

Suleyman Demirel University  
Faculty of Engineering and Natural Sciences  
Department of Mathematics and Natural Sciences



Acting Dean of Faculty

Associate Professor, PhD Zhamanov A.


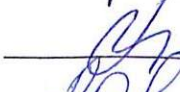

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**Topic of the thesis:**

**Variety of Bicommutative Algebra defined by identities**

$$(ab)c + (ba)c + (ca)b + c(ab) + c(ba) + b(ac) = 0,$$
$$\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + b(ac)] = 0$$

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Head of Department		Madina Abdykarim
Academic Supervisor		Bekzat Zhakhayev
Master student		Zhaniya Akhmetova

Kaskelen, 2023

Ministry of Science and Higher Education  
of the Republic of Kazakhstan  
Suleyman Demirel University



Zhaniya Akhmetova

## Variety of Bicommutative Algebra defined by identities

$$(ab)c + (ba)c + (ca)b + c(ab) + c(ba) + b(ac) = 0, \gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + b(ac)] = 0$$

THESIS

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Faculty of Engineering and Natural Sciences

Supervisor: **Bekzat Zhakhayev**

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# Declaration

I confirm that is my own work and theuse of all material from other sources has been properly and fully acknowledged.

Zhaniya Akhmetova

June 2023

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# Dedication

I would like to express my appreciation to my family. I am grateful to my family for their love, support, and patience. Also, to my family, thank you for empowering me in all of my interests and motivating me to take after my dreams. I am particularly thankful to my parents, who bolstered me sincerely and monetarily. I continuously knew that they always simply believed in me and needed the finest for me. Thank you for instructing me that my work in life was to learn, to be happy, and to know and get it myself. Especially, thank you to my mother, Mendibayeva Kalampyr, for directing me as a person, mathematician.

# Abstract

One of the key inquiries in modern algebra is the investigation of algebras that satisfy specific identities. Within the domain of polynomial identities we have 2 questions. The first is to describe an algebra by a defined identity. The second is to describe identities in algebra. The study of identities will help us in the construction of basis of free algebra, in the study of the Hilbert sequence, the Specht problem and problems of the finite basis. In this work, we used two different research methods. First, the theory of representations of symmetric groups. Second, the theory of representations of linear groups. In this research paper, we have completely described the subvariety of variety of bicommutative algebras defined by the identities  $(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) = 0$ ,  $\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + a(bc)] = 0$ .

# Аңдатпа

Тепе-теңдіктерді қанағаттандыратын алгебраларды зерттеу заманауи алгебраның қызықты сұрақтарының бірі болып табылады. Полиномиал теориясында 2 сұрақ бар. Бірінші сұрағы, тепе-теңдіктермен анықталатын алгебраларды сипаттау. Екінші сұрағы, алгебрадағы тепе-теңдіктерді сипаттау. Тепе-теңдіктерді зерттеу бізге еркін алгебралардың базисін құрастыруға, Гильберт қатарын зерттеуге, Шпехт есебі мен шекті базис сұрақтарына жауап беруге мүмкіндік береді. Есептерді шығару мақсатында екі түрлі зерттеу әдістерін қолдандық. Біріншісі, симметриялық топтардық көрсетілімдер теориясы. Екіншісі, толық сызықтық топтардық көрсетілімдер теориясы. Осы жұмыста  $(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) = 0$ ,  $\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + a(bc)] = 0$  тепе-теңдіктерімен анықталатын бикоммутативті алгебралар көпбейнесінің ішкі-көпбейнелерін толық сипатталды.

# Аннотация

Один из интересных вопросов в современной алгебре — изучение алгебры, удовлетворяющие некоторые тождества. В теории полиномиальных тождеств есть 2 больших вопроса. Первый, описать алгебру, определяемую тождеством. Второй, описать тождества в алгебре. Изучение тождеств поможет нам в конструкции базисов свободной алгебры, в изучении Гильбертовой последовательности, в изучении проблемы Шпехта, а также ответить на вопросы конечного базиса. В этой работе мы использовали два разных метода исследования. Первый, теория представлений симметрических групп. Второй, теория представлений полных линейных групп. В этой исследовательской работе мы полностью описали внутреннее разнообразие коммутативных алгебр, определяемое тождествами  $(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) = 0$ ,  $\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + a(bc)] = 0$ .

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# Chapter 1

## Introduction

One of the significant issues in contemporary algebra is the examination of algebras that satisfy certain identities. Within the theory of algebraic polynomial identities, there exist two primary inquiries:

- 1) explain algebras with identities
- 2) explain identities in algebras.

Delving into these questions necessitates the exploration of free algebras, the construction of their bases, the determination of Hilbert series, the establishment of codimension sequences, the exploration of codimension growth, the calculation of Gelfand-Kirillov dimension, the determination of cocharacter sequences, the evaluation of colengths, the investigation of the Specht problem, and other related topics.

In the theory of polynomial identities, the concepts of identity and algebra are intricately intertwined, and their relationship is established through algebraic varieties. The language of algebraic varieties enables a seamless transition between the notions of identity and algebra, and vice versa. Consequently, the study of algebraic varieties stands as a significant problem within the domain of polynomial identities.

A.I. Malcev [5] and W. Specht [11] studied the representation theory of symmetric group to categorize polynomial identities of algebraic structures for the first time and independently. Every polynomial is known to be equivalent to a finite collection of multilinear polynomials if  $K$  is a field of characteristic 0. As a result, we concentrate on the module structures of multilinear components of free algebras in this paper. Because the multilinear sections of free algebras contain a wealth of relevant and crucial information about various algebras, they are particularly beneficial. The term "module structure" refers to modules over both the symmetric and generic linear groups.

The techniques employed in representation theory, specifically those related to the symmetric group and general linear group, are widely regarded as the most effective approaches for studying the multilinear components of free algebras. These

methods have been successfully utilized to establish the  $S_n$  and  $GL_n$ -module structures of various free algebras. Examples of such algebras include associative algebra, Leibniz algebra, Zinbiel algebra, Lie algebra, right-symmetric algebra, Novikov algebra, and many others.

In many instances, the application of methods from the theory of representations of the general linear group proves to be more advantageous when studying varieties of algebras compared to the representation theory of the symmetric group. This observation is supported by several works. For example, in [9], the varieties of associative algebras with a degree three identity are fully characterized using the methods of the theory of representations of the general linear group. Furthermore, the distributivity criterion for the lattice of subvarieties of associative algebra varieties is established by employing methods from the theory of representations of the general linear group [4]. Similarly, the distributivity criterion for the lattice of subvarieties of alternative algebra varieties is also provided using the methods of the theory of representations of the general linear group [10].

The concept of bicommutative algebras was initially introduced by Dzhumadil'daev and Tulenbaev in their work [1]. In this paper, the authors established that if an algebra  $A$  is bicommutative, then its square  $A^2$  is both commutative and associative. In 2011, Dzhumadil'daev, Ismailov, and Tulenbaev [2] employed combinatorial methods to calculate the multiplicities of irreducible  $S_n$ -representations in the decomposition of the multilinear component. Their work also included the construction of a basis for free bicommutative algebras, the description of multiplication rules for basis elements, the determination of cocharacter and codimension sequences, and the calculation of Hilbert series. Additionally, they demonstrated that the bicommutative operad is not Koszul, and the growth of the codimension sequence for bicommutative algebras is equal to 2. A subsequent study [6] presented an alternative proof of the cocharacter sequence formula for bicommutative algebras. In 2018, Drensky and Zhakhayev [8] provided a positive solution to the Specht problem concerning varieties of bicommutative algebras. The association between bicommutative algebras and the filtration and grading of free Novikov algebras has further motivated a deeper exploration of bicommutative algebra.

In this study, our focus lies in the examination of the variety of bicommutative algebras, which is determined by given identities.

$$\begin{aligned} (ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) &= 0, \\ \gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + a(bc)] &= 0 \end{aligned}$$

where  $\gamma, \delta \in K$ .

One of the significant challenges in the field of polynomial identities in algebra is the description of all varieties of algebras that satisfy a specific set of identities. Our objective is to classify all subvarieties within the realm of bicommutative algebras. Typically, classification is carried out using the language of lattices, and this problem is essentially equivalent to the description of T-ideals. To construct a lattice of subvarieties for a given algebraic variety, the following steps need to be

taken:

- 1) determining the module structure of  $P_n(M)$  over the symmetric group
- 2) identifying the consequences in  $P_{n+1}(M)$  for each irreducible  $S_n$ -module present in  $P_n(M)$ .

# Chapter 2

## Preliminaries

**Definition 2.0.1.** ([7]) *A vector space  $A$  is called an algebra if  $A$  is equipped with a binary operation  $\cdot$ , called multiplication, such that for any  $a, b, c \in A$  and any  $\alpha \in K$*

$$\begin{aligned}(a + b) \cdot c &= a \cdot c + a \cdot b, \\ a \cdot (b + c) &= a \cdot b + a \cdot c, \\ \alpha(a \cdot b) &= (\alpha a) \cdot b = a \cdot (\alpha b).\end{aligned}$$

**Definition 2.0.2.** ([1]) *An algebra  $(A, \cdot)$  is called bicommutative algebra if for any  $a, b, c \in A$  the following identities are satisfied*

$$a \cdot (b \cdot c) = b \cdot (a \cdot c) \tag{2.0.1}$$

$$(a \cdot b) \cdot c = (a \cdot c) \cdot b \tag{2.0.2}$$

**Definition 2.0.3.** ([7]) *Let  $\mathfrak{B}$  be a class of algebras and let  $F(X) \in \mathfrak{B}$  be an algebra generated by a set  $X$ . The algebra  $F(X)$  is called a free algebra in the class  $\mathfrak{B}$ , if for any algebra  $A \in \mathfrak{B}$ , every mapping  $X \rightarrow A$  can be extended to a homomorphism  $F(X) \rightarrow A$ .*

Let  $K\{X\}$  be a free non-associative algebra generated by set  $X = \{x_1, \dots, x_n\}$ .

**Definition 2.0.4.** ([7]) *Let  $f = f(x_1, \dots, x_n) \in K\{X\}$  and let  $A$  be a non-associative algebra. We say that  $f = 0$  is a polynomial identity for  $A$  if  $f(a_1, \dots, a_n) = 0$  for all  $a_1, \dots, a_n \in A$*

**Definition 2.0.5.** ([7]) *Let  $\{f_i(x_1, \dots, x_n) \in K\{X\} \mid i \in I\}$  be a system of polynomials. The class  $\mathfrak{V}$  of all non-associative algebras satisfying the polynomial identities  $f_i$ ,  $i \in I$ , is called the variety defined by the system of polynomial identities  $\{f_i \mid i \in I\}$ . The variety  $\mathfrak{M}$  is called a subvariety of  $\mathfrak{V}$  if  $\mathfrak{M} \subset \mathfrak{V}$ .*

**Definition 2.0.6.** ([7]) *The set  $T(\mathfrak{V})$  of all polynomial identities satisfied by the variety  $\mathfrak{V}$  is called the  $T$ -ideal of  $\mathfrak{V}$ . We say, that the  $T$ -ideal  $T(\mathfrak{V})$  is generated as a  $T$ -ideal by the defining set of identities  $\{f_i \mid i \in I\}$  of the variety  $\mathfrak{V}$ .*

**Definition 2.0.7.** Let  $V$  be a vector space over  $K$  and let  $G$  be a group. Then  $V$  is a  $G$ -module if a multiplication  $gv$  ( $v \in V$ ,  $g \in G$ ) is defined, satisfying the following conditions for all  $u, v \in V$ ,  $\lambda \in K$  and  $g, h \in G$ .

(1)  $gu \in V$ ;

(2)  $(gh)v = g(hv)$ ;

(3)  $1v = v$ ;

(4)  $g(\lambda v) = (g\lambda)v$ ;

(5)  $g(u + v) = gu + gv$ .

**Definition 2.0.8.** ([3]) A polynomial  $f = f(x_1, \dots, x_n)$  is called linear in the variable  $x_i$  if  $x_i$  occurs in every monomial of  $f$  with degree 1. A polynomial  $f$  is called multilinear if it is linear in each variable.

# Chapter 3

## Variety of Bicommutative Algebras defined by identities

$$(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) = 0,$$

$$\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + a(bc)] = 0$$

### 3.1 Main statements

Let  $\mathfrak{M}$  be a variety of bicommutative algebras over a field  $K$  characteristic 0 defined by identities

$$\begin{aligned} (ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) &= 0 \\ \gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + a(bc)] &= 0 \end{aligned}$$

where  $(\gamma, \delta) \neq (0, 0)$ ,  $\gamma, \delta \in K$ .

Let  $F(\mathfrak{M})$  be a free algebra in variety  $\mathfrak{M}$ , and let  $F_n(\mathfrak{M})$  be a free algebra in  $\mathfrak{M}$  generated by  $X = \{x_1, \dots, x_n\}$ .

Let  $P_n$  be a space of multilinear polynomials of  $F_n(\mathfrak{M})$  of degree  $n$ . We will refer to the space of multilinear polynomials as the multilinear component.

There are four possible scenarios that arise based on the coefficients  $\gamma$  and  $\delta$ .

(I)  $\gamma\delta(\gamma - \delta)(\gamma + \delta) \neq 0$

(II)  $\gamma = 0, \delta \neq 0$  ( $\gamma \neq 0, \delta = 0$  analogous)

(III)  $\gamma + \delta = 0$

(IV)  $\gamma - \delta = 0$

### 3.2 Case I: $\gamma\delta(\gamma + \delta)(\gamma - \delta) \neq 0$

Let  $\mathfrak{M}$  be a variety of bicommutative algebras defined by identities

$$(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) = 0 \quad (3.2.1)$$

$$\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + b(ac)] = 0 \quad (3.2.2)$$

where  $\gamma\delta(\gamma + \delta)(\gamma - \delta) \neq 0$ .

**Theorem 3.2.1.** *As  $S_n$ -module*

$$P_1(\mathfrak{M}) \cong S^{(1)}, P_2(\mathfrak{M}) \cong S^{(2)} \oplus S^{(1,1)},$$

$$P_3(\mathfrak{M}) \cong S^{(3)} \oplus S^{(2,1)},$$

$$P_n(\mathfrak{M}) = \{0\} \text{ for } n \geq 4.$$

*Proof.* Let  $n = 3$ .

The number of basis elements of  $F(\mathfrak{B})$  is equal to 6. There are 2 types of elements:

(**) *	*(**)
(ab)c	b(ac)
(ba)c	c(ab)
(ca)b	c(ba)

First we replace  $b$  with  $c$  into (3.2.2):

$$\gamma[(ac)b - 2(ca)b + (ba)c] + \delta[b(ca) - 2b(ac) + c(ab)] = 0 \quad (3.2.3)$$

Then subtract (3.2.2) to (3.2.3):

$$\begin{aligned} & \gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + b(ac)] - \\ & - \gamma[(ac)b - 2(ca)b + (ba)c] - \delta[b(ca) - 2b(ac) + c(ab)] = 0 \\ & \gamma(ab)c - 2\gamma(ba)c + \gamma(ca)b + \delta c(ba) - 2\delta c(ab) + \delta b(ac) - \\ & - \gamma(ac)b + 2\gamma(ca)b - \gamma(ba)c - \delta b(ca) + 2\delta b(ac) - \delta c(ab) = 0 \\ & 3\gamma(ca)b - 3\gamma(ba)c + 3\delta b(ac) - 3\delta c(ab) = 0 \end{aligned}$$

$$3\gamma[(ca)b - (ba)c] = 3\delta[c(ab) - b(ac)]$$

$$(ca)b - (ba)c = \frac{\delta}{\gamma}[c(ab) - b(ac)] \quad (3.2.4)$$

By using (3.2.3):

$$(ca)b = \frac{\delta}{\gamma}[c(ab) - b(ac)] + (ba)c$$

$$(ab)c = \frac{\delta}{\gamma}[c(ab) - b(ac)] + (ba)c$$

By (3.2.1) we know that

$$b(ac) = -[(ab)c + (ba)c + (ca)b + c(ab) + c(ba)]$$

By this new identity we can assert that the number of basis elements of the multilinear part of degree 3 of free algebras is 3.

Let  $n = 4$ . The number of basis elements of  $F(\mathfrak{B})$  is equal to 14. There are 3 types of elements:

$((**)* ) *$	$*((**)* )$	$* (* (**)$
$((ab)c)d$	$a((bc)d)$	$a(b(cd))$
$((bc)d)a$	$a((cb)d)$	$b(c(da))$
$((cd)a)b$	$a((db)c)$	$c(d(ab))$
$((da)b)c$	$b((ca)d)$	$d(a(bc))$
	$b((da)c)$	
	$c((da)b)$	

Let  $a := a$ ,  $b := a$ ,  $c := a$  in the identity (3.2.1):

$$(aa)a + (aa)a + (aa)a + a(aa) + a(aa) + a(aa) = 0$$

$$3(aa)a + 3a(aa) = 0$$

$$(aa)a = -a(aa) \quad (3.2.5)$$

Let  $a := (aa)$ ,  $b := (aa)$ ,  $c := a$  in identity (3.2.2):

$$\gamma[(a(aa))a - 2((aa)a)a + (aa)(aa)] + \delta[a((aa)a) - 2a(a(aa)) + (aa)(aa)] = 0$$

$$\gamma[2(a(aa))a - 2((aa)a)a] + \delta[2a((aa)a) - 2a(a(aa))] = 0$$

$$\gamma[(a(aa))a - ((aa)a)a] + \delta[a((aa)a) - a(a(aa))] = 0$$

Here, by using (3.2.5):

$$\begin{aligned} \gamma[-((aa)a)a - ((aa)a)a] + \delta[-a(a(aa)) - a(a(aa))] &= 0 \\ -2\gamma((aa)a)a &= 2\delta a(a(aa)) \\ ((aa)a)a &= -\frac{\delta}{\gamma}a(a(aa)) = \frac{\delta}{\gamma}a((aa)a) \end{aligned} \quad (3.2.6)$$

Let  $a := a, b := b, c := a$  in the identity (3.2.2):

$$\begin{aligned} 2\gamma[(aa)b - (ba)a] + 2\delta[b(aa) - a(ab)] &= 0 \\ \gamma(aa)b - \gamma(ba)a + \delta b(aa) - \delta a(ab) &= 0 \end{aligned}$$

Here, we multiple by  $a$  from the right-hand side:

$$\gamma((aa)b)a - \gamma((ba)a)a + \delta(b(aa))a - \delta(a(ab))a = 0$$

Then we use (3.2.5):

$$\begin{aligned} \gamma((aa)a)b - \gamma((ba)a)a + \delta b((aa)a) + \delta((aa)a)b &= 0 \\ (\gamma + \delta)((aa)a)b - \gamma((ba)a)a + \delta b((aa)a) &= 0 \end{aligned} \quad (3.2.7)$$

Let  $a := (aa), b := b, c : a$  into (3.2.1):

$$\begin{aligned} \gamma[((aa)b)a - 2(b(aa))a + (a(aa))b] + \delta[a(b(aa)) - 2a((aa)b) + b((aa)a)] &= 0 \\ \gamma[((aa)b)a - 2(b(aa))a - ((aa)a)b] + \delta[b(a(aa)) - 2a((aa)b) - b(a(aa))] &= 0 \\ -2\gamma(b(aa))a - 2\delta a((aa)b) &= 0 \end{aligned}$$

$$(b(aa))a = -\frac{\delta}{\gamma}a((aa)b) = -b(a(aa)) = -\frac{\delta}{\gamma}((aa)a)b \quad (3.2.8)$$

We use (3.2.8) into (3.2.7):

$$\delta b((aa)a) - \gamma((ba)a)a + \frac{\gamma(\gamma + \delta)}{\delta}a((ba)a)$$

$$\gamma((ba)a)a = \left(\delta + \frac{\gamma(\gamma + \delta)}{\delta}\right)a((ba)a) \quad (3.2.9)$$

Let  $a := (ab)$ ,  $b := a$ ,  $c : a$  into (3.2.1):

$$\begin{aligned} \gamma[((ab)a)a - 2(a(ab))a + (a(ab))a] + \delta[a(a(ab)) - 2a((ab)a) + a((ab)a)] &= 0 \\ \gamma[((ab)a)a - (a(ab))a] + \delta[a(a(ab)) - a((ab)a)] &= 0 \\ \gamma[((ab)a)a - (a(ab))a] + \delta[a(a(ab)) - a((ab)a)] &= 0 \\ \gamma((ab)a)a - (\gamma + \delta)(a(ab))a + \delta a(a(ab)) &= 0 \end{aligned}$$

Here, by using (3.2.5):

$$\gamma((aa)a)b + (\gamma + \delta)((aa)a)b + \delta a(a(ab)) = 0$$

$$a(a(ab)) = \frac{2\gamma + \delta}{\delta}((aa)a)b \quad (3.2.10)$$

Let  $a := a$ ,  $b := ba$ ,  $c : a$  into (3.2.1):

$$\begin{aligned} \gamma[(a(ba))a - 2((ba)a)a + (aa)(ba)] + \delta[a((ba)a) - 2a(a(ba)) + a((ba)a)] &= 0 \\ 2\gamma[(a(ba))a - ((ba)a)a] + 2\delta[(a(ba))a - a(a(ba))] &= 0 \\ (2\gamma + 2\delta)(a(ba))a - 2\delta a(a(ba)) - 2\gamma((ba)a)a &= 0 \end{aligned}$$

Here, by using (3.2.5):

$$\begin{aligned} (\gamma + \delta)b((aa)a) + \delta b((aa)a) - \gamma((ba)a)a &= 0 \\ (\gamma + 2\delta)b((aa)a) - \gamma((ba)a)a &= 0 \end{aligned}$$

And we use (3.2.9):

$$(\gamma + 2\delta)b((aa)a) - \left(\delta + \frac{\gamma(\gamma + \delta)}{\delta}\right)b((aa)a) = 0$$

$$\frac{\delta^2 - \gamma^2}{\delta}b((aa)a) = 0$$

$$b((aa)a) = 0 \quad (3.2.11)$$

Let  $a := a$ ,  $b := a$ ,  $c := (bb)$  in the identity (3.2.1):

$$(aa)(bb) + (aa)(bb) + ((bb)a)a + (bb)(aa) + (bb)(aa) + a(a(bb)) = 0$$

$$\begin{aligned}
& b((aa)b) + b((aa)b) + ((bb)a)a + a((bb)a) + a((bb)a) + a(a(bb)) \\
&= 2b((aa)b) + 2a((bb)a) + ((bb)a)a + a(a(bb)) \\
&= 4a((bb)a) + ((bb)a)a + a(a(bb)) = 0
\end{aligned}$$

$$a((bb)a) = -\frac{1}{4}((bb)a)a + a(a(bb)) = 0 \quad (3.2.12)$$

Let  $a := a, b := (bb), c : a$  into (3.2.2):

$$\begin{aligned}
& \gamma[(a(bb))a - 2((bb)a)a + (a(bb))a] + \delta[a((bb)a) - 2a(a(bb)) + a((bb)a)] = 0 \\
& \gamma[2(a(bb))a - 2((bb)a)a] + \delta[2a((bb)a) - 2a(a(bb))] = 0 \\
& \gamma[(a(bb))a - ((bb)a)a] + \delta[a((bb)a) - a(a(bb))] = 0 \\
& (\gamma + \delta)(a(bb))a = \gamma((bb)a)a - \delta a(a(bb))
\end{aligned}$$

$$(a(bb))a = \frac{\gamma}{(\gamma + \delta)}((bb)a)a - \frac{\delta}{(\gamma + \delta)}a(a(bb)) \quad (3.2.13)$$

By (3.2.12) and (3.2.13) we know that:

$$\begin{aligned}
& (a(bb))a = (a(bb))a \\
& -\frac{1}{4}[(a(bb))a + a(a(bb))] = \frac{\gamma}{(\gamma + \delta)}((bb)a)a - \frac{\delta}{(\gamma + \delta)}a(a(bb)) \\
& -\frac{1}{4}((bb)a)a - \frac{1}{4}a(a(bb)) = \frac{\gamma}{(\gamma + \delta)}((bb)a)a - \frac{\delta}{(\gamma + \delta)}a(a(bb)) \\
& -\frac{1}{4}((bb)a)a - \frac{\gamma}{(\gamma + \delta)}((bb)a)a = \frac{1}{4}a(a(bb)) - \frac{\delta}{(\gamma + \delta)}a(a(bb)) \\
& \left(-\frac{1}{4} - \frac{\gamma}{(\gamma + \delta)}\right)((bb)a)a = \left(\frac{1}{4} - \frac{\delta}{(\gamma + \delta)}\right)a(a(bb)) \\
& -\left(\frac{\gamma + \delta + 4\gamma}{4(\gamma + \delta)}\right)((bb)a)a = \left(\frac{\gamma + \delta - 4\delta}{4(\gamma + \delta)}\right)a(a(bb)) \\
& -\left(\frac{5\gamma + \delta}{4(\gamma + \delta)}\right)((bb)a)a = \left(\frac{\gamma - 3\delta}{4(\gamma + \delta)}\right)a(a(bb))
\end{aligned}$$

$$((bb)a)a = -\frac{\gamma - 3\delta}{5\gamma + \delta}a(a(bb)) \quad (3.2.14)$$

Let  $a := a, b := b, c := b$  into (3.2.2):

$$\gamma[(ab)b - 2(ba)b + (ba)b] + \delta[b(ba) - 2b(ab) + b(ab)] = 0$$

$$\gamma[(ab)b - (ba)b] + \delta[b(ba) - b(ab)] = 0$$

Here, we multiple  $a$  from the both side:

$$\gamma[((ab)b)a - ((ba)b)a] + \delta[(b(ba))a - (b(ab))a] = 0$$

$$(b(ba))a - (b(ab))a = \frac{\gamma}{\delta}[((ba)b)a - ((ab)b)a]$$

$$\gamma[a((ab)b) - a((ba)b)] + \delta[a(b(ba)) - a(b(ab))] = 0$$

$$a((ba)b) - a((ab)b) = \frac{\delta}{\gamma}[a(b(ba)) - a(b(ab))]$$

By using this two identity we know that:

$$\frac{\gamma}{\delta}[((ba)b)a - ((ab)b)a] = \frac{\delta}{\gamma}[a(b(ba)) - a(b(ab))]$$

then by (3.2.14):

$$\begin{aligned} \frac{\gamma}{\delta}((ba)b)a - \frac{\gamma}{\delta}((ab)b)a &= \frac{\delta}{\gamma}a(b(ba)) - \frac{\delta}{\gamma}a(b(ab)) \\ -\frac{\gamma - 3\delta}{5\gamma + \delta} \cdot \frac{\gamma}{\delta}a(a(bb)) + \frac{\gamma - 3\delta}{5\gamma + \delta} \cdot \frac{\gamma}{\delta}b(b(aa)) &= \frac{\delta}{\gamma}a(b(ba)) - \frac{\delta}{\gamma}a(b(ab)) \\ -\frac{\gamma - 3\delta}{5\gamma + \delta} \cdot \frac{\gamma}{\delta}a(a(bb)) + \frac{\delta}{\gamma}a(b(ab)) &= \frac{\delta}{\gamma}a(b(ba)) - \frac{\gamma - 3\delta}{5\gamma + \delta} \cdot \frac{\gamma}{\delta}b(b(aa)) \\ \left(\frac{\delta}{\gamma} - \frac{\gamma - 3\delta}{5\gamma + \delta} \cdot \frac{\gamma}{\delta}\right)a(a(bb)) &= \left(\frac{\delta}{\gamma} - \frac{\gamma - 3\delta}{5\gamma + \delta} \cdot \frac{\gamma}{\delta}\right)b(b(aa)) \end{aligned}$$

$$a(a(bb)) = b(b(aa)) \quad (3.2.15)$$

This means that:

$$((bb)a)a = ((aa)b)a \quad (3.2.16)$$

$$a((bb)a) = a((ab)b) \quad (3.2.17)$$

1. In the first type we have two diagrams:

$$((**)*)* = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}$$

$$1.1 \quad ((aa)a)a, \begin{array}{|c|c|c|c|} \hline a & a & a & a \\ \hline \end{array}$$

We multiple (3.2.5) by the left side:

$$\begin{aligned} a((aa)a) + a(a(aa)) &= 0 \\ (a(aa))a + a(a(aa)) &= 0 && \text{(by (2.0.2))} \\ -((aa)a)a + a(a(aa)) &= 0 && \text{(by (3.2.5))} \\ -((aa)a)a - \frac{\gamma}{\delta}((aa)a)a & && \text{(by (3.2.6))} \\ -\frac{(\gamma + \delta)}{\delta}((aa)a)a &= 0 \\ ((aa)a)a &= 0. \end{aligned}$$

$$1.2 \quad ((ab)a)a - ((ba)a)a, \begin{array}{|c|c|c|} \hline a & a & a \\ \hline b & & \\ \hline \end{array}$$

$$\begin{aligned} ((ab)a)a - ((ba)a)a &= \\ = ((ab)a)a - \frac{\delta^2 + 2\delta\gamma}{\gamma^2}(a(ba)a) &&& \text{(by (3.2.9))} \\ = -\frac{\gamma}{\delta}(b(aa))a - \frac{\delta^2 + 2\delta\gamma}{\gamma^2}(a(ba)a) &&& \text{(by (3.2.8))} \\ = 0 - 0 &&& \text{(by (3.2.11))} \\ = 0. &&& \end{aligned}$$

2. In the second type we have three diagrams:

$$*((**))* = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$$

2.1  $a((aa)a)$ , 

a	a	a	a
---	---	---	---

$$a((aa)a) = -a(a(aa)) = \frac{\gamma}{\delta}((aa)a)a = 0$$

$$a((aa)a) = 0.$$

2.2  $a((ab)a) - a((ba)a)$ , 

a	a	a
b		

$$\begin{aligned} & a((ab)a) - a((ba)a) \\ &= -\frac{\gamma}{\delta}(b(aa))a - a((ba)a) && \text{(by (3.2.11))} \\ &= -\frac{\gamma}{\delta}a((ba)a) - a((ba)a) && \text{(by (2.0.2))} \\ &= -\frac{\gamma + \delta}{\delta} \cdot 0 && \text{(by (3.2.8))} \\ &= 0 \end{aligned}$$

2.3  $(ab - ba)(ab - ba)$ , 

a	a
b	b

$$\begin{aligned} & (ab - ba)(ab - ba) \\ &= a((ab)b) - b((ab)a) - a((ba)b) + b((ba)a) && \text{(by (2.0.2))} \\ &= b((ba)a) - 2b((aa)b) + b((ba)a) && \text{(by (3.2.16))} \\ &= 2a((ab)b) - 2b((aa)b) \\ &= 2a((bb)a) - 2b((aa)b) && \text{(by (3.2.17))} \\ &= 0 \end{aligned}$$

3. In the third type we have two diagrams:

$$*((**)) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}$$

$$3.1 \quad a(a(aa)), \begin{array}{|c|c|c|c|} \hline a & a & a & a \\ \hline \end{array}$$

$$a(a(aa)) = -\frac{\gamma}{\delta}((aa)a)a = 0 \quad (\text{by (3.2.3)})$$

$$a(a(aa)) = 0.$$

$$3.2 \quad a(a(ab)) - a(a(ba)), \begin{array}{|c|c|c|} \hline a & a & a \\ \hline b & & \\ \hline \end{array}$$

$$\begin{aligned} & a(a(ab)) - a(a(ba)) \\ &= \frac{2\gamma + \delta}{\delta}((aa)a)b - b(a(aa)) \quad (\text{by (3.2.10)}) \end{aligned}$$

$$= \frac{2\gamma + \delta}{\delta}((aa)a)b + b((aa)a) \quad (\text{by (3.2.5)})$$

$$= -\frac{\gamma(2\gamma + \delta)}{\delta^2}b((aa)a) + b((aa)a) \quad (\text{by (3.2.8)})$$

$$= 0$$

Let  $n = 5$ . The number of base elements of the multilinear component of degree 5 of free bicommutative algebra  $F(\mathfrak{B})$  is equal to 30. There are 4 types of the base elements.

First type

$$V_5((((**)*)**) = \{(((ab)c)d)e, (((ba)c)d)e, (((ca)b)d)e, (((da)b)c)e, (((ea)b)c)d\}.$$

Second type

$$\begin{aligned} V_5(*(((**)*)*)) &= \{a(((bc)d)e), a(((cb)d)e), a(((db)c)e), a(((eb)c)d), b(((ca)d)e), \\ & \quad b(((da)c)e), b(((ea)c)d), c(((da)b)e), c(((ea)b)d), d(((ea)b)c)\}. \end{aligned}$$

Third type

$$\begin{aligned} V_5(*(*(((**)*))) &= \{a(b((cd)e)), a(b((dc)e)), a(b((ec)d)), a(c((db)e)), a(c((eb)d)), \\ & \quad a(d((eb)c)), b(c((da)e)), b(c((ea)d)), b(d((ea)c)), c(d((ea)b))\}. \end{aligned}$$

Fourth type

$$V_5(*(*(*(**)))) = \{a(b(c(de))), b(c(d(ea))), c(d(e(ab))), d(e(a(bc))), e(a(b(cd)))\}.$$

In [2] proved that  $V_5((((**)*)**)$ ,  $V_5(*(((**)*)*))$ ,  $V_5(*(*((**)*))$  and  $V_5(*(*(*(**))))$  are invariant under action of symmetric group  $S_5$  and as  $S_5$ -module

$$\begin{aligned} V_5((((**)*)**) &\cong S^{(5)} \oplus S^{(4,1)}, \\ V_5(*(((**)*)*)) &\cong S^{(5)} \oplus S^{(4,1)} \oplus S^{(3,2)}, \\ V_5(*(*((**)*)) &\cong S^{(5)} \oplus S^{(4,1)} \oplus S^{(3,2)}, \\ V_5(*(*(*(**)))) &\cong S^{(5)} \oplus S^{(4,1)}, \end{aligned}$$

and

$$\begin{aligned} P_5(\mathfrak{B}) &\cong V_5((((**)*)**) \oplus V_5(*(((**)*)*)) \oplus V_5(*(*((**)*)) \oplus V_5(*(*(*(**)))) \cong \\ &4S^{(5)} \oplus 4S^{(4,1)} \oplus 2S^{(3,2)}. \end{aligned}$$

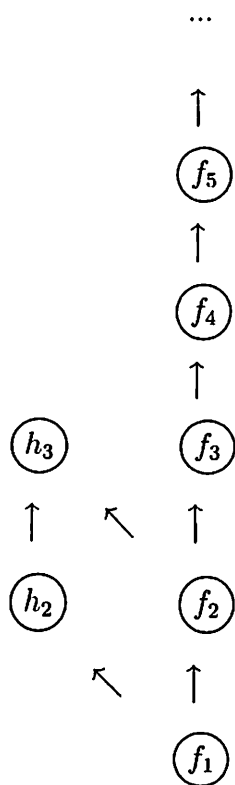
As we have already proven that in  $n = 4$  we do not have base elements, here we use the same methods and this will be easy to prove it.

□

**Theorem 3.2.2.** *Let a linearization of  $f$  generates an irreducible  $S_n$ -submodule of  $P_n(\mathfrak{M})$ . Then the consequences of higher degrees from the  $f$  are equivalent to the following identities*

- (a)  $f_{n+1}$  if  $f = f_n$ ,  $n \geq 1$ ;
- (b)  $h_{n+1}$  if  $f = f_n$ ,  $n = 2$ ;
- (c)  $h_{n+1}$  if  $h = h_n$ ,  $n = 1, 2$ .

To illustrate this theorem we can use this form::



*Proof.* Take note that  $f_1 = f'_1 = h_1$  and  $f_2 = f'_2$ .

(a) For  $n = 1$  :  $f_1 = z = 0 \rightarrow f_2 = zz = 0 \cdot z = 0$ .

For  $n = 2$  :  $f_2 = zz = 0 \rightarrow f_3 = (zz)z = 0 \cdot z = 0$ .

For  $n = m$

$$f_m = \underbrace{\left( \dots \left( \underbrace{zz}_{m-2 \text{ times}} \right) z \right) \dots z}_{m-2 \text{ times}} z = 0 \rightarrow$$

$$f_{m+1} = \underbrace{\left( \dots \left( \underbrace{zz}_{m-1 \text{ times}} \right) z \right) \dots z}_{m-1 \text{ times}} z = f_m \cdot z = 0 \cdot z = 0.$$

(b) For  $n = 1$  :  $f_1 = z = 0 \rightarrow h_2 = zw - wz = 0 \cdot w - w \cdot 0 = 0$ .

For  $n = 2$  :  $f_2 = zz = 0 \rightarrow h_3 = (zw)z - (wz)z$ .

Through the process of substitution  $z := z + w$  into  $f_2$  we get:

$$zz = (z + w)(z + w) = zz + x + zw + wz + ww = 0$$

$ww=zz=0$  since  $f_2 = 0$ . This implies that  $zw + wz = 0$  and we get:

$$zw = -wz$$

Then  $h_3 = (zw)z - (wz)z = (zw)z + (zw)z = 2(zz)w = 2f_2 \cdot y = 0$

(c) For  $n = 2$ :  $h_2 = zw - wz = 0 \rightarrow h_3 = (zw)z - (wz)z = (zw - wz)z = h_2 \cdot z = 0 \cdot z = 0$ .

□

### 3.3 Case II: $\gamma = 0, \delta \neq 0$

Let  $\mathfrak{M}$  be a variety of bicommutative algebras defined by identities

$$\begin{aligned} (ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) &= 0 \\ (ab)c - 2(ba)c + (ca)b &= 0 \end{aligned} \tag{3.3.1}$$

**Theorem 3.3.1.** *As  $S_n$ -module*

$$\begin{aligned} P_1(\mathfrak{M}) &\cong S^{(1)}, P_2(\mathfrak{M}) \cong S^{(2)} \oplus S^{(1,1)}, \\ P_3(\mathfrak{M}) &\cong S^{(3)} \oplus S^{(2,1)}, \\ P_n(\mathfrak{M}) &= 0 \text{ for } n \geq 4. \end{aligned}$$

*Proof.* By subtracting (3.2.1) to (3.3.1) we get:

$$\begin{aligned} (ab)c + (ba)c + (ca)b + c(ab) + c(ba) + a(cb) - (ab)c + 2(ba)c - (ca)b &= 0 \\ c(ba) + c(ab) + b(ac) &= -3(ba)c \end{aligned}$$

We simplify the (3.3.1):

$$2(ba)c = (ab)c + (ca)b$$

These two expressions give us the following system of identities:

$$-3(ba)c = c(ab) + c(ba) + b(ac) \tag{3.3.2}$$

$$2(ba)c = (ab)c + (ca)b \tag{3.3.3}$$

$n = 3$

By using (3.3.2) we get:

$$(ab)c = -\frac{1}{3}[c(ba) + c(ab) + a(bc)]$$

$$(ba)c = -\frac{1}{3}[c(ab) + c(ba) + b(ac)]$$

$$(ca)b = -\frac{1}{3}[b(ac) + b(ca) + c(ab)]$$

So all elements of the type  $(**)^*$  can be removed by the  $*(**)$  typed elements.

Based on the investigation and outcoming result we conclude that the  $P_3(\mathfrak{M})$  is equal to 3.

$$\{b(ac); b(ca); c(ab)\}$$

$$n = 4$$

$n = 4$  with  $((**)^*)^*$  typed elements Using (3.3.3) gives us:

$$2d((ba)c) = d[(ca)b + (ab)c] = d((ca)b) - d(ab)c = (ca)(db) + (ac)(db)$$

Here, we take  $c := b, d := a, (ab) := b$ , then using (3.3.3) gives us:

$$(ca)(db) + (ac)(db) = 2((db)a)c.$$

Which means:

$$((db)a)c = d((ba)c) \tag{3.3.4}$$

Consequently,  $((**)^*)^*$  are described by  $*(**)^*$  elements. So, we cancel them from the basis.

$n = 4$  with  $*(**)$  typed elements By simplifying (3.3.2):

$$c(ab) = 3(ba)c + c(ba) + b(ac)$$

We use this identity to  $a(b(cd))$  (here,  $a := c, b := a, (cd) := b$ ) and get:

$$a(b(cd)) = 3((cd)b)a + a((cd)b) + (cd)(ba)$$

$$a(b(cd)) = 3c((db)a) + a((cd)b) + c((db)a)$$

$$a(b(cd)) = 4c((db)a) + a((cd)b) \tag{3.3.5}$$

It means that  $*(**)$  typed elements can be described by the elements typed  $*(**)*$ . So, we cancel them from the basis.

$n = 4$  with  $*(**)*$  typed elements: Let's  $a((bc)d)$  express by (3.3.2):

$$c(ab) = 3(ba)c + c(ba) + b(ac)$$

Here,  $c := a, a := (bc), b := d$ :

$$a((bc)d) = 3(d(bc))a + a(d(bc)) + d((bc)a)$$

$$a((bc)d) = 4(d(bc))a + a(d(bc))$$

Then, use (3.3.5):

$$a((bc)d) = 4(d(bc))a + a(d(bc))$$

$$a((bc)d) = 4(d(bc))a + 4b((cd)a) + a((bc)d)$$

Here, (3.3.4) gives:

$$b((cd)a) = ((bc)d)a = ((bd)c)a = b((dc)a) = (d(bc))a$$

That means:

$$a((bc)d) = 4(d(bc))a + 4(d(bc))a + a((bc)d)$$

$$a((bc)d)a((bc)d) = 8(d(bc))a$$

$$(d(bc))a = 0$$

Consequently,  $b((cd)a) = ((bc)d)a = ((bd)c)a = b((dc)a) = (d(bc))a = 0$  By calculating we obtain that in case  $n = 4$  there is no basis element.

$n = 5$

When  $n = 5$ , we have 30 basis elements of multilinear part of free bicommutative algebra. There are 4 type of elements:

$((**)*)*$	$*(***)$	$*(((**)*)*$	$*(*(**))$
$((ab)c)d$	$a(b(cd)e)$	$a(((bc)d)e)$	$a(b(c(de)))$
$((bc)d)e$	$a(b((dc)e))$	$a(((cb)d)e)$	$b(c(d(ea)))$
$((cd)e)a$	$a(b((ec)d))$	$a(((db)c)e)$	$c(d(e(ab)))$
$((de)a)b$	$a(c((db)e))$	$a(((eb)c)d)$	$d(e(a(bc)))$
$((ea)b)c$	$a(c((eb)d))$	$b(((ca)d)e)$	$e(a(b(cd)))$
	$a(d((eb)c))$	$b(((da)c)e)$	
	$b(c((da)e))$	$b(((ea)c)d)$	
	$b(c((ea)d))$	$c(((da)b)e)$	
	$b(d((ea)c))$	$c(((ea)b)d)$	
	$c(d((ea)b))$	$d(((ea)b)c)$	

$((**)*)*$  and  $*(**)**$  type of elements we have diagrams like:

$$(((**)*)*)* = *(**)** = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array}$$

$*((**)**)$  and  $*(((**)*)*)$  type of elements we have diagrams like:

$$*(((**)*)*)* = *(**)** = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}$$

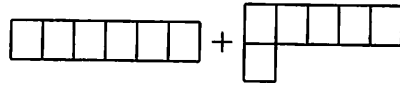
Let  $n = 6$ . The number of basis elements of multilinear part of degree 6 of free bicommutative algebras is equal to 62. There are 5 types of elements:

$$\begin{aligned} & (((**)*)*)* * *(((**)*)*)* * (*(**)**) * (*(**)**) * (*(**)**) \\ & (((ab)c)d)e)f \quad a(((bc)d)e)f \quad a(b((cd)e)f) \quad a(b(c(de)f)) \quad a(b(c(d(ef)))) \\ & (((bc)d)e)f)a \quad a(((cb)d)e)f \quad a(b((dc)e)f) \quad a(b(c(ed)f)) \quad b(c(d(e(fa)))) \\ & (((cd)e)f)a)b \quad a(((db)c)e)f \quad a(b((ec)d)f) \quad a(b(c(fd)e)) \quad c(d(e(f(ab)))) \\ & (((de)f)a)b)c \quad a(((eb)c)d)f \quad a(b((fc)d)e) \quad a(b(d(ec)f)) \quad d(e(f(a(bc)))) \\ & (((ef)a)b)c)d \quad a(((fb)c)d)e \quad a(c((db)e)f) \quad a(b(d(fc)e)) \quad e(f(a(b(cd)))) \\ & (((fa)b)c)d)e \quad b(((ca)d)e)f \quad a(c((eb)d)f) \quad a(b(e((fc)d))) \quad f(a(b(c(de)))) \\ & \quad b(((da)c)e)f \quad a(c((fb)d)e) \quad a(c(d((eb)f))) \\ & \quad b(((ea)c)d)f \quad a(d((eb)c)f) \quad a(c(d((fb)e))) \\ & \quad b(((fa)c)d)e \quad a(d((fb)c)e) \quad a(c(e((fb)d))) \\ & \quad c(((da)b)e)f \quad a(e((fb)c)d) \quad a(d(e((fb)c))) \\ & \quad c(((ea)b)d)f \quad b(c((da)e)f) \quad b(c(d((ea)f))) \\ & \quad c(((fa)b)d)e \quad b(c((ea)d)f) \quad b(c(d((fa)e))) \\ & \quad d(((ea)b)c)f \quad b(c((fa)d)e) \quad b(c(e((fa)d))) \\ & \quad d(((fa)b)c)e \quad b(d((ea)c)f) \quad b(d(e((fa)c))) \\ & \quad e(((fa)b)c)d \quad b(d((fa)c)e) \quad c(d(e((fa)b))) \\ & \quad \quad b(e((fa)c)d) \\ & \quad \quad c(d((ea)b)f) \\ & \quad \quad c(d((fa)b)e) \end{aligned}$$

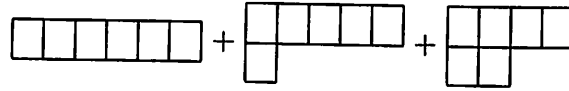
$$c(e(((fa)b)d))$$

$$d(e(((fa)b)c))$$

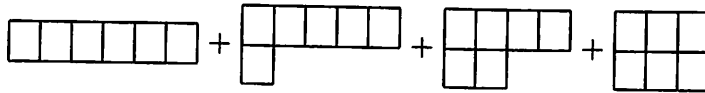
$(((((**)*)*)*)*$  and  $*(*((**)*))$  type of elements we have diagrams like:



$*(((**)*)*)*$  and  $*(*((**)*))$  type of elements we have the next diagrams:



And the last  $*(*(((**)*)*))$  type of elements we have diagrams like:



As we have already proven that in  $n \cong 4$  we do not have base elements, here we use the same methods and this will be easy to prove it.

Starting with the  $n = 5, 6, 7, \dots$  we do not have a basic element.

In this way our theorem is proved. □

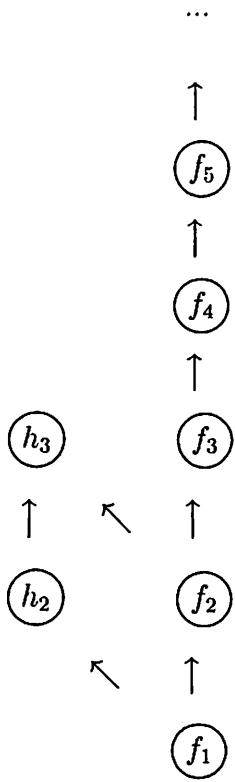
Dimension of  $P_n(\mathfrak{M})$  is equal to 0, for  $n \geq 4$

$n$	1	2	3	4	5	6	...	$n$	...
$dim(P_n(\mathfrak{M}))$	1	2	3	0	0	0	...	0	...

**Theorem 3.3.2.** *Let a linearization of  $f$  generates an irreducible  $S_n$ -submodule of  $P_n(\mathfrak{M})$ . Then the consequences of higher degrees from the  $f$  are equivalent to the following identities*

- (a)  $f_{n+1}$  if  $f = f_n$ ,  $n \geq 1$ ;
- (b)  $h_{n+1}$  if  $f = f_n$ ,  $n = 2$ ;
- (c)  $h_{n+1}$  if  $h = h_n$ ,  $n = 1, 2$ .

To illustrate this theorem we can use this form::



*Proof.* Take note that  $f_1 = f'_1 = h_1$  and  $f_2 = f'_2$ .

(a) For  $n = 1$  :  $f_1 = z = 0 \rightarrow f_2 = zz = 0 \cdot z = 0$ .

For  $n = 2$  :  $f_2 = zz = 0 \rightarrow f_3 = (zz)z = 0 \cdot z = 0$ .

For  $n = m$

$$f_m = \underbrace{\left( \dots \left( \underbrace{zz}_{m-2 \text{ times}} \right) z \dots z \right)}_{m-2 \text{ times}} z = 0 \rightarrow$$

$$f_{m+1} = \underbrace{\left( \dots \left( \underbrace{zz}_{m-1 \text{ times}} \right) z \dots z \right)}_{m-1 \text{ times}} z = f_m \cdot z = 0 \cdot z = 0.$$

(b) For  $n = 1$  :  $f_1 = z = 0 \rightarrow h_2 = zw - wz = 0 \cdot w - w \cdot 0 = 0$ .

For  $n = 2$  :  $f_2 = zz = 0 \rightarrow h_3 = (zw)z - (wz)z$ .

Through the process of substitution  $z := z + w$  into  $f_2$  we get:

$$zz = (z + w)(z + w) = zz + x + zw + wz + ww = 0$$

$ww=zz=0$  since  $f_2 = 0$ . This implies that  $zw + wz = 0$  and we get:

$$zw = -wz$$

$$\text{Then } h_3 = (zw)z - (wz)z = (zw)z + (zw)z = 2(zz)w = 2f_2 \cdot y = 0$$

(c) For  $n = 2$ :  $h_2 = zw - wz = 0 \rightarrow h_3 = (zw)z - (wz)z = (zw - wz)z = h_2 \cdot z = 0 \cdot z = 0$ .

□

### 3.4 Case III: $\gamma + \delta = 0$

Let  $\mathfrak{M}$  be a variety of bicommutative algebras defined by identities

$$\begin{aligned} (ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) &= 0, \\ (ab)c - 2(ba)c + (ca)b - c(ba) + 2c(ab) - a(bc) &= 0 \end{aligned} \quad (3.4.1)$$

**Theorem 3.4.1.** *As  $S_n$ -module*

$$P_1(\mathfrak{M}) \cong S^{(1)}, P_2(\mathfrak{M}) \cong S^{(2)} \oplus S^{(1,1)},$$

$$P_3(\mathfrak{M}) \cong S^{(3)} \oplus S^{(2,1)},$$

$$P_n(\mathfrak{M}) = 0 \text{ for } n \geq 4.$$

*Proof.* Let  $n = 3$ .

By adding (3.2.1) to (3.4.1) we get:

$$\begin{aligned} &(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) + \\ &+ (ab)c - 2(ba)c + (ca)b - c(ba) + 2c(ab) - a(bc) = 0 \\ &2(ab)c - (ba)c + 2(ca)b + 3c(ab) = 0 \\ &-3c(ab) = 2(ab)c - (ba)c + 2(ca)b \end{aligned}$$

By subtracting (3.2.1) to (3.4.1) we get:

$$\begin{aligned}
& (ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) - \\
& (ab)c + 2(ba)c - (ca)b + c(ba) - 2c(ab) + a(bc) = 0 \\
& 3(ba)c + 2c(ba) - c(ab) + 2b(ac) = 0 \\
& -3(ba)c = 2c(ba) - c(ab) + 2b(ac)
\end{aligned}$$

These two expressions give us the following system of identities:

$$-3c(ab) = 2(ab)c - (ba)c + 2(ca)b \quad (3.4.2)$$

$$-3(ba)c = 2c(ba) - c(ab) + 2b(ac) \quad (3.4.3)$$

By using (3.4.3) we get:

$$(ab)c = -\frac{1}{3}[2c(ab)c(ba) + 2a(bc)]$$

$$(ba)c = -\frac{1}{3}[2c(ba)c(ab) + 2b(ac)]$$

$$(ca)b = -\frac{1}{3}[2b(ca)b(ac) + 2c(ab)]$$

So all elements of the type  $(**)^*$  can be removed by the  $*(**)$  typed elements.

Let  $n = 4$ .

$n = 4$  with  $((**)^*)^*$  typed elements

Using (3.4.3) gives us:

$$\begin{aligned}
[(ab)c]d &= -\frac{1}{3}[2c(ab) - c(ba) + 2a(bc)]d = \\
&= -\frac{1}{3}[2(c(ab))d - (c(ba))d + 2(a(bc))d]
\end{aligned}$$

Consequently,  $((**)^*)^*$  are described by  $*(**)$  elements. So, we cancel them from the basis.

$n = 4$  with  $*(**)$  typed elements By (3.4.2) we get:

$$d[c(ab)] = -\frac{1}{3}d[2(ab)c - (ba)c + 2(ca)b] =$$

$$= -\frac{1}{3}[2d((ab)c) - d((ba)c) + 2d((ca)b)]$$

That means:  $*((**))*$  can express elements typed  $*((**))$ . That is why we eliminate it from basis elements.

$n = 4$  with  $*((**))*$  typed elements

First, we express  $a((bc)d)$  using (3.4.1). Then, signify  $(**)*$  typed elements by (3.4.2) identity:

$$\begin{aligned} a[(bc)d] &= \frac{1}{2}a[(cb)d + (dc)b - d(bc) + 2d(cb) - b(cd)] \\ 2a[(bc)d] &= a((cb)d) + a((dc)b) - a(d(bc)) + 2a(d(cb)) - a(b(cd)) \\ 2a[(bc)d] &= a((cb)d) + a((dc)b) + \frac{1}{3}[2a((bc)d) - a((cb)d) - a((db)c)] + \\ &+ \frac{1}{3}[2a((cb)d) - a((bc)d) - a((dc)b)] + \frac{1}{3}[2a((cd)b) - a((dc)d) - a((bc)d)] \\ 6a[(bc)d] &= 6a((cb)d) + 6a((dc)b) + 6a((bc)d) \\ &6a((cb)d) + 6a((dc)b) = 0 \end{aligned}$$

That means

$$a((cb)d) = -a((dc)b) \quad (3.4.4)$$

$$1. a((bc)d)$$

$$2. a((cb)d) = a((bc)d)$$

$$3. a((dc)b) = a((db)c) = a((bd)c) = a((bc)d)$$

$$4. b((ca)d) = b((ac)d) = a((bc)d)$$

$$5. b((da)c) = b((ad)c) = a((bc)d)$$

$$6. c((da)b) = c((ad)b) = a((cb)d) = a((bc)d)$$

The unique element  $a((bc)d)$  can express all others.

Let  $c := a, a := (bc), b := d$  into (3.4.3):

$$-3(d(bc))a = 2a(d(bc)) - a((bc)d) + 2(bc)(da)$$

$$a((bc)d) = 3(d(bc))a + 2a(d(bc)) + 2(bc)(da)$$

$$a((bc)d) = 3(d(bc))a + 2a(d(bc)) + 2(b(da))c$$

$$a((bc)d) = 5(d(bc))a + 2a(d(bc))$$

$$a((bc)d) = 5(d(bc))a + 2[2a((bc)d) + 2a((db)c) - a((cb)d)]$$

$$a((bc)d) = 5(d(bc))a + 4a((bc)d) - 4a((bc)d) - a((cb)d)$$

$$a((bc)d) + a((cb)d) = 5(d(bc))a$$

$$(d(bc))a = 0$$

By calculating we obtain that in case  $n = 4$  there is no basis element.

□

In other cases it is trivial.

Dimension of  $P_n(\mathfrak{M})$  is equal to 0, for  $n \geq 4$ .

$n$	1	2	3	4	5	...	$n$	...
$\dim(P_n(\mathfrak{M}))$	1	2	0	0	0	...	0	...

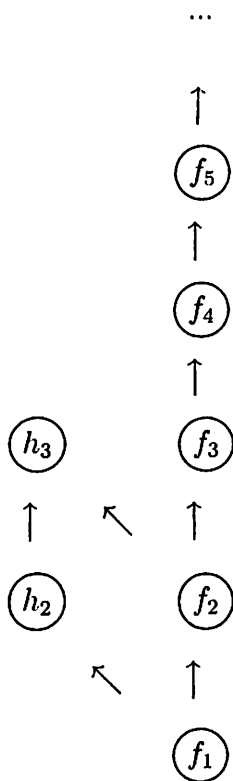
**Theorem 3.4.2.** *Let a linearization of  $f$  generates an irreducible  $S_n$ -submodule of  $P_n(\mathfrak{M})$ . Then the consequences of higher degrees from the  $f$  are equivalent to the following identities*

(a)  $f_{n+1}$  if  $f = f_n$ ,  $n \geq 1$ ;

(b)  $h_{n+1}$  if  $f = f_n$ ,  $n = 2$ ;

(c)  $h_{n+1}$  if  $h = h_n$ ,  $n = 1, 2$ .

To illustrate this theorem we can use this form::



*Proof.* Take note that  $f_1 = f'_1 = h_1$  and  $f_2 = f'_2$ .

(a) For  $n = 1$ :  $f_1 = z = 0 \rightarrow f_2 = zz = 0 \cdot z = 0$ .

For  $n = 2$ :  $f_2 = zz = 0 \rightarrow f_3 = (zz)z = 0 \cdot z = 0$ .

For  $n = m$

$$f_m = \underbrace{(\dots((}_{m-2 \text{ times}} zz \underbrace{)z)\dots z)}_{m-2 \text{ times}} z = 0 \rightarrow$$

$$f_{m+1} = \underbrace{(\dots((}_{m-1 \text{ times}} zz \underbrace{)z)\dots z)}_{m-1 \text{ times}} z = f_m \cdot z = 0 \cdot z = 0.$$

(b) For  $n = 1$ :  $f_1 = z = 0 \rightarrow h_2 = zw - wz = 0 \cdot w - w \cdot 0 = 0$ .

For  $n = 2$ :  $f_2 = zz = 0 \rightarrow h_3 = (zw)z - (wz)z$ .

Through the process of substitution  $z := z + w$  into  $f_2$  we get:

$$zz = (z + w)(z + w) = zz + x + zw + wz + ww = 0$$

$ww=zz=0$  since  $f_2 = 0$ . This implies that  $zw + wz = 0$  and we get:

$$zw = -wz$$

$$\text{Then } h_3 = (zw)z - (wz)z = (zw)z + (zw)z = 2(zz)w = 2f_2 \cdot y = 0$$

(c) For  $n = 2$ :  $h_2 = zw - wz = 0 \rightarrow h_3 = (zw)z - (wz)z = (zw - wz)z = h_2 \cdot z = 0 \cdot z = 0$ .

□

### 3.5 Case IV: $\gamma - \delta = 0$

Let  $\mathfrak{M}$  be a variety of bicommutative algebras defined by identities

$$\begin{aligned} (ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) &= 0, \\ (ab)c - 2(ba)c + (ca)b + c(ba) - 2c(ab) + a(bc) &= 0 \end{aligned} \quad (3.5.1)$$

where  $\gamma = \delta$ .

**Theorem 3.5.1.** *As  $S_n$ -module*

$$\begin{aligned} P_1(\mathfrak{M}) &\cong S^{(1)}, P_2(\mathfrak{M}) \cong S^{(2)} \oplus S^{(1,1)}, \\ P_3(\mathfrak{M}) &\cong S^{(3)} \oplus S^{(2,1)}, \\ P_n(\mathfrak{M}) &= 0 \text{ for } n \geq 4. \end{aligned}$$

*Proof.* Let  $n = 3$ .

First we subtract (3.2.1) to (3.5.1) and get:

$$\begin{aligned} (ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) - \\ - (ab)c + 2(ba)c - (ca)b - c(ba) + 2c(ab) - a(bc) &= 0 \\ 3(ba)c &= -3c(ab) \end{aligned}$$

$$(ba)c = -c(ab) \quad (3.5.2)$$

By using (3.5.2):

$$\begin{aligned} (ab)c &= -c(ba) \\ (ba)c &= -c(ab) \\ (ca)b &= -b(ac) \end{aligned}$$

So all elements of the type  $*(**)$  can be removed by the  $(**)*$  typed elements.

According to the findings dimension of  $P_3(\mathfrak{M})$  is equal to 3.  
 $n = 4$ .

Let  $a := a, b := a, c := a$  in the identity (3.2.1):

$$\begin{aligned}(aa)a + (aa)a + (aa)a + a(aa) + a(aa) + a(aa) &= 0 \\ 3(aa)a &= -3a(aa) \\ (aa)a &= -a(aa)\end{aligned}$$

So we get identity:

$$(aa)a = -a(aa) \quad (3.5.3)$$

When we multiply a  $b$  to the left of identity (3.5.3):

$$\begin{aligned}b((aa)a) + b(a(aa)) &= 0 \\ a((ba)a) + a(a(ba)) &= 0 \\ a((ba)a) &= -a(a(ba))\end{aligned}$$

Then we get a new identity:

$$a((ba)a) = -a(a(ba)) \quad (3.5.4)$$

And when we multiply a  $b$  to the right of identity (3.5.3):

$$\begin{aligned}((aa)a)b + (a(aa))b &= 0 \\ ((ab)a)a + (a(ab))a &= 0 \\ ((ab)a)a &= -(a(ab))a\end{aligned}$$

Then we get a new identity:

$$((ab)a)a = -(a(ab))a \quad (3.5.5)$$

When we multiply an  $a$  to the left and right of identity (3.5.3), we can get two new identities such that:

$$a((aa)a) = -a(a(aa)) \quad (3.5.6)$$

$$((aa)a)a = -(a(aa))a \quad (3.5.7)$$

Let  $a := (aa), b := a, c := a$  in the identity (3.2.1):

$$((aa)a)a + (a(aa))a + (a(aa))a + a(a(aa)) + a((aa)a) + a((aa)a) = 0$$

$$\begin{aligned}
&= ((aa)a)a - ((aa)a)a - ((aa)a)a + a(a(aa)) - a(a(aa)) - a(a(aa)) = 0 \\
&= -((aa)a)a - a(a(aa)) = 0
\end{aligned}$$

So we get the following identity:

$$((aa)a)a = -a(a(aa)) \quad (3.5.8)$$

Let  $a := (aa), b := a, c := b$  in the identity (3.2.1):

$$\begin{aligned}
&((aa)a)b + (a(aa))b + (b(aa))a + b(a(aa)) + b((aa)a) + a((aa)b) = 0 \\
&((aa)a)b - ((aa)a)b + b((aa)a) - b((aa)a) + b((aa)a) + (a(aa))b = 0 \\
&\quad b((aa)a) + (a(aa))b = 0 \\
&((aa)a)b = -b(a(aa)) \quad (3.5.9)
\end{aligned}$$

Let  $a := a, b := a, c := b$  in identity (3.2.1):

$$2(aa)b + 2b(aa) + (ba)a + a(ab) = 0 \quad (3.5.10)$$

Then we multiple an  $a$  at the right of the identity (3.5.9):

$$\begin{aligned}
&2((aa)b)a + 2(b(aa))a + ((ba)a)a + (a(ab))a = 0 \\
&\quad 2((aa)a)b + 2b((aa)a) + ((ba)a)a + a((aa)b) \\
&= 2((aa)a)b + 2b((aa)a) + ((ba)a)a + (a(aa))b
\end{aligned}$$

That means

$$\begin{aligned}
&= 2((aa)a)b - 2b(a(aa)) + ((ba)a)a - ((aa)a)b \quad (\text{by (3.5.3)}) \\
&= ((aa)a)b - 2b(a(aa)) + ((ba)a)a \\
&= ((aa)a)b - b(a(aa)) - b(a(aa)) + ((ba)a)a = 0 \\
&\quad ((aa)a)b - a(a(ab)) = ((ba)a)a - b(a(aa))
\end{aligned}$$

And multiple an  $a$  on the left of the identity (3.5.9):

$$2a((aa)b) + 2a(b(aa)) + a((ba)a) + a(a(ab)) = 0$$

By using (3.5.3)

$$2a((aa)b) + 2b(a(aa)) + b((aa)a) + a(a(ab))$$

$$\begin{aligned}
&= 2a((aa)b) + 2b(a(aa)) - b(a(aa)) + a(a(ab)) \\
&= 2(a(aa))b + b(a(aa)) + a(a(ab)) \\
&= -2((aa)a)b + b(a(aa)) + a(a(ab)) \\
&= -((aa)a)b + b(a(aa)) - ((aa)a)b + a(a(ab)) = 0 \\
&((aa)a)b - a(a(ab)) = -((aa)a)b + b(a(aa))
\end{aligned}$$

From these two equality we see that the left side is the same, so we can write as follows:

$$\begin{aligned}
((ba)a)a - b(a(aa)) &= ((aa)a)b - a(a(ab)) \\
((ba)a)a + a(a(ab)) &= ((aa)a)b + b(a(aa))
\end{aligned}$$

$$((ba)a)a = -a(a(ab)) \quad (\text{by (3.5.6)})$$

So we get new identity:

$$((ba)a)a = -a(a(ab)) \quad (3.5.11)$$

Let  $a := a, b := (aa), c := b$  in the identity (3.2.1):

$$\begin{aligned}
(a(aa))b + ((aa)a)b + (ba)(aa) + b((aa)a) + b(a(aa)) + (aa)(ab) &= 0 \\
(a(aa) + (aa)a)b + b((aa)a + a(aa)) + (ba)(aa) + (aa)(ab) &= 0
\end{aligned}$$

By using (3.5.3)

$$\begin{aligned}
(ba)(aa) + (aa)(ab) &= 0 \\
a((ba)a) + (a(ab))a &= 0 \\
b((aa)a) + (a(aa))b &= 0 \\
-b(a(aa)) - ((aa)a)b &= 0
\end{aligned}$$

$$b(a(aa)) = -((aa)a)b \quad (3.5.12)$$

Let  $a := a, b := a, c := (bb)$  in the identity (3.2.1):

$$\begin{aligned}
(aa)(bb) + (aa)(bb) + ((bb)a)a + (bb)(aa) + (bb)(aa) + a(a(bb)) &= 0 \\
b((aa)b) + b((aa)b) + ((bb)a)a + a((bb)a) + a((bb)a) + a(a(bb)) & \\
= 2b((aa)b) + 2a((bb)a) + ((bb)a)a + a(a(bb)) & \\
= 4a((bb)a) + ((bb)a)a + a(a(bb)) &= 0
\end{aligned}$$

$$4a((bb)a) + ((bb)a)a + a(a(bb)) = 0 \quad (3.5.13)$$

Let  $a := a, b := b, c := (cd)$  in the identity (3.2.1):

$$\begin{aligned} (ab)(cd) + (ba)(cd) + ((cd)a)b + (cd)(ba) + (cb)(ab) + b(a(cd)) &= 0 \\ c((ab)d) + c((ba)d) + ((cd)a)b + b((cd)a) + a((cd)b) + b(a(cd)) &= 0 \\ 2c((ab)d) + 2c((ba)d) + ((cd)a)b + b(a(cd)) &= 0 \end{aligned}$$

Then we multiple a  $b$  to the right of identity (3.2.1):

$$\begin{aligned} ((ab)c)d + ((ba)c)d + ((ca)b)d + (c(ba))d + (c(ab))d + (b(ac))d &= 0 \\ ((ab)c)d + ((ba)c)d + ((ca)b)d + c((ba)d) + c((ab)d) + b((ac)d) &= 0 \end{aligned}$$

and to the left:

$$d((ab)c) + d((ba)c) + d((ca)b) + d(c(ba)) + d(c(ab)) + d(b(ac)) = 0$$

$$\begin{cases} -((da)b)c + c((ba)d) + c((ab)d) + b((ac)d) = 0 \\ -c(b(ad)) + d((ab)c) + d((ba)c) + d((ca)b) = 0 \end{cases}$$

So we obtain new identity:

$$-((da)b)c = c(b(ad)) \quad (3.5.14)$$

In the identity (3.5.12) by identity (3.5.13) we can take new identity:

$$b((aa)b) = 0 \quad (3.5.15)$$

We multiple a  $b$  to the right of identity (3.5.10):

$$\begin{aligned} ((aa)b)b + ((aa)b)b + ((ba)a)b + (b(aa))b + (b(aa))b + (a(ab))b &= 0 \\ 2((aa)b)b + ((ba)a)b + 2(b(aa))b + (a(ab))b &= 0 \end{aligned}$$

and we multiple a  $b$  to the left of the identity (3.5.10):

$$\begin{aligned} b((aa)b) + b((aa)b) + b((ba)a) + b(b(aa)) + b(b(aa)) + b(a(ab)) &= 0 \\ 2b((aa)b) + 2b(b(aa)) + b((ba)a) + b(a(ab)) &= 0 \end{aligned}$$

Then we add this two 2 identities

$$2((aa)b)b + ((ba)a)b + 2(b(aa))b + (a(ab))b$$

$$\begin{aligned}
& 2b((aa)b) + 2b(b(aa)) + b((ba)a) + b(a(ab)) \\
&= 4b((aa)b) + ((bb)a)a + a(a(bb)) + 2((aa)b)b \\
&\quad + 2b(b(aa)) + a((ab)b) + b((ba)a) = 0
\end{aligned}$$

By identity (3.5.13), we obtain following identity:

$$2((aa)b)b + 2b(b(aa)) + a((ab)b) + b((ba)a) = 0$$

By identity (3.5.14), we get the next identity :

$$a((ab)b) + b((ba)a) = 0 \tag{3.5.16}$$

When we change  $a := a, b := b, c := a$  into (3.2.1) and (3.5.1), and by subtracting this two identities we obtain:

$$(ab)a = -a(ab) \tag{3.5.17}$$

By multiply (3.5.17) from the right and left side, we get:

$$((ab)a)a = -(a(ab))a \tag{3.5.18}$$

$$a((ab)a) = -a(a(ab)) \tag{3.5.19}$$

1. In the first type we have two diagrams:

$$((**))* = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}$$

1.1  $((aa)a)a, \begin{array}{|c|c|c|c|} \hline a & a & a & a \\ \hline \end{array}$

$$\begin{aligned}
a((aa)a) + a(a(aa)) &= 0 && \text{(by (3.5.6))} \\
(a(aa))a + a(a(aa)) &= 0 && \text{(by (2.0.2))} \\
- ((aa)a)a + a(a(aa)) &= 0 && \text{(by (3.5.3))} \\
- ((aa)a)a - ((aa)a)a &= 0 && \text{(by (3.5.8))} \\
- 2((aa)a)a &= 0 \\
((aa)a)a &= 0.
\end{aligned}$$

$$1.2 \quad ((ab)a)a - ((ba)a)a, \begin{array}{|c|c|c|} \hline a & a & a \\ \hline b & & \\ \hline \end{array}$$

$$\begin{aligned} & ((ab)a)a - ((ba)a)a = \\ & = -b(a(aa)) - ((ba)a)a && \text{(by (3.5.12))} \\ & = -b(a(aa)) - (a(ba))a && \text{(by (3.5.17))} \\ & = -b(a(aa)) - b((aa)a) && \text{(by (2.0.1))} \\ & = -b[a(aa) + (aa)a] && \text{(by (3.5.3))} \\ & = 0 \end{aligned}$$

2. In the second type we have three diagrams:

$$*((**))* = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

$$2.1 \quad a((aa)a), \begin{array}{|c|c|c|c|} \hline a & a & a & a \\ \hline \end{array}$$

$$\begin{aligned} a((aa)a) &= -a(a(aa)) \\ a((aa)a) &= 0. \end{aligned}$$

$$2.2 \quad a((ab)a) - a((ba)a), \begin{array}{|c|c|c|} \hline a & a & a \\ \hline b & & \\ \hline \end{array}$$

$$\begin{aligned} & a((ab)a) - a((ba)a) \\ & = a((aa)b) - a((ba)a) && \text{(by (2.0.1))} \\ & = (a(aa))b - b((aa)a) && \text{(by (2.0.2))} \\ & = -((aa)a)b + b(a(aa)) && \text{(by (3.5.3))} \\ & = 0 \end{aligned}$$

$$2.3 \quad (ab - ba)(ab - ba), \begin{array}{|c|c|} \hline a & a \\ \hline b & b \\ \hline \end{array}$$

$$\begin{aligned} & (ab - ba)(ab - ba) \\ & = (ab)(ab) - (ab)(ba) - (ba)(ab) + (ba)(ba) \\ & = a((ab)b) - b((ab)a) - a((ba)b) + b((ba)a) && \text{(by (2.0.2))} \\ & = a((ab)b) - 2b((aa)b) + b((ba)a) && \text{(by (2.0.1))} \\ & = a((ab)b) + b((ba)a) && \text{(by (3.5.15))} \\ & = 0 && \text{(by (3.5.16))} \end{aligned}$$

3. In the third type we have two diagrams:

$$*(*(**)) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}$$

$$3.1 \quad a(a(aa)), \begin{array}{|c|c|c|c|} \hline a & a & a & a \\ \hline \end{array}$$

$$((aa)a)a = -a(a(aa)) \quad (\text{by (3.5.8)})$$

$$a(a(aa)) = 0.$$

$$3.2 \quad a(a(ab)) - a(a(ba)), \begin{array}{|c|c|c|} \hline a & a & a \\ \hline b & & \\ \hline \end{array}$$

$$\begin{aligned} & a(a(ab)) - a(a(ba)) \\ &= a(a(ab)) - b(a(aa)) && (\text{by (2.0.2)}) \\ &= -((ba)a)a - b(a(aa)) && (\text{by (3.5.11)}) \\ &= -((ba)a)a + ((aa)a)b && (\text{by (3.5.12)}) \\ &= 0 \end{aligned}$$

By calculating we obtain that in case  $n = 4$  there is no basis element.

□

In other cases it is trivial.

Dimension of  $P_n(\mathfrak{M})$  is equal to 0, for  $n \geq 4$ .

$n$	1	2	3	4	5	...	$n$	...
$\dim(P_n(\mathfrak{M}))$	1	2	0	0	0	...	0	...

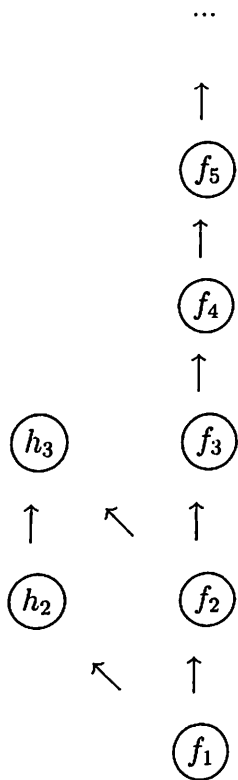
**Theorem 3.5.2.** *Let a linearization of  $f$  generates an irreducible  $S_n$ -submodule of  $P_n(\mathfrak{M})$ . Then the consequences of higher degrees from the  $f$  are equivalent to the following identities*

(a)  $f_{n+1}$  if  $f = f_n$ ,  $n \geq 1$ ;

(b)  $h_{n+1}$  if  $f = f_n$ ,  $n = 2$ ;

(c)  $h_{n+1}$  if  $h = h_n$ ,  $n = 1, 2$ .

To illustrate this theorem we can use this form::



*Proof.* Take note that  $f_1 = f'_1 = h_1$  and  $f_2 = f'_2$ .

(a) For  $n = 1$ :  $f_1 = z = 0 \rightarrow f_2 = zz = 0 \cdot z = 0$ .

For  $n = 2$ :  $f_2 = zz = 0 \rightarrow f_3 = (zz)z = 0 \cdot z = 0$ .

For  $n = m$

$$f_m = \underbrace{\left( \dots \left( \underbrace{zz}_{m-2 \text{ times}} \right) z \right) \dots z}_{m-2 \text{ times}} z = 0 \rightarrow$$

$$f_{m+1} = \underbrace{\left( \dots \left( \underbrace{zz}_{m-1 \text{ times}} \right) z \right) \dots z}_{m-1 \text{ times}} z = f_m \cdot z = 0 \cdot z = 0.$$

(b) For  $n = 1$ :  $f_1 = z = 0 \rightarrow h_2 = zw - wz = 0 \cdot w - w \cdot 0 = 0$ .

For  $n = 2$ :  $f_2 = zz = 0 \rightarrow h_3 = (zw)z - (wz)z$ .

Through the process of substitution  $z := z + w$  into  $f_2$  we get:

$$zz = (z + w)(z + w) = zz + x + zw + wz + ww = 0$$

$ww=zz=0$  since  $f_2 = 0$ . This implies that  $zw + wz = 0$  and we get:

$$zw = -wz$$

$$\text{Then } h_3 = (zw)z - (wz)z = (zw)z + (zw)z = 2(zz)w = 2f_2 \cdot y = 0$$

(c) For  $n = 2$ :  $h_2 = zw - wz = 0 \rightarrow h_3 = (zw)z - (wz)z = (zw - wz)z = h_2 \cdot z = 0 \cdot z = 0$ .

□

# Chapter 4

## Conclusions

### 4.1 Conclusions

The major goal of this thesis is to classifying all subvarieties of the bicommutative algebras described by the given identities.

$$(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + a(bc) = 0,$$

$$\gamma[(ab)c - 2(ba)c + (ca)b] + \delta[c(ba) - 2c(ab) + b(ac)] = 0.$$

In this thesis work we used 2 methods of classifying:

- The methods of linear algebra;
- The methods of the representation theory of the symmetric groups.