure $\mathcal{H}^t(F) = 0$. Thus, the transition from ∞ to 0 as s increases must occur sharply at a specific critical value.

This transition point is known as the Hausdorff dimension.

Definition 4.3.2 (Hausdorff Dimension)**.** *The Hausdorff dimension of a subset $F \subset \mathbb{R}^n$, denoted $\dim_H F$, is defined by:*

$$\dim_H F := \inf\{s > 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

(With the convention that the supremum of an empty set is taken as zero.)

The behavior of $\mathcal{H}^s(F)$ is then characterized as:

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } 0 \leq s < \dim_H F, \\ 0 & \text{if } s > \dim_H F. \end{cases}$$

At the exact value $s = \dim_H F$, the measure $\mathcal{H}^s(F)$ could take any value in $[0, \infty]$, including finite nonzero quantities:

$$0 < \mathcal{H}^s(F) < \infty.$$

This construction and its sharp transition point provide a powerful way to quantify the "size" of fractals and irregular sets beyond traditional integer-valued dimensions.

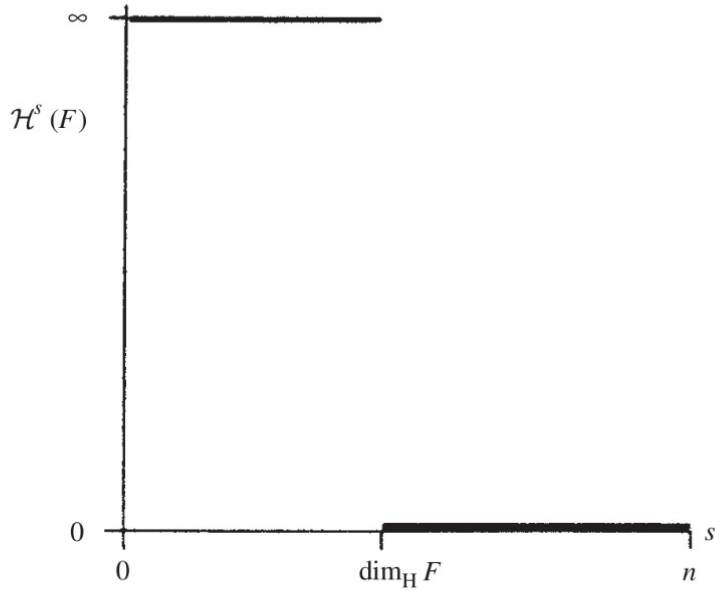


Figure 4.5: *This graph shows how $\mathcal{H}^s(F)$ changes with s. The Hausdorff dimension is the point where the value suddenly drops from ∞ to 0.*

4.4 Hausdorff dimension of middle-third Cantor set

The Cantor middle-third set $C \subset [0,1]$ is built step by step by removing the open middle third from each interval at every stage. After n steps, the set is made up of 2^n intervals, and each one has length 3^{-n}.

To find the Hausdorff dimension of the set C, we use the definition of the Hausdorff s-measure. In a metric space (X,d), the Hausdorff s-measure of a set $E \subset X$ is defined as follows:

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \inf \left\{ \sum_i (\operatorname{diam}(U_i))^s : E \subset \bigcup_i U_i,\ \operatorname{diam}(U_i) < \delta \right\}.$$

The Hausdorff dimension of a set E is then given by

$$\dim_H(E) = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = \infty\}.$$

For the Cantor set, we can use the covering at the nth step of its construction: 2^n intervals of length 3^{-n}. This gives the estimate

$$\mathcal{H}^s(C) \leq \lim_{n\to\infty} 2^n \cdot (3^{-n})^s = \lim_{n\to\infty} \left(2 \cdot 3^{-s}\right)^n.$$

Let $f(s) = (2 \cdot 3^{-s})^n$. The behavior of this expression determines the Hausdorff dimension:

- If $2 \cdot 3^{-s} < 1$, then $f(s) \to 0$ as $n \to \infty$, so $\mathcal{H}^s(C) = 0$.
- If $2 \cdot 3^{-s} > 1$, then $f(s) \to \infty$, so $\mathcal{H}^s(C) = \infty$.

The critical value occurs when $2 \cdot 3^{-s} = 1$, which gives

$$3^s = 2 \quad \Rightarrow \quad s = \frac{\log 2}{\log 3}.$$

Therefore, the Hausdorff dimension of the Cantor middle third set is

$$\dim_H(C) = \frac{\log 2}{\log 3} \approx 0.6309,$$

and the Hausdorff measure in this dimension is finite and positive:

$$0 < \mathcal{H}^{\log 2/\log 3}(C) < \infty.$$

5. Proof of Main Results

In Section Introduction, we showed our main results: Theorem 1.0.1 and Theorem 1.0.2.

Question 1: For a given $\sigma \in \{-1, 1\}^{\mathbb{N}}$, is it true that the set of σ-badly approximable numbers, BA^{σ}, has Lebesgue measure zero?

In this generality, the theorem is sharp in the sense that the converse is not always true. In other words, BA^{σ} may not be identical to BA. To justify this, we consider a special case: the backward continued fractions. An SRCF is called a *backward continued fraction* if σ consists of a constant sequence of negative ones. We let BA^{-} denote the set of badly approximable numbers with respect to backward continued fractions.

Theorem 1.0.2 provides an affirmative answer to Question 1 in the special case of backward continued fractions. We pose another related question.

Question 2: For a given $\sigma \in \{-1, 1\}^{\mathbb{N}}$, what is the Hausdorff dimension of the set of points in $BA^{\sigma} \cap BA^{c}$?

While Theorem 1.0.2 provides a partial answer for the backward case, the general case remains open.

The remainder of this paper is organized as follows: In the next section, we examine transformations of badly approximable numbers through specific operations, focusing on converting regular continued fractions into their semi-regular counterparts. This includes proving key results, such as Theorem 1.0.1, via lemmas addressing insertion and singularization. The section *Well-approximable numbers with bounded backward continued fraction quotients* investigates well-approximable numbers with bounded backward continued fraction quotients, culminating in the proof of Theorem 1.0.2 and employing fractal geometric techniques to derive lower bounds on their Hausdorff dimensions. Finally, in the section *Numerical Analysis*, we present numerical analyses exploring various choices of σ, highlighting the differences between semi-regular and regular continued fractions, and providing insights into the interplay between the structure of σ and the boundedness of partial quotients.

5.1 Transforming badly approximable numbers

In this section, we examine transformations of badly approximable numbers through specific operations. These transformations play a crucial role in converting regular continued fractions into semi-regular ones, a process essential for understanding the behavior of such numbers in different contexts. We begin by proving the main result of the section, Theorem 1.0.1, through a series of lemmas that address insertion and singularization. These transformations allow us to manipulate the continued fraction expansions in a controlled manner, ensuring that certain properties, such as the boundedness of partial quotients, are preserved. Through these transformations, we can gain deeper insights into the geometric properties of these sets and their relationship to regular continued fractions.

Lemma 5.1.1. *[7] The following transformations hold:*

1. *Insertion:*
$$a + \frac{\varepsilon}{1 + \frac{1}{b+\eta}} = a + \varepsilon + \frac{-\varepsilon}{b + 1 + \eta}. \tag{5.1.1}$$

2. *Singularization:*
$$a + \frac{1}{b + \eta} = a + 1 + \frac{-1}{1 + \frac{1}{b-1+\eta}}. \tag{5.1.2}$$

Proof. We prove each transformation separately.

For insertion, simplify the inner term:

$$a + \frac{\varepsilon}{1 + \frac{1}{b+\eta}} = a + \frac{\varepsilon}{\frac{b+\eta+1}{b+\eta}} = a + \frac{\varepsilon(b+\eta)}{b+\eta+1} = a + \varepsilon + \frac{-\varepsilon}{b+1+\eta}.$$

For singularization, starting with the left-hand side:

$$a + \frac{1}{b+\eta} = a + 1 + \left(-1 + \frac{1}{b+\eta}\right).$$

Since

$$-1 + \frac{1}{b+\eta} = \frac{-b-\eta+1}{b+\eta} = \frac{-1}{\frac{b+\eta}{b+\eta-1}} = \frac{-1}{1 + \frac{1}{b-1+\eta}},$$

we obtain the desired result. □

In transforming a regular continued fraction into a semi-regular, the idea is to implement insertion when $B = 1$ and otherwise apply singularization. For any

$M > 1$ we let BA_M denote the real numbers with partial quotients bounded by M. The following lemma will be used to utilize the induction argument.

Lemma 5.1.2. *Let a be a positive integer and $x, M > 1$. If $a + \frac{1}{x} \in BA_M$, then*

$$a + \frac{1}{x} = a + 1 + \frac{-1}{x'}$$

with $x' \in BA_M$ and $1 \leq [x'] \leq M + 1$.

Proof. Since $1 + \frac{1}{x} \in BA_M$, the continued fraction $[a_0(x); a_1(x), a_2(x), \dots]$ of x satisfies $a_i(x) \leq M$ for all $i \geq 0$.

If $a_0(x) = 1$, then we apply insertion to obtain

$$a + \frac{1}{x} = a + \frac{1}{a_0(x) + \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{\ddots}}}} = a + 1 + \frac{-1}{a_1(x) + 1 + \frac{1}{a_2(x) + \frac{1}{\ddots}}}.$$

So, clearly $x' = a_1(x) + 1 + \frac{1}{a_2(x) + \frac{1}{\ddots}} \in BA_M$ and $[x'] = a_1(x) + 1 \in [1, M+1]$..

If $a_0(x) \neq 1$ then $x > 1$ implies $a_0(x) = [x] > 1$. Thus, we may apply singularization to obtain

$$a + \frac{1}{x} = a + \frac{1}{a_0(x) + \frac{1}{a_1(x) + \frac{1}{\ddots}}} = a + 1 + \frac{-1}{1 + \frac{1}{a_0(x) - 1 + \frac{1}{a_1(x) + \frac{1}{\ddots}}}}.$$

In this case we have

$$x' = 1 + \frac{1}{a_0(x) - 1 + \frac{1}{a_1(x) + \frac{1}{\ddots}}} \in BA_M \subset BA_M.$$

In either case we get $x' \in BA_M$ with $[x'] = a_0 - 1 \in [1, M-1]$.

$\square$

Lemma 5.1.3. *For a given $x \in BA$, assume that the regular continued fraction expansion of x is*

$$[a_0; a_1(x), a_2(x), a_3(x), \dots],$$

with $a_i \leq M$ for some $M > 0$. Then, for any choice of σ, the corresponding semi-regular continued fraction

$$[a_0(x, \sigma); (\sigma_1, a_1(x, \sigma)), (\sigma_2, a_2(x, \sigma)), \dots, (\sigma_n, a_n(x, \sigma))]$$

satisfies $a_i(x,\sigma) \leq M+2$ *for all* $i \geq 0$.

We note that in the special case of backward continued fractions this is already established in[7]. In fact, Dajani and Kraaikamp provide precise formula for converting regular continued fractions into backward continued fractions, see[7, Proposition 2].

Proof. We prove this by induction on the index n, considering the process of transforming a regular continued fraction into a semi-regular continued fraction.

Base Case ($n = 0$):

For $x \in (0,1)$, we have $a_0(x) = 0$. If $\sigma_1 = 1$, we retain $a_0(x,\sigma) = a_0(x) = 0$. Otherwise, if $\sigma_1 = -1$, we need to adjust the representation of x based on the value of $a_1(x)$:

If $a_1(x) = 1$, we apply insertion:

$$x = 0 + \frac{1}{1 + \frac{1}{a_2(x)+\xi}} = 1 + \frac{-1}{a_2(x) + 1 + \xi}.$$

Thus, in the semi-regular representation, we get $a_1(x,\sigma) = a_2(x) + 1 \leq M+1$.

If $a_1(x) > 1$, we apply singularization:

$$x = 0 + \frac{1}{a_1(x) + \xi} = 1 + \frac{-1}{1 + \frac{1}{a_1(x)-1+\xi}}.$$

In this case, we see that $a_1(x,\sigma) = a_1(x) - 1 \leq M+1$.

In either case, we find that $a_1(x,\sigma) \leq M+1$, completing the base case.

Inductive Step ($n > 0$):

Assume that for all $i \leq n-1$, the partial quotients satisfy $a_i(x,\sigma) \leq M+2$. We now prove that $a_n(x,\sigma) \leq M+2$.

At each step, the transformation from regular to semi-regular form involves either insertion or singularization, as described in the preceding lemma. Let us consider the general semi-regular transformation for $a + \frac{e}{b+\xi}$:

During insertion, the transformation increases the first partial quotient by 1, while leaving the rest of the continued fraction unchanged:

$$a + \frac{e}{1 + \frac{1}{b+\xi}} = a + e + \frac{-e}{b+1+\xi}.$$

During singularization, the transformation similarly adjusts the partial quotients, with the second partial quotient being increased by at most 1:

$$a + \frac{1}{b+\xi} = a + 1 + \frac{-1}{1 + \frac{1}{b-1+\xi}}.$$

By the inductive hypothesis, we know that prior partial quotients remain bounded by $M + 2$. When a transformation is applied to x, the following effects are observed: 1. The first relevant partial quotient $a_n(x, \sigma)$ is increased by at most 1. 2. Any subsequent partial quotients $a_{n+1}(x, \sigma), a_{n+2}(x, \sigma), \ldots$ remain bounded by $M + 2$.

Since each step increases any given partial quotient by at most 1, and we begin with partial quotients bounded by M, it follows that all semi-regular partial quotients satisfy $a_i(x, \sigma) \leq M + 2$.

Thus, we conclude that for all $n \geq 0$, the semi-regular continued fraction satisfies $a_i(x, \sigma) \leq M + 2$. This completes the proof.

□

Proof of Theorem 1.0.1. The first part of the theorem follows from Lemma 5.1.3. The second part is a consequence of the fact that $\dim BA = 1$[2], which implies that

$$1 = \dim_H BA \leq \dim_H BA^\sigma.$$

□

5.2 Well-approximable numbers with bounded backward continued fraction quotients

In this section, we want to prove Theorem 1.0.2. To do this, it is enough to find a lower bound for the Hausdorff dimension of the set we are studying. We use a well-known method from fractal geometry, explained in [1, Example 4.6]. This method looks at a sequence of sets made from intervals. We start with $E_0 = [0, 1]$, and then build new sets $E_1, E_2, \ldots$ by dividing the intervals into smaller pieces. By knowing how many pieces we have and how big the gaps between them are, we can apply a general result that gives us a lower bound for the Hausdorff dimension of the final set. This helps us estimate the dimension needed to prove Theorem 1.0.2.

Theorem 5.2.1. *Suppose we have a sequence of sets $\{E_k\}$, where we start with*

$E_0 = [0, 1]$, *and each new set is made by taking smaller and smaller intervals:*

$$E_0 \supset E_1 \supset E_2 \supset \cdots$$

Each set E_k contains a limited number of separate, closed intervals. Assume the following:

- *Every interval in E_{k-1} contains at least $m_k \geq 2$ non-overlapping intervals from E_k.*
- *The intervals in E_k are separated by gaps, and the size of these gaps is at least δ_k, with each δ_k getting smaller as k increases.*

Now, let $F = \bigcap_{k=0}^{\infty} E_k$ be the set that remains after infinitely many steps. Then the Hausdorff dimension of F is at least:

$$\dim_H(F) \geq \liminf_{k\to\infty} \frac{\log(m_1 m_2 \cdots m_{k-1})}{-\log(\delta_k m_k)}.$$

Lemma 5.2.2. *Let x be a real number in the interval $[0, 1)$, and let (a_n) be the sequence of partial quotients in the regular continued fraction expansion of x. Then, x belongs to the set BA^- if and only if the sequence made from the even-positioned terms (a_{2n}) stays within a fixed upper limit (i.e., is bounded).*

The proof follows easily from [7, Proposition 2] and is therefore omitted.

We now proceed to prove Theorem 1.0.2. We recall that, see e.g. [3, 8], for the regular continued fractions the convergents $\frac{p_n}{q_n}$ satisfy

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2}, \\ q_n &= a_n q_{n-1} + q_{n-2}, \end{aligned} \tag{5.2.1}$$

for $n \geq 1$ with initial values $p_0 = a_0, p_{-1} = 1, q_0 = 1, q_{-1} = 0$, and

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^n. \tag{5.2.2}$$

To evaluate the Hausdorff dimension, we utilize a specific family of nested intervals determined by continued fraction expansions. For each $n \in \mathbb{N}$, define:

$$I_n(a_1, a_2, \dots, a_n) = \{x \in (0, 1) : a_1(x) = a_1, \dots, a_n(x) = a_n\},$$

which identifies the set of real numbers in $(0, 1)$ having fixed first n partial quotients in their continued fraction representation. As established in the literature (see, e.g., [3, 9]), each such set forms a closed interval. Specifically:

$$I_n(a_1,\ldots,a_n) = \begin{cases} \left[\dfrac{p_n}{q_n}, \dfrac{p_n+p_{n-1}}{q_n+q_{n-1}}\right) & \text{if } n \text{ is even,} \\ \left(\dfrac{p_n+p_{n-1}}{q_n+q_{n-1}}, \dfrac{p_n}{q_n}\right] & \text{if } n \text{ is odd,} \end{cases} \tag{5.2.3}$$

where p_n/q_n denotes the nth convergent associated to the continued fraction with digits $a_1,\ldots,a_n$.

Proof of Theorem 1.0.2. According to Lemma 5.2.2, a number $x \in (0,1)$ lies in the set $BA^- \cap BA^c$ precisely when the subsequence of even-indexed partial quotients $(a_{2n}(x))$ is unbounded.

To bound from below the Hausdorff dimension of $BA^- \cap BA^c$, it suffices to construct a family of subsets whose dimensions can be made arbitrarily close to 1. Fixing any integer $M \geq 2$, define:

$$F(M) := \left\{ x \in (0,1) \mid \forall n \geq 1, \begin{cases} n \leq a_n \leq n+1 & \text{if } n \text{ is a power of 2,} \\ 1 \leq a_n \leq M & \text{otherwise} \end{cases} \right\}.$$

Using Lemma 5.2.2, we confirm $F(M) \subseteq BA^- \cap BA^c$ for all M.

To analyze $\dim_H F(M)$, we invoke Theorem 5.2.1. Let $E_n(M)$ be the nth-level truncation:

$$E_n(M) := \{x \in (0,1) \mid \text{conditions on } a_k \text{ for all } k \leq n \text{ as above}\},$$

and clearly:

$$F(M) = \bigcap_{n=1}^{\infty} E_n(M).$$

Denote the parent interval at level n by $J_n(a_1,\ldots,a_n)$, representing the closure of all descendant intervals:

$$J_n(a_1,\ldots,a_n) := \overline{\bigcup_{a_{n+1}} I_{n+1}(a_1,\ldots,a_{n+1})},$$

where a_{n+1} belongs to a small or bounded range depending on whether $n+1$ is a power of two.

From this recursive structure, we compute:

$$m_k = \begin{cases} 2 & \text{if } k \text{ is a power of 2,} \\ M & \text{otherwise,} \end{cases} \tag{5.2.4}$$

where m_k is the number of subintervals at stage k.

To bound the separation gap δ_k, we consider the minimal length of subintervals $J_n(\cdot)$. When $n+1$ is a power of two, we use explicit bounds:

$$|J_n| < \frac{2}{(n+1)(n+3)q_n^2},$$

and otherwise:

$$|J_n| < \frac{M}{(M+1)q_n^2}.$$

Hence, conservatively:

$$\delta_k \geq \frac{1}{k^3(M+1)^{2(k+1)}}.$$

Combining with (5.2.4), we compute:

$$\liminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(\delta_k m_k)} = \frac{\log M}{2\log(M+1)}.$$

Consequently, by Theorem 5.2.1:

$$\dim_H(F(M)) \geq \frac{\log M}{2\log(M+1)}.$$

Since $F(M) \subseteq BA^- \cap BA^c$, we conclude:

$$\dim_H(BA^- \cap BA^c) \geq \frac{1}{2}.$$

□

5.3 Numerical analysis

We explore various σ values to provide numerical evidence regarding the potential contents of $BA^\sigma \cap BA^c$. Figure 5.1 presents the density distribution of the differences in the maximum values of the partial quotients for two different continued fraction representations of rational numbers. Specifically, we fix 10,000 rational numbers in the interval $(0,1)$ with denominators up to 1000. For each rational number x, we generate a random $\sigma \in \{-1,1\}^{30}$. For each x, we compute the difference $\max(a_n(x)) - \max(a_n(x,\sigma))$, and each plot shows the density of these differences for a given σ. The results confirm that these differences are bounded below by -2, consistent with Lemma 5.1.3. This observation provides numerical validation of the boundedness of semi-regular continued fraction partial quotients relative to their regular counterparts, highlighting the stability introduced by the semi-regular framework.

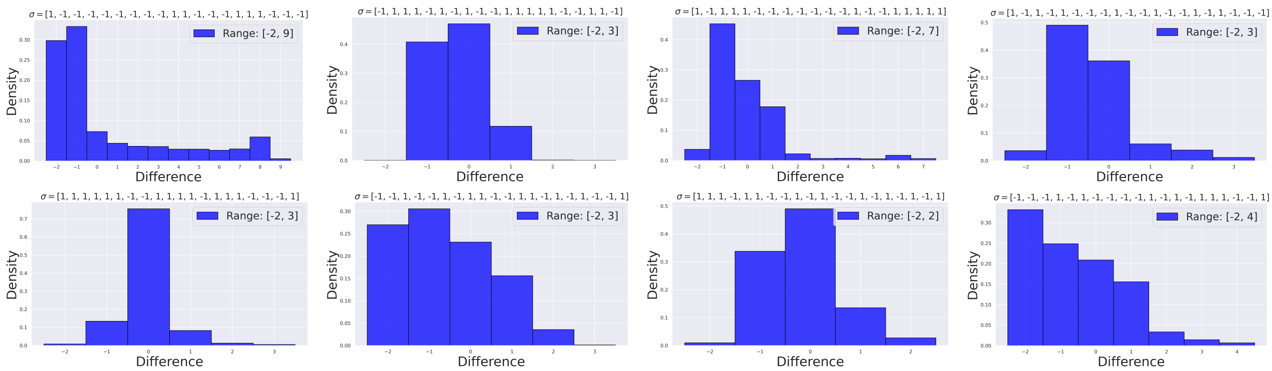
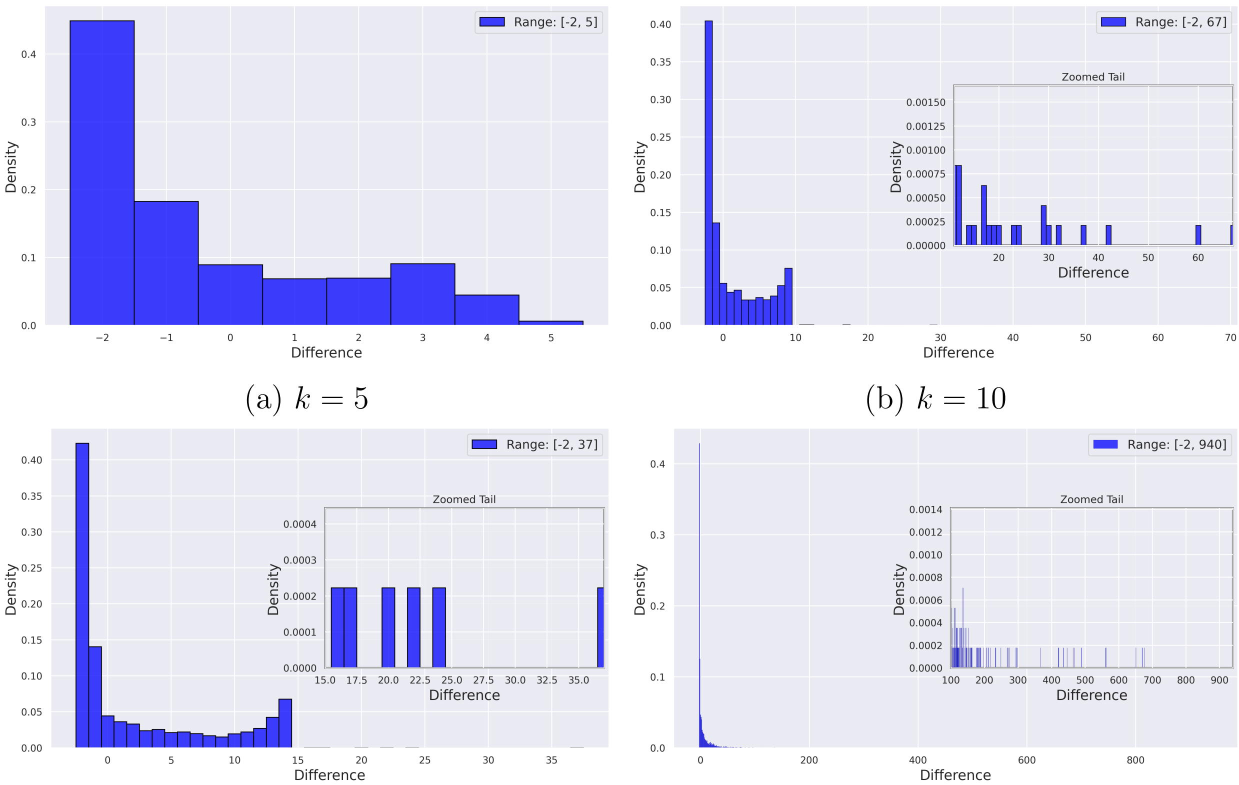


Figure 5.1: Distribution of the differences between the maximum values of two continued fraction representations, $max((a_n(x)) - max(a_n(x, \sigma))$

Figure 5.2 investigates periodic sequences σ, where σ alternates with k consecutive -1's followed by 1. In particular, Figure 5.2d gives the distribution for backward continued fraction. The distribution of differences between maximum partial quotients demonstrates how the periodicity of σ influences these values. As $k \to \infty$, the differences align closely with the backward continued fraction framework, emphasizing the importance of sequence periodicity in shaping the behavior of partial quotients. Figure 5.3 explores the impact of random variations in σ on fixed rational numbers x. It compares distributions of differences for distinct x, with each value analyzed under 10,000 random σ. The results show variability in the distributions, underscoring the dependency of the difference magnitude on the specific properties of x. These findings suggest a complex interplay between the structure of σ and the inherent characteristics of x in determining the maximal partial quotients.

The numerical analysis highlights that the existence of long consecutive -1's in the sequence σ can introduce notable differences between semi-regular and regular continued fraction representations. This observation is particularly evident in periodic sequences with extended stretches of -1, as these tend to deviate significantly from the behavior seen in regular continued fractions. While Theorem 1.0.2 establishes such a difference in the specific case of backward continued fractions, the broader question remains unresolved: whether there exists a sequence σ and a well-approximable number x such that x has bounded partial quotients in its σ-semi-regular continued fraction representation. This open problem underscores the need for further investigation into the interaction between the structure of σ and the properties of the corresponding semi-regular continued fraction, as well as its implications for the classification of well-approximable numbers.



(a) $k = 5$ (b) $k = 10$

(c) $k = 15$ (d) $k = \infty$

Figure 5.2: Distribution of the differences for a periodic sequence σ, consisting of k consecutive -1's followed by 1

5.4 Methodology numerical fractal analysis of exceptional sets in the Lehner expansion

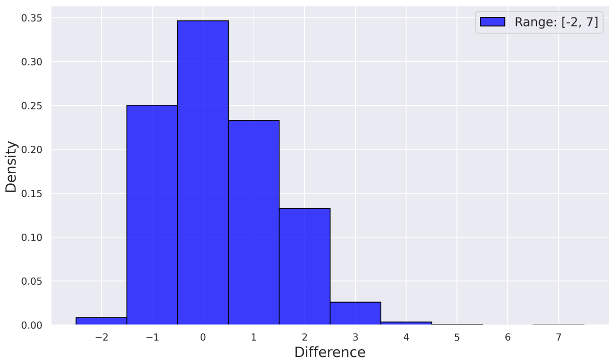
The box dimension of a set G is defined as

$$\dim_B(G) = \lim_{\varepsilon \to 0} \frac{\log M_\varepsilon(G)}{-\log \varepsilon},$$

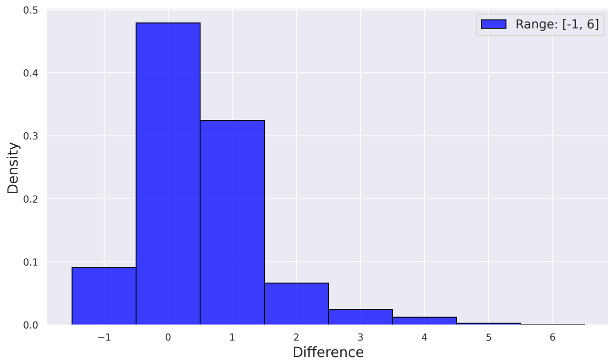
where $M_\varepsilon(G)$ is the number of small boxes of size ε needed to cover the set G. This value d tells us how the set behaves when we look at it more closely using smaller and smaller boxes.

where $N(\delta)$ is the number of boxes size δ required to cover set S. This dimension s characterizes the fractal scaling behavior of the set as the solution δ decreases.

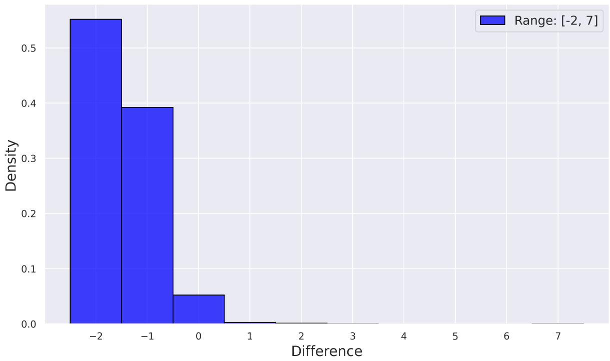
To estimate the box dimension numerically, we analyze the scaling behavior of unique truncated binary words derived from points in a given set. Each point in the set is mapped to a binary expansion with fixed precision by repeatedly multiplying the point by two. If the result is at least one, '1' is appended to the binary string, and one is subtracted from the point; otherwise, '0' is appended. This process is repeated for the desired precision; see Figure 5.4



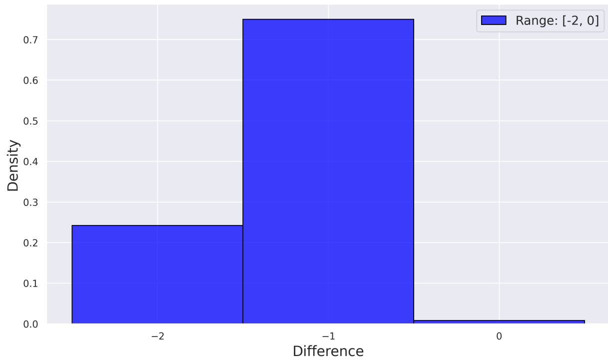
(a) $x = [0; 5, 7, 8, 6, 6, 10, 5]$



(b) $x = [0; 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$



(c) $x = [0; 10, 1, 10, 1, 10, 1, 10, 1, 10]$



(d) $x = [0; 4, 4, 4, 4, 4, 4, 4, 4, 4, 4]$

Figure 5.3: Distribution of the differences for a fixed rational x and 10000 varying σ.

To estimate the complexity of the given set, each binary expansion is truncated to a fixed word length, meaning that only the first few digits of the binary representation are considered. For each chosen word length, the number of unique binary words (subsequences of that length) is counted. This process is repeated for multiple word lengths, allowing us to analyze how the number of distinct binary words grows as the word length increases.

Next, the base-2 logarithm of the number of unique binary words is computed for each word length. This step helps to transform the data into a form that reveals scaling properties. The resulting data points, which represent the relationship between word length and the logarithm of the unique word count, are then analyzed using linear regression. Linear regression is used to fit a trend line to the data, which captures the overall pattern of growth.

Once the trend line is obtained, the box dimension of the set is determined by the slope of the regression line. This slope quantifies how the number of unique binary words scales with word length and provides a numerical measure of the set's complexity. A higher slope indicates greater complexity, while a lower slope suggests a more structured or predictable pattern in the binary expansions.

```
Input:
    x - a real number in [0,1)
    precision - number of bits in the binary expansion

Output:
    binary_expansion - a string representing the binary expansion

Initialize binary_expansion as an empty string

For i from 1 to precision do
  x ← 2x
  If x ≥ 1 then:
    Append "1" to binary_expansion
    x ← x − 1
  Else:
    Append "0" to binary_expansion

Return binary_expansion
```

Figure 5.4: Algorithm to compute the binary expansion of real numbers

The computational procedure follows these steps: The Set points are first converted to binary expansions of a specified precision. For each word length, the binary expansions are truncated, and the number of unique words is counted. The $\log_2$ of of the unique word count is computed and stored. Then a linear regression is performed on the relationship between word length and $\log_2$ count, and the slope of the regression line is returned as the estimated box dimension. The results are visualized through a regression plot that shows the relationship between word length and $\log_2$ of unique word counts, where the slope of the fitted trend line provides an approximation of the box dimension of the underlying fractal set.The following pseudocode Figure 5.5 summarizes the computational procedure:

To carry out the experiments we generated one million points uniformly from the interval [1,2]. The distribution of denominator averages of these numbers are depicted in Figure 5.6

Input: Set points, precision p, max word length L

Output: Estimated box dimension

1. Convert each set point to a binary expansion of length p.

2. Initialize an empty list for log_2 counts.

3. For each word length l from 1 to L:

   a. Truncate each binary expansion to the first l bits.

   b. Count the number of unique truncated words.

   c. Compute log_2 of the unique word count and store it.

4. Perform linear regression on (word length, log_2 count) pairs.

5. Return the slope of the regression line as the estimated box dimension.

Figure 5.5: Algorithm to numerically compute the box dimension

5.5 Numerical fractal analysis of exceptional sets in the Lehner expansion

Figure 5.7 provide numerical results for estimating the Box dimension of $S(\epsilon, c)$ for fixed ϵ=0.01 and c ranging from 1.60 to 1.95.Our numerical investigation of the box dimension of $S(\epsilon, c)$ reveals a clear dependence on c.Using the binary word-based box-counting method, we estimated the box dimension of these exceptional sets. Linear regression of log-transformed unique binary word counts against word length yielded a slope that characterizes the fractal scaling behavior of $S(\epsilon, c)$.

The Figure 5.8 suggests that as c increase, the box dimension stabilizes, reinforcing the theoretical expectation that the set of exceptions forms a measure-zero yet structurally complex subset.

Figure 5.9(a) depicts a fractal structure generated from continued fraction expansions with a restricted digit set { 1, 2, 3, 4 }. The x and y coordinates correspond to values derived from odd- and even-indexed terms of randomly generated

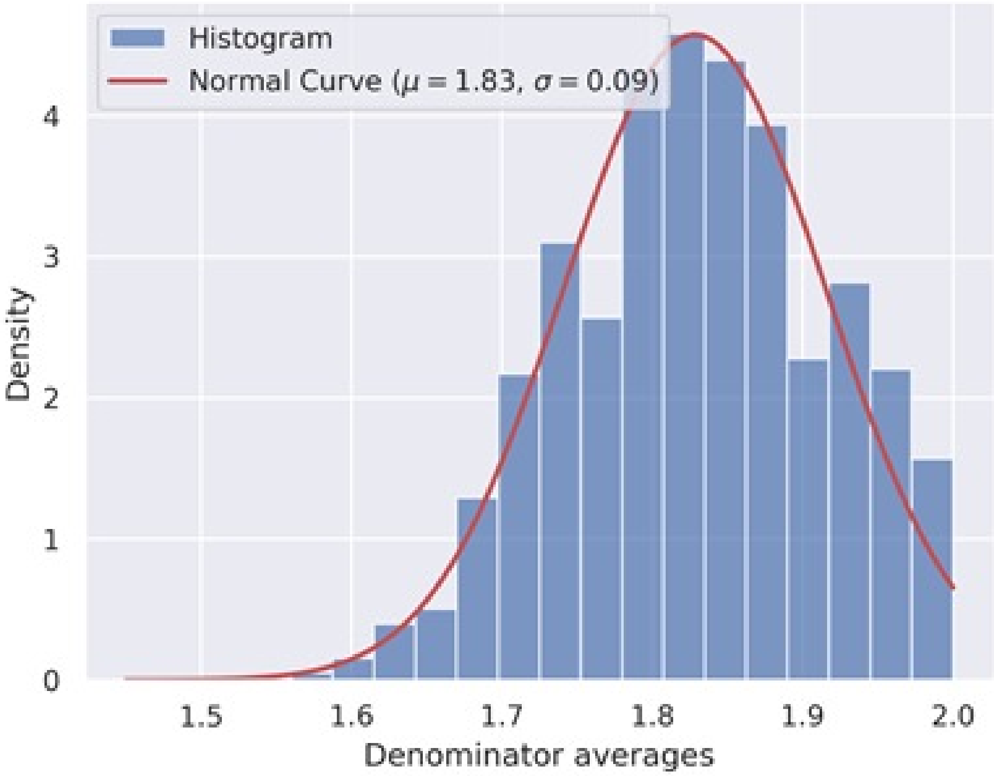
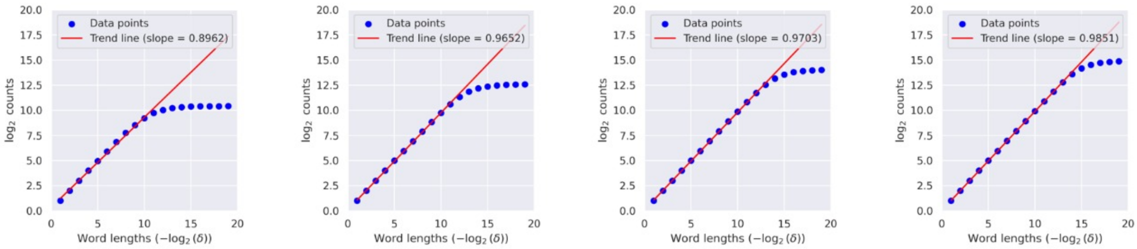
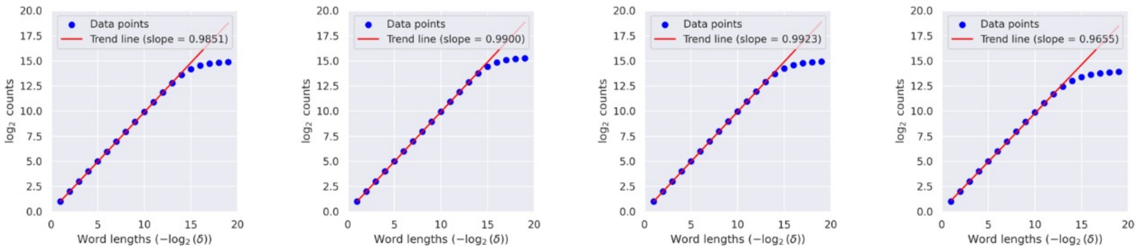


Figure 5.6: Histogram plot of relative frequency distribution of average denominators of Lehner expansion

continued fraction sequences. The resulting structure reveals an intricate, self-similar distribution within the unit square, illustrating how different digit choices influence the fractal pattern. Figure 5.9 (b) shows a similar fractal formation, but based on the Lehner expansion, a variant of continued fraction representation defined for numbers in the interval [1,2]. Here, the x and y coordinates are determined by evaluating the odd- and even-indexed Lehner terms as continued fractions. The clustering and density variations within the bounded region reflect the distinctive number-theoretic properties of the Lehner transformation and its role in generating self-similar structures.



(a) $S(0.01,1.60)$ (b) $S(0.01,1.65)$ (c) $S(0.01,1.70)$ (d) $S(0.01,1.75)$



(e) $S(0.01,1.80)$ (f) $S(0.01,1.85)$ (g) $S(0.01,1.90)$ (h) $S(0.01,1.95)$

Figure 5.7: Numerical box dimension estimates of $S(\epsilon, c)$ for ϵ=0.01 and varying c.

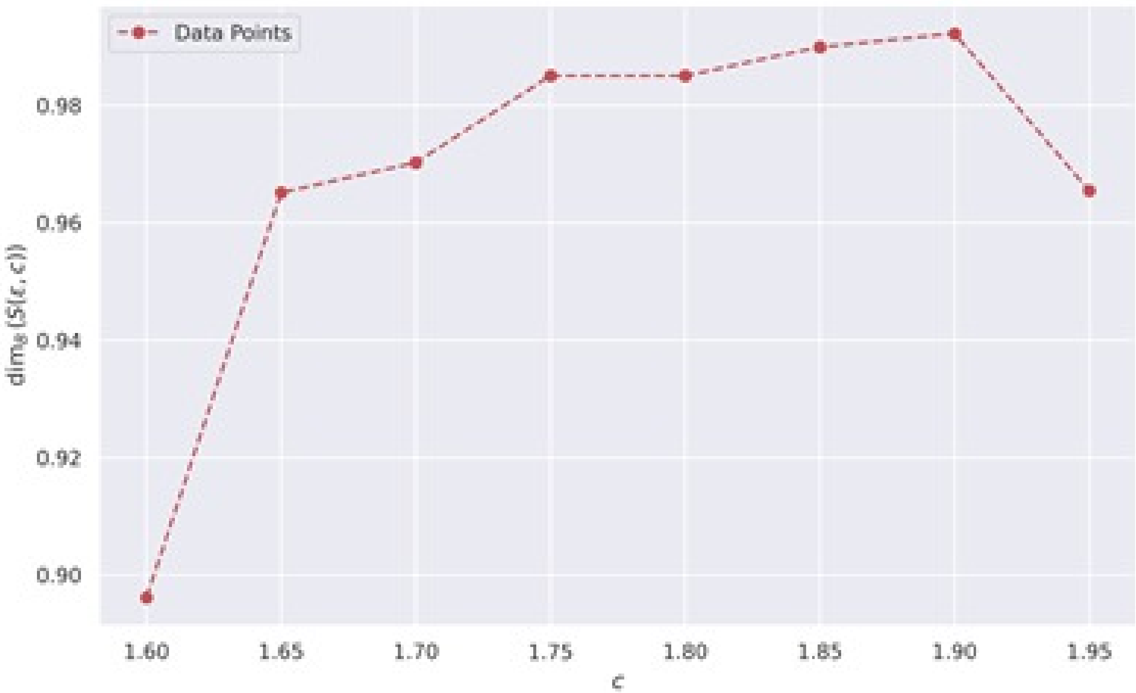
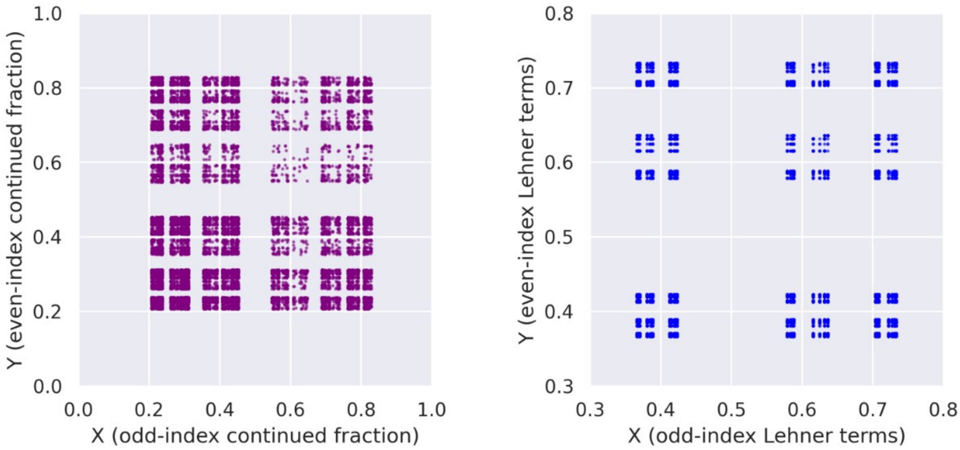


Figure 5.8: The graph of box dimensions of $S(\epsilon, c)$ as c changes from 1.60 to 1.95



(a) 2D fractal structure of regular continued fractions

(b) 2D fractal structure of Lehner continued fractions

Figure 5.9: Two-Dimensional Fractal Structures from Continued Fraction Expansions

6. Conclusion

In this thesis, we explored the concept of σ-badly approximable numbers in the setting of semi-regular continued fractions and investigated the fractal geometry of exceptional sets arising from Lehner expansions. By extending the classical notion of badly approximable numbers, we defined the class of σ-badly approximable numbers and proved that these sets always have full Hausdorff dimension, equal to 1. This result highlights their rich geometric structure, despite having zero Lebesgue measure, and establishes a strong connection to regular badly approximable numbers, which they strictly contain.

Complementing this theoretical framework, we studied the Lehner continued fraction expansion and examined how the average value of the digit sequence b_n influences the box dimension of associated exceptional sets $S(\epsilon, c)$. Using a binary word-based box-counting method, our numerical experiments revealed that the box dimension stabilizes as the average c increases, indicating a non-trivial and self-similar fractal structure. These findings reinforce the utility of continued fraction expansions, including non-classical forms like the Lehner expansion, in modeling complex geometric behaviors in number theory.

Together, these investigations demonstrate that continued fractions—both semi-regular and Lehner types—serve as a powerful framework for understanding the interplay between Diophantine approximation and fractal geometry. Theoretical results on σ-badly approximable numbers, combined with numerical analysis of Lehner expansions, reveal deep structural insights into exceptional sets of real numbers.

Future work may involve a deeper measure-theoretic analysis of σ-badly approximable sets, further study of the transition behavior in box dimension for varying ϵ and c, and comparisons across different classes of continued fractions such as backward and even multi-dimensional variants. This thesis contributes to the broader understanding of how arithmetic complexity translates into geometric irregularity, offering both rigorous results and computational perspectives that can inform ongoing research in number theory and dynamical systems.

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Article

Fractal Dimension of Exceptional Sets in Semi-Regular Continued Fractions

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Abstract

This paper examines how the average value of the sequence b_n in the Lehner expansion of a real number x influences its box dimension. Our primary objective is to analyze how variations in the average of b_n impact the box dimension, which serves as a measure of the complexity of the sequence. Using the box-counting method, we numerically estimate the box dimension and explore its relationship with the fractal nature of Lehner expansions.

Keywords:Regular continued fraction, Lehner expansion, semi-regular continued fraction, box dimension.

I. Introduction

Continued fractions [1]–[4] play important role in number theory,this way of writing numbers is very useful in number theory because it helps us understand how well we can approximate real numbers using fractions.A semi-regular continued fraction [7] is a special type of continued fraction that extends classical regular continued fractions by allowing a broader set of partial quotients while keeping important mathematical properties.One such semi-regular expansion is the Lehner continued fraction, which has been investigated for its unique convergence behavior and number-theoretic significance (Lehner, 1949).
Any irrational number $x \in [1, 2]$ has a unique Lehner expansion of the form

$$b_0 + \cfrac{\sigma_1}{b_1 + \cfrac{\sigma_2}{b_2 + \cdots + \cfrac{\sigma_n}{b_n + \dots}}} = [b_0; \sigma_1/b_1, \sigma_2/b_2, \dots, \sigma_n/b_n, \dots] \tag{1}$$

where (b_i, σ_{i+1}) equals (1,1) or (2,-1).We call these continued fractions Lehner fractions or Lehner expansions. Every rational number has two different finite Lehner expansions.

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Lehner expansions can be found using this map $L : [1, 2) \to [1, 2)$, which is defined as follows.

$$Lx := \begin{cases} \frac{1}{2-x}, & 1 \le x < \frac{3}{2}, \\ \frac{1}{x-1}, & \frac{3}{2} \le x < 2. \end{cases}$$

Notice that in this expansion for $x \in [1, 2)$ one has

$$(b_i, \sigma_{i+1}) = \begin{cases} (1, 1), & \text{if } L^i(x) \in \left[\frac{3}{2}, 2\right), \\ (2, -1), & \text{if } L^i(x) \in \left[1, \frac{3}{2}\right). \end{cases}$$

Lehner expansions are a type of semi-regular continued fraction. A semi-regular continued fraction can be either a finite or an infinite fraction.

The study of exceptional sets in continued fractions has been a focus of recent research, exploring their fractal properties and Hausdorff dimensions. Fang et al. [8] determined the Hausdorff dimension of a set related to the growth rate of continued fraction coefficients. Kazin and Kadyrov [5] extended Good's work on fractal geometry in continued fractions, establishing new bounds on Hausdorff dimensions of level sets formed by restricting partial quotients. Bakhtawar et al. [9] calculated the Hausdorff dimension of a set defined by conditions on ratios of consecutive continued fraction coefficients, contributing to the metrical theory of continued fractions. While not directly addressing continued fractions, Parsell and Wooley [10] investigated exceptional sets for Diophantine inequalities, showing that under certain conditions, the measure of the exceptional set in an interval is bounded. These studies collectively advance our understanding of exceptional sets in number theory and their geometric properties fractal properties of these sets, particularly their Hausdorff and box dimensions, have been the subject of extensive research [11]. For regular continued fractions, the dimension of sets defined by constraints on their partial quotients has been thoroughly examined [4], [12]. However, for semi-regular expansions such as the Lehner continued fraction, a comprehensive understanding of these exceptional sets remains incomplete.

Fractal dimension measures how completely a fractal fills space as one zooms in on finer scales. Unlike traditional Euclidean dimensions, which take integer values (e.g., a line has dimension 1, a plane has dimension 2), fractal dimensions can be non-integer, reflecting the complexity and self-similarity of fractal structures. It quantifies how detail in a pattern changes with the scale at which it is measured, making it useful for characterizing irregular shapes in nature, such as coastlines, clouds, and turbulent flows. The Hausdorff dimension and box dimension are both types of fractal dimensions. For more information on how these various notions of dimension are related, we refer to [6]. In this paper, we focus only on the box dimension.The box dimension of set S is defined as

$$\dim_B(S) = \lim_{\delta \to 0} \frac{\log N(\delta)}{-\log \delta},$$

where $N(\delta)$ is the number of boxes size δ required to cover set S.

If this limit exists. This dimension captures how the number of covering elements scales with their size and provides a practical way to estimate fractal complexity.

Theorem [7,theorem4] *For almost all real numbers $x \in (1, 2)$, we have that their Lehner expansions*

$$x = [b_0; \sigma_1/b_1, \sigma_2/b_2, \cdots, \sigma_n/b_n, \cdots]$$

$$\lim_{n \to \infty} \frac{b_1 + b_2 + \cdots + b_n}{n} = 2.$$

In this work, we focus on the set of numbers for which the sequence of partial quotients b_n in the Lehner expansion exhibits an anomalous growth pattern, specifically cases where the long-term average deviates from its expected limit. Such deviations are known to correspond to fractal-like structures, whose complexity can be quantified using box dimension [13].Our objective is to determine how the box dimension of these exceptional sets depends on the asymptotic behavior of b_n, extending results known for regular continued fractions []. We consider those real numbers x for which the above limit is not equal to 2. By the theorem we know that this set has Lebesgue measure zero. However, it may have a complex structure from a fractal geometry point of view. To understand, for any $\epsilon > 0$ we define sets

$$S(\epsilon, c) = \left\{ x \in (1, 2) : \lim_{n \to \infty} \frac{b_1 + b_2 + \cdots + b_n}{n} \in (c - \epsilon, c + \epsilon) \right\}.$$

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Our research question is to numerically investigate how box dimension of $S(\epsilon, c)$ depends on ϵ. For the definition of box dimension, see the next section.

To achieve this, we employ a computational approach based on binary word representations, adapting established methods from multifractal analysis (Barreira & Schmeling, 2000). By numerically estimating the box dimension for different classes of exceptional sets, we provide new insights into the geometric complexity of Lehner continued fraction expansions. Our findings contribute to the broader understanding of fractal structures in number theory and highlight the rich interplay between continued fractions and dynamical systems.

The structure of the paper is as follows. In Section 2, we introduce the mathematical framework of continued Lehner fractions and review key definitions. Section 3 describes the methodology for computing the box dimension, detailing the binary word-based approach. Section 4 presents numerical results and discusses the implications of our findings. Finally, in Section 5, we summarize our conclusions and suggest directions for future research.

II. Methodology

The box dimension of set S is defined as

$$\dim_B(S) = \lim_{\delta \to 0} \frac{\log N(\delta)}{-\log \delta},$$

where $N(\delta)$ is the number of boxes size δ required to cover set S. This dimension characterizes the fractal scaling behavior of the set as the solution δ decreases.

To estimate the box dimension numerically, we analyze the scaling behavior of unique truncated binary words derived from points in a given set. Each point in the set is mapped to a binary expansion with fixed precision by repeatedly multiplying the point by two. If the result is at least one, '1' is appended to the binary string, and one is subtracted from the point; otherwise, '0' is appended. This process is repeated for the desired precision; see Fig.1.

```
Input:
    x - a real number in [0,1)
    precision - number of bits in the binary expansion

Output:
    binary_expansion - a string representing the binary expansion

Initialize binary_expansion as an empty string

For i from 1 to precision do
  x ← 2x
  If x ≥ 1 then:
    Append "1" to binary_expansion
    x ← x − 1
  Else:
    Append "0" to binary_expansion

Return binary_expansion
```

Fig. 1. Algorithm to compute the binary expansion of real numbers

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To estimate the complexity of the given set, each binary expansion is truncated to a fixed word length, meaning that only the first few digits of the binary representation are considered. For each chosen word length, the number of unique binary words (subsequences of that length) is counted. This process is repeated for multiple word lengths, allowing us to analyze how the number of distinct binary words grows as the word length increases.

Next, the base-2 logarithm of the number of unique binary words is computed for each word length. This step helps to transform the data into a form that reveals scaling properties. The resulting data points, which represent the relationship between word length and the logarithm of the unique word count, are then analyzed using linear regression. Linear regression is used to fit a trend line to the data, which captures the overall pattern of growth.

Once the trend line is obtained, the box dimension of the set is determined by the slope of the regression line. This slope quantifies how the number of unique binary words scales with word length and provides a numerical measure of the set's complexity. A higher slope indicates greater complexity, while a lower slope suggests a more structured or predictable pattern in the binary expansions.

The computational procedure follows these steps: The Set points are first converted to binary expansions of a specified precision. For each word length, the binary expansions are truncated, and the number of unique words is counted. The $\log_2$ of the unique word count is computed and stored. Then a linear regression is performed on the relationship between word length and $\log_2$ count, and the slope of the regression line is returned as the estimated box dimension. The results are visualized through a regression plot that shows the relationship between word length and $\log_2$ of unique word counts, where the slope of the fitted trend line provides an approximation of the box dimension of the underlying fractal set.The following pseudocode Fig.2. summarizes the computational procedure:

To carry out the experiments we generated one million points uniformly from the interval [1,2]. The distribution of denominator averages of these numbers are depicted in Fig.3.

III. Results and Discussion

Fig.4 provide numerical results for estimating the Box dimension of $S(\epsilon, c)$ for fixed ϵ=0.01 and c ranging from 1.60 to 1.95.Our numerical investigation of the box dimension of $S(\epsilon, c)$ reveals a clear dependence on c.Using the binary word-based box-counting method, we estimated the box dimension of these exceptional sets. By comparing the log of unique binary word counts to word length using linear regression, we found a slope that shows how $S(\epsilon, c)$ scales in a fractal way.

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Input: Set points, precision p, max word length L

Output: Estimated box dimension

1. Convert each set point to a binary expansion of length p.
2. Initialize an empty list for log_2 counts.
3. For each word length l from 1 to L:
   a. Truncate each binary expansion to the first l bits.
   b. Count the number of unique truncated words.
   c. Compute log_2 of the unique word count and store it.
4. Perform linear regression on (word length, log_2 count) pairs.
5. Return the slope of the regression line as the estimated box dimension.

Fig. 2. Algorithm to numerically compute the box dimension

The Fig. 5 suggests that as c increase, the box dimension stabilizes, reinforcing the theoretical expectation that the set of exceptions forms a measure-zero yet structurally complex subset.

Fig.6(a) depicts a fractal structure generated from continued fraction expansions with a restricted digit set $\{1, 2, 3, 4\}$. The x and y coordinates correspond to values derived from odd- and even-indexed terms of randomly generated continued fraction sequences. The resulting structure reveals an intricate, self-similar distribution within the unit square, illustrating how different digit choices influence the fractal pattern. Fig.6 (b) shows a similar fractal formation, but based on the Lehner expansion, a variant of continued fraction representation defined for numbers in the interval [1,2]. Here, the x and y coordinates are determined by evaluating the odd- and even-indexed Lehner terms as continued fractions. The clustering and density variations within the bounded region reflect the distinctive number-theoretic properties of the Lehner transformation and its role in generating self-similar structures.

IV. Conclusion

In this paper, we investigated the fractal properties of exceptional sets in the Lehner expansion by examining how the average value of the sequence b_n affects the box dimension. By employing numerical fractal analysis, we computed the box dimensions of

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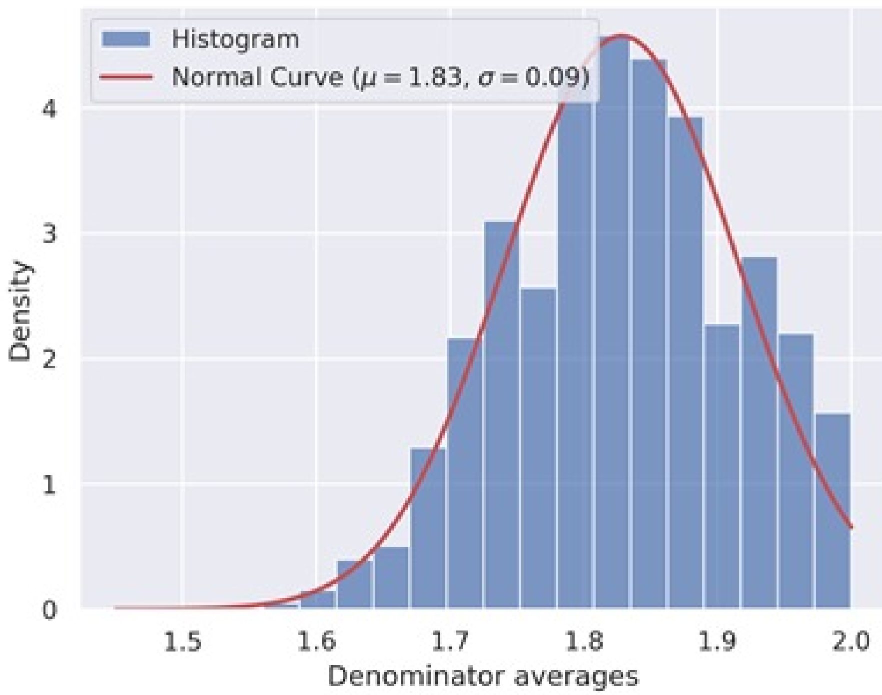

Fig. 3. Histogram plot of relative frequency distribution of average denominators of Lehner expansion

these exceptional sets using a binary word-based box-counting method. Our findings demonstrate that the box dimension of $S(\epsilon, c)$ exhibits a clear dependence on c, with the box dimension stabilizing as c increases. This aligns with the theoretical expectation that these sets, despite having Lebesgue measure zero, exhibit intricate fractal structures.

Our numerical results provide evidence that the exceptional sets in the Lehner expansion possess a non-trivial fractal nature, reinforcing the idea that continued fraction expansions offer a rich framework for studying complex structures in number theory. The observed self-similar patterns in Fig 6(a) and 6(b) further illustrate how the Lehner expansion differs from regular continued fractions while maintaining its own unique fractal characteristics. The histogram of denominator averages (Fig.3) and the scaling behavior of box dimensions (Fig.5) suggest that the complexity of these sets is deeply tied to the digit distributions in their continued fraction representations.

Future work could extend this study by exploring different ranges of ϵ and c to further characterize the transition behaviors of fractal dimensions. Additionally, a theoretical analysis of the scaling behavior observed in our numerical experiments could provide deeper insights into the number-theoretic properties of Lehner expansions. Overall, this study contributes to the growing body of research on the fractal geometry of exceptional sets in continued fraction theory, offering new perspectives on their complexity and structure.

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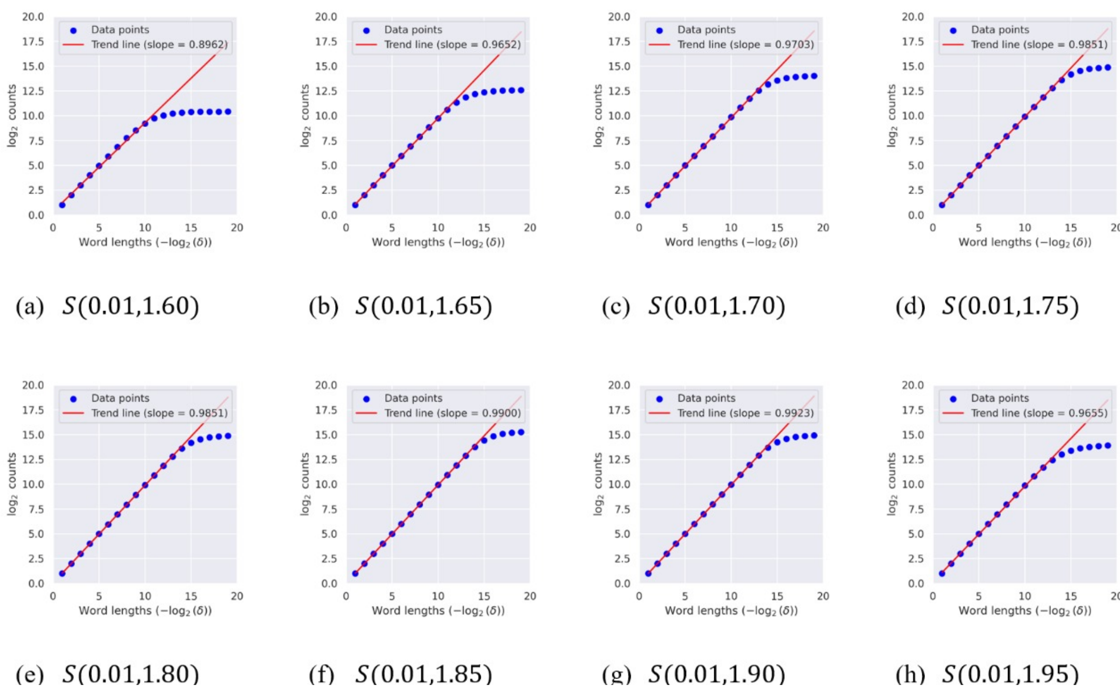

(a) $S(0.01,1.60)$ (b) $S(0.01,1.65)$ (c) $S(0.01,1.70)$ (d) $S(0.01,1.75)$

(e) $S(0.01,1.80)$ (f) $S(0.01,1.85)$ (g) $S(0.01,1.90)$ (h) $S(0.01,1.95)$

Fig. 4. Numerical box dimension estimates of $S(\epsilon, c)$ for ϵ=0.01 and varying c.

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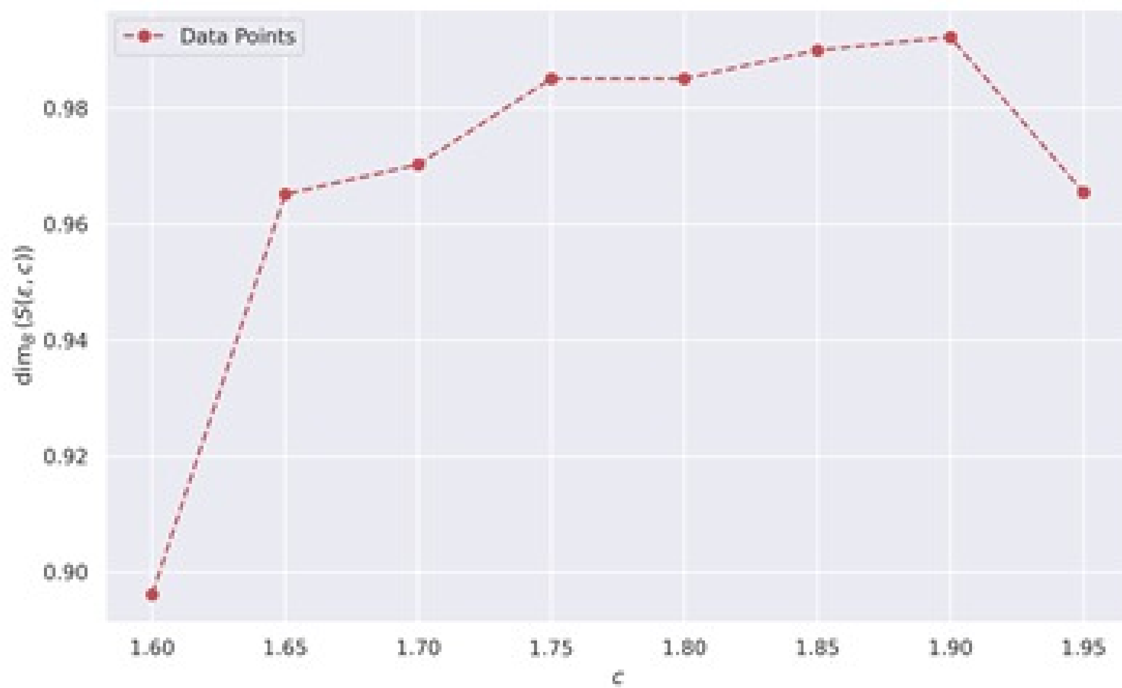
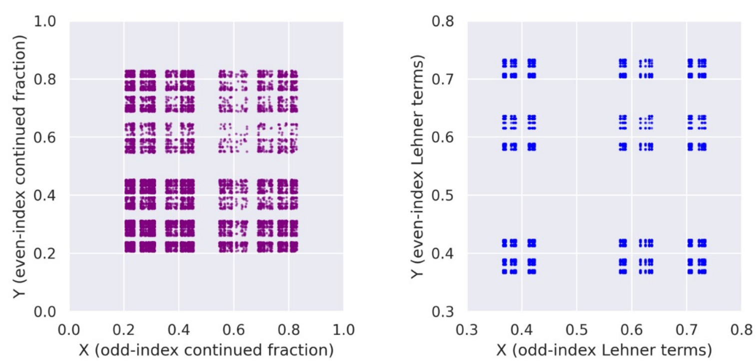


Fig. 5. The graph of box dimensions of $S(\epsilon, c)$ as c changes from 1.60 to 1.95



(a) 2D fractal structure of regular continued fractions

(b) 2D fractal structure of Lehner continued fractions

Fig. 6. Two-Dimensional Fractal Structures from Continued Fraction Expansions

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