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λ - CANTOR SETS AND TOTAL SELF-SIMILARITY

Abstract: *We study overlapping Cantor sets with parameter λ and classify the situations when these fractal sets are totally self-similar. More precisely, we consider iterated function system consisting of three functions $f_1(x) = x/3$, $f_\lambda(x) = (x + \lambda)/3$ and $f_2(x) = (x + 2)/3$ and the fractal set E_λ it generates in the real line. We define what totally self-similar means and show for $1 < \lambda < 2$ that the fractal set E_λ is totally self-similar if and only if it is in the form $\lambda = 1 + 3^{-n}$ for some positive integer n. We mainly rely on the recent work of Dajani, Kong, and Yao where they consider the analogous problem for $0 < \lambda < 1$.*

Keywords*: total self-similarity, Cantor sets, fractals, conjugacy, iterated function systems.*

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λ - МНОЖЕСТВА КАНТОРА И ПОЛНОЕ САМОПОДОБИЕ

Аннотация*: Мы изучаем перекрывающиеся множества Кантора с параметром λ и классифицируем ситуации, когда эти фрактальные множества полностью автомодельны. Точнее, мы рассматриваем итерационную функциональную систему, состоящую из трех функций $f_1(x) = x/3$, $f_\lambda(x) = (x + \lambda)/3$ и $f_2(x) = (x + 2)/3$ и фрактального множества E_λ, которое она генерирует в реальной линии. Мы определяем, что означает полностью автомодельное и показываем для $1 < \lambda < 2$, что фрактальное множество E_λ полностью автомодельно тогда и только тогда, когда оно находится в форме $\lambda = 1 + 3^{-n}$ для некоторого положительного целого числа n. Мы в основном полагаемся на недавнюю работу Дайани, Конга и Яо, где они рассматривают аналогичную задачу для $0 < \lambda < 1$. Ключевые слова. полное самоподобие, Канторово множество, фрактал, сопряжённость, итерационная функциональная система.*

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1 Introduction

A fractal is a set in Euclidean space that has self-similar structure, that is, it repeats itself as you zoom in its image. The famous example of a fractal is the well-known middle third Cantor set which is obtained by removing the middle third of the preceding interval starting with [0,1]. Some of the characteristics of the middle third Cantor set are that it is totally disconnected, uncountable, compact, and has zero Lebesgue measure, fractal dimension $\log 2/\log 3$. We recall that the fractal dimension, e.g. Box dimension, Hausdorff dimension, Capacity, is a way to measure the complexity of the set. In general, it is difficult to characterize fractal sets.

One interesting approach to study and construct fractals is by means of Iterated Function Systems (IFS). Consider two functions $f_0, f_2\colon [0,1] \to [0,1]$ given by

$$f_0(x) = \frac{x}{3} \text{and} f_2(x) = \frac{x+2}{3}.$$

These are contractions and the well-known Contraction Mapping Theorem, see e.g. cite [1], the contraction mappings have a unique fixed point in complete metric spaces. In particular, the IFS $\{f_0, f_2\}$ gives rise to a unique fractal and by iteratively applying these functions to the unit interval it is easy to see that in the end we obtain the middle third Cantor set.

If so called Open Set Condition (OSC) holds for an IFS with contraction rates $r_1, r_2, \ldots, r_n$ then [1] cite Theorem implies that the fractal dimension d satisfies $r_1^d + r_2^d + \cdots + r_n^d = 1$. In our example of $\{f_0, f_2\}$ we have $r_1 = r_2 = 1/3$, thus $1/3^d + 1/3^d = 1$ yields $d = \log 2/\log 3$, the dimension of the middle third Cantor set.

In this article we consider a family of IFSs $\{f_0, f_\lambda, f_2\}$ parametrized by $\lambda \in [0,2]$ defined as

$$f_0(x) = \tfrac{x}{3}, f_\lambda(x) = \tfrac{x+\lambda}{3}, \text{and} f_2(x) = \tfrac{x+2}{3}. \quad (1)$$

Unless $\lambda = 1$ these IFSs produce overlapping images of the unit interval and as such it is difficult to see for which λ it satisfies OSC, hence it is difficult to say when one can use the above equation to compute the fractal dimension, or more specifically Hausdorff dimension. In this direction B. Solomyak and P. Shmerkin proved that the Hausdorff dimension of the fractal for λ irrational is one, for the proof see [4]. This established a conjecture of Furstenberg from 1970s, see also [5],[6] for partial results in this direction.

We now introduce a finer notion of self-similarity due to Broomhead, Montaldi and Sidorov [2], called totally self-similarity. To this end, we follow the notation from [3]. For any $\lambda \in [0,2]$ we let E_λ denote the compact set (fractal) in [0,1] generated by the IFS $\{f_0, f_\lambda, f_2\}$ and $\Omega_\lambda = \{0, \lambda, 2\}$ is an alphabet consisting of 3 letters, $\Omega_\lambda^{\mathbb{N}}$ is the space of infinite words from the alphabet Ω_λ, for any natural number n, Ω_λ^n is the set of all finite words from alphabet Ω_λ of length n, and $\Omega_\lambda^* = \cup_{n=1}^{\infty} \Omega_\lambda^n$ is the space of all finite words. For any finite word $d_1 d_2 \ldots d_n \in \Omega_\lambda^n$ we denote $f_{d_1 d_2 \ldots d_n} = f_{d_1} \circ \cdots \circ f_{f_n}$. The space of infinite words $\Omega_\lambda^{\mathbb{N}}$ gives us a symbolic representation of the fractal E_λ: For any $x \in E_\lambda$ there exists an infinite word (sequence) $(d_n) \in \Omega_\lambda^{\mathbb{N}}$ such that

$$x = \lim_{n\to\infty} f_{d_1 d_2 \ldots d_n}(0) = \sum_{i=1}^{\infty} \frac{d_i}{3^i}.$$

Here one can take any other number instead of zero.

Let $I := [0,1]$. We say that E_λ is *totally self-similar* if

$$f_d(E_\lambda) = f_d(I) \cap E_\lambda$$

for any finite word $d \in \Omega_\lambda^*$.

One of the main results of [3] is the following.

Theorem. *For $\lambda \in (0,1)$, the set E_λ is totally self similar if and only if $\lambda = 1 - 3^{-n}$ for some natural number n.*

Our goal is to obtain the similar result for other values of λ. Our main result is the following.

Theorem 1. *For $\lambda \in (1,2)$, the set E_λ is totally self similar if and only if $\lambda = 1 + 3^{-n}$ for some natural number n.*

The next section is devoted to proving the main result.

2 Proof of Theorem 1

In this section we prove Theorem 1. For any $\lambda \in (1,2)$ we set $\bar{\lambda} = 2 - \lambda \in (0,1)$. We prove

Proposition 1. *For any $\lambda \in (1,2)$, the set E_λ is totally self-similar if and only if the corresponding set $E_{\bar{\lambda}}$ is totally self-similar.*

The proof of Theorem 1 now easily follows from Proposition 1.

Proof of Theorem 1. It follows from result of [3], the theorem stated above, that $E_{\bar{\lambda}}$ is totally self-similar if and only if $\bar{\lambda} = 1 - 3^{-n}$ for some natural number n. Hence, for $\lambda \in (1,2)$ using Proposition 1 we get that E_λ is totally self-similar if and only if $2 - \lambda = \bar{\lambda} = 1 - 3^{-n}$ for some natural number n. Therefore, we see that $\lambda = 1 + 3^{-n}$ for some natural number n which finishes the proof.

So, it remains to prove Proposition 1. The idea of the proof is to realize that the two IFSs are topologically conjugate. To this end, we let $\phi: [0,1] \to [0,1]$ via

$$\phi(x) = 1 - x.$$

Lemma 1. *For any $\lambda \in [1,2]$ we have $f_{\bar{\lambda}} = \phi^{-1} \circ f_\lambda \circ \phi = \phi \circ f_\lambda \circ \phi$.*

Proof. Recall that $f_\lambda(x) = \frac{x+\lambda}{3}$ which is also true for $\lambda = 0{,}2$. A simple calculation yields

$$f_\lambda \circ \phi(x) = \frac{\phi(x) + \lambda}{3} = \frac{1 - x + \lambda}{3} = 1 - \frac{x + (2 - \lambda)}{3} = \phi \circ f_{\bar{\lambda}}(x),$$

so that $f_{\bar{\lambda}} = \phi^{-1} \circ f_\lambda \circ \phi$. The last assertion follows since $\phi^{-1} = \phi$, that is, $\phi^2(x) = x$.

Lemma 2. *For any $\lambda \in (1,2)$ we have $\phi(E_\lambda) = E_{\bar{\lambda}}$*

Proof. Recall that $x \in E_\lambda$ if and only if there exists $(d_i) \in \Omega_\lambda^{\mathbb{N}}$ such that $x = \lim_{n\to\infty} f_{d_1 d_2 \dots d_n}(0)$. Then, Lemma 1 gives

$$\begin{aligned} \phi(x) &= \lim_{n\to\infty} \phi \circ f_{d_1 d_2 \dots d_n}(0) \\ &= \lim_{n\to\infty} (\phi \circ f_{d_1} \circ \phi^{-1}) \circ (\phi \circ f_{d_2} \circ \phi^{-1}) \cdots (\phi \circ f_{d_n} \circ \phi^{-1}) \circ \phi(0) \\ &= \lim_{n\to\infty} f_{\bar{d}_1 \bar{d}_2 \dots \bar{d}_n}(1). \end{aligned}$$

As all $\bar{d}_i \in \Omega_{\bar{\lambda}}$ we see that $\phi(x) \in E_{\bar{\lambda}}$. Hence, $\phi(E_\lambda) = E_{\bar{\lambda}}$.

Proof of Proposition 1.

Let us assume that $E_{\bar{\lambda}}$ is totally self-similar. We claim that E_λ is also totally self-similar. For this we need to check that $f_d(E_\lambda) = f_d(I) \cap E_\lambda$ for any finite word $d = (d_1 d_2 \dots d_n) \in \Omega_\lambda^*$. We already know that $f_{\bar{d}_1 \dots \bar{d}_n}(E_{\bar{\lambda}}) = f_{\bar{d}_1 \dots \bar{d}_n}(I) \cap E_{\bar{\lambda}}$. Thus, from Lemma 2 we see that

$$\phi \circ f_{\bar{d}_1 \dots \bar{d}_n}(\phi(E_\lambda)) = \phi \circ f_{\bar{d}_1 \dots \bar{d}_n}(I) \cap \phi \circ E_{\bar{\lambda}}. \qquad (2)$$

Note that from Lemma 2 we have $\phi(E_{\bar{\lambda}}) = \phi^2(E_\lambda) = E_\lambda$. Using Lemma 1 repeatedly we get $\phi \circ f_{\bar{d}_1 \dots \bar{d}_n}(\phi(E_\lambda)) = f_{d_1 \dots d_n}(E_\lambda)$ and $\phi \circ f_{\bar{d}_1 \dots \bar{d}_n}(I) \cap E_\lambda = f_{d_1 \dots d_n}(I) \cap E_\lambda$ as $\phi(I) = I$. Hence, (2) implies that $f_{d_1 \dots d_n}(E_\lambda) = f_{d_1 \dots d_n}(I) \cap E_\lambda$, for any finite word $d_1 \dots d_n$. Thus, E_λ is totally self-similar. The converse is analogous.

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