

QUASI-ASSOCIATIVE ALGEBRAS

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ABSTRACT. We consider class of algebras $\mathfrak{A}_{ass}^{(q)}$ constructed by associative algebras under q -commutator $a \circ_q b = a \circ b + qb \circ a$, where $q \in K, q^2 \neq 0, 1$ and $p = \text{char } K \neq 2, 3$. We establish that this class forms variety and satisfies the identity $ass^{(q)} = 0$, where $ass^{(q)} = (q-1)^2(t_1, t_3, t_2) + q[t_3, [t_1, t_2]]$. If $q^2 - 4q + 1 \neq 0$ the variety of q -associative algebras is generated by this identity. If $q^2 - 4q + 1 = 0$ this identity is not enough to generate $\mathfrak{A}_{ass}^{(q)}$ and it should be supplemented by the Lie-Admissibility identity $[[t_1, t_2], t_3] + [[t_2, t_3], t_1] + [[t_3, t_1], t_2] = 0$. In exceptional case a variety generated by the identity $ass^{(q)} = 0$ is equivalent to the variety of alternative algebras.

Well known that any associative algebras under commutator became Lie and under anticommutator became Jordan. In our paper we study class of algebras generated by associative algebras endowed by q -commutator, where $q^2 \neq 0, 1$.

Let $A = (A, \circ)$ be an algebra with a multiplication \circ . All vector spaces are considered over a field K of characteristic $p \neq 2, 3$. Denote by \circ_q a q -commutator of the multiplication \circ ,

$$a \circ_q b = a \circ b + qb \circ a.$$

For example, $[a, b] = a \circ_{-1} b$ is a Lie commutator and $\{a, b\} = a \circ_1 b$ is an Jordan commutator. Let

$$ass(a, b, c) = (a, b, c) = a \circ (b \circ c) - (a \circ b) \circ c$$

be an associator. For $q \in K$ denote by $A^{(q)} = (A, \circ_q)$ an algebra with a vector space A and a multiplication \circ_q .

For a non-commutative non-associative polynomial $g = g(t_1, t_2, \dots)$ denote by $F_q g$ a polynomial obtained from g using multiplication $t_i \cdot_q t_j = t_i t_j + q t_j t_i$. For example,

$$F_q(t_1(t_2 t_3)) = t_1(t_2 t_3) + q t_1(t_3 t_2) + q(t_2 t_3)t_1 + q^2(t_3 t_2)t_1, \quad (1)$$

$$F_q((t_1 t_2)t_3) = (t_1 t_2)t_3 + q t_3(t_1 t_2) + q(t_2 t_1)t_3 + q^2 t_3(t_2 t_1) \quad (2)$$

and

$$F_q \text{ass}(t_1, t_2, t_3) = \text{ass}(t_1, t_2, t_3) + \tag{3}$$

$$q(\text{ass}(t_1, t_3, t_2) - \text{ass}(t_3, t_1, t_2) - \text{ass}(t_2, t_3, t_1) +$$

$$\text{ass}(t_2, t_1, t_3) + [[t_1, t_3]t_2]) - q^2 \text{ass}(t_3, t_2, t_1).$$

Similarly, for algebra $A = (A, \circ)$ denote by $F_q A$ the algebra (A, \circ_q) .

For $q \in K, q^2 \neq 0, 1$, define q -associativity polynomial $\text{ass}^{(q)}$ by

$$\text{ass}^{(q)}(t_1, t_2, t_3) = (q - 1)^2(t_1, t_3, t_2) + q[t_3, [t_1, t_2]].$$

Then

$$\text{ass}^{(0)} = \text{ass}.$$

Let lia be Lie-Admissibility polynomial

$$lia(t_1, t_2, t_3) = [[t_1, t_2], t_3] + [[t_2, t_3], t_1] + [[t_3, t_1], t_2].$$

Recall that here $(t_i, t_j, t_s) = t_i(t_j t_s) - (t_i t_j)t_s$ is an associator and $[t_i, t_j] = t_i t_j - t_j t_i$ is a commutator. An algebra A is called Lie-Admissible, if it satisfies the identity $lia = 0$.

For a non-commutative, non-associative polynomial with k -variables $f = f(t_1, \dots, t_k)$ say that $f = 0$ is identity on (A, \circ) if $f(a_1, \dots, a_k) = 0$ for any substitutions $t_1 = a_1, \dots, t_k = a_k \in A$. Here multiplications are done in terms of \circ . For polynomials f_1, f_2, \dots, f_r denote by $Var(f_1, f_2, \dots, f_r)$ a variety generated by identities $f_1 = 0, f_2 = 0, \dots, f_r = 0$. Let \mathfrak{Ass} be a category of associative algebras and $\mathfrak{Ass}^{(q)}$ be a category of algebras of a form (A, \circ_q) , where (A, \circ) is associative algebra. Well known, that $\mathfrak{Ass} = Var(\text{ass})$ is a variety and $\mathfrak{Ass}^{(-1)}$ is a variety generated by anti-commutativity and Jacobi polynomials (Lie algebras). As far as $\mathfrak{Ass}^{(1)}$ (special Jordan algebras) it is not a variety,

$$\mathfrak{Ass}^{(1)} \subset Var(\text{acom}, \text{jor}), \quad \mathfrak{Ass}^{(1)} \neq Var(\text{acom}, \text{jor}),$$

where

$$\text{acom} = t_1 t_2 - t_2 t_1, \quad \text{jor} = (t_1^2, t_2, t_1).$$

Details see for example, [4]. Let

$$\text{lalt} = (t_1, t_2, t_3) - (t_1, t_3, t_1),$$

$$\text{ralt} = (t_1, t_2, t_3) - (t_2, t_1, t_3),$$

are left and right-alternative polynomials and $\mathfrak{Alt} = Var(\text{lalt}, \text{ralt})$ is a variety of alternative algebras.

In our paper we study a category $\mathfrak{Ass}^{(q)}$ for $q^2 \neq 0, 1$. Recall that q -algebra (A, \circ_q) of associative algebra (A, \circ) is called quasi-associative. We call it q -associative. Recall also that flexible algebra with Jordan identity $(t_1^2, t_2, t_1) = 0$ is called non-commutative Jordan. Connection

between quasi-associative algebras, associative algebras and noncommutative Jordan algebras was studied in [1],[2],[3].

Let \mathfrak{L}° and \mathfrak{L}^* are categories of algebras. Objects of \mathfrak{L}° are algebras (A, \circ) with multiplication \circ and objects of \mathfrak{L}^* are algebras (A, \star) with multiplication \star . Morphisms between objects are usual homomorphisms between corresponding algebras. Recall that a map between categories

$$F : \mathfrak{L}^\circ \rightarrow \mathfrak{L}^*, \quad F(A, \circ) = (A, \star)$$

is called morphism of categories if any morphism $\alpha \in Mor((A, \circ), (B, \circ))$ in category \mathfrak{L}° pass to a morphism in category \mathfrak{L}^* . In our case this means that, if $\alpha : (A, \circ) \rightarrow (B, \circ)$ homomorphism of algebras, then

$$\alpha : (A, \star) \rightarrow (B, \star)$$

is also a homomorphism of algebras. In this sense the map

$$F_q : \mathfrak{L}^\circ \rightarrow \mathfrak{L}^{\circ q}$$

is a morphism of categories.

Say that F is an *equivalence* of categories if there exists a morphism

$$G : \mathfrak{L}^* \rightarrow \mathfrak{L}^\circ$$

such that compositions

$$GF : \mathfrak{L}^\circ \rightarrow \mathfrak{L}^\circ, \quad FG : \mathfrak{L}^* \rightarrow \mathfrak{L}^*,$$

are identity morphisms,

$$GF(A, \circ) = (A, \circ), \quad FG(A, \star) = (A, \star).$$

Theorem 1. *Class of q -associative algebras $\mathfrak{Ass}^{(q)}$ forms a variety if $q^2 \neq 0, 1$. Namely,*

$$\mathfrak{Ass}^{(q)} = Var(ass^{(q)}) \text{ if } q^2 - 4q + 1 \neq 0,$$

$$\mathfrak{Ass}^{(q)} = Var(ass^{(q)}, lia) \text{ if } q^2 - 4q + 1 = 0.$$

Moreover, the following categories (varieties) are equivalent

$$Var(ass^{(q)}) \sim \mathfrak{Ass}, \text{ if } q^2 - 4q + 1 \neq 0,$$

$$Var(ass^{(q)}, lia) \sim \mathfrak{Ass}, \text{ if } q^2 - 4q + 1 = 0,$$

$$Var(ass^{(q)}) \sim \mathfrak{Alt}, \text{ if } q^2 - 4q + 1 = 0.$$

Equivalence morphisms are given by the map F_{-q} :

$$F_{-q} : Var(ass^{(q)}) \rightarrow Var(ass), \text{ if } q^2 - 4q + 1 \neq 0,$$

$$F_{-q} : Var(ass^{(q)}) \rightarrow Var(ralt, lalt), \text{ if } q^2 - 4q + 1 = 0,$$

$$F_{-q} : Var(ass^{(q)}, lia) \rightarrow Var(ass), \text{ if } q^2 - 4q + 1 = 0.$$

Inverse morphisms are given by the map $(1 - q^2)^{-2} F_q$.

Notice that in exceptional case $q^2 - 4q + 1 = 0$, one can present

$$ass^{(q)} = q ass_0^{(q)},$$

where

$$ass_0^{(q)} = 2(t_1, t_3, t_2) + [t_3, [t_1, t_2]].$$

Corollary 2. *The following identities are consequences of the identity $ass^{(q)} = 0$:*

$$elast(t_1, t_2, t_3) = (t_1, t_2, t_3) + (t_3, t_2, t_1),$$

$$assq_1(t_1, t_2, t_3, t_4) = (t_1 t_2)(t_3 t_4) + (t_1 t_3)(t_4 t_2) + (t_1 t_4)(t_2 t_3) - (t_1(t_3 t_4))t_2 - (t_1(t_4 t_2))t_3 - (t_1(t_2 t_3))t_4,$$

$$assq_2(t_1, t_2, t_3, t_4) = (t_1 t_2)(t_3 t_4) + (t_2 t_3)(t_1 t_4) + (t_3 t_1)(t_2 t_4) - t_3((t_1 t_2)t_4) - t_1((t_2 t_3)t_4) - t_2((t_3 t_1)t_4),$$

$$assq_3(t_1, t_2, t_3, t_4) = t_1([t_2, t_4]t_3) - (t_3 t_1)[t_2, t_4] + [t_3, (t_1 t_2)t_4 - (t_1 t_4)t_2].$$

In particular, any q -associative algebra is non-commutative Jordan.

One can show that identities $elast = 0, assq_1 = 0, assq_2 = 0, assq_3 = 0$ are independent.

Lemma 3. *If $g = g(t_1, \dots, t_l)$ is a homogeneous non-commutative, non-associative polynomial of degree k , then*

$$F_{-q} F_q g (= F_q F_{-q} g) = (1 - q^2)^{k-1} g.$$

Proof. There exist $\binom{2(n-1)}{n-1}$ bracketing types on n letters. For example, there exist two bracketing types on three letters: $t_1(t_2 t_3)$ and $(t_1 t_2)t_3$.

It is enough to prove that

$$F_q F_{-q} \sigma = (1 - q^2)^{k-1} \sigma.$$

for any bracketing type σ on k letters.

We use induction on k .

In case $k = 2$ we have only one bracketing type: $t_1 t_2$. In this case our statement is true:

$$F_q(t_i t_{i_2}) = t_i t_{i_2} + q t_{i_2} t_i,$$

and

$$F_{-q} F_q(t_i t_{i_2})$$

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$$\begin{aligned}
 &= t_{i_1} t_{i_2} - q t_{i_2} t_{i_1} + q t_{i_2} \cdot t_{i_1} - q^2 t_{i_1} t_{i_2} \\
 &= (1 - q^2) t_{i_1} t_{i_2} = (1 - q^2) (t_{i_1} t_{i_2}).
 \end{aligned}$$

Suppose that for $k - 1$ our statement is true. Let σ be one of bracketing types on k letters. It can be presented in a form

$$\sigma(t_{i_1}, \dots, t_{i_k}) = \sigma'(t_{i_1}, \dots, t_{i_{k'}}) \sigma''(t_{i_{k'+1}}, \dots, t_{i_k})$$

for some bracketing types σ' and σ'' on k' and $k'' = k - k'$ letters, $1 \leq k' \leq k$.

Then

$$F_q \sigma(t_{i_1}, \dots, t_{i_k})$$

$$\begin{aligned}
 &= F_q \sigma'(t_{i_1}, \dots, t_{i_{k'}}) F_q \sigma''(t_{i_{k'+1}}, \dots, t_{i_k}) \\
 &+ q F_q \sigma''(t_{i_{k'+1}}, \dots, t_{i_k}) F_q \sigma'(t_{i_1}, \dots, t_{i_{k'}})
 \end{aligned}$$

and

$$F_{-q} F_q \sigma(t_{i_1}, \dots, t_{i_k})$$

$$\begin{aligned}
 &= F_{-q} F_q \sigma'(t_{i_1}, \dots, t_{i_{k'}}) F_{-q} F_q \sigma''(t_{i_{k'+1}}, \dots, t_{i_k}) \\
 &- q F_{-q} F_q \sigma''(t_{i_{k'+1}}, \dots, t_{i_k}) F_{-q} F_q \sigma'(t_{i_1}, \dots, t_{i_{k'}}) \\
 &+ q F_{-q} F_q \sigma''(t_{i_{k'+1}}, \dots, t_{i_k}) F_{-q} F_q \sigma'(t_{i_1}, \dots, t_{i_{k'}}) \\
 &- q^2 F_{-q} F_q \sigma'(t_{i_1}, \dots, t_{i_{k'}}) F_{-q} F_q \sigma''(t_{i_{k'+1}}, \dots, t_{i_k})
 \end{aligned}$$

$$\begin{aligned}
 &= F_{-q} F_q \sigma'(t_{i_1}, \dots, t_{i_{k'}}) F_{-q} F_q \sigma''(t_{i_{k'+1}}, \dots, t_{i_k}) \\
 &- q^2 F_{-q} F_q \sigma'(t_{i_1}, \dots, t_{i_{k'}}) F_{-q} F_q \sigma''(t_{i_{k'+1}}, \dots, t_{i_k})
 \end{aligned}$$

By inductive suggestion

$$\begin{aligned}
 &F_{-q} F_q \sigma'(t_{i_1}, \dots, t_{i_{k'}}) \\
 &= (1 - q^2)^{k'-1} \sigma'(t_{i_1}, \dots, t_{i_{k'}}),
 \end{aligned}$$

$$\begin{aligned}
 &F_{-q} F_q \sigma''(t_{i_{k'+1}}, \dots, t_{i_k}) \\
 &= (1 - q^2)^{k-k'-1} \sigma''(t_{i_{k'+1}}, \dots, t_{i_k}).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 &F_{-q} F_q \sigma'(t_{i_1}, \dots, t_{i_{k'}}) F_{-q} F_q \sigma''(t_{i_{k'+1}}, \dots, t_{i_k}) \\
 &= (1 - q^2)^{k-2} \sigma(t_{i_1}, \dots, t_{i_k}),
 \end{aligned}$$

$$\begin{aligned}
 &- q^2 F_{-q} F_q \sigma'(t_{i_1}, \dots, t_{i_{k'}}) F_{-q} F_q \sigma''(t_{i_{k'+1}}, \dots, t_{i_k}) \\
 &= -q^2 (1 - q^2)^{k-2} \sigma(t_{i_1}, \dots, t_{i_k})
 \end{aligned}$$

and

$$\begin{aligned} & F_{-q}F_q\sigma(t_{i_1}, \dots, t_{i_k}) \\ &= F_{-q}F_q\sigma'(t_{i_1}, \dots, t_{i_{k'}})F_{-q}F_q\sigma''(t_{i_{k'+1}}, \dots, t_{i_k}) \\ &-q^2F_{-q}F_q\sigma'(t_{i_1}, \dots, t_{i_{k'}})F_{-q}F_q\sigma''(t_{i_{k'+1}}, \dots, t_{i_k}) \\ &= (1 - q^2)^{k-1}\sigma(t_{i_1}, \dots, t_{i_k}). \end{aligned}$$

Lemma is proved.

Lemma 4. *If $q^2 - 4q + 1 \neq 0$, then the identity $lia = 0$ follows from the identity $ass^{(q)} = 0$.*

If $q^2 - 4q + 1 = 0$ and $p \neq 2, 3$, then the identities $ass^{(q)} = 0$ and $lia = 0$ are independent.

Proof. Takes place the following relation

$$\begin{aligned} & ass^{(q)}(t_1, t_2, t_3) + ass^{(q)}(t_2, t_3, t_1) + ass^{(q)}(t_3, t_1, t_2) - \\ & ass^{(q)}(t_1, t_3, t_2) - ass^{(q)}(t_2, t_1, t_3) - ass^{(q)}(t_3, t_2, t_1) = \\ & (q^2 - 4q + 1)lia(t_1, t_2, t_3). \end{aligned}$$

Therefore,

$$ass^{(q)} = 0, q^2 - 4q + 1 \neq 0, \Rightarrow lia = 0.$$

Let \mathbf{Ca} be a 8-dimensional Cayley-Dickson algebra corresponding to Cayley octonians (algebra $\mathbf{C}(-4/5, -1, -1)$ in notations of [4], chapter 2, p.48, exercise 3). Recall that \mathbf{Ca} is alternative algebra. Take base e_0, e_1, \dots, e_7 on \mathbf{Ca} with multiplication table as it given in [4], p.48, exercise 3.

One checks that three-linear maps lia and $ass^{(q)}$ on \mathbf{Ca} are skew-symmetric. Moreover,

$$(q - 1)^2(q^2 - 4q + 1)lia(e_i, e_j, e_s) - 6ass^{(q)}(e_i, e_j, e_s) = 0.$$

It is easy to check that

$$lia(e_1, e_2, e_4) = 12e_7,$$

$$ass^{(q)}(e_1, e_2, e_4) = 2(q - 1)^2(q^2 - 4q + 1)e_7.$$

Thus, octonians algebra is not Lie-admissible and satisfies the identity $ass^{(q)} = 0$ if $q^2 - 4q + 1 = 0$ and $p \neq 2, 3$. This means that identities $ass^{(q)} = 0$ and $lia = 0$ are independent if $q^2 - 4q + 1 = 0$.

Lemma 5. *Let $q^2 - 4q + 1 = 0$. Then*

$$\begin{aligned} & 2F_{-q}ralt(t_1, t_2, t_3) = \\ & (1 - q)(ass^{(q)}(t_1, t_2, t_3) + ass^{(q)}(t_1, t_3, t_2)) \\ & + (1 - 3q)(ass^{(q)}(t_2, t_1, t_3) + ass^{(q)}(t_3, t_1, t_2)), \end{aligned}$$

$$2F_{-q}lalt(t_1, t_2, t_3) = (1-3q)ass^{(q)}(t_1, t_2, t_3) + 2qass^{(q)}(t_1, t_3, t_2) + (-1+3q)ass^{(q)}(t_2, t_1, t_3) + 2(1-2q)ass^{(q)}(t_2, t_3, t_1) + 2(1-3q)ass^{(q)}(t_3, t_1, t_2)$$

and

$$ass^{(q)}(t_1t_2, t_3) = -2lalt(t_1, t_2, t_3) + 6qlalt(t_1, t_2, t_3) + 4qlalt(t_1, t_3, t_2) + 2ralt(t_1, t_2, t_3) - 4qralt(t_1, t_2, t_3) + 2ralt(t_2, t_1, t_3) - 8qralt(t_2, t_1, t_3) - 2qralt(t_3, t_1, t_2).$$

Proof. In the following calculations we use (1), (2) and (3). We have

$$ass^{(q)}(t_1, t_2, t_3) + ass^{(q)}(t_1, t_3, t_2) = t_1(t_2t_3) - 2qt_1(t_2t_3) + q^2t_1(t_2t_3) + t_1(t_3t_2) - 2qt_1(t_3t_2) + q^2t_1(t_3t_2) + qt_2(t_1t_3) - qt_2(t_3t_1) + qt_3(t_1t_2) - qt_3(t_2t_1) - (t_1t_2)t_3 + q(t_1t_2)t_3 - q^2(t_1t_2)t_3 - (t_1t_3)t_2 + q(t_1t_3)t_2 - q^2(t_1t_3)t_2 + q(t_2t_1)t_3 + q(t_3t_1)t_2,$$

$$ass^{(q)}(t_2, t_1, t_3) + ass^{(q)}(t_3, t_1, t_2) = -qt_2(t_1t_3) + t_2(t_3t_1) - qt_2(t_3t_1) + q^2t_2(t_3t_1) - qt_3(t_1t_2) + t_3(t_2t_1) - qt_3(t_2t_1) + q^2t_3(t_2t_1) + q(t_1t_2)t_3 + q(t_1t_3)t_2 - q(t_2t_1)t_3 - (t_2t_3)t_1 + 2q(t_2t_3)t_1 - q^2(t_2t_3)t_1 - q(t_3t_1)t_2 - (t_3t_2)t_1 + 2q(t_3t_2)t_1 - q^2(t_3t_2)t_1$$

and

$$F_{-q}ralt(t_1, t_2, t_3) = t_1(t_2t_3) - qt_1(t_2t_3) + t_1(t_3t_2) - qt_1(t_3t_2) + qt_2(t_1t_3) - q^2t_2(t_3t_1) + qt_3(t_1t_2) - q^2t_3(t_2t_1) - (t_1t_2)t_3 - (t_1t_3)t_2 + q(t_2t_1)t_3 - q(t_2t_3)t_1 + q^2(t_2t_3)t_1 + q(t_3t_1)t_2 - q(t_3t_2)t_1 + q^2(t_3t_2)t_1.$$

Thus,

$$(1-q)(ass^{(q)}(t_1, t_2, t_3) + ass^{(q)}(t_1, t_3, t_2)) + (1-3q)(ass^{(q)}(t_2, t_1, t_3) + ass^{(q)}(t_3, t_1, t_2)) - 2F_{-q}ralt(t_1, t_2, t_3) = 2(1-4q+q^2)(ass(t_2, t_3, t_1) + ass(t_3, t_2, t_1)).$$

Similarly,

$$(1-3q)ass^{(q)}(t_1, t_2, t_3) + 2qass^{(q)}(t_1, t_3, t_2) + (-1+3q)ass^{(q)}(t_2, t_1, t_3)$$

$$+2(1-2q)ass^{(q)}(t_2, t_3, t_1)+2(1-3q)ass^{(q)}(t_3, t_1, t_2)-2F_{-q}lalt(t_1, t_2, t_3) = 2(1-4q+q^2)(ass(t_3, t_1, t_2) + ass(t_3, t_2, t_1))$$

and

$$-2lalt(t_1, t_2, t_3) + 6qlalt(t_1, t_2, t_3) + 4qlalt(t_1, t_3, t_2) + 2ralt(t_1, t_2, t_3) - 4qralt(t_1, t_2, t_3) + 2ralt(t_2, t_1, t_3) - 8qralt(t_2, t_1, t_3) - 2qralt(t_3, t_1, t_2) - F_q ass(t_1, t_2, t_3) =$$

$$(1-4q+q^2)(2ass(t_2, t_3, t_1) + [[t_1, t_2], t_3]).$$

Lemma is proved.

We omit calculations of the following lemmas. They are similar to calculations given above.

Lemma 6.

$$F_q ass^{(q)}(t_1, t_2, t_3) = (q-1)^2(qass(t_1, t_2, t_3) + ass(t_1, t_3, t_2)) - q(q-1)^2(ass(t_2, t_1, t_3) + qass(t_2, t_3, t_1)) + q(q-1)^2(ass(t_3, t_1, t_2) - ass(t_3, t_2, t_1)).$$

Lemma 7.

$$(q-1)(q^2-4q+1)F_{-q}ass(t_1, t_2, t_3) = (3q-1)ass^{(q)}(t_1, t_3, t_2) - q(q-1)ass^{(q)}(t_1, t_2, t_3) + q(q-1)ass^{(q)}(t_2, t_1, t_3) + q(1-q)ass^{(q)}(t_2, t_3, t_1) + 3q^2ass^{(q)}(t_3, t_1, t_2) - q^3ass^{(q)}(t_3, t_1, t_2) - qass^{(q)}(t_3, t_2, t_1) + q^2ass^{(q)}(t_3, t_2, t_1).$$

Lemma 8. Let $q^2 - 4q + 1 = 0$. Then

$$2qF_{-q}ass(t_1, t_2, t_3) = (1-3q)ass^{(q)}(t_3, t_1, t_2) + (1-q)ass^{(q)}(t_1, t_3, t_2) - 2q^2lia(t_1, t_2, t_3).$$

Proof of Theorem 1. By Lemma 4, $lia = 0$ is a consequence of identity $ass^{(q)} = 0$, if $q^2 - 4q + 1 \neq 0$. In other words,

$$\mathfrak{Ass}^{(q)} = \mathfrak{Ass}^{(q, lia)} \text{ if } q^2 - 4q + 1 \neq 0.$$

By lemma 6, if (A, \circ) associative algebra, then the algebra (A, \circ_q) satisfies the identity $ass^{(q)} = 0$.

Let $q^2 - 4q + 1 \neq 0$. By Lemma 7, if an algebra (A, \circ) satisfies the identity $ass^{(q)} = 0$, then the algebra (A, \circ_{-q}) satisfies the identity $ass = 0$.

Now consider the case $q^2 - 4q + 1 = 0$. By Lemma 8, if (A, \circ) satisfies the identities $ass^{(q)} = 0$ and $lia = 0$, then (A, \circ_{-q}) satisfies the

identity $ass = 0$. By Lemma 5, if (A, \circ) satisfies the identity $ass^{(q)} = 0$, then (A, \circ_{-q}) is alternative and, inversly, if (A, \circ) is alternative then (A, \circ_q) satisfies the identity $ass^{(q)} = 0$.

So, if (A, \circ) is associative algebra, then (A, \circ_q) satisfies the identities $ass^{(q)} = 0, lia = 0$ and vice-versa, if (A, \star) satisfies the identities $ass^{(q)} = 0, lia = 0$ then (A, \star_{-q}) is associative. Thus, by Lemma 3 the functor

$$F_q : \mathfrak{Ass} \rightarrow \mathfrak{Ass}^{(q)}, \quad (A, \circ) \rightarrow (A, \circ_q)$$

is well defined and has inverse

$$(1 - q^2)^{-2} F_{-q} : \mathfrak{Ass}^{(q)} \rightarrow \mathfrak{Ass},$$

if $q^2 - 4q + 1 \neq 0$. Similarly, the functors

$$F_q : \mathfrak{Ass} \rightarrow \mathfrak{Ass}^{(q, lia)}, \quad (A, \circ) \rightarrow (A, \circ_q),$$

$$(1 - q^2)^{-2} F_{-q} : \mathfrak{Ass}^{(q, lia)} \rightarrow \mathfrak{Ass}, \quad (1 - q^2)^{-2} F_{-q} : \mathfrak{Ass}^{(q)} \rightarrow \mathfrak{Alt},$$

are also well-defined and

$$(1 - q^2)^{-2} F_{-q} F_q = id,$$

if $q^2 - 4q + 1 = 0$.

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Резюме.

Рассматриваются классы алгебр полученных из ассоциативных с помощью q -коммутаторов. Установлены, что такие алгебры бывают четырех типов:

- 1) ассоциативные
- 2) Лиевы (в случае $q = -1$)
- 3) Иордановы (в случае $q = 1$)
- 4) альтернативные (в случае $q = 2 \pm \sqrt{3}$)

Түйін

Ассоциативті алгебралардан q -коммутатор арқылы шығатын алгебралар класы қарастырылады. Бұндай алгебралардың 4 түрі болатыны дәлелденген. Олар мынадай алгебралар:

- 1) ассоциативті
- 2) Ли ($q = -1$ болғанда)
- 3) Иордан ($q = 1$ болғанда),
- 4) альтернативті ($q = 2 \pm \sqrt{3}$ болғанда)

Lie-algebras of Inertial Navigation

Алгебры Ли инерциальной навигации.

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Аннотация

Различные задачи инерциальной навигации приводят к вопросам изучения следующих групп преобразований: 1-параметрическая группа переносов во времени; 4-параметрическая группа, включающая переносы во времени и пространственные повороты; 10-параметрическая группа Галилея, включающая переносы во времени и в пространстве, пространственные повороты и преобразования Галилея (нерелятивистские бусты); 13-параметрическая группа включающая все преобразования 10-параметрической группы Галилея и преобразования, позволяющие описывать гравитационные ускорения; 15-параметрическая конформная группа, учитывающая релятивистские эффекты.

Согласно обозначениям, принятым в [2], I – инерциальная система отсчета, $E(t)$ – текущая инерциальная система отсчета, связанная с объектом. $\Lambda_{E(t)}$ – текущее преобразование, связывающее базовую систему отсчета с текущим положением объекта. Общее уравнение имеет вид:

$$\Lambda_{E(t+d\tau)} = \Lambda_{E(t)} \circ \Lambda_{E(t)E(t+d\tau)} \quad (1)$$

Для переноса по времени имеем следующее уравнение:

$$T_{t+dt} = T_t \circ T_{\delta t} \quad (2)$$

Уравнение, учитывающее инерциальную ориентацию:

$$T_{t+dt} \circ \Theta_{\delta t} = (T_t \circ \Theta_t) \circ (T_{\delta t} \circ \Theta_{\delta t}) \quad (3)$$

Инерциальная навигация без гравитации подчинено соотношению:

$$T_{t+dt} R_{t+dt} V_{v+dv} \circ \Theta_{\delta t} = (T_t R_t V_v \circ \Theta_t) \circ (T_{\delta t} V_{\delta v} \circ \Theta_{\delta t}) \quad (4)$$

Для нерелятивистской инерциальной навигации имеем тождество:

$$T_{t+dt} R_{t+dt} V_{v+dv} \circ \Theta_{\delta t} \circ G_{g+dg} = (T_t R_t V_v \circ \Theta_t \circ G_g) \circ (T_{\delta t} V_{\delta v} \circ \Theta_{\delta t} \circ G_{\delta g}) \quad (5)$$

Уравнение, учитывающее релятивистскую инерциальную навигацию:

$$T_{t+dt} R_{t+dt} V_{v+dv} \circ \check{A}_{\gamma+d\gamma} \circ \Theta_{\delta t} \circ G_{g+dg} W_{w+dw} = (T_t R_t V_t \circ \check{A}_\gamma \circ \Theta_t \circ G_g W_w) \circ (T_{\delta t} V_{\delta v} \circ \check{A}_{\delta\gamma} \circ \Theta_{\delta t} \circ G_{\delta g} W_{\delta w}) \quad (6)$$

Идея изучения навигации с помощью теории групп приводит к задаче изучения алгебр Ли, соответствующим алгебраическим группам. Очевидно, что данные группы преобразований являются алгебраическими.

Пусть G – алгебраическая группа и $A=K[G]$ – соответствующая групповая алгебра.

$\text{Der } A$ – алгебра Ли дифференцирований A относительно операции коммутирования.

Подпространство $L(G)=\{\delta \in \text{Der } A \mid \delta \lambda_x = \lambda_x \delta, \text{ где } \lambda_x f(y) = f(x^{-1}y)\}$ является алгеброй

Ли группы G . С точки зрения алгебр Ли, данные задачи навигации обладают следующими свойствами: вещественность (алгебры рассматриваются как векторные пространства над \mathbb{R}), объекты космической природы (планеты, кометы, астероиды и т. д.), объекты человеческого прогресса (спутники, ракеты и т. д.). Основная идея состоит не в изучении групп, а изучение соответствующим им алгебр Ли.

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