Ministry of Education and Science of the Republic of Kazakhstan
Suleyman Demirel University

Nazipa Aitu

Cardinality of survivor sets in open dynamical systems

THESIS

Presented in Partial Fulfillment for the
Degree of Master of Science in Mathematics
(degree code: 6M060100)
Department of Natural sciences
Faculty of Engineering and Natural Sciences

Supervisor: **Shirali Kadyrov**

Kaskelen, 2020

Abstract

In this thesis, our goal is to learn about open dynamical systems corresponding interval maps. We study the class of dynamical systems with holes: Expanding maps of the interval. In detail, We consider symbolic dynamics with holes. Let H-hole lies in the interval $[0, 1)$ and let $T : [0, 1) \longrightarrow [0, 1)$ be a self map. The survivor set $\Omega(H) := \{x \in [0, 1) : \ T^n x \notin H, \ n \geq 0\}$. Depending on location and size of the holes we will characterize and study the survivor set $\Omega(H)$ infinite or finite, uncountable or countable and survivor set $\Omega(H)$ has positive entropy.

Аңдатпа

Бұл диссертациялық жұмыста біздің мақсатымыз интервалды карталарға сәйкес келетін ашық динамикалық жүйелер туралы білу. Тесіктері бар динамикалы жүйелер класын зерттейміз: Аралық карталарын кеңейту. Толығырақ, біз саңылаулары бар символикалық динамиканы қарастырамыз. H шұңқыр $[0, 1]$ аралықта орналассын және $T : [0, 1) \longrightarrow [0, 1)$ өзіндік карта болсын. Аман қалған элементтер $\Omega(H) := \{x \ -i[0, 1]- : \ T^n x \ H, \ n \geq 0\}$. Тесіктердің орналасуы мен мөлшеріне байланысты біз аман қалған адамды $Omega(H)$ шексіз немесе шексіз, есептелмейтін немесе есептелетін және аман қалған $\Omega(H)$ жиынтығының оң энтропиясын сипаттаймыз және зерттейміз.

Аннотация

В этом тезисе наша цель - узнать об открытых динамических системах соответству интервальных отображений. Мы изучаем класс динамических систем с дырками: Расширяющиеся отображения отрезка. Подробно рассмотрим символическую динамику с дырками. Пусть H-дыра лежит в интервале $[0,1)$, и пусть T : $[0,1) \longrightarrow [0,1)$ является собственным отображением. Набор выживших $\Omega(H) := \{x \ in[0,1) : \ T^n x \notin H, \ n \geq 0\}$. В зависимости от расположения и размера дырок мы будем характеризовать и изучать множество выживших $\Omega(H)$, бесконечное или конечное, неисчисляемое или счетное, а множество выживших $\Omega(H)$ имеет положительную энтропию.

Acknowledgements

Thanks to the government of Republic of Kazakhstan for providing a grant and free education. Thanks to my supervisor for giving a great deal of knowledge of Dynamical systems. And for the fact that he professionally supervised the writing of the thesis, for his patience and for making my thesis understandable. Thanks for correcting me when I made mistakes and for motivating when there was no time. thank you for doing your job honestly. Thanks to my group mates for being with me on this path. And for my family for believing in me.

Contents

1. Introduction

In the following thesis, we are studying the properties of open dynamical systems, maps with holes. Pianigiani and Yorke[14] in slightly different way first proposed the notion of dynamical systems with holes, also called open dynamical systems. First, we want to give short definitions and history of dynamical systems in next sections.

1.1 Dynamical systems

Dynamics is the mechanism of time growth. It can be either deterministic, or stochastic. Any devices also are difficult to forecast longer term. Only the normal geometry can't reflect its trajectories. there is also unpredictability inside other natural and social phenomenon. Unpredictability is also an inherent attribute that is found at the very phenomena. It will have significant influence on human society and on science notion. Many problems remain throughout the human imagination, for example, how could a deterministic course be unpredictable? What are the reasons of symmetrical ice and snow flakes developing in Natural world? How does one call trajectories chaotic? May random be a deterministic trajectory?

How can one characterize fluid motion and justify it? Would there be any order in core Chaos? How does one introduce chaotic dynamics to fractals of objects? There is no other direction we can address such concerns except studying nonlinear dynamics.

Dynamic structures are usually defined by differential or equations of difference. Applications of differential equations in math have been dedicated for over two decades primarily to discovering analytical implementations to equations. But empirical or closed-form methods can not necessarily decide the dynamical activities of a device. Furthermore, it is impossible to achieve empirical formulation to nonlinear equations other than in a some particular cases. At the late nineteenth century, the topic dynamic structures had developed and made some important additions to explaining certain nonlinear phenomenas.

A system's dynamics can be either articulated as a periodic-time or as a distinct-time-evolutionary process. The easiest continuous system theoretical models were those composed of differential equations of that first degree. In an autonomy device of the first order (explicit in time) the dynamics is a rather limiting device type because its movement is still in the scientific field. In another side, in straightforward non-autonomous situations, the dynamics are very complex.

Throughout the fields of the natural and social sciences, nonlinear nature and its dynamics is of considerable significance. Case studies are including physical science (e.g., planet's atmosphere, gamma ray, digital loop, quantum entanglement, static downdrafts, and so on.), chemistry (Belousov – Zhabotinsky response, Brusselator system, etc.), biology (neurological and heart systems, biological procedures), ecology and social sciences (receding rise, infection circulating, industry and stock exchange price volatility, and so on.), to name a few examples. Nonlinear processes are more challenging to solve (if not always difficult) versus linear processes, since the latter obey the superposition principle and can be divided into parts. Every factor may be solved separately, and a combination of them all gives the end result. Hence, linear structure structures are useful in determining nonlinear processes.

The distinctive feature of open dynamical systems is that the orbits on X through H hole, whereas in dynamical system (X,T) the orbits maintain their image on the space for every time n. Open dynamical Systems have specific applications, including understanding of oceanic and atmospheric systems, space objects, planetary motions, or models of biological or medical processes.[3]

Nowadays, open dynamical systems is an active area of research. General properties of this class of dynamical systems have been studied for hyperbolic diffeomorphisms[4], [5], [6], [7] and billiards[8],[11], [10],[13]. In particular, the properties of the escape rate function have been studied extensively.

In general structure terms, Let (X, T) be dynamical system, where X is a compact metric space and $T : X \to X$ is positive topological entropy and continuous map. Let H be an open joined subset of X, named as a hole. The map $T : X\backslash H \to X$ is called as open system since $X\backslash H$ may not be an invariant set covered T . Let $\omega(H)$ be the maximal T -invariant subset of $X\backslash H$. Clearly

$$\Omega(H) = \{x \in X|T^n x \notin H, n \geq 0\} = X \bigcup_{n \leq 0} T^n(H)$$

In dynamical systems the survivor set mentioned as the set $\Omega(H)$.

The provided definition requires a topological structure on (X,T). The study of dynamical systems with topological structure is called *topological dynamics*. Nonetheless, to study dynamical systems in general, the topological structure of (X,T) a different structure. For example, we can consider $(X, B(X), \mu)$ a measure space and a transformation $T : X \longrightarrow X$ to be measurable. The study of the dynamics with measure theoretic structure is known as *Ergodic Theory*

However, some essential and certainly interesting questions around dynamical properties of (X_Ω, T) still have unknown answers. For more details, if X_Ω has positive Hausdorff dimension we may ask if (X_Ω, T) transitive, or if (X_Ω, T) has the specification property. Another natural and interesting question is if (X_Ω, T) is intrinsically ergodic.[15]

And here we listed some interesting research questions regarding the system $T|\Omega(H)$ such as its ergodic properties. For example, for piecewise C_2 expanding maps on an interval, Bedem and Chernov [16] proved that if combined the lengths of the holes are small and they are in general positions, then there is a conditionally invariant measure on X H equivalent to Lebesgue measurement. They also proved that this drug is unique and giving we move to another measure that is unique but unchanging. Chernov and Markanian[9] proved that for the generic dimensions of Lebesgue, $T|\Omega(H)$ is a fine-type sub-change if T is a hyperbolic algebraic automorphism in X = Tn (= [0, 1) n), and the hole H is a parallelepiped constructed along stable and unstable foliation of T passing through 0 of some dimensions.

Topological Entropy Topological entropy measures the development of distinct orbits over time , giving an idea of the complexity of your system's orbit structure. Entropy distinguishes a dynamic system where points closely related to the dynamic system where groups of points move further.
[1] analogous to the definition of measured theoretical enteropy invited by Shannon and Kolmogorov and Sinai, for general compact dynamic systems, was defined as the topologic of entropy. We do not provide a relatively general description since the particular case of symbolic dynamic structures is far easier to formulate. In terms of information theory, this wording can be motivated.
Think of a permitted n-word of a subshift X as the knowledge we get when we look at the clock's symbolic dynamic system. If given X space is full two shape, the n word can be any of the two binary n streams so we get n item of knowledge or one item for every symbol by recording which specific n-word occurs. However, since X is the mean perfect transformation, it is not feasible to have all binary lines. We receive less information from observing a specific word, because we could have excluded many words in advance. The N_n number of n binary strings without 1 consecutive fulfills the recurrence relationship of Fibonacci

$$N_{n+2} = N_{n+1} + N_n,$$

As we can either build a permissible block (n + 2), by pushing a 0 on a permissible word (n + 1)-word or 01 after an n-word has been announced. N_n is the number (n + 2)-th Number of Fibonacci, growing asymptotic as $C\gamma^n$ where C is constant, and golden mean $\gamma = (1 + \sqrt{5})/2$. We will tell how many we have information we obtain is the golden mean shift by observing a special n-word about $nlog_2(C\gamma^n) = nlog_2\gamma + log_2C$ bits, or generally $log_2(\gamma)$ bits every symbol. We interpret the topological entropy of a shift space X to be the limit

$$h(X) = \lim_{n\to\infty} \frac{1}{n} logN_n$$

Where $N_n = N_n(X)$ is the number of granted n-words of X. $h(X)$ can be defined as the rate of change per sign, or the exponential growth rate of the n-word number. The observation that $N_{m+n} \leq N_m \Delta N_n$ can establish that this limit exists (and is equal to the infinum of the sequence). It is a matter of personal preference whether we use natural or Simple two logarithms; for the natural log we calculate knowledge in *nats* instead of in bits.

If Y is a element of X, as a sliding image of the m-word language, it is difficult to surpass the amount of n-word of Y (n + m)-words of X. Thus

$$h(Y) = \lim_{n\to\infty} \frac{1}{n} logN_n(Y) \leq \lim_{n\to\infty} \frac{1}{n} logN_{n+m}(X) = (\lim_{n\to\infty} \frac{n+m}{n})h(X) = h(X). \tag{1.1}$$

Because conjugate shifts are mutually influences, entropy is the same.

We can see immediately that k-shift entropy is logk, because kn words are long n. The number of allowable n-words is the sum of the entries of A^n for a shift of fintitis type X_A given by a transition matrix A. As a consequence called the Perron-Frobenius theorem, increasing square, not negative matrix A has a nonnegative real proper value of equal to or above the Perron-Frobenius own value λ_A element. For the mean golden change $\lambda_A = \gamma$. In general, $h(X_A) = log = \lambda_A$ can be seen.

for more detailed definitions see 2.

Research into dynamic systems with holes, for instance, points that do not fit into some predetermined constructs iterating through a map, makes us pose questions about both arithmetic properties and dynamic interpretation of the points.[15]

Our main research questions related to the survivor set $\Omega(H)$. For us interesting size of survivor set in open dynamical systems. We will find that survivor set is finite depending on location and size. And also we will consider the survivor set subshifts of finite type uncountable and has positive topological entropy in Theorem 3.1.

Doubling Map Consider the expanding map $E_k : [0, 1) \to [0, 1)$ with expansion constant $k \geq 2$ ($k \in N$) defined as

$$E_k(x) = kxmod1.$$

For the doubling map ($k = 2$), Glendinning and Sidorov in [12] considered interval holes symmetric about the point $\frac{1}{2}$, and asymmetric interval holes in [12]. In our case, We will consider expanding map with $k = 2$(Doubling map). And we will use its canjugacy with shift map.

Shift map For integer $k > 1$, let Σ_k be the set of one-sided sequences with entries from the set $\Lambda_k = \{0, 1, \ldots, k-1\}$, excluding the sequences ending with $(k-1)^\infty$. For a finite length word w consisting of symbols $0, 1, \ldots, k-1$, we denote its length by $|w|$. Every such finite word w can be represented as $\omega 0^\infty \in \Sigma_k$. Set

$$B_k = \{\frac{\ell}{k^n} | \ell = 0, 1, .., k^n - 1, n \in \mathbb{N}\} \tag{1.2}$$

. Let $\sigma_k : \Sigma_k \to \Sigma_k$ be the one-sided shift defined as

$$\sigma_k(a_1, a_2, a_3, ...) = (a_2, a_3, ...).$$

We identify Σ_k with the interval [0,1) defined as

$$\pi_k(a_1, a_2, a_3, ...) = \sum_{n=1}^{\infty} \frac{a_n}{k^n},$$

for $(a_1, a_2, a_3, ...) \in \Sigma_k$. The map π_k is a bijection, and the inverse image of any element of B_k is a sequence in Σ_k ending with 0^∞. Representations of real numbers with an arbitrary base k $>$ 1 . There $\pi_k^{-1} x \in \Sigma_k$ gives the k-expansion of x $\in$ [0, 1). Note that the points (in B_k) have two k-expansions, one ending with 0^∞, and other ending with $(k-1)^\infty$. It is well-known that the diagram given below (Figure 1.1) commutes. That is, $E_k \pi_k = \pi_k \sigma_k$, for all k $\geq$ 2. A partial order $\prec$ can be defined on Σ_k as follows: u $\prec$ v if and only if either $u_1 < v_1$, or there exists $\ell \geq 2$ such that $u_i = v_i$, for $i = 1, \ldots, \ell - 1$, and $u_\ell < v_\ell$. For $u, v \in \Sigma_k$, we denote the set of all sequences $w \in \Sigma_k$ such that u $\prec$ w and w $\prec$ v, including u and v, by [u, v], which is called an interval. for more definitions see 2.1.1

***k*-transformations and symbolic space** In the rest of the paper we fix, a positive integer $k \geq 2$. Let $X = [0, 1)$ and k-transformation $T_k : X \to X$ given

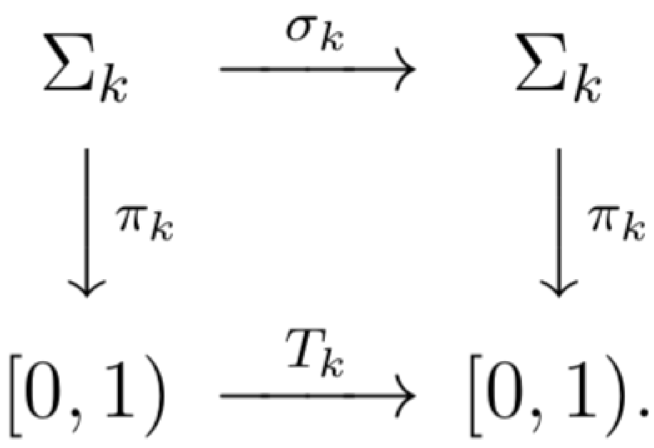

Figure 1.1: Commuting diagram.

by $T(x) = kx \mod 1$. More specifically,

$$T_k(x) = \begin{cases} kx, \text{ if } kx < 1, \\ kx - 1, \text{ if } 1 \leq kx < 2, \\ \cdots \\ kx - (k-1), \text{ if } k-1 \leq kx < k. \end{cases}$$

Let $H = (a, b)$ be an open interval in X. The case of $k = 2$ was studies by Glendinning and Sidorov in [**GS01**, **GS15**]. They prove,

Theorem *The Hausdorff dimension of the survivor set $\Omega_H(T_k)$ is positive and in particular it is uncountable if $b - a < 1 - 2a_*$ where $a_* \approx 0.41245$ is the Thue–Morse constant. Moreover, if the hole H contains the midpoint 0.5, then $\Omega_H(T_k)$ has positive Hausdorff dimension if and only if $b < \chi(a)$, where $\chi(\cdot)$ is given in [**cite {glendinning2015doubling}**].*

The above theorem is recently generalized to k-transformations for arbitrary $k \geq 2$ by Agarwal in [2].

The main idea of the proofs in the above mentioned results are to transfer the problem to symbolic space, namely the full shift on k letters.

The chaotic tent map Yoshida T [17] Analytical studies of chatic tent map behaviors of invariant density and power continuum were undertaken in its messy area. Chaotic tent map is a piece by piece linear map with a single maximum point. If the target height is reduced, consecutive split transitions occur within the chaotic region and at the transfer stage are aggregated into the nonchaotic area. At band splitting points and in the proximity of some points, the time-relation function of nonperiodic orbits and their power distribution are determined precisely. Topologically, the tent map is conjugated and the comportements of the map under iteration in this sense are therefore identical. The chaotic tent map is given by:

$$T_c = \begin{cases} cx & \text{if } x < \frac{1}{2}, \\ c(1-x) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Where $x \in [0,1]$. This map transforms an interval [0 , 1] on itself and only contains one parameter of control c. We can get this orbit for each x. The system 1.1 shows a range of dynamic behaviours, from predictable to chaotic, according to the control parameter c.

If c=0,1,1,1.5,2 the chaotic tent map image (Figure **??**) will be displayed. In Figure **??**, the dynamics of the $c < 1$ tent map family are shown. For us (figure **??**) interesting if $c > 2$.

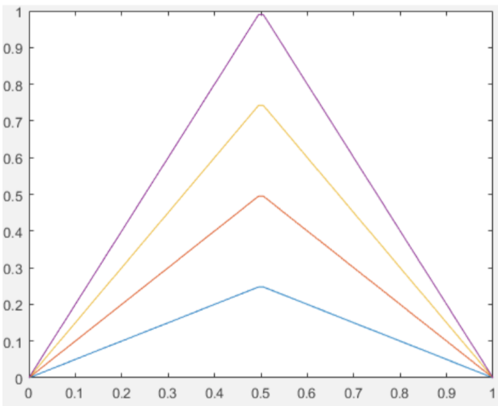

Figure 1.2: Graphs of the Tent map family when c=0.5,1,1.5,2.

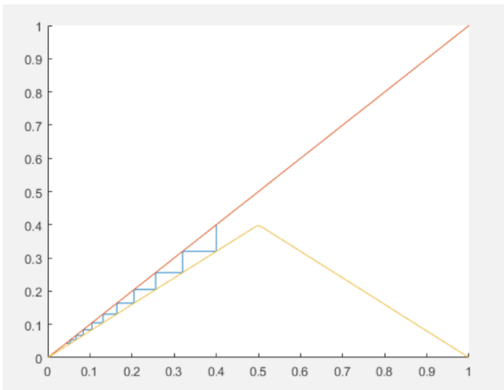

Figure 1.3: The dynamics of Tent map family for $c < 1$.

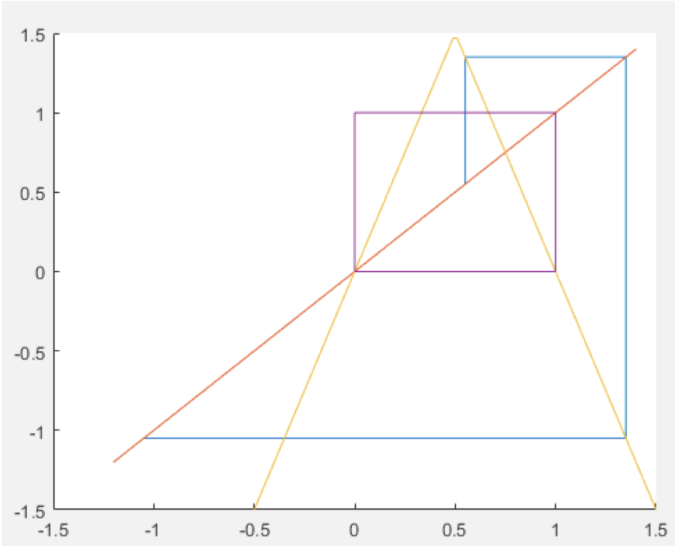

Figure 1.4: Dynamics of the Tent map family for $c > 2$. The purple square indicates the interval [0,1] and the horizontal T(x) = 0 and T(x) = 1.

for more detailed definitions see **??**.

1.2 Statement of Results

As it is seen from the literature that the previous work related to the survivor set was mainly to understand the cardinally and fractal dimension of survivor set given the open dynamical system. We would like to extend these results with a slight twist. Mainly, we ask the following question:

Main question: For a given interval $H = (a, b)$, which intervals maps with hole H induce the survivor set Ω_H with positive fractal dimension?

We state our result in symbolic space and leave it to the interested reader to rephrase the theorem in terms of interval k-transformation using the above mentioned commuting diagram.

Theorem 1.1. *Let $k \geq 2$ be an integer, A a $k \times k$ transition matrix, and (Σ_A, σ_k) is the induced subshift of finite type. Assume that there exist two distinct symbols $i, j \in \Lambda_k$ such that $A_{ii} = A_{jj} = A_{ij} = A_{ji} = 1$. For any $\ell \geq 0$ we define the subset of Σ_A*

$$S_\ell(i,j) := \{a_1 a_2 \cdots \in \{i,j\}^{\mathbb{N}} : a_m = i, \implies a_{m+n} = j, n = 1, 2, \ldots, \ell\}. \quad (1.3)$$

If the hole H in Σ_A is disjoint from S_ℓ for some ℓ, then the survivor set satisfies

- *$\Omega_H(\sigma_k)$ is uncountable,*
- *$\Omega_H(\sigma_k)$ has topological entropy at least $\frac{\log 2}{\ell+1}$,*
- *$\Omega_H(\sigma_k)$ has Hausdorff dimension at least $\frac{\log 2}{\ell \log k}$.*

Lemma 1. *Let function $S : \Sigma_2 \to [0,1]$ is homeomorphism defined as*

$$S(a_1, a_2, a_3, ...) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

Then there is conjugacy between T_3 and σ such that $S \circ \sigma = T_3 \circ S$ There is a function $f : [0,1] \to [0,1]$ such that $f = S \circ \pi$.

Theorem 1.2. $\Omega(I_a) = \infty$ *if and only if* $\Omega(f(I_a)) = \infty$

2. Preliminaries

The main purpose of this section is to recall some necessary theorems, lemmas, definitions and results that will be helpful in this paper. In this way, we will focus on Symbolic dynamics, topological entropy, Hausdorff dimension, chaos and Invariants.

2.1 Symbolic dynamics

Symbolic dynamics is a increasingly growing component of complex structures. Since it emerged as a tool for the analysis of generalized dynamic structures, the methods and concepts find major uses in database processing and storage and also linear algebra.

The topic is firstly mentioned in Jacques Hadmard's 1898 article on the subject of geodesy on curved surfaces negatively. Marston Morse implemented the idea in 1921 to build unconventional repetitive geodesy. Emil Artin(the system is now called Artin Billiard), Peka Myberg, Paul Kobe, Jacob Nielsen, G. A.Hedlund did similar works in 1924.

The first official procedure was published by Morse and Hedlundin 1938. George Birkhoff, Norman Levinson and partner Mary Cartwrightand J. E. Littlewood used methods similar to the qualitative analysis of second order non-automatic differential equations.

Claude Shannon applied symbolic sequences and modifications to the finiteness in his work "A mathematical theory of communication" that brought information theory.

In the late 1960s, the method of symbolic dynamics evolved into a hyperbolic turbomorphism by Royadler and Benjamin Weiss and in a differential phase of Anosov by Yakovsinai using a symbolic model to construct a Gibbs measure. In the early 1970s, this theory was extended to the Anosov flow of Marina Ratner and the difeomorphism and Axiom A flow of Rufus Bowen.

2.1.1 Shift spaces

To abstract mechanics, shift spaces are what operations such as addition, subtraction, multiplication and division are to algebra. We begin by presenting such spaces, and explain a number of examples to direct the intuition of readers. We'll later reflect on different groups of move rooms. As the name may imply, there is a shift map of the change from the space to itself on each Shift space. Together such form a "dynamic shift space." They should concentrate on these dynamic structures, their relations and their implementations.

Full Shifts Knowledge is in more cases described as well as a series of discrete symbols come from a finite set that elements considered as fixed.For starters, this thesis is really a long line of letters, punctuation and other symbols from the common stock of the typist. The infinite sequence of symbols in its decimal expansion represents a real number. Data are stored in computers as O's and l's sequences. Signal Samples are used by compact audio disks to digitally record Beethoven symphonics, using blocks of O's and l's.
In each of these examples, there is a finite set Λ of symbols which we named the alphabet. Elements of Λ are also called letters, and they will typically be denoted by $a,\ b,\ c, \ldots$ or sometimes by digits like 0, 1, 2, ..., when this is more meaningful. Decimal expansions, for example, use the alphabet $\Lambda =$ 0,1,...,9.
While symbols in real life are finite, long series in both directions (or bi-infinite) are also extremely valuable. It is related with the usage of real numbers, consistency and other research concepts to explain the actual measurements that can only be calculated with small precision.

The operation σ, pictured in (Figure 2.1), maps the full shift $\Lambda^{\mathbb{Z}}$ onto itself.

There is also the inverse operation σ^{-1} of shifting one place to the right, so it means σ is both onto and one-to-one map. If we take an integer number $k > 0$, Composition of shift map with goes to itself k-times $\sigma^k = \sigma \circ \cdots \circ \sigma$, it shifts sequence k places to the left, and the same number shifting with inverse of map to the right while $(\sigma^{-1})^k = \sigma^{-k}$. That's why $\Lambda^{\mathbb{Z}}$ is called a full shift.

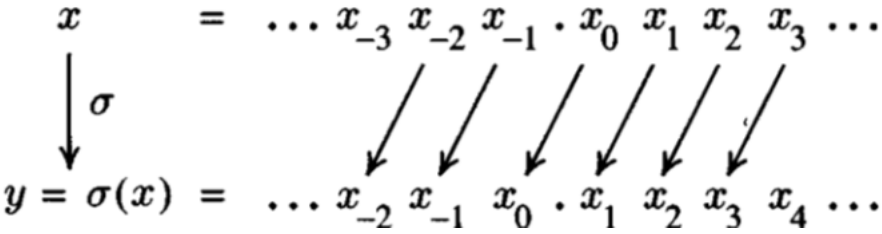

Figure 2.1: Representing points symbolically.

Definition 2.1. A point ω is periodic for σ if $\sigma^n(\omega) = \omega$ for some $n \geq 1$, and we say that ω has period n under σ If ω is periodic, the smallest positive integer n for which $\sigma^n(\omega) = \omega$ is the least period of ω. If $\sigma(\omega) = \omega$, then ω is called fixed point for σ.

Shift spaces The sequences of symbols that we are researching are always constrained. The popular alphabet Morse is conveyed by a dot word and stitches with a total duration of at most six, such that a duration word of at least seven with no split is prohibited to happen (the one exception being the SOS signals). This ensures that the SOS signals are not used for all the symbols "line," "mark," and "stop." In programming language python, a code line such as $cos(x) **2 == y$ is not allowed, nor are unbalanced line axes, as Python's syntax laws ignore them. Different kinds of binary sequences have been used to fix major errors in compact audio discs, defined by a certain number of conditions In this segment we discuss the fundamental notion of shift space, which is a subset of points that follow constraints in complete shift.

List F can be infinite or minimal. In any case it is countable as the members may be grouped in a sequence (only join the length 1 words, instead the length 2 words, etc.). Many collections F may define a given shift space. Remember that empty set $\emptyset$ is a moving space, as $F = \Lambda$ excludes all products. When an X is in a Y-shift, we say that X is a Y-shifting subshift.

The X=X_F equation relates to how a move space has been created while X relates to the corresponding set. X refers to the X Related typographical variations between an action and its outcome are used often. By utilizing such variations, we expect that the sort of incomprehensible equations like "y = y(x)" you have seen in calculus lessons can be prevented.

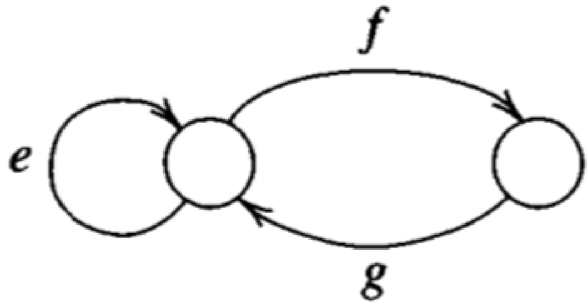

Figure 2.2: A graph defining a shift space.

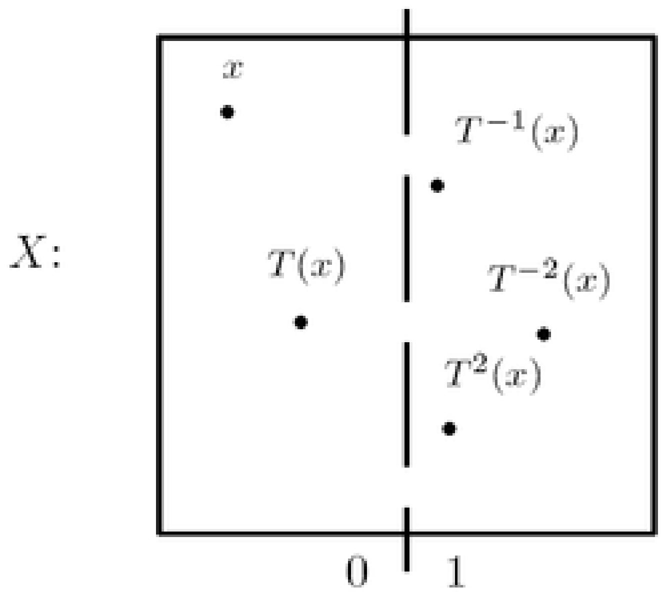

Figure 2.3: Representing points symbolically.

Coding and isomorphism The term code is different for mapping of different kinds between symbolic or general dynamic systems in the context of symbolic dynamiques and related fields. Symbolic system to one. The one-sided golden mean shift, for example, could be described as a coding of the interval map $x \rightarrow \alpha x$. It may be remembered that a mapping may be called an encoder in the coding and knowledge theory, and its picture may be called a file.

We are generally interested in maps that maintain conceptual dynamics, at the topology and dynamics of the change space are less than to some degree. We want nearby points to be sent to nearby points and when x is sent to y, its shift to σy should be sent. A continual map σ commuting with a shift is a continuing homomorphism from one subshift to the other, i.e. for which $h \circ \sigma = \sigma \circ h$. A factor map, a concept often used in ergodic theory, is historically regarded as homomorphism, even though the concept quotient map may correlate with the usage of many mathematical fields. An isomorphism is also referred to as a theoretical conjugacy, as is common in general theory of dynamic systems, or an invertible homomorphism. from one subshift to another.

From the golden median shift to the even shift we can define the factor map h: map $y = (y_j)$ where $x_j = 1 - (y_j + y_{j+1})$ for all j.Because every 1 in y is preceded and followed directly by 0, every 0s in x in pairs is produced. Clearly,$\sigma h(x) = h(\sigma x)$. The map is continuous, as a local rule determines a central word of x by a slightly longer central block of y. Generally speaking, it is the case a sliding word code from a subshift X to a subshift Y is a map h given by a local rule $(h(x))_i = h(x_{i-n} \dots x_{i+a})$, where a and m are from integer numbers with $-n \leq a$ and h is a map from the $n + a + 1-$ words of X which goes to the symbols of Y . Memory and anticipation are respectively the numbers m and a, normally considered to be non-negative. The Curtis-Hedlund-Lyndon theorem comes from an statement that uses the compactness of theX: any homomorphism is supplied by a sliding-word code.

The low block codes are particularly important. We can define this a homomorphism from any subshift X into the full $\Lambda_n(X)$-shift by the sliding word code $h(x_0x_1 \dots x_{n-1}) = [x_0x_1 \dots x_{n-1}]$. Here, we use the square brackets to stress that a single symbol in a new alphabet is the enclosed word.Thus when $n = 2$, the sequence $\dots x_{-1}x_0x_1x_2 \dots$ is sent to

$$\dots [x_{-1}x_0][x - 0x - 1][x - 1x_2] \dots$$

This homeomorphism is clearly one-to-one. Its image is the m-word representation of X, denoted by $X^{[m]}$.

Graphs and matrices Care that a first step subshift of finite type X is described by its set of allowed second words, it means, by a collection of symbols that could be preceded in our sequences of symbols. We may represent the scheme

via a guidance graph: the vertices are the alphabet Λ signs, and if the term ab is allowed, there is an edge from a to b. Each element (x_i) of X is a two-infinite step on the graph from vertex edges to edges. On the opposite, a line G (finite) with the shift of finite type X_G with alphabet equals the set of no parallel edges $G's$ vertices. This is also the G-associated vertex move. The diagram of (2.4) demonstrates the golden mean move. Chart still denotes a from here on the graph directed.
A graph G which has n vertices is practically represented by giving its trabsition matrix, the n by n matrix $A = (a_{ij})$ such that a_{ij} is the number of edges to jth vertex from ith vertex.Thereby every vertex change is equal to a 0 square matrix A And $1's$, the transformation matrix of move vertex also named. The transition golden middle shift Matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Figure 2.4: Vertex graph of the golden mean shift.

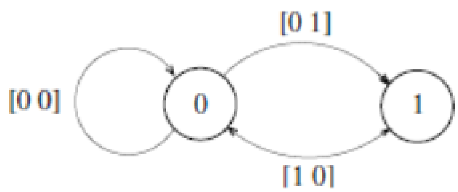

Figure 2.5: second words presenatation of the golden mean shift.

Vertex transfers are attractively simple to capture the constraints of a first stage shift of finite type. These graphs are used to define Markov chain structures in the context of stochastic processes. The vertices are system states, and the edge from one to b is marked by the likelihood of a transfer from state a to state b, which is considered stationary and independent from previous states. The absence of a boundary suggests the possibility of zero transfer. The topological space that supports the Markov chain then constitutes the vertex shift given by

the graph. This is why first step finite type shifts are also referred to as Markov topological chains.
Even if there are parallel edges in graph G, we can always see it as a first stage shift of finite type if the set of G edges is taken as the sign instead of the vertices. The X_G boundary transition is the set of bi-infinite walks on the edges of G, which implies the set of boundary parts (x_i) to make the terminal top of x_i the first vertex of $x_i + 1$ for all i. X_A denotes the edge change, where A is the matrix for G as previously.

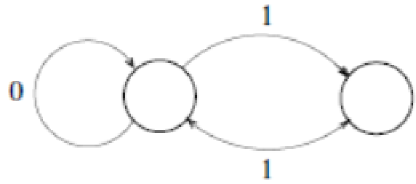

Figure 2.6: Graph of even shift.

The second word code from the golden mean shift to the same shift described in the previous section produces the edge marking in (2.6). It is easy to see that the prime gap system can not be represented by a finite graph labeled, so it is not a sophistication shift.
Sofic systems have a normal automated theory relation. We find the vertices of the diagram to be inner states for a computer and the marker at the edge between v and v' to indicate that if in Indicate v the system is interpreted as input it will switch into State v'. A NFA is just a guided edge marking map of one or two initial states and permissible States. NFA is a non-deterministic Finite state automaton. The vehicle accepts or recognizes a word $b_1 \dots b_n$ if this word marks a path to the accepting condition from an initial state. The set is a regular language for all words accepted in an NFA. Within this term, the set of words of all lengths in a sofisticated structure Y is a normal language given by an NFA where both original and appropriate countries are described.

Combinatorics on words There we want to give some definitions with results which are taken from combinatorics on words for more and detailed exposition. There are two finite words $u = u_1 ... u_k$ and $v = v_1 ... v_n$ we look at uv for their concatenation $u = u_1 ... u_k \; v = v_1 ... v_n$. Specificly, $u^m = u ... u$(m times) and $u^\infty =$

$uuu\cdots = lim_{n\to\infty}u^n$, Limit is from topology of coordinate-wise convergence. From here a "word" means a word that has 0s and 1s letters. Let ω be a finite or infinite word. It will be a finite word ω is a factor to u where for $u = \omega_k...\omega_{k+n}$ there exists k for some $n \geq 0$. For a finite word ω let $|\omega|$ equal to its length and $|\omega|_1$ stand for the number of $1s$ in ω. The 1-ratio of ω is provided as $|\omega|_1/|\omega|$. For $\omega_1\omega_2...$ is an infinite word the 1-ratio is provided as $lim_{n\to\infty}|\omega_1...\omega_n|_1/n$.

We will show that a finite or infinite word ω is balanced if for any $n \geq 1$ and any two u, v factors of ω which is with n length we get $||u|_1 - |v|_1 \leq 1$. Sturmian is the name of an infinite word if it is balanced and not eventually periodic. A finite word ω is cyclically balanced if ω_2 is balanced. It is well known that if u and v are two cyclically balanced words with $|u| = |v| = q$ and $|u|_1 = |v|_1 = p$ and gcd(p, q) = 1, then v is a permutation of cyclic for u . Thus, there are only q distinct cyclically balanced words of length q with p 1s. word of length q with 1-ratio r beginning with 1.

2.2 Invariants

One of the basic questions concerning abstract complex structures may be posed Is how we can learn the conjugate between two subshifts. The changes of the finite form, a total and efficient classification remain elusive except for the simplest class of structures.

In the other side, it is very difficult for us to suggest that there are two subshifts. By an invariant of conjugation, we are referring to any amount or mathematical object that is unchanged by subshifting by a conjugal subshift. Invariants can be described in general for dynamic systems and others which relate only to shift spaces, or for the smaller shift of finite type class.

Periodic points If f is a compact spaces X homeomorphism, We are marking by $Fix_n(X)$ the set of all $x \in X$ with $T^n(x) = x$. These are the n points in the dynamic system that we call period. Note that the period n point with this terminology is also a period m point if m is several n. It is the least period of x that we distinguish the least n such as this. A conjugation of dynamic processes holds points intervals such that conjugate changes have the same cardinality of n points per increasing n.

As shown above, the period n points are the sequences for a shift area (x_i) for all I $x_{i+n} = x_i$. If the alphabet Λ has k symbols, the maximum kn period is n points. It is known that the number of walks from top i to top j along the edges of the graph in a graph with adjacency matrix A is simply the ij entry of A^n. For a shift of the rim X_A, period n points match length n walks with the same initial and final state i. Thus, it is easy to calculate the number of periods n points as the diagonal entries or sum of the A^n trace.
The periodic count of the points in X Artin-Masur function can be encoded where the $p_n(X)$ of period n points of X is finite for all n. That is the below given power series

$$\Xi_X(t) = e^{\sum_{n=1}^{\infty} \frac{p_n(X)}{n} t^n}. \tag{2.1}$$

This function takes the remarkably easy form for the shift of finite type X_A given by a $k \times k$ to the transition matrix A

$$\Xi_A(t) = \frac{1}{det(I - tA)} = \frac{1}{t^k X_A(t^{-1})}, \tag{2.2}$$

Where iron A is a feature of the polynomial A. This formula, called Bowen This formula the formula Lanford can be verified by putting A in its usual form and by using it the above-mentioned trace result. More generally, Manning theorem shows that each sofistic system has a rational zeta function.

Topological Entropy In relation to the number of applications on our map T, we consider the exponential growth of distinguishable orbits. We will first take into account that segments of finite orbit of length n are differentiable at a certain finite resolution. Compact granted place metric $(X; d)$ and map $T : Y \to Y$, we interpret the $d_n : Y^{-}X \times X \to R$ by

$$d_n(x, y) = max_{0 \le k \le n} d(T^k(x), T^k(y)).$$

d_n is a metric of X for each n as d is a metric of X already. With this modern notion of space, points are as near as possible if they are as similar as possible to n iterates of T. Tomorrow should be viewed as a settlement, the smallest distance from which two items can be distinguished.
Next we will have a quantity which counts these distinctive orbits. In other words,

the points which are close relative to d_n-metric and how many of these collections we have, we are collecting indistinguishable orbits. However, we would prefer to eliminate the risk that the samples themselves would be similar, because that will mean the differentiable orbits to be measured. Therefore, we are considering an X-cover that contains as little collection ϵ as possible.

Definition 2.2. Let *epsilon* ≥ 0 constantly. Let $cov(n, \epsilon, T)$ stand for the minimal X scope cardinality given by sets of d_n-diameter lower than ϵ, the supremum of distances between pairs of points in the system shows the diameter of the collection.

Remark 2.3. Because compactness means that each X open cover has a finite subcover, $cov(n; \epsilon; T)$ follows that it is a finite.

Definition 2.4. Here given

$$h_(\epsilon)(T) = \limsup_{n\to\infty} \frac{1}{n} ln(cov(n, \epsilon, T)). \tag{2.3}$$

This limit gives the $cov(n; \epsilon; T)$ exponential growth with a fixed resolution as the length of the orbits we see in infinity tends to be. We assume that the maximum is greater since we do not know if the cap occurs. We require the following lemma from the calculus to show this.

Lemma 2.5. *Given a sequence $b_{nn\geq 1}$ be a sequence of subadditives, which is $b_{m+n} \leq b_m + b_n$ for all m and n. Then* $\lim_{n\to\infty} \frac{a_n}{n}$ *occur and it is equal to* $inf_n \frac{a_n}{n}$.

Proposition 2.6. *The given limit* $\lim_{n\to\infty} \frac{1}{n} ln(cov(n, \epsilon, T)) = h_\epsilon(T) is\ finite\ and$ *occur.*

Proof. Given the above lemma, the $ln(cov(n, \epsilon, T))$ sequence must be shown as subadditive. Since then it must be finite, because $ln(cov(n, \epsilon, T))$ is lower than 0, $infln \frac{(cov(n,\epsilon,T))}{n}$. Let $\epsilon \geq 0$. Assume that A is an X cover with d_n-diameter sets less than ϵ the uterus and B an X cover with d_m-diameter sets less than the uterus. Then $cov(n; \epsilon; T) \leq |A|$ and $cov(m; \epsilon; T) \leq |B|$, as a result of cov assign to the lower minimum cardinality. Let $V \in A$ and $U \in B$. If $V \cap T^{-n}(U) \neq \emptyset$, then show $a, b \in V \cap T_{-n}(U)$. For us given $a, b \in V$, then

$$\max_{0 \le i \le n} d(T^i(a), T^i(b)) \le \epsilon.$$

A;so $a, b \in T^{-n}(U)$ that means $T^n(a),\ T^(n)(b) \in U$. Since U has d_m-diameter lower than ϵ, given that

$$\max_{0 \le i \le n} d(T^i(T^n(a)), T^i(T^n(b))) \le \epsilon.$$

Then given above two inequalities give us

$$\max_{0 \le i \le m+n} d(T^i(a), T^i(b)) \le \epsilon.$$

Then the set $V \cap T^{-n}(U)$ gives us d_{n+m}-diameter which is lower than ϵ. Suppose $D = V \cap T_{-n}(U) : V \in A, U \in B$. So $|D| \le |A||B|$, When each crossroad is non-empty, equity is observed. In addition , we show that D has a diameter of d_{n+m} less than ϵ so $cov(m + n; \epsilon; T)$. Remembering the $cov(n; \epsilon; T)$ and $cov(m; \epsilon; T)$ that we have we initially bound:

$$cov(m + n; \epsilon; T) \le |D| \le |A||B| \le cov(m; \epsilon; T).$$

Reminding the basic logarithmic properties, we have the $ln(cov(n; \epsilon; T))$ sequence, which completes the evidence, as an added element. We want the resolution to be reduced next. It is obvious that if we lower a score of less than a size to cover X can only increase the number of sets of diameter. Thus $cov(n; \epsilon; T)$, is monotonous, and $h_\epsilon(T)$ means that it is limited to 0.

Definition 2.7. The function T's topological entropy is given by $h_T = \lim_{\epsilon \to 0^+} h_\epsilon(T)$.

This definition can be used to illustrate a variety of properties concerning topological entropy, but before we take examples, the next segment includes more important definitions.

Spanning And Separated Sets We must again acknowledge the metric d_n for a fixed resolution to count distinguishable orbit fragments.

Definition 2.8. Take fixed $\epsilon \geq 0$. Let $n \in \mathbb{N}$. The set $K \in X$ is an (n, ϵ)-spanning set if for all $x \in X$, there exist $y \in K$ such that $d_n(x, y) \leq \epsilon$.

Remark 2.9. Note that $d_n's$ definition depends on T.

The last definition is a advantageous characterization for the reason that we can construct easily as a sets for obvious dynamical systems. Firstly that we will show type of dynamical system is doubling map $T : S^1 \to S^1$ the function is $Tx = 2xmod1$. Here we use S^1 as $[0; 1]$ with identity endpoints. The normal alternative of unit circle S^1 metric is

$$d(x, y) = min(|x - y|, 1 - |x - y|)$$

T is constant in respect to this. When thinking about the doubling map, we should still find this measure. We need the following lemma before we can construct a (n, ϵ) collection for the doubling map.

Lemma 2.10. *Let T be the doubling map on S^1. Then it follows we have*

$$d(x, y) \leq \frac{1}{4} itgivesd(T(x), T(y)) = 2d(x, y).$$

Proof. How we know, $d(x, y) = |x - y|$ for $|x - y| \leq \frac{1}{2}$. Let $x,\ y$ are given by $d(x, y) \leq \frac{1}{4}$. It follows $|x - y| \leq \frac{1}{4}$. By using definition od doubling map T, we get

$$d(Tx, Ty) = d(2xmod1, 2ymod1) = min(|2x - 2ymod1|).$$

Remark that $|2x - 2y| \leq \frac{1}{2}$, and $2x - 2ymod1 = 2x - 2y$.
Accordingly

$$d(Tx, Ty) = 2|x - y| = 2d(x, y).$$

Notation. Let S_n assign the set of dyadic rationals with denominator 2^n, which is

$$S_n = \{\frac{i}{2^n}, where 0 \leq i \leq 2^n - 1\}.$$

Proposition 2.11. *The fractions S_{n+k} set is a doubling map set $(n; \epsilon)$-span set.*

Definition 2.12. Take fixed $\epsilon \geq 0$. Let $n \in \mathbb{N}$. A set $G \in X$ is an (n, ϵ)-separated set if for all elements $x, y \in G$ where they are not equal to each other, we get $d_n(x, y) \geq \epsilon$.

Hausdorff dimension In mathematics, Hausdorff dimension is a ruggedness or, in particular, the fractal dimension that the mathematician Felix Hausdorff first introduced in 1918. For example, the Hausdorff dimension is zero, one section is 1 and a cube is 3. In other words, the Hausdorff dimension is an integer which agrees with the normal sense of dimension also known as the topological dimension for sets of items defining a smooth form or a shape that has a small number of corners – the forms of traditional geometry and science.Nevertheless, formulas have also been developed that enable calculation of the dimension of other less simple objects, in which it is concluded that particular objects — including fractals — has non-integer dimensions of Hausdorff only because of their scaling and self similarity qualities. This is also commonly referred to as the Hausdorff-Besicovitch dimension because of the important technical advances made by Abram Samoilovitch Besicovitch to enable measured dimensions to be calculated for highly irregular or "rough" sets.
More precisely, the Hausdorff dimension is another dimension number connected with a particular group, which determines the distinctions between all representatives of that group. Such a set is known as a gallery. In contrast to the more logical aspect that is not connected with common metric spaces and only takes values in non-negative integers, the aspect of the enlarged true numbers, $\overline{\mathbb{R}}$, is identified.

The Hausdorff dimension generalizes the concept of a real vector space in mathematical terms. That is, the n-dimensional interior product field of Hausdorff is equivalent to n. This is why the Hausdorff dimension is zero, one line, etc., and irregular sets can have dimensions of the Hausdorff non-integer. This is based on the previous statement. In each iteration their component line segments are divided into 3 unit segments, the newly formed middle segment is used as a base of a new equilateral triangle pointing outward and then this base segment is deleted to leave a final object from the iteration of unit length of4. For instance, the snowflake of the Koch shown right is made up of an equilateral triangle. It implies that each initial section of the line was replaced with N=4 after the first repeat, with each replica being 1 / S = 1/3 long as the first. We have taken another approach, an entity with the Euclidean dimensions, D, and in each direction, decreased its linear scale by 1/3, to N = SD its length. This equation is easylly fixed for D, resulting in the ratio of logarithms (or natural logarithms) in the figures and the non-integer dimensions for these objects in the Koch and other fractal cases.

The Hausdorff factor is the product of a simplified box-counting or Minkowski-Bouligand factor, but generally similar. The intuitive concept for the dimension of a geometric object X is the number of separate parameters to choose from. Any point indicated by two parameters may, however, be indicated by one point, as the real plane's cardinality is equal to the true line's cardinality (this can be seen in an argument that intertwines two-digit figures in order to produce a single number encrypting the same information). The example of a curve for space-fullness shows that the true line can even be mapped to the true plane surjectively (taking one real number into a pair of real numbers so that all pairs of numbers are covered), so that one-dimensional object fills up a higher-dimensional object completely.

Every space filled curve has multiple points so there is no continuing reverse. Two dimensions can not be mapped on one in a way that is indefinitely and continuously invertible. The topological dimension, also known as Lebesgue, illustrates why. This size is n if there is at least one point where n+1 balls overlap in each X covering with small open balls. For eg, if a line with short open intervals is filled, other points have to be filled twice, giving dimension n=1.

But topological dimension (distance near one point) is a very rough measure of a space 's local dimensions. Although it fills most of the region, a curve which is almost space-filling can continue to have a topological dimension. A fractal has an integral topological dimension, but it acts like a higher dimension in terms of the amount of space it takes up.

The Hausdorff dimension takes into consideration the distance between points and the metric the local scale of the field. Take into account the N(r) amount of radius balls at maximum r needed to cover X entirely. N(k) grows polynomially with 1/k when k is very small. The Hausdorff dimension is the only number d for a sufficiently well-behaved X, so N(k) grows to 1 / k^d when k is approaching zero. It determines more specifically the box accounting factor, equivalent to the Hausdorff component, where value d is a crucial limit between space-insufficient growth levels and overabundant growth rates.

The Hausdorff dimension is an integer that is associated with the topological dimension for smooth forms or forms at a limited number of angles, aspects in conventional geometry and physics. But Benoit Mandelbrot found that fractals are found all over the nature in setting with noninteger Hausdorff dimensions. He noticed that the best idealization of the most raw types you can find around him is not in smooth idealized forms, but in fractal idealized forms:

Clouds don't have circles, mountains don't have cones and oceans, wood isn't flat or lighting flies in straight lines.

The Hausdorff and the box-counting element correspond with fractals that exist in nature. The packaging dimension is yet another similar idea that gives many shapes the same value, but there are well-documented exceptions where they vary. *definitions.* Let X space be a metric. If $U \in X$ and $d \in [0, \infty)$, the d-dimensional Hausdorff dimension that unlimited of U is described by

$$C_H^d(U) = inf\{\sum_j k_j^d : hereacoverofUbyballswithradiusk_i \geq 0\}.$$

In general words, $C_H^d(U)$ is the lowest number of the series $\delta \geq 0$ which is some collection of balls $\{D(x_j, k_j) : j \in J\}$ covering U with the value $k_j \geq 0$ for every $j \in J$ that satisfies $\sum_{j \in J} k_j^d \leq \delta$. (In this case, we are using the standard agreement $inf\emptyset = \infty$.)

Hausdorff measure

Instead of contemplating every potential coverage of U, we will see what happens when the size of the balls goes to zero, as opposed to the infinite Hausdorff material. The d-Dimensional Hausdorff external measure of U is described in the case of the $d \geq 0 d \geq 0$

$$H^d(U) = \lim_{k \to 0} inf \sum_j k_j^d : hereacoverofUbyballswithradius 0 \leq k_i \leq k$$

Hausdorff dimension So our general definition and formula of Hausdorff dimension defined as

$$dim_H(X) = inf d \geq 0 : H^d(X) = 0.$$

$Dim_H(X)$ can also be described as the minimum of a set d'alternatively $[0, \infty)$ to zero for d-dimensional Hausdorff measures. The same is valid for the supremum of $d \in [0, \infty)$ such that the d-dimensional Hausdorff calculation of X is infinite (unless the Hausdorff dimension is zero because this above number range is empty, d).

2.3 Cardinality

The cardinality of the set is a measure of the "number of elements in the set". Since the end of the nineteenth century, this concept has been generalized into infinite sets, making it possible to distinguish different phases of the infinite and perform arithmetic calculations there. There are two approaches to cardinality: one that directly compares sets using injections and injections and the other using primes. The cardinality of a set is also called dimension when it cannot be confused with other dimension concepts.

Any set X that has the same cardinality as a set of natural numbers, or $|X| = |N| = \aleph_0 \aleph_0$, is said to be a countably infinite set.

Any set X with cardinality greater than natural numbers or $|X > N|$, for example $|R = \mathfrak{c}\mathfrak{c} > |N|$ is considered countless.

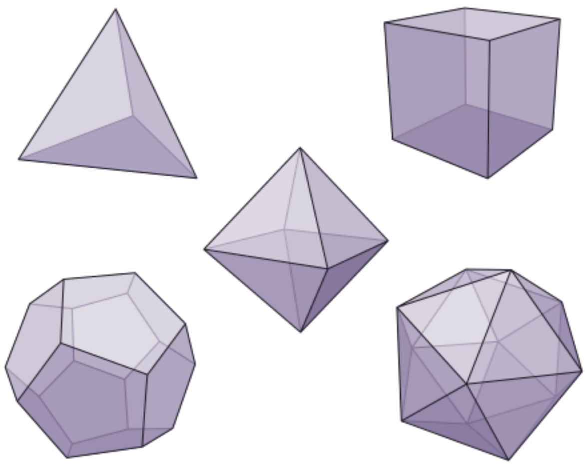

Figure 2.7: The set S of all Platonic solids has 5 elements. Thus |S|=5.

Theorem 2.13. *Cantor's diagonal argument*

2.4 Chaos theory

Chaos theory is the field of mathematical science that focuses on the analysis of chaos – structures of fluid systems that appear to be arbitrarily caused by distur-

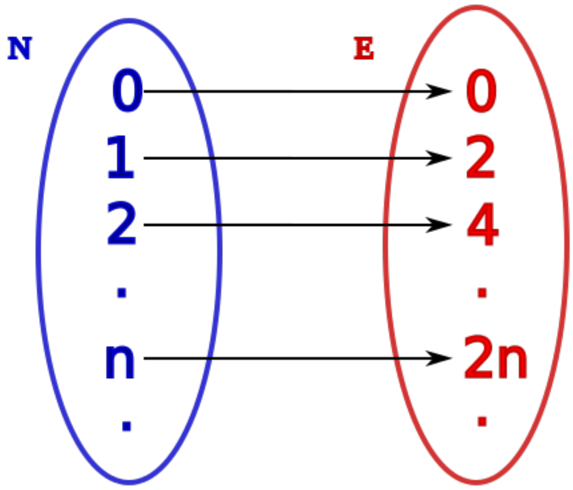

Figure 2.8: Bijective function from N to the set E of even numbers. Although E is a proper subset of N, both sets have the same cardinality.

bances and irregularities.[1] The chaos theory is an interdisciplinary hypothesis, because in the apparent randomness of disorderly, complicated processes there is a historical structure underlying theory. The butterfly-related effect, a fundamental principle of chaos, describes how a slight change to one state in the non-linear deterministic system could lead to large discrepancies in a subsequent state, (this is, sensitive dependence on initial conditions).
Small differences in initial condition, such that due to mistakes in measurements, or due to rounding errors in numerical computing, can generate broadly diverging results for these dynamical systems and, generally speaking, make long-term prediction of their behavior unenlightened. Edward Lorenz summed up the theory as:

Chaos: If the present determines the future, but the present approximates does not determine the future approximately.

There are chaotic behaviour, including fluid flow, heartbeat irregularities, weather and climate, in many natural systems. In certain structures of artificial components, such as storage and road traffic, it often happens randomly. This behaviour, by analyzing the mathematical model or by analytical techniques such as recurrence plots and Poincaré maps, can be investigated. Chaos theory has a wide range of applications in meteorology, anthropology , sociology, physics, environment sciences, informatics, engineering , economics and biology , ecology. For such fields of study, theory formed the basis of complex dynamic systems,

$$
\begin{array}{ll}
s_1 & = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\
s_2 & = 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \ldots \\
s_3 & = 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \ldots \\
s_4 & = 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_5 & = 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_6 & = 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0 \ldots \\
s_7 & = 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0 \ldots \\
s_8 & = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1 \ldots \\
s_9 & = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0 \ldots \\
s_{10} & = 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1 \ldots \\
s_{11} & = 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0 \ldots \\
\vdots & \quad \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots
\end{array}
$$

$$
s = 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1 \ldots
$$

Figure 2.9: An illustration of Cantor's diagonal argument (in base 2) for the existence of uncountable sets. The sequence at the bottom cannot occur anywhere in the enumeration of sequences above.

chaos theory edge, and self-assembly processes.
The theory of chaos is about deterministic systems which in principle are predictable for behaviour. Chaotic systems are for some time predictable and then appear random. The time required to effectively predict the behavior of a chaotic system depends on three things: how much uncertainty the forecast can accept, how precisely it can measure its present state and the time-scale according to the system's dynamics, called time for Lyapunov. Some examples of the times in Lyapunova are: chaotic electrical circuits, about 1 millisecond; some days of weather (unproven); 4 million to 5 million years of internal Solar System. Uncertainty grows exponentially when time is running out in chaotic systems. There is also the proportionate volatility in the forecast mathematically doubling the forecast time. In fact, this implies that a reliable forecast can not be rendered more than 2-3 periods the lyapunov periods over an period. The system appears random if meaningful forecasts can not be made.
chaotic dynamics "chaos" means "a state of disorientation" in common use. But the concept is described more specifically in chaos theory. While there is no universally accepted definition of chaos, a common definition, originally formulated by Robert L. Devaney, states that it should have these properties in order to classify a dynamic system as chaotic:

1. Sensitive to initial conditions, must be
2. It must be transitive topologically
3. Dense regular orbits are required.

In some cases, the last two properties showed that they were sensitive to initial conditions. In this instance "sensitivity towards initial conditions" does not have to be indicated in the definition, although it is often the most practically significant property.

The second property implies both other properties if the attention is limited to intervals. Only the first two properties in the list above use an alternative and a weaker definition of chaos.

Chaos as a spontaneous breakdown of topological supersymmetry In complex structures in time, instability is an inherent feature of engineering staff in both

stochastic and deterministic (partial) differential equations, as a consequence of a random breake-down of topological supersymmetry. This picture of dynamic chaos not only works for deterministic models, but also for models with external noise, which is a major physical generalization since in fact all dynamic systems influence their stochastic surroundings. This picture refers to Goldstone 's theorem — in application to the spontaneous topological supersymmetry rupture — the long-range dynamical behavior associated to chaotic dynamics (e.g., butterfly action).
Initial sensitivity means that every point in a chaotic system is approximated by other points with significantly different future paths or paths arbitrarily. This may contribute to dramatically changed behaviors, with no improvement in or alteration in existing trajectories.
Sensitivity to the initial condition is commonly called "the butterfly effect," known as the 'Predictability' of the Flap of a Butterfly's Wings in Brazil, which was given by Edward Lorenz in 1972 to the American Association for the Advancement of Science in Washington, D.C. The fluttering wing represents a small change in the system 's initial condition which leads to a chain to prevent large-scale phenomena from becoming predictable. The path of the overall system might have been vastly different if the butterfly didn't float its wings.

As a consequence of the sensitivity to initial conditions, the system would be no longer possible after a certain time if we started by a limited quantity of system information (as it is usually the case). This is particularly common in environment, which is normally only around a week in advance predictive. This doesn't mean that in the future you can't make any claim on events only some system restrictions. We know , for example, from time to time that the natural temperature is not 100 C or 130 C (in the current geological age), but this doesn't mean we can exactly predict the day with the heat of the year.
In maths, in the form of an exponential divergence rate from disrupted initial conditions, the Lyapunov exponent measures the sensitivity to initial conditions. More particularly in the phase space, where the initial separation $\delta \mathbf{Z}$ is infinitesimally near, the two trajectories finally diverge at the rate indicated by

$$|\delta\mathbb{Z}(t)| = exp(\lambda t)|\delta\mathbb{Z}_0|$$

Where t is the time and where Lyapunov is the exponent of λ. The separate intensity of the Lyapunov exponents is based on the direction of the initial separating vector. The number of Lyapunov exponents is equivalent to the number of phase space lengths, but it is sometimes referred to as the highest. The maximum exponent Lyapunov (MLE), as a result of determining the total system predictability, is most often used. For example. A good MLE is generally considered as indication of a chaotic system.

Many characteristics relevant to vulnerability to initial conditions often occur in addition to this property. These include measurement-theoretical mixing and the properties of a K system (as described in the ergodic theory). *Non-periodicity* A chaotic system can have value sequences that precisely repeat themselves for the evolving variable, giving a periodic behavior from anywhere in that sequence. However, these sequences are repellent instead of enticing, which implies that if the changing variable is beyond the sequence but near, it will not join the sequence and therefore differentiates from it. So the variable develops chaotically with non-periodic behavior for nearly all initial conditions.

Topological mixing Topological mixing (or the weaker state of topological transitivity) means that the system is evolving in time so as to overlap any region or open phase with any other region. The concept of "mixing" is the mathematics of normal intuition and the mixture of colored dyes or fluids illustrates a chaotic system.The famous accounts of disorder, which associate disorder with only vulnerability to first conditions, frequently exclude topological involvement. Responsive dependency alone on initial circumstances does not, therefore, trigger confusion. Take the simple dynamical system generated by doubling the initial value repeatedly, for example. This device is prone to initial circumstances all over the world because each pair of points in the region is inevitably widespread. But there is no topological mixing in this example and therefore no chaos. In fact, the behavior is extremely simple: with the exception of 0 all points tend to be either positive or negative.

Topological transitivity A topologically transitive map $f : X \to X$ occurs, if $k > 0$ for some pair of free sets $U, V \in X$ A weaker form of topological mixing is topological transitivity. There is a dot y near x with the orbit passing through V intuitively if a map is topologically transitive then given a dot x and a region V. That means that it can not be dived into two open sets.

The Birkhoff theorem on transitivity is an important related theorem. It is clear to grasp that the presence, in topological transit, of a dense orbit means. In the Birkhoff Transitivity theory, the existence of dense points in X which have dense orbits implies that X is a second countable, complete metric space, but then a topological transitivity. *Strange attractors* Dynamic structures like $x \to 4x(1-x)$ are chaotic everywhere but chaotic actions can also only be seen in a subset of phase space, such as the one-dimensional logistic maps. The most interesting instances arise when a chaotic attractor is used to perform, since a large number of initial circumstances lead to orbits converging into this chaotic region.

An simple way to imagine a chaotic attractor is to begin with a point in the attractor's basin and then clearly draw the next orbit. It is likely to result in a image of the entire final attractor because of the topological transitivity state, and both orbits in the figure on the right provide a glimpse of the general form of the attributor Lorenz. This attractor comes from the Lorenz weather system's basic 3D model. Perhaps the Lorenz Attractor is one of the best popular chaotic structure diagrams as it's actually not just one, it's also one of the most complicated, so as such a very curious pattern is generated that looks like a butterfly's wings with a little creativity.

The attractors that arise from unstable processes called strange attractors, are very informative and dynamic, as opposed to fixed-point attractors and bounding cycles. Continuous fluid processes (for example the Lorenz system) and isolated processes (for example the Hénon map) generate odd attractors. Other discreet dynamic systems have a repulsive structure called a Julia set, forming at the frontier of the fixed-point attraction basins. Sets of Julia may be known as strange repellents. Both international attractors and sets of Julia usually have a fractal form and can be measured for them with a fractal element.

History Henri Poincaré was an early advocate of chaos theory. In the 1880s, when researching the three-body problem, he observed that orbits could not expand or reach a fixed point indefinitely. Jacques Hadamard conducted an groundbreaking thesis in 1898 on a constantly negatively curved board, which became known as "Hadamard the billiard," on the disorderly motion of a free glyding atom. Hadamard was able to prove that every trajectory is unstable, because all trajectories of particles diverge exponentially, with a positive exponent of Lyapunov.

In the area of ergodic theory, chaos theory started. Subsequent studies were conducted by George David Birkhoff, Andrey Nikolaevich Kolmogorov and Mary Lucy Cartwright, and John Edensor Littlewood and Stephen Smale, also on the topic of nonlinear differential equations. Except Smale, the three-body problems for Birkhoff, the turbulence, astronomical problems for Kolmogorov and radio engineering for Cartwright and Littlewood, all of these studies have been inspired by physics directly. Although there was no observation about chaotic planetary motion, experimentalists encountered fluid motion turbulences and the non-periodic radio circuit oscillation with no theory to explain what they saw.

Despite initial observations from the first half of the 20th century, chaos theory only became formalized as such after the middle of the centuries when certain scientists could not understand the observable behaviour of some experiments including logistic charts, for instance because it was apparent that linear theory, the dominant systems theory at the period. Chaos theorists regarded that the measurement of inaccuracy and simple "noise" as a full component of the systems studied.

The electronic computer was the main catalyst for developing chaos theory. Part of the mathematics of the theory of disorder includes the regular application of basic and hand-made mathematical formulas. These repeated equations were made realistic by electronic machines, and such structures could be visualized by figures and pictures. Yoshisuke Ueda worked with analog computers when, on 27th November 1961, he was a student in the Laboratory of Chihiro Hayashi at Kyoto University and discovered what he termed 'extreme transitional phénomènes.' However, his advisor was at that point not in agreement with his findings and allowed him not to report his findings before 1970.

The early pioneer of theory was Edward Lorenz. He accidentally became interested in chaos through his work in 1961 on weather prediction[12]. Lorenz used his weather simulation with a simple digital computer, Royal McBee LGP-30. He needed to see another series of data once more, so in the center of his course he began simulating. He accomplished so by entering in the center of the initial simulation a printout of the data that fit state. The weather the machine started to predict was, to his surprise, totally different. Lorenz followed it up to the printout in the computer. The machine was 6-digit, but the printer rounded the variables to a 3-digit number and printed a value of 0.506127 to 0.506. This gap is minimal and it was the opinion that it was not realistic at the point. Nevertheless,

Lorenz notes that minor adjustments in initial conditions have resulted in major improvements in the long-term result[68] The observation of Lorenz, which gives Lorenz 's name, demonstrates that even comprehensive air simulation can not, in reality, necessarily exclude accurate weather forecasts in the long run.
In 1963, at every scale of data on cotton prices Benoit Mandelbrot found repeating patterns. In the past, it had examined the theory of knowledge with the result that noise was represented as a series of Cantors: on any scale the share of noise-containing times for cycles free from errors was constant. Mandelbrot described the "Noah effect" (which can lead to sudden discontinuous changes) and the "Joseph effect" (which can produce a value for some time but then change suddenly). The belief that market increases are usually spread was disputed. In 1967, he published "The Coast of England? Statistical self-similarity and fractional dimension" showing that the length of a coastline varies with the measuring device's scale, resembles itself on all scales and is an endless measuring device for an infinitely small measurement device. Arguing that a twin ball appears to be viewed from distant (0-dimensional), a close-by ball (3-dimensional), or a flattened (1-dimensional) line, he suggests that an object's scale is proportional to the viewer and can be fractional.A fractal object with a constant irregularity on a variety of scales ("self-similarity") (examples are the Menger sponge, Sierpiński gauge and the cooking curve or snowflake, infinitely long but containing a finite area with a fractal dimension of around 1,2619). The Fractal Geometry of Nature which was converted into a classical theory for chaos was published by the artist in 1982. A fractal model has proved appropriate for biological systems like the ramifying of circulatory and bronchial systems.
The first Symposium on Chaos, which included David Ruelle, Robert May, James A. Yorke (coiner of the word chaos as used in mathematics), Robert Shaw and the meteorologist Edward Lorenz, was organized by the New York Academy of Sciences in December 1977. After a three-year referee rejection, Pierre Coullet and Charles Tresser published "Iterations d'endomorphismes et groupe de renormalisation" in the following year, and Mitchell Feigenbaum published an article called "Quantitative universality for a nonlinear transformation class." Thus Feigenbaum (1975) and Coullet Tresser (1978) discovered a universality in chaos that allowed many different phenomena to apply the theory of chaos.
Albert J. Libchaber presented his experimental observation of the fork cascade,

which leads to chaos and turbulence in the convectional systems from Rayleigh to Benard in 1979 during a symposium hosted by Pierre Hohenberg in Aspen. In 1986, together with Mitchell J. Feigenbaum, he and his inspirational work received the Wolf Prize in Physics.

The New York Academy of Sciences co-organized the first significant conference on disorder in biology and medicine in 1986, along with the National Center of Mental Health and the Office of Naval Studies. Bernardo Huberman presented a statistical model of a schizophrenic eye disease there. This led to a physiology renewal in the 1980s, for instance in the study of pathological heart cycles, by application of the chaos theory.

In 1987, in Physical Review Letters, Per Bak, Chao Tang and Kurt Wiesenfeld published the first paper which described SOC as one of the means under which complexity occurs in the nature. The paper was first described.

Almost all other investigations have focused on large-scale natural or social system which are (or are) known to show scale-invarying behavior, alongside approaches based largely on laboratory work, such as the Bak – Tange – Wiesfeld sandpile. While specialists for the areas examined have not (at least initially) always welcomed these approaches, the SOC has nevertheless emerged as a strong candidate for explaining a number of natural phenomena, including earthquakes (whose scales invariant behavior, like the Richter – Gutenberg law, describing a statistical difference long before the discovery of the SOC were found) and biological evolution (Where, for instance, SOC has been invoked as the dynamical mechanism behind Niles Eldredge and Stephen Jay Gould's theory of 'pointing equilibrium'). Given the implications of a non-scale distribution of event sizes, some researchers suggested that the occurrence of wars is another phenomenon that must be seen as an example of SOC. These SOC investigations included both attempts at modeling and comprehensive analyzes of the existence or characteristics of natural scaling laws (either developing new models or adapting existing ones to the specifics of a given natural system).

In the same year, Gleick published Chaos: Making a New Science which, although its story underlined significant contributions from the Soviet Union, was a best-seller and introduced to the public the general principles and the history of chaos theory.Chaos theory started as a transdisciplinary field and as an academic one, mainly under the name of nonlinear structure analysis, and was established in the

sphere of certain single individuals. In reference to Thomas Kuhn 's concept of a paradigm shift in Scientific Revolutions Structure (1962), many "chaologists," as certain described themselves, claimed to be a model of such a shift in this new theory and Gleick 's thesis.
Cheaper , more efficient machines are accessible to increase the applicability of the principle of chaos. The chaos theory continues to be a field of active research in many diverse disciplines, including mathematics, topology, physics, social systems, demographic modelling, biology, meteorology, astrophysics, information theory, computer neuroscience and management of the pandemic crisis.
Applications Although the principle of disorder was born from weather prediction, it has been generalized to a number of other contexts. Geology, mathematics, microbiology, genetics, IT, business, electronics, finance, software, meteorology, ecology, physics, politics , economics, the society, psychology and robotics are fields that support today 's theory of chaos. Some categories with examples are listed below, but not a full list as new applications appear.

Cryptography The theory of chaos has been used in cryptography for many years. In the past few Decades hundreds of cryptographic primitives were generated with instability and nonlinear dynamics. These algorithms include algorithms for image encryption, hash functions, secure generators of pseude random numbers, stream cyphers, watermarking, style. Most of these algorithms are based on uni-modal chaotic maps and a great number of these algorithms use their control and initial condition.The main motif for the creation of chaotic cryptographic algorithms from a broader perspective, without loss of generality, is similarity between chaotic maps and cryptographic systems. Another form of encryption is based on the diffusion and ambiguity that is well influenced by chaos theory, as is the hidden key or symmetric key. DNA computing provides a way of encrypting images and other information when coupled to chaos theory. Many cryptographic DNA-Chaos algorithms are either not secure, or it is suggested that the technique used is not effective.
Robotics Robotics is another sector in which chaos theory has recently benefited. Installation of a predictive model was used by chaos theory, instead of robots acting in a refined trial / error type for interacting with their environment. The passive walking biped robots have shown chaotic dynamics.

Biology Biologists have been tracking colonies of multiple animals through community simulation for over a hundred years. Most models are constant, but scientists have recently been able in certain populations to implement chaotic templates. For examples, a analysis of Canadian lynx models found that population growth was chaotic. In ecological systems, like hydrology, chaos can also be seen. While there is much to learn from the analysis of data via the lens of the theory of chaos in a chaotic pattern in hydrology. Cardiotocography contains another biological application. Fetal monitoring is a delicate balance for accurate and noninvasive information. Using disorderly modeling, improved fetal hypoxia alarm structures may be obtained.

Other areas In chemistry, gas solubility is essential to the production of polymers, but PSO models tend to converge on incorrect points. An upgraded version of PSO was generated with chaos that prevents simulations from stuck. The application of chaos theory leads to better predictions when those objects will approach the earth and other planet in celestial mechanics especially when asteroids are being observed.Pluto rotates chaotically four of the five moons. The analysis of Josephson 's wide sets of junctions significantly gained from chaos theory in quantum mechanics and electrical engineering. Coal mines were also unsafe in the area where repeated spills of natural gas caused many deaths. Until recently, when they would occur was not reliable. However, these gas leaks have disorderly characteristics that can be anticipated relatively accurately when properly designed.

While disorder may be utilized beyond the sciences, in historical terms nearly all these studies suffer from lack of reproductiveness; weak external validity; and/or inattention to cross validation, contributing to low predictive accuracy. Glass and Mandell and Selz showed that no EEG studies have shown that weird attractors or other signs of chaotic behaviour.

Researchers continued to apply psychology to chaos theory. For instance , researchers have shown that group dynamics result in the individual dynamics of the members in modeling group action in which heterogeneous members may behave, as if competing in different grades was a basic assumption in Wilfred Bion 's theory: each individual reproduces the group dynamics in a specific scale, and in each group's chaotic actions.

Redington and Reidbord (1992) tried to prove the chaotic characteristics of the human heart. During a counseling session, they tracked the shifts in inter-heartbeat cycles within a particular individual as it went through various phases in emotional stress. Admittedly, the results were unfinished. In the various plots produced by authors, which allegedly showed evidence for caotic dynamics (spectral analysis, phase trajectory and autocorrelation plots), the writers found that it could not be done reliably if they tried to calculate the Lyapunov exponent as more definite confirmation of chaotic behavior.

Metcalf and Allen claimed in their paper of 1995 that they revealed a pattern of time that doubles into chaos in their animal behaviour. The authors investigated the well-known reaction called expected polydipsis, which causes an animal without food to consume excessive amounts of water for some period before it is eventually introduced. The control parameter (r) here was once resumed the interval length between the feeds. The authors carefully tested and included a large number of replications and designed their experiments to rule out the likelihood that changes in response patterns are caused by various locations in r.

Time series and first delayed plots provide the optimum support for the claims, which show that the feeding times have increased quite clearly from periodicity to irregularity. On the other hand, the various plots and spectral analyses in different phases do not correspond sufficiently well with the other charts or with the theory as a whole to produce a chaotic diagnosis inexorable. For starters, there is no gradual advancement towards increased complexity (and aside from periodicity) in the step trajectories; the mechanism seems very puzzled. There is also room for alternate explanations where Metcalf and Allen saw cycles of two and six in their spectral plots. All this uncertainty needs any post-hoc serpentine interpretation whose findings match into a chaotic paradigm.

Aniundson and Light also shown that stronger recommendations may be created for individuals who have issues with work choices by modifying a style of employment therapy to provide a complex view of workplace and job market relationships. Modern institutions, with critical non-linear social mechanisms and subject to internal and external factors that lead to disorder, are increasingly seen as transparent, dynamic adaptive processes. For example , team building and group development are increasingly being researched as an absolutely impredictable system, because the uncertainty of the meetings of various individuals makes the team's trajectory unknown for the first time.

Some claim the disorder concept — used in verbals — focused on statistical and psychological theories of human activity gives valuable insight into the dynamics of limited units at staff, which go beyond the statistical concept itself.

It may also be possible to improve economic models by applying the theory of chaos, but the health prediction and the factors that influence the system most are extremely difficult tasks.Because of the interactions of the human being, the economic and financial systems are fundamentally different from those in classical natural sciences, and so pure deterministic models are unlikely for accurate representations of the data. Economic and financial chaos testing empirical literature showed very mixed results, partly due to the confusion of specific chaos testing and more general non-linear relationship testing.

Applications in chaos theory may benefit from traffic forecasts. Better forecasts of the occurrence of traffic could allow for action to be taken before it occurred. Combining the principles of chaos theory with some other methods led to a better short-term model of prediction.

Data from environmental water cycles (such as hydrological data) such as rainfall and streamflow have been used for chaos. These studies have produced controversial results because of the relativly subjective methods for the detection of chaotic signatures. Early studies tended "to succeed" in chaos, while subsequent studies and meta-analyzes called these studies in question and explained why these datasets are unlikely to have chaotic dynamics of a low dimension.

3. Main Results with subshifts of finite type

Main question: For a given interval $H = (a, b)$, which intervals maps with hole H induce the survivor set Ω_H with positive fractal dimension?

We state our result in symbolic space and leave it to the interested reader to rephrase the theorem in terms of interval k-transformation using the above mentioned commuting diagram.

Theorem 3.1. *Let $k \geq 2$ be an integer, A a $k \times k$ transition matrix, and (Σ_A, σ_k) is the induced subshift of finite type. Assume that there exist two distinct symbols $i, j \in \Lambda_k$ such that $A_{ii} = A_{jj} = A_{ij} = A_{ji} = 1$. For any $\ell \geq 0$ we define the subset of Σ_A*

$$S_\ell(i, j) := \{a_1 a_2 \cdots \in \{i, j\}^{\mathbb{N}} : a_m = i, \implies a_{m+n} = j, n = 1, 2, \dots, \ell\}. \quad (3.1)$$

If the hole H in Σ_A is disjoint from S_ℓ for some ℓ, then the survivor set satisfies

- *$\Omega_H(\sigma_k)$ is uncountable,*
- *$\Omega_H(\sigma_k)$ has topological entropy at least $\frac{\log 2}{\ell+1}$,*
- *$\Omega_H(\sigma_k)$ has Hausdorff dimension at least $\frac{\log 2}{\ell \log k}$.*

3.1 Proof of main results

To prove 3.1, we show that the sets $S_\ell(i, j)$ have the desired properties. We recall that an infinite work is in $S_\ell(i, j)$ if between two symbols 'i', there should be symbols 'j' repeated at least ℓ times, see Figure 3.1.

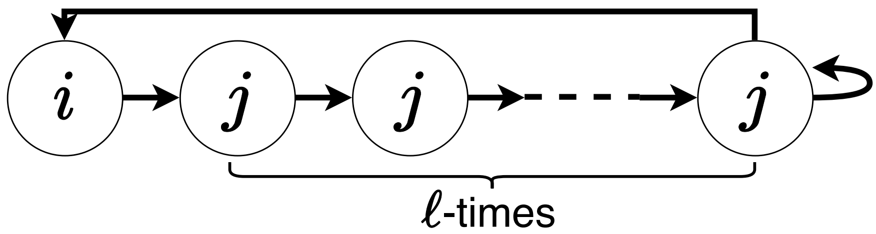

Figure 3.1: Letter transitions in $S_\ell(i,j)$ illustrated.

Proof of Theorem 3.1. From the assumption on transition matrix we see that the subshift of finite type $\{i,j\}^{\mathbb{N}}$ is embedded in Σ_A. Let ℓ be a positive integer such that $S_\ell(i,j)$ is disjoint from H, as $S_\ell(i,j)$ is σ_k-invariant and it is contained in $\{i,j\}^{\mathbb{N}}$ we conclude that $S_\ell(i,j) \subset \Sigma_H(\sigma_k)$. Thus, it suffices to show the desired properties for the dynamical system $(S_\ell(i,j), \sigma_k)$. The next three lemmas finishes the proof. □

Lemma 3.2. *Let i,j be two distinct non-negative integers and ℓ a positive integer. Then, the set $S_\ell(a,b)$ is uncountable.*

Proof. The proof of the lemma is analogous to Cantor's diagonal argument from set theory. For the sake of completeness we will reproduce the proof here.

Assume by contradiction that for some i,j,ℓ, we have $S_\ell(i,j)$ (at most) countable. Let x and y be two finite words of length $\ell+1$ and ℓ, respectively, given by

$$y = \underbrace{jjj\dots j}_{\ell-\text{times}} \text{ and } x = yi. \tag{3.2}$$

Let us take a set $U(x,y) \subset S_\ell(i,j)$ given by $U(x,y) = \{x,y\}^{\mathbb{N}}$. Then, $U(x,y)$ must be (at most) countable too, say $U(x,y) = \{\omega_1, \omega_2, \dots\}$. We now regard, x and y as letters. For any i,j we let ω_i^j denote the jth letter of ω_i and define a new word $\omega \in U(x,y)$ as follows, for any $j = 1,2,3\dots$, the jth letter is given by

$$\omega^j = \begin{cases} x & \text{if } w_j^j = y, \\ y & \text{if } w_j^j = x. \end{cases}$$

Since $\omega_j^j \neq w^j$ for any j we conclude that $\omega \neq \omega_j$, that is, $\omega \notin U(x,y)$ a contradiction. Thus, $U(x,y)$ is uncountable and so is $S_\ell(i,j)$. This concludes the lemma. □

As noted before the sets $S_\ell(a,b)$ are shift-invariant, that is, $\omega \in S_\ell(a,b)$ implies $\sigma_k(\omega) \in S_\ell(a,b)$. Hence, it makes sense to study the topological entropy of the set, which is another way to measure the complexity of a set. A set of positive entropy is necessarily uncountable, but the converse is false. Namely, the uncountability of sets $S_\ell(a,b)$ is not sufficient to deduce positive topological entropy. The next lemma exactly does this.

Lemma 3.3. *Let i, j be two distinct non-negative integers and ℓ a positive integer. The set $S_\ell(i,j)$ has positive topological entropy. In fact*

$$h(S_\ell(i,j), \sigma_k) \geq \frac{\log 2}{\ell + 1}$$

Proof. We note that for the topological entropy can be computed using the number of periodic orbits. More specifically, let $N(n)$ denote the number of periodic orbits in $S_\ell(a,b)$ with period n. Then, the topological entropy $h(\sigma_k, S_\ell(a,b))$ satisfies

$$h(\sigma_k, S_\ell(a,b)) = \lim_{n\to\infty} \frac{\log N(n)}{n}. \tag{3.3}$$

To finish the proof, we estimate $N(n)$. As in (3.2) with slight modification we let x and y be two finite words of lengths $\ell + 1$, given by

$$y = \underbrace{jjj\ldots j}_{\ell+1-\text{times}} \text{ and } x = \underbrace{jjj\ldots j}_{\ell-\text{times}} i.$$

Analogously, set $U(x,y) = \{x,y\}^{\mathbb{N}}$ which is a subset of $S_\ell(x,y)$. We will use the periodic orbits of $U(x,y)$ to estimate $N(n)$ from below. To this end, we note that there are at least 2 periodic orbits of length $\ell + 1$, namely x^∞ and y^∞, so $N(\ell + 1) \geq 2$. inductively, we see that $N((\ell + 1)n) \geq 2^n$. Let m be any large integer, we may find a positive integer n such that $(\ell+1)n \leq m < (\ell+1)(n+1)$. For any periodic orbit z^∞ in $U(x,y)$ of length $(\ell+1)n$ we may associate a periodic orbit ω of $U(x,y) \subset S_\ell(x,y)$ of length m we letting

$$\omega = (z \quad \underbrace{jjj\ldots j}_{(m-(\ell+1)n)-\text{times}} \quad)^\infty.$$

Thus, $N(m) \geq 2^n$ for $(\ell + 1)n \leq m < (\ell + 1)(n + 1)$. Using the formula (3.3) we arrive at

$$h(\sigma_k, S_\ell(a,b)) = \lim_{m\to\infty} \frac{\log N(m)}{m} \geq \lim_{n\to\infty} \frac{\log 2^n}{(\ell+1)(n+1)} = \frac{\log 2}{\ell+1}.$$

This finishes the proof. □

Lemma 3.4. *Let i, j be two distinct non-negative integers and ℓ a positive integer. The set $S_\ell(i,j)$ has positive Hasudroff dimension. In fact*

$$\dim_H(S_\ell(i,j)) \geq \frac{\log 2}{\ell \log k}$$

Proof. To prove the lemma, we make use of Mass Distribution Principle, see e.g. [**Fal04**]. As in (3.2) with slight modification we let x and y be two finite words of lengths $\ell+1$, given by

$$y = \underbrace{jjj\dots j}_{\ell+1-\text{times}} \text{ and } x = \underbrace{jjj\dots j}_{\ell-\text{times}} i.$$

Analogously, set $U(x,y) = \{x,y\}^{\mathbb{N}}$ which is a subset of $S_\ell(x,y)$. Thus, it suffices to estimate the Hausdorff dimension of $U(x,y)$. To this end, we inductively define a probability measure μ on Σ_A supported on $U(x,y)$ as follows:

For any finite word z of length ℓ we call $C(z) = \{a_0a_1\dots \in \Sigma_k \mid a_0a_1\dots a_{\ell-1} = z\}$ *a cylinder set.*

We let $\mu(\Sigma_k) = 1$ and set $E_0\{\Sigma_k\}$. There are $k^{\ell+1}$ finite words of length $\ell+1$.

In the first step, we may split Σ_k into $k^{\ell+1}$ cylinder sets of equal length only pick two of them, namely, $E_1 = \{C(x), C(y)\}$ and let $\mu(C(x)) = \mu(C(y)) = 1/2$.

Next, in step 2, we split each $C(x)$ and $C(y)$ into $k^{\ell+1}$ cylinder sets of equal length and pick the four $E_2 = \{C(xx), C(xy), C(yx), C(yy)\}$ and set $\mu(C(xx)) = \mu(C(xy)) = \mu(C(yx)) = \mu(C(yy)) = 1/4$.

Inductively, in step n we further split each previously obtained 2^{n-1} cylinder sets into $k^{\ell+1}$ smaller cylinder sets. We note that exactly 2^n of these cylinder sets are defined using x, y and we place them into E_n and assign measure 2^{-n} to each.

This inductively defines a probability measure supported in $U(x,y)$.

We notice that $d(x,y) = k^{-\ell}$ and inductively one can show that for any two distinct cylinder sets $C, C' \in E_n$ one has $d(C,C') \geq k^{-\ell n}$. Now, let U be any subset of Σ_A with diameter $\delta > 0$. We may find non-negative integer n such that

$k^{-(n+1)\ell} \leq \delta < k^{-n\ell}$, that is, Clearly, on E_n there exists at most one cylinder set C that intersects with U. Hence, the measure of U satisfies $\mu(U) \leq \mu(C) = 2^{-n}$. That is,

$$\mu(U) \leq 2^{-n} = (k^{-n\ell})^{\log 2/\ell \log k} \leq (k^{\ell}\delta)^{\log 2/\ell \log k} \ll \delta^{\log 2/\ell \log k}.$$

Hence, it follows from the Mass Distribution Principle that the Hausdorff dimension of the support $U(x, y)$ satisfies

$$\dim_H(U(x, y)) \geq \frac{\log 2}{\ell \log k}.$$

Hence, $\dim_H(S_\ell(x, y)) \geq \dim_H(U(x, y)) \geq \log 2/\ell \log k$ which finishes the proof.

□

3.2 Cardinality of survivor set for chaotic tent map with holes

Lemma 3.5. *Let function $S : \Sigma_2 \to [0, 1]$ is homeomorphism defined as*

$$S(a_1, a_2, a_3, ...) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

Then there is conjugacy between T_3 and σ such that $S \circ \sigma = T_3 \circ S$

Theorem 3.6. $\Omega(I_a) = \infty$ *if and only if* $\Omega(f(I_a)) = \infty$

3.3 Proof of main results

Proof of Lemma 3.5 We need to prove that S is bijection. We identify Σ_2 with the interval [0,1] defined as $S(a_1, a_2, a_3, ...) = \sum_{n=0}^{\infty} a_n/3^n$, for $(a_1, a_2, a_3, ...) \in \Sigma_2$. The map S is a bijection, and the inverse image of any element of B_2 is a sequence in Σ_2 ending with 0^∞. There $\pi_k^{-1}x \in \Sigma_k$ gives the k-expansion of $x \in [0, 1)$. Note that the points (in B_2) have two k-expansions, one ending with 0^∞, and other ending with 1^∞. It is well-known that is, $S \circ \sigma = T_3 \circ S$.

Proof of Theorem 3.6. To prove this theorem we need to show that $f : \Omega(I_a) \to \Omega(f(I_a))$ is a bijection. If $f : A \to B$ is bijection then $|A| = |B|$. $f : \Omega(I_a) \to \Omega(f(I_a))$. $x \in \Omega(I_a)$ and $f(x) \in \Omega(f(I_a))$ it follows $x, E_2(x), \ldots, E_2^n(x) \notin I_a \Leftrightarrow x \in \bigcap_{n=0}^{\infty} E_2^{-n}(I_a)^c$. $f(x) \in \bigcap_{n=0}^{\infty} f \circ E_2^n(I_a)^c = \bigcap_{n=0}^{\infty} f \circ f^{-1} \circ T_3^{-1} \circ f(I_a)^c = \bigcap_{n=0}^{\infty} T_3^{-n} \circ f(I_a)^c \Leftrightarrow f(x) \in \Omega(f(I_a))$.

4. Conclusion

We studied open dynamical systems in symbolic dynamics. It was very interesting branch of dynamical systems for us. Our results motivate us to continue and get a better results. In our study of open dynamical systems in subshifts of finite type we obtained sufficient condition when the survivor set $\Omega_H(\sigma_k)$ is uncountable and estimated from below the Hausdorff dimension and topological entropy. We make no claim on the sharpness of our estimates. Indeed, it is an interesting question to find exact Hausdorff dimension and topological entropy of survivor sets. Another interesting question is to investigate the necessary condition for the survivor sets in the subshifts of finite type to be uncountable. And we studied open dynamical systems in tent maps and we obtained that conjugacy between Doubling map and T_3 tent map. And we proved that if Survival set infinite in Doubling map then it is also infinite in T_3 tent map. Indeed, there is real interesting research question that what about real cardinality of survivor set. Is it depends on interval location and size? Another interesting question is to investigate the necessary condition for the survivor sets in T_k Tent map to be uncountable.

To continue our thesis work we can answer to questions listed below and it will be good result.

1. What is the Hausdorff dimension of Survivor set?;
2. What about the speed of orbits escape to the hole?

References

[1] Roy L Adler, Alan G Konheim, and M Harry McAndrew. "Topological entropy". In: *Transactions of the American Mathematical Society* 114.2 (1965), pp. 309–319.

[2] Nikita Agarwal. "The k-Transformation on an Interval with a Hole". In: *Qualitative Theory of Dynamical Systems* 19.1 (2020), p. 30.

[3] Wael Bahsoun, Christopher Bose, and Gary Froyland. *Ergodic Theory, Open Dynamics, and Coherent Structures.* Springer, 2014.

[4] Henk Bruin, Mark Demers, and Ian Melbourne. "Existence and convergence properties of physical measures for certain dynamical systems with holes". In: *Ergodic Theory and Dynamical Systems* 30.3 (2010), pp. 687–728.

[5] Stefan Bundfuss, Tyll Krüger, and Serge Troubetzkoy. "Topological and symbolic dynamics for hyperbolic systems with holes". In: *Ergodic Theory and Dynamical Systems* 31.5 (2011), pp. 1305–1323.

[6] Leonid A Bunimovich and Alex Yurchenko. "Where to place a hole to achieve a maximal escape rate". In: *Israel Journal of Mathematics* 182.1 (2011), pp. 229–252.

[7] N Chernov and R Markarian. "Anosov maps with rectangular holes. Nonergodic cases". In: *Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilia Mathematical Society* 28.2 (1997), pp. 315–342.

[8] N Chernov, R Markarian, and S Troubetzkoy. "Invariant measures for Anosov maps with small holes". In: *Ergodic Theory and Dynamical Systems* 20.4 (2000), pp. 1007–1044.

[9] Nikolai Chernov and Roberto Markarian. *Chaotic billiards.* 127. American Mathematical Soc., 2006.

[10] Mark F Demers, Paul Wright, and Lai-Sang Young. "Entropy, Lyapunov exponents and escape rates in open systems". In: *Ergodic Theory and Dynamical Systems* 32.4 (2012), pp. 1270–1301.

[11] Mark Demers, Paul Wright, and Lai-Sang Young. "Escape rates and physically relevant measures for billiards with small holes". In: *Communications in Mathematical Physics* 294.2 (2010), pp. 353–388.

[12] Paul Glendinning and Nikita Sidorov. "The doubling map with asymmetrical holes". In: *Ergodic Theory and Dynamical Systems* 35.4 (2015), pp. 1208–1228.

[13] Artur Lopes and Roberto Markarian. "Open Billiards: Invariant and Conditionally Iinvariant Probabilities on Cantor Sets". In: *SIAM Journal on Applied Mathematics* 56.2 (1996), pp. 651–680.

[14] Giulio Pianigiani and James A Yorke. "Expanding maps on sets which are almost invariant. Decay and chaos". In: *Transactions of the American Mathematical Society* 252 (1979), pp. 351–366.

[15] Nikita Sidorov. "Arithmetic dynamics". In: *Topics in dynamics and ergodic theory* 310 (2003), pp. 145–189.

[16] H Van den Bedem and N Chernov. "Expanding maps of an interval with holes". In: *Ergodic Theory and Dynamical Systems* 22.3 (2002), pp. 637–654.

[17] T Yoshida, H Mori, and H Shigematsu. "Analytic study of chaos of the tent map: band structures, power spectra, and critical behaviors". In: *Journal of statistical physics* 31.2 (1983), pp. 279–308.