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**THE SOLUTION OF PROBLEMS WITH DISCONTINUOUS
COEFFICIENTS OF THERMAL CONDUCTIVITY BY THE INTEGRAL
OF THE ERROR FUNCTION**

**TERM PAPER
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Abstract.

The present paper attempts to investigate a new effective method of solving problems of thermal conductivity, new methods of solving parabolic equations with moving boundaries. In this paper it was tried to show the use of interdisciplinary connection on the example of Mathematical Physics course. Using Integral Error Function a new effective method was developed that positively effects on mathematical achievement of students.

Approximate and analytical solutions of the boundary-value problems is found using Integral Error Functions and their properties or by IEF method, which enable to solve wide range of heat equations with fixed and moving boundaries.

Analytical solution of heat equation with discontinuous coefficients for the thermal conductivity by IEF method is found in this term paper.

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DEFINITIONS AND ABBREVIATIONS

IEF: Integral Error Function

HP: Heat potential

FTB: First type boundary

STB: Second type boundary

TTB: Third type boundary

BVP: Boundary value problem

INTRODUCTION

As is well known, methods of solving Heat Problems with moving boundaries are based on the reduction of Heat Equation to a system of integral equations which cause great difficulties. Daunting methods of solving heat equations with moving boundaries, lead to the development of new methods which have great theoretical and practical significance.

The term paper is based on the investigation of new methods of solving parabolic equations with moving boundaries, investigate the influence on improvement of Master degree and 3-rd grade university students in solving Heat problems by the Integral Error Functions method (IEF method) which is considered to implement into the teaching of "Partial Differential Equations" course in the university, investigate use of IEF method in mathematical modeling of arc phenomena and erosion occurring during arcing.

. Existing methods for solving boundary value problems of heat and mass transfer in domains with free boundaries, based on their reduction to the systems of integral equations, originally accepted non degenerate at the initial time. In the case of degeneracy of the integral equations, singularity occurs and development of new methods for the solution required. Some solutions can be obtained by constructing heat potentials; however they can give only a qualitative description of the process and cannot be used for adequate quantitative calculation of for example temperature fields and erosion of electrical contacts. The practical significance of the work goes far beyond the processes occurring in the electric-contacts. The developed methods for problems with free boundaries can be effectively used in other fields of engineering, which require consideration of the phenomena of phase transformation. They can also be used in the plasma, laser and thermochemical technologies (heat treatment, plating, cutting), in powder metallurgy, electric welding and other areas, for the calculation of the processes of phase transformations of substance.

The method of solving heat transfer problems with free boundaries and phase transformation is represented by Integral Error Functions and its properties.

The results indicate that Integral Error Functions enable to solve, many practical problems described above in the easier way than classical methods, and could be implemented into the course of teaching mathematical physics, as special methods of solving heat transfer problems with moving boundaries.

Practical significance

Important role in the study of the electric-effects problems is mathematical modeling of the electric-processes, which in some cases, it is the only source of information about their dynamics due to the extremely short time of their occurrence, which lies in the nanosecond range, which causes considerable experimental difficulties. Mathematical models describing the transient electromagnetic and temperature fields in areas with phase transitions, the dynamics of interaction of an electric arc with electrodes and other phenomena can clarify many issues related to the study of processes in the electrical contacts.

Since most mathematical models describing above phenomena are based on Stefan type problems, it is of great theoretical and practical interest to solve these problems in the easier way, which Integral Error Functions and its properties provide.

Aim of the term paper is to investigate and develop new methods of solving Heat Equations in the domains with moving boundaries by the help of Integral Error Functions, investigate influence of IEF method on improvement of students in solving heat problems with moving boundaries and investigate use of IEF method in solving practical problems.

Scientific novelty

- New methods of solving heat equations with moving boundaries, which are degenerate at the initial time are investigated. Solution of Stefan type Heat equations with moving unknown boundaries are investigated.

- IEF method positively influence on improvement of students in solving Heat equations with moving boundaries.

Outline of statements and aspects brought to defense

-Analytic and approximate solutions of Heat Equations with moving boundaries by IEF method. Insolvability of heat equations in the domains with moving boundaries with degenerate boundaries at the initial time by Picard's iteration method. Comparison of IEF method with traditional methods.

- Study of influence of IEF method on improvement of students in solving Heat Problems with moving boundaries,

Outline of the term paper

First part consists of first chapter with three topics which are explaining general theory of Integral Error Functions, its properties and corollaries necessary to solve Heat equations analytically and approximately in the domains with given and unknown, fixed and moving boundaries are represented. In the fifth part of dissertation quantitative research of influence of IEF method on improvement Master Degree students in solving heat problems in the domains with moving boundaries was represented.

Second part consists of second chapter with 2 topics and 4 subtopics. In the first 2 subtopics of the chapter approximate solutions of Heat Equations in the domains with fixed boundaries by IEF method are considered. Second two subtopics about solutions of Heat Equations in the domains with moving, given boundaries by IEF method.

In the third chapter of the third part has the 1 topic.

Analytical solution of heat equation with discontinuous coefficients for the first type boundary conditions by IEF method is considered.

THEORY OF INTEGRAL ERROR FUNCTION

1.METHOD OF SOLVING PROBLEMS

There are traditional methods of solving Heat Equations with fixed and moving boundaries. By the help of simple examples it was shown that sometimes it is not preferable to use traditional methods. In traditional methods like Laplace and Fourier transform it is hard or sometimes impossible to find inverse transforms which makes us refer to numerical methods. It was shown that mostly solution can't give quantitative solution that could be helpful and used in applications. Moreover solutions are complicated by bulky integrals that are in most cases hard to calculate.

As for solutions of problems with discontinuous coefficients of thermal conductivity by the integral error function.

1.1 Integral Error Function Method

Heat equations are solved by the help of Integral Error Functions (IEF method) and its properties, which were introduced by Hartree in 1935 and reasonably sometimes called Hartree functions. As it will be shown in further paragraphs, method can be used to solve first, second and third boundary value problems for Heat Equations with fixed and moving finite, semi-infinite and infinite boundaries. Even though it is not the most powerful side of IEF method for the domains with fixed boundaries and it is hard to say that the introduced method is more advantageous than classical Fourier, Laplace transforms and Heat Potentials except the difficulties that were mentioned in above, that it is sometimes hard to find inverse transformation and almost every time impossible to evaluate and find quantitative values of bulky integrals, however it is possible to see that in the domains with moving boundaries especially in the domains with degenerating boundary conditions at the initial time, IEF method is much more preferable than classical methods, as from theoretical same from practical points of view.

The integral error functions determined by recurrent formulas

$$i^n \operatorname{erfc} x = \int_x^\infty i^{n-1} \operatorname{erfc} v dv, \quad n=1,2,\dots, \quad i^0 \operatorname{erfc} x \equiv \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-v^2) dv \quad (1)$$

$$\operatorname{erf} x = 1 - \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-v^2) dv$$

Where

One can obtain from (2)

$$i^n \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \frac{1}{n!} \int_x^\infty (v-x)^n \exp(-v^2) dv \quad (3)$$

Expressions (1) satisfy the differential equation

$$\frac{d^2}{dx^2} i^n \operatorname{erfc} x + 2x \frac{d}{dx} i^n \operatorname{erfc} x - 2ni^n \operatorname{erfc} x = 0 \quad (4)$$

and recurrent formulas

$$2ni^n \operatorname{erfc} x = i^{n-2} \operatorname{erfc} x - 2xi^{n-1} \operatorname{erfc} x \quad (5)$$

Integral Error Functions are very useful for investigation of heat transfer, diffusion and other phenomena which can be described by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (6)$$

in a region $D(t > 0, 0 < x < \alpha(t))$ with free boundary $x = \alpha(t)$, since the functions

$$u_n(\pm x, t) = t^{\frac{n}{2}} i^n \operatorname{erfc} \frac{\pm x}{2a\sqrt{t}}$$

suffice the equation (6) as well as their linear combination or even series

$$u(x, t) = \sum_{n=0}^{\infty} [A_n u_n(x, t) + B_n u_n(-x, t)]$$

For any constants A_n, B_n . We can choose these constants to satisfy the boundary conditions at

$x = 0$ and $x = \alpha(t)$, if given boundary functions can be expanded into Taylor series with powers

t or \sqrt{t} .

1.2 Properties of Integral Error Function

It is possible to derive new properties of Integral Error Functions.

1. If n is an integer, then

$i^n \operatorname{erfc}(-x) + (-1)^n i^n \operatorname{erfc} x = \frac{1}{2^{n-1} n! i^n} H_n(ix) = \frac{1}{2^{n-1} n!} e^{-x^2} \frac{d^n}{dx^n} e^{x^2}$ with $i = \sqrt{-1}$ and Hermite polynomials $H_n(x)$ in the right side. Indeed, using formula (3) one can write

$$\begin{aligned} i^n \operatorname{erfc}(-x) + (-1)^n i^n \operatorname{erfc} x &= \frac{2}{\sqrt{\pi}} \frac{1}{n!} \int_{-x}^{\infty} (v+x)^n \exp(-v^2) dv + \\ & \frac{(-1)^n 2}{n! \sqrt{\pi}} \int_x^{\infty} (v-x)^n \exp(-v^2) dv = \frac{2}{n! \sqrt{\pi}} \int_{-\infty}^{\infty} (v+x)^n \exp(-v^2) dv = \frac{1}{2^{n-1} n! i^n} H_n(ix) \quad (7) \end{aligned}$$

2. Using formula for Hermite polynomials one can derive

$$i^n \operatorname{erfc}(-x) + (-1)^n i^n \operatorname{erfc} x = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2m}}{2^{2m-1} m! (n-2m)!} \quad (8)$$

If $n = 2k$, then

$$i^{2k} \operatorname{erfc} x + i^{2k} \operatorname{erfc}(-x) = \sum_{m=0}^k \frac{x^{2(k-m)}}{2^{2m-1} m!(2k-2m)!}$$

In particular

$$\operatorname{erfc} x + \operatorname{erfc}(-x) = 2,$$

$$i^2 \operatorname{erfc} x + i^2 \operatorname{erfc}(-x) = \frac{1}{2} + x^2,$$

$$i^4 \operatorname{erfc} x + i^4 \operatorname{erfc}(-x) = \frac{1}{8} + \frac{1}{4}x^2 + \frac{1}{12}x^4.$$

If $n = 2k + 1$, then

$$i^{2k+1} \operatorname{erfc}(-x) - i^{2k+1} \operatorname{erfc} x = \sum_{m=0}^k \frac{x^{2(k-m)+1}}{2^{2m-1} m!(2k-2m+1)!} \quad (9)$$

In particular

$$i \operatorname{erfc}(-x) - i \operatorname{erfc} x = 2x,$$

$$i^3 \operatorname{erfc}(-x) - i^3 \operatorname{erfc} x = \frac{1}{2}x + \frac{1}{3}x^3,$$

$$i^5 \operatorname{erfc}(-x) - i^5 \operatorname{erfc} x = \frac{1}{2^3 \cdot 2!}x + \frac{1}{2 \cdot 2! \cdot 3!}x^3 + \frac{2}{5!}x^5.$$

The proof of the formula

$$i^n \operatorname{erfc}(-x) - (-1)^n i^n \operatorname{erfc} x = \frac{1}{2^{n-1} n!} e^{-x^2} \frac{d^n}{dx^n} (e^{x^2} \operatorname{erfc} x) \quad (10)$$

where

$$\operatorname{erfc} x = 1 - \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-v^2) dv$$

can be obtained by mathematical induction method using recurrent formula (2.3).

3. Differentiating the right side of formula (10), we obtain

$$i^n \operatorname{erfc}(-x) - (-1)^n i^n \operatorname{erfc} x = P_n(x) \operatorname{erfc} x - Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2), \quad (11)$$

where polynomials $P_n(x)$ and $Q_n(x)$ are defined by formulas

$$P_n(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2m}}{2^{2m-1} m!(n-2m)!}, \quad Q_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^{n-k} H_{n-k-1}(x)}{2^{n-k} (n-k)!} P_k(x)$$

4. From (10), (11) we can obtain the explicit expressions for Integral Error Functions of an integer index

$$i^n \operatorname{erfc} x = \frac{(-1)^n}{2} [P_n(x) \operatorname{erfc} x + Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2)] \quad (12)$$

$$i^n \operatorname{erfc}(-x) = \frac{1}{2} [P_n(x) \operatorname{erfc}(-x) - Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2)] \quad (13)$$

5. Using L'Hopital rule and representation (1.1), it is not difficult to show that

$$\lim_{x \rightarrow \infty} \frac{i^n \operatorname{erfc}(-x)}{x^n} = \frac{2}{n!} \quad (14)$$

1.3 Corollaries for Integral Error Function Method

Following corollaries will be helpful to so solve Heat equations.

1. Using property 2 one can derive following formula

$$u(x, t) = \sum_{n=0}^k \left\{ A_{2n} \sum_{m=0}^n x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^n x^{2n-2m+1} t^m \beta_{2n+1,m} \right\}$$

Where $u(x, t)$ is a solution of Heat Equation in polynomial form and

$$\beta(n, m) := \frac{1}{2^{n+m-1} \cdot m! \cdot (n-2m)!}$$

2. Expression $u(x, t)$ can be expanded in the following form

$$u(x, t) = A_0 \beta_{0,0} +$$

$$+ A_2 (x^2 \beta_{2,0} + t \beta_{2,1}) +$$

$$+ A_4 (x^4 \beta_{4,0} + x^2 t \beta_{4,1} + t \beta_{4,2}) + \dots +$$

$$+ A_{2k} (x^{2k} \beta_{2k,0} + x^{2k-2} t \beta_{2k,1} + \dots + x^2 t^{k-1} \beta_{2k,k-1} + t^k \beta_{2k,k}) +$$

$$+ A_1 x \beta_{1,0} +$$

$$+ A_3 (x^3 \beta_{3,0} + x t \beta_{3,1}) +$$

$$+ A_5 (x^5 \beta_{5,0} + x^3 t \beta_{5,1} + x t^2 \beta_{5,2}) + \dots +$$

$$+ A_{2k+1} (x^{2k+1} \beta_{2k+1,0} + x^{2k-1} t \beta_{2k+1,1} + \dots + x^3 t^{k-1} \beta_{2k+1,k-1} + x t^k \beta_{2k+1,k})$$

3. Following expression will be frequently used in solutions of Heat Equations

$$\frac{du}{dx} = 2A_2 x \beta_{2,0} +$$

$$+ A_4 (4x^3 \beta_{4,0} + 2xt \beta_{4,1}) +$$

$$+ A_6 (6x^5 \beta_{6,0} + 4x^3 t \beta_{6,1} + 2xt^2 \beta_{6,2}) + \dots +$$

$$+ A_{2k} (2kx^{2k-1} \beta_{2k,0} + (2k-1)x^{2k-3} t \beta_{2k,1} + \dots + 2xt^k \beta_{2k,k}) +$$

$$+ A_1 \beta_{1,0} +$$

$$+ A_3 (3x^2 \beta_{3,0} + t \beta_{3,1}) +$$

$$\begin{aligned}
& +A_5 \left(5x^4\beta_{5,0} + 3x^2t\beta_{5,1} + t^2\beta_{5,2} \right) + \dots + \\
& +A_{2k+1} \left((2k+1)x^{2k}\beta_{2k+1,0} + (2k-1)x^{2k-2}t\beta_{2k+1,1} + \dots \right. \\
& \quad \left. + 3x^2t^{k-1}\beta_{2k+1,k-1} + t^k\beta_{2k+1,k} \right)
\end{aligned}$$

4. Expression $u(x, t) = \sum_{n=0}^k (\sqrt{t})^n \left[A_n i^n \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^n \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right]$ can be expanded

$$\begin{aligned}
u(x, t) &= (\sqrt{t})^0 \left[A_0 i^0 \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_0 i^0 \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right] + \\
&+ (\sqrt{t})^1 \left[A_1 i^1 \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_1 i^1 \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right] + \\
&+ (\sqrt{t})^2 \left[A_2 i^2 \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_2 i^2 \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right] + \dots + \\
&\quad + (\sqrt{t})^n \left[A_n i^n \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^n \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right]
\end{aligned}$$

5. Partial derivative of $u(x, t)$ can be written as

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{2a\sqrt{t}} \left[-A_0 \exp\left(\frac{x^2}{4a^2t}\right) + B_0 \exp\left(\frac{-x^2}{4a^2t}\right) \right] + \\
&+ \frac{1}{2a} \left[-A_1 i^0 \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_1 i^0 \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right] + \\
&+ \left(\frac{\sqrt{t}}{2a} \right)^1 \left[-A_2 i^1 \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_2 i^1 \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right] + \dots + \\
&+ \left(\frac{\sqrt{t}}{2a} \right)^{n-1} \left[-A_n i^{n-1} \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^{n-1} \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right]
\end{aligned}$$

PART II

SOLUTIONS OF PROBLEMS FOR HEAT EQUATIONS

2. ANALYTIC AND APPROXIMATE SOLUTIONS OF HEAT EQUATIONS BY IEF METHOD

2.1 Approximate solutions of Heat Equations in the domains with fixed boundaries by IEF method

It is possible to observe in further paragraphs and examples that IEF method enables to solve boundary value problems for Heat equations in domains with fixed boundaries analytically and approximately.

2.1.1 Approximate solution of the first boundary value

Problem

The solution of the problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0 \quad (15)$$

Subject to

$$\text{I.C:} \quad u(x,0) = 0, \quad (16)$$

$$\text{B.C:} \quad u(0,t) = \varphi(t), \quad (17)$$

$$u(l,t) = \phi(t), \quad (18)$$

$$u(0,0) = 0, \quad (19)$$

where $\varphi(t), \phi(t)$ are analytical functions, can be represented in the form

$$u(x,t) = \sum_{n=0}^k \left\{ A_{2n} \sum_{m=0}^n x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^n x^{2n-2m+1} t^m \beta_{2n+1,m} \right\}$$

or

This function satisfies the heat equation (15), as it was mentioned above and coefficients A_{2n}, A_{2n+1} has to be found. To satisfy the initial and boundary conditions (17) and (18) we expand the functions $\varphi(t), \phi(t)$ in Maclaurin's series:

$$\varphi(t) = \sum_{n=0}^{\infty} \frac{\varphi^n(0)}{n!} \cdot t^n, \quad (20)$$

and

$$\phi(t) = \sum_{k=0}^{\infty} \frac{\phi^n(0)}{n!} \cdot t^n \quad (21)$$

for $t=0, A_0 = 0$

and for $x=0$, even coefficients A_{2k} can be found from

$$A_0\beta_{0,0} + A_2t\beta_{2,1} + A_4t^2\beta_{4,2} + \dots + A_{2k}t^{2k}\beta_{2k,k} = \sum_{k=0}^{\infty} \frac{\phi^{2k}(0)}{(2k)!} t^{2k}$$

substituting even coefficients into the boundary condition for $x=l$ and combining like terms, odd coefficients A_{2k+1} can be found from

$$\begin{aligned} & A_0\beta_{0,0} + \\ & + A_2 \left(l^2\beta_{2,0} + t\beta_{2,1} \right) + \\ & + A_4 \left(l^4\beta_{4,0} + l^2t\beta_{4,1} + t\beta_{4,2} \right) + \dots + \\ & + A_{2k} \left(l^{2k}\beta_{2k,0} + l^{2k-2}t\beta_{2k,1} + \dots + l^2t^{k-1}\beta_{2k,k-1} + t^k\beta_{2k,k} \right) + \\ & + A_1l\beta_{1,0} + \\ & + A_3 \left(l^3\beta_{3,0} + lt\beta_{3,1} \right) + \\ & + A_5 \left(l^5\beta_{5,0} + l^3t\beta_{5,1} + lt^2\beta_{5,2} \right) + \dots + \\ & + A_{2k+1} \left(l^{2k+1}\beta_{2k+1,0} + l^{2k-1}t\beta_{2k+1,1} + \dots + l^3t^{k-1}\beta_{2k+1,k-1} \right. \\ & \quad \left. + lt^k\beta_{2k+1,k} \right) = \sum_{k=0}^{\infty} \frac{\phi^{2k+1}(0)}{(2k+1)!} t^{2k+1} \end{aligned}$$

2.1.2 Approximate solution of the first boundary value problem for the Heat Equation in the half infinite bar

The solution of the problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0 \quad (22)$$

$$u(x,0) = \varphi(x) \quad (23)$$

$$u(0,t) = f(t) \quad (24)$$

$$u(\infty,t) = 0 \quad (25)$$

$$\varphi(0) = f(0), \quad \varphi(\infty) = 0 \quad (26)$$

where $\varphi(x)$ and $f(t)$ are analytical functions, can be represented in the form

$$u(x,t) = \sum_{n=0}^{\infty} (2a\sqrt{t})^n \left[A_n i^n \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^n \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right] \quad (27)$$

2.2 Solution of Heat Equations in the domains with moving, given boundaries by IEF method

As it was told before Integral Error Functions enable investigate heat transfer, diffusion and other phenomena which can be described by the equation

$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ In a region $D(t > 0, 0 < x < \alpha(t))$ with free boundary $x = \alpha(t)$, and solution is considered in the following form

$$u(x,t) = \sum_{n=0}^{\infty} [A_n u_n(x,t) + B_n u_n(-x,t)]$$

For any constants A_n, B_n . Which have to be found from boundary conditions at $x=0$ and $x=\alpha(t)$, if given boundary functions can be expanded into Taylor series with powers t or \sqrt{t} . It is possible to see in the following examples (where all types of boundary value problems considered) and paragraphs, that analytic solutions of Heat Equations are found.

2.2.1 Analytic solution of Heat Equation with the first type boundary conditions in the $\beta\sqrt{t} < x < \alpha\sqrt{t}$ domain by IEF method

Analytical solution of Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad \beta\sqrt{t} < x < \alpha\sqrt{t}, \quad t > 0 \quad (28)$$

Subject to

$$\text{I.C:} \quad u(x,0) = 0, \quad (29)$$

$$\text{B.C:} \quad u(0,t) = \varphi(t), \quad (30)$$

$$u(l,t) = \phi(t), \quad (31)$$

$$u(0,0) = 0, \quad (32)$$

If functions $\varphi(t), \phi(t)$ are definite functions given in the form $\varphi(t) = \sum_{n=0}^k \mu_n t^{\frac{n}{2}}, \phi(t) = \sum_{n=0}^m \nu_n t^{\frac{n}{2}}$ Then solution can be represented in the form

$$u(x,t) = \sum_{n=0}^{\infty} (\sqrt{t})^n \left[A_n i^n \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^n \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right]$$

Substituting expression into the boundary conditions for $x = \beta\sqrt{t}$

$$u(\beta\sqrt{t}, t) = \sum_{n=0}^{\infty} (\sqrt{t})^n \left[A_n i^n \operatorname{erfc} \frac{\beta}{2a} + B_n i^n \operatorname{erfc} \frac{-\beta}{2a} \right]$$

where $\gamma = \sup\{m, n\}$

for $x = \alpha\sqrt{t}$

Finally coefficients $A_0, A_1, A_2, \dots, A_n$ and $B_0, B_1, B_2, \dots, B_n$ are determined from system of linear equations

$$A_0 i^0 \operatorname{erfc} \frac{\beta}{2a} + B_0 i^0 \operatorname{erfc} \frac{-\beta}{2a} = \mu_0$$

$$A_0 i^0 \operatorname{erfc} \frac{\alpha}{2a} + B_0 i^0 \operatorname{erfc} \frac{-\alpha}{2a} = \nu_0$$

$$A_1 i^1 \operatorname{erfc} \frac{\beta}{2a} + B_1 i^1 \operatorname{erfc} \frac{-\beta}{2a} = \mu_1$$

$$A_1 i^1 \operatorname{erfc} \frac{\alpha}{2a} + B_1 i^1 \operatorname{erfc} \frac{-\alpha}{2a} = \nu_1$$

$$A_2 i^2 \operatorname{erfc} \frac{\beta}{2a} + B_2 i^2 \operatorname{erfc} \frac{-\beta}{2a} = \mu_2$$

$$A_2 i^2 \operatorname{erfc} \frac{\alpha}{2a} + B_2 i^2 \operatorname{erfc} \frac{-\alpha}{2a} = \nu_2$$

.....

$$A_i i^\gamma \operatorname{erfc} \frac{\beta}{2a} + B_i i^\gamma \operatorname{erfc} \frac{-\beta}{2a} = \mu_\gamma$$

$$A_i i^\gamma \operatorname{erfc} \frac{\alpha}{2a} + B_i i^\gamma \operatorname{erfc} \frac{-\alpha}{2a} = \nu_\gamma$$

where $i^\gamma \operatorname{erfc} \frac{\beta}{2a}, i^\gamma \operatorname{erfc} \frac{-\beta}{2a}, i^\gamma \operatorname{erfc} \frac{\alpha}{2a}, i^\gamma \operatorname{erfc} \frac{-\alpha}{2a}, \gamma=0,1,2,\dots$ are identified from tables.

Remark: One of key points in solving Heat Equations in the domains with moving boundaries of the first type is to correctly identify value of γ , which takes maximum value between m and k in the boundary conditions.

2.2.2 Analytic solution of Heat Equation with the second type boundary conditions in the $\beta\sqrt{t} < x < \alpha\sqrt{t}$ domain by IEF method

Analytic solution of Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad \beta\sqrt{t} < x < \alpha\sqrt{t}, \quad t > 0 \quad (33)$$

subject to

$$\text{I.C:} \quad u(x, 0) = 0, \quad (34)$$

$$\text{B.C:} \quad \left. \frac{\partial u}{\partial x} \right|_{x=0} = \phi(t), \quad (35)$$

$$\left. \frac{\partial u}{\partial x} \right|_{x=\alpha(t)} = \phi(t), \quad (36)$$

$$u(0,0) = 0, \quad (37)$$

where $\omega(t) = \sum_{n=0}^k \mu_n t^{\frac{n}{2}}$, $\phi(t) = \sum_{n=0}^k \nu_n t^{\frac{n}{2}}$ analytical functions, can be represented in the form

$$u(x, t) = \sum_{n=0}^k (\sqrt{t})^n \left[A_n i^n \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^n \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right] \quad (38)$$

Substituting expression (38) into the boundary conditions (35) and (36) and applying UC method (undetermined coefficients method) for $x = \beta\sqrt{t}$ we have:

$$\begin{aligned} \frac{\partial u}{\partial x} \Big|_{x=\beta\sqrt{t}} &= \frac{1}{2a\sqrt{t}} \left[-A_0 \exp\left(\frac{\beta^2}{4a^2}\right) + B_0 i^0 \exp\left(-\frac{\beta^2}{4a^2}\right) \right] + \\ &+ \frac{1}{2a} \left[-A_1 i^0 \operatorname{erfc} \frac{\beta}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\beta}{2a} \right] + \\ &+ \left(\frac{\sqrt{t}}{2a} \right)^1 \left[-A_2 i^1 \operatorname{erfc} \frac{\beta}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\beta}{2a} \right] + \dots + \\ &+ \left(\frac{\sqrt{t}}{2a} \right)^k \left[-A_{k+1} i^k \operatorname{erfc} \frac{\beta}{2a} + B_{k+1} i^k \operatorname{erfc} \frac{-\beta}{2a} \right] = \\ &= \sum_{n=0}^k \mu_n t^{\frac{n}{2}} \end{aligned}$$

and for $x = \alpha\sqrt{t}$ we have

$$\begin{aligned} \frac{\partial u}{\partial x} \Big|_{x=\alpha\sqrt{t}} &= \frac{1}{2a\sqrt{t}} \left[-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right) + B_0 i^0 \exp\left(-\frac{\alpha^2}{4a^2}\right) \right] + \\ &+ \frac{1}{2a} \left[-A_1 i^0 \operatorname{erfc} \frac{\alpha}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\alpha}{2a} \right] + \\ &+ \left(\frac{\sqrt{t}}{2a} \right)^1 \left[-A_2 i^1 \operatorname{erfc} \frac{\alpha}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\alpha}{2a} \right] + \dots + \\ &+ \left(\frac{\sqrt{t}}{2a} \right)^k \left[-A_{k+1} i^k \operatorname{erfc} \frac{\alpha}{2a} + B_{k+1} i^k \operatorname{erfc} \frac{-\alpha}{2a} \right] = \\ &= \sum_{n=0}^k \nu_n t^{\frac{n}{2}} \end{aligned}$$

Thus following system obtained where coefficient $A_0, A_1, A_2, \dots, A_k$ and $B_0, B_1, B_2, \dots, B_k$ can be determined

$$-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right) + B_0 \exp\left(-\frac{\alpha^2}{4a^2}\right) = 0$$

$$-A_0 \exp\left(\frac{\beta^2}{4a^2}\right) + B_0 \exp\left(-\frac{\beta^2}{4a^2}\right) = 0$$

$$-A_1 i^0 \operatorname{erfc} \frac{\beta}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\beta}{2a} = \mu_1$$

$$-A_1 i^0 \operatorname{erfc} \frac{\alpha}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\alpha}{2a} = \nu_1$$

$$-A_2 i^1 \operatorname{erfc} \frac{\beta}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\beta}{2a} = \mu_2$$

$$-A_2 i^1 \operatorname{erfc} \frac{\alpha}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\alpha}{2a} = \nu_2$$

.....

$$-A_k i^{k-1} \operatorname{erfc} \frac{\beta}{2a} + B_k i^{k-1} \operatorname{erfc} \frac{-\beta}{2a} = \mu_k$$

$$-A_k i^{k-1} \operatorname{erfc} \frac{\alpha}{2a} + B_k i^{k-1} \operatorname{erfc} \frac{-\alpha}{2a} = \nu_k$$

If the functions $\varphi(t), \phi(t)$ are given in the form

$$\varphi(t) = \sum_{n=0}^k \mu_n t^{\frac{n}{2}}, \phi(t) = \sum_{n=0}^m \nu_n t^{\frac{n}{2}}$$

then for $m \geq k$,

solution will be considered in the form

$$u(x, t) = \sum_{n=0}^m (\sqrt{t})^n \left[A_n i^n \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^n \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right] \quad (39)$$

where $y = m + 1$

for $x = \beta\sqrt{t}$ we have

$$\begin{aligned} \frac{\partial u}{\partial x} \Big|_{x=\beta\sqrt{t}} &= \frac{1}{2a\sqrt{t}} \left[-A_0 \exp\left(\frac{\beta^2}{4a^2}\right) + B_0 i^0 \exp\left(-\frac{\beta^2}{4a^2}\right) \right] + \\ &+ \frac{1}{2a} \left[-A_1 i^0 \operatorname{erfc} \frac{\beta}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\beta}{2a} \right] + \\ &+ \left(\frac{\sqrt{t}}{2a} \right)^1 \left[-A_2 i^1 \operatorname{erfc} \frac{\beta}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\beta}{2a} \right] + \dots + \\ &+ \left(\frac{\sqrt{t}}{2a} \right)^k \left[-A_{k+1} i^k \operatorname{erfc} \frac{\beta}{2a} + B_{k+1} i^k \operatorname{erfc} \frac{-\beta}{2a} \right] = \\ &= \sum_{n=0}^k \mu_n t^{\frac{n}{2}} \end{aligned}$$

For $x = \alpha\sqrt{t}$

$$\frac{\partial u}{\partial x} \Big|_{x=\alpha\sqrt{t}} = \frac{1}{2a\sqrt{t}} \left[-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right) + B_0 i^0 \exp\left(-\frac{\alpha^2}{4a^2}\right) \right] +$$

$$\begin{aligned}
& + \frac{1}{2a} \left[-A_1 i^0 \operatorname{erfc} \frac{\alpha}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\alpha}{2a} \right] + \\
& + \left(\frac{\sqrt{t}}{2a} \right)^1 \left[-A_2 i^1 \operatorname{erfc} \frac{\alpha}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\alpha}{2a} \right] + \dots + \\
& + \left(\frac{\sqrt{t}}{2a} \right)^{m+1} \left[-A_{m+1} i^m \operatorname{erfc} \frac{\alpha}{2a} + B_{m+1} i^m \operatorname{erfc} \frac{-\alpha}{2a} \right] = \\
& = \sum_{n=0}^{m+1} v_n t^{\frac{n}{2}}
\end{aligned}$$

Finally coefficients $A_0, A_1, A_2, \dots, A_m$ and $B_0, B_1, B_2, \dots, B_m$ are determined from system of linear equations

$$\begin{aligned}
-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right) + B_0 \exp\left(-\frac{\alpha^2}{4a^2}\right) &= 0 \\
-A_0 \exp\left(\frac{\beta^2}{4a^2}\right) + B_0 \exp\left(-\frac{\beta^2}{4a^2}\right) &= 0 \\
-A_1 i^0 \operatorname{erfc} \frac{\beta}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\beta}{2a} &= \mu_1 \\
-A_1 i^0 \operatorname{erfc} \frac{\alpha}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\alpha}{2a} &= v_1 \\
-A_2 i^1 \operatorname{erfc} \frac{\beta}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\beta}{2a} &= \mu_2 \\
-A_2 i^1 \operatorname{erfc} \frac{\alpha}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\alpha}{2a} &= v_2 \\
&\dots \dots \dots \\
&\dots \dots \dots \\
-A_{m+1} i^m \operatorname{erfc} \frac{\beta}{2a} + B_{m+1} i^m \operatorname{erfc} \left(-\frac{\beta}{2a}\right) &= 0 \\
-A_{m+1} i^m \operatorname{erfc} \frac{\alpha}{2a} + B_{m+1} i^m \operatorname{erfc} \frac{-\alpha}{2a} &= v_m
\end{aligned}$$

Form $< k$,

solution is considered in the form

$$u(x, t) = \sum_{n=0}^y (\sqrt{t})^n \left[A_n i^n \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^n \operatorname{erfc} \frac{-x}{2a\sqrt{t}} \right]$$

where $y = k + 1$

for $x = \beta\sqrt{t}$

$$\left. \frac{\partial u}{\partial x} \right|_{x=\beta\sqrt{t}} = \frac{1}{2a\sqrt{t}} \left[-A_0 \exp\left(\frac{\beta^2}{4a^2}\right) + B_0 i^0 \exp\left(-\frac{\beta^2}{4a^2}\right) \right] +$$

$$\begin{aligned}
& + \frac{1}{2a} \left[-A_1 i^0 \operatorname{erfc} \frac{\beta}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\beta}{2a} \right] + \\
& + \left(\frac{\sqrt{t}}{2a} \right)^1 \left[-A_2 i^1 \operatorname{erfc} \frac{\beta}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\beta}{2a} \right] + \dots + \\
& + \left(\frac{\sqrt{t}}{2a} \right)^k \left[-A_{k+1} i^k \operatorname{erfc} \frac{\beta}{2a} + B_{k+1} i^k \operatorname{erfc} \frac{-\beta}{2a} \right] = \\
& = \sum_{n=0}^{k+1} \mu_n t^{\frac{n}{2}}
\end{aligned}$$

and for $x = \alpha\sqrt{t}$ we have

$$\begin{aligned}
\frac{\partial u}{\partial x} \Big|_{x=\alpha\sqrt{t}} &= \frac{1}{2a\sqrt{t}} \left[-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right) + B_0 i^0 \exp\left(-\frac{\alpha^2}{4a^2}\right) \right] + \\
& + \frac{1}{2a} \left[-A_1 i^0 \operatorname{erfc} \frac{\alpha}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\alpha}{2a} \right] + \\
& + \left(\frac{\sqrt{t}}{2a} \right)^1 \left[-A_2 i^1 \operatorname{erfc} \frac{\alpha}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\alpha}{2a} \right] + \dots + \\
& + \left(\frac{\sqrt{t}}{2a} \right)^k \left[-A_{k+1} i^k \operatorname{erfc} \frac{\alpha}{2a} + B_{k+1} i^k \operatorname{erfc} \frac{-\alpha}{2a} \right] = \\
& = \sum_{n=0}^{k+1} \nu_n t^{\frac{n}{2}}
\end{aligned}$$

Thus following system obtained where coefficient $A_0, A_1, A_2, \dots, A_k$ and $B_0, B_1, B_2, \dots, B_k$ can be determined

$$-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right) + B_0 \exp\left(-\frac{\alpha^2}{4a^2}\right) = 0$$

$$-A_0 \exp\left(\frac{\beta^2}{4a^2}\right) + B_0 \exp\left(-\frac{\beta^2}{4a^2}\right) = 0$$

$$-A_1 i^0 \operatorname{erfc} \frac{\beta}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\beta}{2a} = \mu_1$$

$$-A_1 i^0 \operatorname{erfc} \frac{\alpha}{2a} + B_1 i^0 \operatorname{erfc} \frac{-\alpha}{2a} = \nu_1$$

$$-A_2 i^1 \operatorname{erfc} \frac{\beta}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\beta}{2a} = \mu_2$$

$$-A_2 i^1 \operatorname{erfc} \frac{\alpha}{2a} + B_2 i^1 \operatorname{erfc} \frac{-\alpha}{2a} = \nu_2$$

.....

$$-A_{k+1} i^k \operatorname{erfc} \frac{\beta}{2a} + B_{k+1} i^k \operatorname{erfc} \frac{-\beta}{2a} = \mu_{k+1}$$

$$-A_{k+1}i^k \operatorname{erfc} \frac{\alpha}{2a} + B_{k+1}i^k \operatorname{erfc} \frac{-\alpha}{2a} = v_{k+1}$$

where $i^{\gamma} \operatorname{erfc} \frac{\alpha}{2a}, i^{\gamma} \operatorname{erfc} \frac{-\alpha}{2a}, i^{\gamma} \operatorname{erfc} \frac{\alpha}{2a}, i^{\gamma} \operatorname{erfc} \frac{-\alpha}{2a}, \gamma=0,1,2,\dots$ are treated as constants which can be determined from erfc tables.

PART III

3. SOLUTIONS OF PROBLEMS FOR HEAT EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

3.1 Analytical solution of heat equation with discontinuous coefficients for the first type boundary conditions by IEF method

$$\frac{\partial u_1}{\partial t} = a^2 \frac{\partial^2 u_1}{\partial x^2}, \quad \alpha\sqrt{t} < x < \beta\sqrt{t}, \quad t > 0$$

$$\frac{\partial u_2}{\partial t} = b^2 \frac{\partial^2 u_2}{\partial x^2}, \quad \beta\sqrt{t} < x < \delta\sqrt{t}, \quad t > 0$$

$$u_1(0,0) = 0, \quad (1)$$

$$u_1(\alpha\sqrt{t}, t) = \varphi(t), \quad t > 0 \quad (2)$$

$$u_1(\beta\sqrt{t}, t) = \varnothing(t), \quad t > 0 \quad (3)$$

$$u_2(\beta\sqrt{t}, t) = \psi(t), \quad t > 0 \quad (4)$$

$$u_2(\delta\sqrt{t}, t) = \rho(t), \quad t > 0 \quad (5)$$

Where $\varphi(t)$, $\varnothing(t)$, $\psi(t)$, $\rho(t)$ are analytical functions and can be represented in the following form

$$\varphi(t) = \sum_{n=0}^k \mu_n t^{\frac{n}{2}}; \quad (6)$$

$$\varnothing(t) = \sum_{n=0}^m \vartheta_n t^{\frac{n}{2}}; \quad (7)$$

$$\psi(t) = \sum_{n=0}^l q_n t^{\frac{n}{2}}; \quad (8)$$

$$\rho(t) = \sum_{n=0}^s r_n t^{\frac{n}{2}}; \quad (9)$$

The solution of heat equation can be written as followings:

$$u_1(x, t) = \sum_{n=0}^{\infty} t^{\frac{n}{2}} \left\{ A_n i^n \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^n \operatorname{erfc} \left(\frac{-x}{2a\sqrt{t}} \right) \right\} \quad (10)$$

$$u_2(x, t) = \sum_{n=0}^{\infty} t^{\frac{n}{2}} \left\{ C_n i^n \operatorname{erfc} \frac{x}{2b\sqrt{t}} + D_n i^n \operatorname{erfc} \left(\frac{-x}{2b\sqrt{t}} \right) \right\} \quad (11)$$

$$u_1(x, t) = \sum_{n=0}^k t^{\frac{n}{2}} \left\{ A_n i^n \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^n \operatorname{erfc} \left(\frac{-x}{2a\sqrt{t}} \right) \right\}$$

$$u_1(\alpha\sqrt{t}, t) = \sum_{n=0}^{\gamma} t^{\frac{n}{2}} \left\{ A_n i^n \operatorname{erfc} \frac{\alpha}{2a} + B_n i^n \operatorname{erfc} \left(\frac{-\alpha}{2a} \right) \right\}$$

where for $x = \alpha\sqrt{t}$,
 $\gamma = \sup\{m, k\}$

$$\begin{aligned} & A_0 i^0 \operatorname{erfc} \frac{\alpha}{2a} + B_0 i^0 \operatorname{erfc} \left(\frac{-\alpha}{2a} \right) + \\ & + (\sqrt{t})^1 \left\{ A_1 i^1 \operatorname{erfc} \frac{\alpha}{2a} + B_1 i^1 \operatorname{erfc} \left(\frac{-\alpha}{2a} \right) \right\} + \\ & + (\sqrt{t})^2 \left\{ A_2 i^2 \operatorname{erfc} \frac{\alpha}{2a} + B_2 i^2 \operatorname{erfc} \left(\frac{-\alpha}{2a} \right) \right\} + \\ & \dots \\ & + (\sqrt{t})^{\gamma} \left\{ A_{\gamma} i^{\gamma} \operatorname{erfc} \frac{\alpha}{2a} + B_{\gamma} i^{\gamma} \operatorname{erfc} \left(\frac{-\alpha}{2a} \right) \right\} = \sum_{n=0}^{\gamma} \mu_n t^{\frac{n}{2}}; \end{aligned}$$

$$u_1(x, t) = \sum_{n=0}^k t^{\frac{n}{2}} \left\{ A_n i^n \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^n \operatorname{erfc} \left(\frac{-x}{2a\sqrt{t}} \right) \right\}$$

for $x = \beta\sqrt{t}$

$$\begin{aligned}
u_1(\beta\sqrt{t}, t) &= \sum_{n=0}^{\gamma} t^{\frac{n}{2}} \left\{ A_n i^n \operatorname{erfc} \frac{\beta}{2a} + B_n i^n \operatorname{erfc} \left(\frac{-\beta}{2a} \right) \right\} = \\
&A_0 i^0 \operatorname{erfc} \frac{\beta}{2a} + B_0 i^0 \operatorname{erfc} \left(\frac{-\beta}{2a} \right) + \\
&+(\sqrt{t})^1 \left\{ A_1 i^1 \operatorname{erfc} \frac{\beta}{2a} + B_1 i^1 \operatorname{erfc} \left(\frac{-\beta}{2a} \right) \right\} + \\
&+(\sqrt{t})^2 \left\{ A_2 i^2 \operatorname{erfc} \frac{\beta}{2a} + B_2 i^2 \operatorname{erfc} \left(\frac{-\beta}{2a} \right) \right\} + \\
&+ \dots \\
&+(\sqrt{t})^{\gamma} \left\{ A_{\gamma} i^{\gamma} \operatorname{erfc} \frac{\beta}{2a} + B_{\gamma} i^{\gamma} \operatorname{erfc} \left(\frac{-\beta}{2a} \right) \right\} = \sum_{n=0}^{\gamma} v_n t^{\frac{n}{2}};
\end{aligned}$$

Then we get coefficients $A_0, A_1, A_2, \dots, A_{\gamma}$ and $B_0, B_1, B_2, \dots, B_{\gamma}$ are determined from system of linear equations.

$$\begin{cases} A_0 i^0 \operatorname{erfc} \frac{\alpha}{2a} + B_0 i^0 \operatorname{erfc} \left(\frac{-\alpha}{2a} \right) = \mu_0 \\ A_0 i^0 \operatorname{erfc} \frac{\beta}{2a} + B_0 i^0 \operatorname{erfc} \left(\frac{-\beta}{2a} \right) = \vartheta_0 \end{cases}$$

$$\begin{cases} A_1 i^1 \operatorname{erfc} \frac{\alpha}{2a} + B_1 i^1 \operatorname{erfc} \left(\frac{-\alpha}{2a} \right) = \mu_1 \\ A_1 i^1 \operatorname{erfc} \frac{\beta}{2a} + B_1 i^1 \operatorname{erfc} \left(\frac{-\beta}{2a} \right) = \vartheta_1 \end{cases}$$

$$\begin{cases} A_2 i^2 \operatorname{erfc} \frac{\alpha}{2a} + B_2 i^2 \operatorname{erfc} \left(\frac{-\alpha}{2a} \right) = \mu_2 \\ A_2 i^2 \operatorname{erfc} \frac{\beta}{2a} + B_2 i^2 \operatorname{erfc} \left(\frac{-\beta}{2a} \right) = \vartheta_2 \end{cases}$$

...

$$\begin{cases} A_\gamma i^\gamma \operatorname{erfc} \frac{\alpha}{2a} + B_\gamma i^\gamma \operatorname{erfc} \left(\frac{-\alpha}{2a} \right) = \mu_\gamma \\ A_\gamma i^\gamma \operatorname{erfc} \frac{\beta}{2a} + B_\gamma i^\gamma \operatorname{erfc} \left(\frac{-\beta}{2a} \right) = \vartheta_\gamma \end{cases}$$

Where $i^\gamma \operatorname{erfc} \frac{\alpha}{2a}$, $i^\gamma \operatorname{erfc} \left(\frac{-\alpha}{2a} \right)$, $i^\gamma \operatorname{erfc} \frac{\beta}{2a}$, $i^\gamma \operatorname{erfc} \left(\frac{-\beta}{2a} \right)$, $\gamma = 0, 1, 2, \dots$ are identified from tables.

From (11)

$$u_2(x, t) = \sum_{n=0}^l t^{\frac{n}{2}} \left\{ C_n i^n \operatorname{erfc} \frac{x}{2b\sqrt{t}} + D_n i^n \operatorname{erfc} \left(\frac{-x}{2b\sqrt{t}} \right) \right\}$$

where $\gamma = \sup\{l, s\}$

for $x = \beta\sqrt{t}$

$$u_2(\beta\sqrt{t}, t) = \sum_{n=0}^{\gamma} t^{\frac{n}{2}} \left\{ C_n i^n \operatorname{erfc} \frac{\beta}{2b} + D_n i^n \operatorname{erfc} \left(\frac{-\beta}{2b} \right) \right\} =$$

$$= C_0 i^0 \operatorname{erfc} \frac{\beta}{2b} + D_0 i^0 \operatorname{erfc} \left(\frac{-\beta}{2b} \right) +$$

$$+(\sqrt{t})^1 \left\{ C_1 i^1 \operatorname{erfc} \frac{\beta}{2b} + D_1 i^1 \operatorname{erfc} \left(\frac{-\beta}{2b} \right) \right\} +$$

$$+(\sqrt{t})^2 \left\{ C_2 i^2 \operatorname{erfc} \frac{\beta}{2b} + D_2 i^2 \operatorname{erfc} \left(\frac{-\beta}{2b} \right) \right\} +$$

...

$$+(\sqrt{t})^\gamma \left\{ C_\gamma i^\gamma \operatorname{erfc} \frac{\beta}{2b} + D_\gamma i^\gamma \operatorname{erfc} \left(\frac{-\beta}{2b} \right) \right\} = \sum_{n=0}^l q_n t^{\frac{n}{2}}$$

for $x = \delta\sqrt{t}$

$$\begin{aligned}
 u_2(\delta\sqrt{t}, t) &= \sum_{n=0}^{\gamma} t^{\frac{n}{2}} \left\{ C_n i^n \operatorname{erfc} \frac{\delta}{2b} + D_n i^n \operatorname{erfc} \left(\frac{-\delta}{2b} \right) \right\} = \\
 &= C_0 i^0 \operatorname{erfc} \frac{\delta}{2b} + D_0 i^0 \operatorname{erfc} \left(\frac{-\delta}{2b} \right) + \\
 &+ (\sqrt{t})^1 \left\{ C_1 i^1 \operatorname{erfc} \frac{\delta}{2b} + D_1 i^1 \operatorname{erfc} \left(\frac{-\delta}{2b} \right) \right\} + \\
 &+ (\sqrt{t})^2 \left\{ C_2 i^2 \operatorname{erfc} \frac{\delta}{2b} + D_2 i^2 \operatorname{erfc} \left(\frac{-\delta}{2b} \right) \right\} + \\
 &\dots \\
 &+ (\sqrt{t})^{\gamma} \left\{ C_{\gamma} i^{\gamma} \operatorname{erfc} \frac{\delta}{2b} + D_{\gamma} i^{\gamma} \operatorname{erfc} \left(\frac{-\delta}{2b} \right) \right\} = \sum_{n=0}^{\gamma} r_n t^{\frac{n}{2}}
 \end{aligned}$$

Finally we get coefficients $C_0, C_1, C_2, \dots, C_{\gamma}$ and $D_0, D_1, D_2, \dots, D_{\gamma}$ are determined from system of linear equation.

$$C_0 i^0 \operatorname{erfc} \frac{\beta}{2b} + D_0 i^0 \operatorname{erfc} \left(\frac{-\beta}{2b} \right) = q_0$$

$$C_0 i^0 \operatorname{erfc} \frac{\delta}{2b} + D_0 i^0 \operatorname{erfc} \left(\frac{-\delta}{2b} \right) = r_0$$

$$C_1 i^1 \operatorname{erfc} \frac{\beta}{2b} + D_1 i^1 \operatorname{erfc} \left(\frac{-\beta}{2b} \right) = q_1$$

$$C_1 i^1 \operatorname{erfc} \frac{\delta}{2b} + D_1 i^1 \operatorname{erfc} \left(\frac{-\delta}{2b} \right) = r_1$$

...

$$C_{\gamma} i^{\gamma} \operatorname{erfc} \frac{\beta}{2b} + D_{\gamma} i^{\gamma} \operatorname{erfc} \left(\frac{-\beta}{2b} \right) = q_{\gamma}$$

$$C_{\gamma} i^{\gamma} \operatorname{erfc} \frac{\delta}{2b} + D_{\gamma} i^{\gamma} \operatorname{erfc} \left(\frac{-\delta}{2b} \right) = r_{\gamma}$$

where $i^{\gamma} \operatorname{erfc} \frac{\beta}{2b}$, $i^{\gamma} \operatorname{erfc} \left(\frac{-\beta}{2b} \right)$, $i^{\gamma} \operatorname{erfc} \frac{\delta}{2b}$, $i^{\gamma} \operatorname{erfc} \left(\frac{-\delta}{2b} \right)$

$\gamma = 0, 1, 2, \dots$ are identified from tables.

CONCLUSION

By the term paper the Integral Error Function method is applied to analytical and approximate solutions for heat equations, and students had test; according to these procedures we can see that Integral Error Function method is really effective.

Physics problems are solved by mathematical approach easily, it was shown in solution of problems solved by IEF method.

The most powerful side of IEF method for the domains with fixed boundaries and it is hard to say that the introduced method is more advantageous than classical Fourier, Laplace transforms and Heat Potentials except the difficulties that were mentioned in above, that it is sometimes hard to find inverse transformation and almost every time impossible to evaluate and find quantitative values of bulky integrals, however it is possible to see that in the domains with moving boundaries especially in the domains with degenerating boundary conditions at the initial time, IEF method is much more preferable than classical methods, as from theoretical same from practical points of view.

By using following forms of solution of heat problems with discontinuous coefficients:

$$u_1(x, t) = \sum_{n=0}^{\infty} t^{\frac{n}{2}} \left\{ A_n i^n \operatorname{erfc} \frac{x}{2a\sqrt{t}} + B_n i^n \operatorname{erfc} \left(\frac{-x}{2a\sqrt{t}} \right) \right\}$$

$$u_2(x, t) = \sum_{n=0}^{\infty} t^{\frac{n}{2}} \left\{ C_n i^n \operatorname{erfc} \frac{x}{2b\sqrt{t}} + D_n i^n \operatorname{erfc} \left(\frac{-x}{2b\sqrt{t}} \right) \right\}$$

we are getting coefficients.

As it is shown above in problems of heat equation with discontinuous coefficients for thermal conductivity Integral Error Function method also is used. And it gives good results.

So, in the future we recommend using this method not only in mathematics, but also in physics.

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APPENDIX

1. L'Hopital rule :

$$\lim_{x \rightarrow \infty} \frac{i^n \operatorname{erfc}(-x)}{x^n} = \frac{2}{n!}$$

2. Taylor series:

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi^n(a)}{n!} (x - a)^n$$