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SIERPINSKI TRIANGLE AND OPEN SET CONDITION

Abstract. In terms of the development of modern technology, the use of computer programs has become normal. Even in the field of mathematics, using computer programs, we have opened a new path to the development of science. In this article we used the iterated function systems, calculating the Box dimension of the Serpinski triangle and using the arguments of the Serpinski triangle functions, calculated the computer - software Bisection method and collected the analyses. We determine when it is possible to correctly use the methods to calculate the dimensions of fractal sets. Comparing the results collected in this article, we determine the effectiveness-inefficiency of both methods and incorrectly determined values of methods, and show them in the form of a table.

Key words:iterated function systems,open set condition,Sierpinski triangle, Box dimension, matlab, computer programming calculations.

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Андатпа. Қазіргі технологияның дамыған заманында компютерлік бағдарламаларды қолдану қалыпты жағдайға айналды. Тіпті математика саласында да есептеу бағдарламаларын қолдану арқылы ғылымның дамуына жаңа жол аштық. Біз аталмыш мақаладаитерациялық функционалдық жүйесін қолданып, компьютерлік Matlab бағдарламасымен Серпин үшбұрышының Box өлшемін есептедік және Серпин үшбұрышы функцияларының аргументтерін компютерлік - бағдарламалық Бисекция әдісімен есептеп, анализдер жинадық. Фракталды жиындардың өлшемін есептеуге арналған әдістерді қай кезде дұрыс қолдануға болатынын анықтаймыз.Біз бұл мақалада жиналған нәтижелерді салыстыра отырып екі әдістің де тиімді-тиімсіздігін, әрі әдістердің дұрыс анықталмайтын мәндерін анықтап оны кесте түрінде көрсетеміз.

Түйін сөздер: итерациялық функционалдық жүйелер, ашық жиын шарты, Серпин үшбұрышы, Минков өлшемі (Box өлшемі), матлаб, компьютерлік бағдарлама есептеулері.

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Аннотация. В условиях развития современной технологии применение компьютерных программ стало нормальным. Даже в области математики, используя вычислительные программы, мы открыли новый путь к развитию науки. Мы в данной статье использовали систему итерационные функций, рассчитав размерность Минковского(Box) треугольника Серпина и используя аргументы функций Серпинского треугольника, рассчитали компьютерного - программного методом Бисекция и собирали анализы. Определим, когда можно правильно использовать методы для расчета размеров фрактальных множеств. Сравнивая результаты, собранные в данной статье, мы определяем эффективность-неэффективность обоих методов и неправильно определяемые значения методов, и показываем их в виде таблицы.

Ключевые слова: итерационные функциональные системы, условия открытых множеств, Серпинский треугольник, меры Минковского (размерностьBox), матлав, вычисления компьютерного программирования.

Introduction

In geometry, fractals are very natural sets that have self-similarity. Certain fractals are more complex in nature and it is the important question to measure the complexity of these objects. One such option is by means of their box dimension.

In some cases it is possible to analytically compute exact Box dimension. One such general case is when the fractal is obtained as the fixed point of certain conformal Iterated Function Systems (IFSs) that satisfies the so called Open Set Condition (OSC). For all these notions are recalled in the next section.

However, in general analytic exact solution for box dimension problem is a very difficult task. In particular, for IFS if the OSC fails, then ethere is no guarantee that the formul will work.

In this paper, we want to consider a special family of fractals, called Sierpenski Traingles. For various parameters the fractal structore changes and so does the box dimension. Our goal is to consider various parameter values that determine the IFS for Sierpenski Triangle and try to estimate their box dimensions. On one hand, we use the formula from the literature that is valid when OSC holds and on the other hand we use computer software to numerically estimate the dimension. Then, we compare the two values to get an idea when the OSC seems to hold true.

Scientists such as A. A. Vinogradova, D. N. Kaliteevsky, talked about getting a dimension from the Sierpinski triangle. Other scientists were able to calculate it's dimension in an equilateral triangle only in a single position. After that "why not bring the sizes of different kinds of serpin triangle?" the question arose. So I studied all the other situations and found all the dimensions.

First of all, we describe the systems and functions that we use. Familiarize yourself with this system because the function was used through the Iterated function system. An iterated function system (IFS) is a set of abbreviations

$\{S_1, S_2, \ldots, S_m\}$, with *m*≥2, on a closed subset *D* of R^n. A nonempty compact subset *F* of *D*is an IFS attractor if

$$F = \bigcup_{i=1}^{m} S_i(F) \qquad (1)$$

Is a method of creating fractals this is an Iterative functional system in mathematics.Fractal-a mathematical set that has the properties of self-identification (object, exactly or approximately coincides with one part, it is a form as a whole one or more parts).

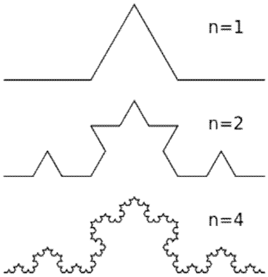

Image 1

The function we use is a function of the Sierpinski triangle. There are several ways to obtain the function of this Sierpinski triangle.For the first time such concept of a triangle was introduced in 1915 by the Polish mathematician Vaclav Sierpinski. To get it, you need to take an (equilateral) triangle from the inside, hold it along the middle line and throw out the Central of the four formed small triangles. Then the same actions should be repeated with each of the remaining three triangles, etc. The 2-figure shows the first four steps, and in the flash demonstration you can practice and take a step to an infinite step.

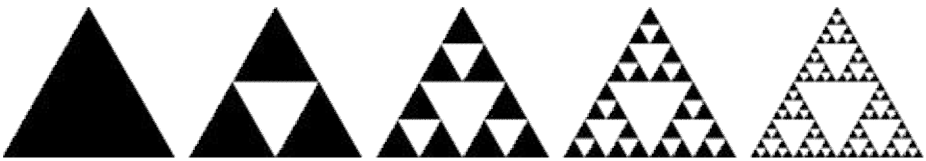
Image 2

A dimension is the distance in one direction, in this case, width. Fractal dimension is a fundamental concept of geometry. It is known that we usually know a straight line or curves is dimension of the 1- dimensional, surface is2-dimensional and figure in planimetriesis3- dimensional. The definition of measurement is mainly based on the idea of "measuring sets at scaleδ". We can measure every geometric pattern. For each δ, we measure the set in such a way that we detect inhomogeneities of the delta of size δ, and we see how these measurements behave like $\delta \to 0$.Thus, one of the methods of computation of dimension is the Box dimension.Kenneth Falconer[1] fully specified in his work the generalization and calculation of the formula.

When F is a limited subset of R^n, then Box dimension of F is defined as

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \qquad (2)$$

Here, $N(F)$ be the least number of sets of diameter at most δ which can cover F.Based on this formul, we will calculate the dimension of the Serpinski triangle.

One of the most important concepts in this article is the open set condition (OSC). The f_i are said to satisfy the OSC if there exists a nonempty open set $V \subset R^n$ such that

$$\bigcup_{i=1}^{m} f_i(V) \subseteq V \text{ and } f_i(V) \cap f_j(V) = \emptyset \text{ for } i \neq j. \qquad (3)$$

We can also calculate Hausdorff and box dimensions using the OSC for self-similar set F. For this it is necessary to use the theorem dimensions of self-similar sets in the scientific work[2] of Kenneth Falconer.As shown in the theorem,suppose S_i on R^n with radius $0<r_i<1$ for $1\leqslant i\leqslant m$ is satisfied for similarity the open set conditionand if there is

$$F = \bigcup_{i=1}^{m} S_i(F) \qquad (4)$$

so$F$ is the attractor of the IFS $\{S_1,\dots,S_m\}$, then the formulation $\dim_B F=s$ is correct.Where s is obtained by the equation

$$\sum_{i=1}^{m} r_i^s = 1 \qquad (5)$$

at $0 < H^s(F) < \infty$.The proof of this can be seen in this article.

1. Main part

Let's take its graphical image using the Sierpinski triangle function, using the version of Paulo Silva in Matlab For $a = \overline{2:N}$, N=10 000, $\alpha_i, \beta_i \epsilon (0;1)$, $i = \overline{1;3}$Sierpinski's triangle depends on the arguments $\alpha_i$andβ_i:

$$f_1(\alpha_1,\beta_1) = \begin{cases} x(a) = \alpha_1 * x * (a-1) \\ y(a) = \beta_1 * y * (a-1) \end{cases};$$

$$f_2(\alpha_2,\beta_2)=\begin{cases}x(a)=\alpha_2 * x * (a-1)+0.25\\ y(a)=\beta_2 * y * (a-1)+\frac{\sqrt{3}}{4}\end{cases};$$
$$f_3(\alpha_3,\beta_3)=\begin{cases}x(a)=\alpha_3 * x * (a-1)+0.5\\ y(a)=\beta_3 * y * (a-1)\end{cases}; \qquad (6)$$

Here α and β consider the arguments (0;1) in the interval. Moving on step +0.1 each you can extract the image of Matlab (729 pieces). These collected graphs can be used to study this function.Of these, choose a random three images (image 3-5):

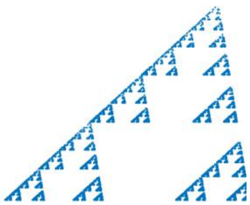

Image 3
$\alpha_1=\beta_1=0.4$
$\alpha_2=\beta_2=0.6$
$\alpha_3=\beta_3=0.3$

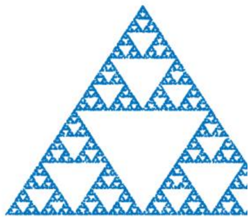

Image 4
$\alpha_1=\beta_1=0.5$
$\alpha_2=\beta_2=0.5$
$\alpha_3=\beta_3=0.5$

Image 5
$\alpha_1=\beta_1=0.7$
$\alpha_2=\beta_2=0.8$
$\alpha_3=\beta_3=0.3$

Find these images by the Box dimension method, i.e. with the formula (2) – the dimension of the Serpinski triangle. To do this, we calculate the dimension using the Matlab code[3] that was written by Frederic Moisy in this way, i.e. Box dimension. Thus, the dimension of the Serpinski triangle can be found in the Matlab with a very small error accuracy (for example: $\alpha_1=\beta_1=0.6$, $\alpha_2=\beta_2=0.4$, $\alpha_3=\beta_3=0.5 \Rightarrow \dim_B F=1.4534\pm 0.18967$).

If you are running open set condition on a set of function points, the following method for calculating dimension - the Bisection method.If for a set of points functions performed open set condition, then the following method to calculate dimension-the method of Bisection**.** In this method the arguments $\alpha_1=\beta_1\equiv\gamma_1$, $\alpha_2=\beta_2\equiv\gamma_2$, $\alpha_3=\beta_3\equiv\gamma_3$ to the (6) – function for

$$\gamma_1^{x}+\gamma_2^{x}+\gamma_3^{x}=1 \qquad (7)$$

the value x in the interval $0<x<2$ is the dimension in the given (6) - function argument and will be equal to the Box dimension. And in order to find the value of x, we use the Bisection method to solve equation (7).This method searches for a value by selecting from the given interval [a; b] to find the value of *x* in the Matlab.To solve the Brato Chakrabarti's Matlabalgorithm[4]using the

Bisection method, we introduce the function "f=@(x)-1+γ1^x+γ2^x+γ3^x" looking for *x* with the interval [a; b]=[0; 20]. Where*x* is the dimension of the Sierpinski triangle in $\gamma_1, \gamma_2, \gamma_3 \epsilon (0;1)$ values.

With the above two ways you can get the dimension of all the Sierpinski triangles. We consider the coefficients of the (6) function as $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$ identical.Compared to these initial results (table 1-3), we see that the dimensions occurring by the two methods have mutually incompatible values. Of course, the size of one graphic image can not be of two kinds. The results of the Box Dimension obtained by the method obtained using the figure are absolutely correct. However, some values obtained by Bisection method which were easy and efficient to calculate are false. Because, as we said above, it is necessary to fulfill the open set condition (OSC) for the correct execution of Formula (5). For a function(6), you can view the following tables, which don't always meet the conditions.

Summarizing this, we can analyze the dimensions of the Sierpinski triangle obtained by two methods.First of all, we analyze which methods are used correctly and which methods are effective.Let's group the results and highlight:

- **BOLD NUMBER** – the dimensions obtained by the Bisection method and the $\dim_B F$ ± approximation error obtained by the image are equal to each other,
- NORMAL NUMBER – the dimensions obtained by the Bisection method and the $\dim_B F$ ± approximation error obtained by the image are not equal to each other,
- *ITALIC NUMBER*–the values of the invalid dimension obtained through the Bisection method, which were greater than two.

It is known that the dimension of the Sierpinski triangle should not exceed two, since the image of the function that we study is in the plane (2D). The table shows the values of numbers obtained through the Bisection method.

$\alpha_i = \beta_i \equiv \gamma_i$, $i = \overline{1;3}$, $\gamma_i \epsilon (0;1)$

Table 1: $\gamma_1 = 0.1$, γ_2 –vertically, γ_3 –horizontally

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	**0.4771**	**0.5381**	**0.5931**	**0.6493**	**0.7110**	**0.7830**	**0.8734**	**1.0000**	1.2197
0.2	**0.5381**	**0.6071**	**0.6702**	**0.7353**	**0.8073**	**0.8922**	**1.0000**	1.1527	1.4223
0.3	**0.5931**	**0.6702**	**0.7418**	**0.8165**	**0.9002**	**1.0000**	**1.1285**	1.3139	1.6499
0.4	**0.6493**	**0.7353**	**0.8165**	**0.9024**	**1.0000**	**1.1181**	1.2728	1.5003	1.9239

0.5	**0.7110**	**0.8073**	**0.9002**	**1.0000**	**1.1151**	**1.2569**	1.4458	1.7295	*2.2699*
0.6	**0.7830**	**0.8922**	**1.0000**	**1.1181**	**1.2569**	**1.4309**	1.6672	*2.0288*	*2.7292*
0.7	**0.8734**	**1.0000**	**1.1285**	1.2728	1.4458	1.6672	1.9733	*2.4491*	*3.3818*
0.8	**1.0000**	**1.1527**	1.3139	1.5003	1.7295	*2.0288*	*2.4491*	*3.1098*	*4.4246*
0.9	**1.2197**	1.4223	1.6499	1.9239	*2.2699*	*2.7292*	*3.3818*	*4.4246*	*6.5788*

Table 2: $\gamma_1 = 0.5$, γ_2 –vertically, γ_3 –horizontally

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	**0.7110**	**0.8073**	**0.9002**	**1.0000**	**1.1151**	**1.2569**	1.4458	1.7295	*2.2699*
0.2	**0.8073**	**0.9051**	**1.0000**	**1.1019**	**1.2187**	**1.3612**	1.5489	1.8267	*2.3484*
0.3	**0.9002**	**1.0000**	**1.0984**	**1.2046**	**1.3264**	**1.4747**	1.6689	1.9536	*2.4792*
0.4	**1.0000**	**1.1019**	**1.2046**	**1.3166**	**1.4459**	**1.6035**	1.8100	*2.1118*	*2.6645*
0.5	**1.1151**	**1.2187**	**1.3264**	**1.4459**	**1.5850**	**1.7559**	1.9809	*2.3113*	*2.9178*
0.6	**1.2569**	**1.3612**	**1.4747**	**1.6035**	**1.7559**	1.9454	*2.1978*	*2.5727*	*3.2716*
0.7	1.4458	**1.5489**	1.6689	1.8100	1.9809	*2.1978*	*2.4921*	*2.9389*	*3.7987*
0.8	1.7295	1.8267	1.9536	*2.1118*	*2.3113*	*2.5727*	*2.9389*	*3.5162*	*4.6899*
0.9	*2.2699*	*2.3484*	*2.4792*	*2.6645*	*2.9178*	*3.2716*	*3.7987*	*4.6899*	*6.6723*

Table 3: $\gamma_1 = 0.9$, γ_2 –vertically, γ_3 –horizontally

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.2197	1.4223	1.6499	1.9239	*2.2699*	*2.7292*	*3.3818*	*4.4246*	*6.5788*
0.2	1.4223	1.5915	1.7903	*2.0342*	*2.3484*	*2.7763*	*3.4027*	*4.4298*	*6.5791*
0	1.	1.	1.	*2.*	*2.*	*2.*	*3.*	*4.*	*6.5*

.3	6499	7903	9674	*1899*	*4792*	*8763*	*4658*	*4558*	*822*
0.4	1.9239	*2.0342*	*2.1899*	*2.3943*	*2.6645*	*3.0369*	*3.5912*	*4.5308*	*6.6012*
0.5	*2.2699*	*2.3484*	*2.4792*	*2.6645*	*2.9178*	*3.2716*	*3.7987*	*4.6899*	*6.6723*
0.6	*2.7292*	*2.7763*	*2.8763*	*3.0369*	*3.2716*	*3.6098*	*4.1190*	*4.9768*	*6.8675*
0.7	*3.3818*	*3.4027*	*3.4658*	*3.5912*	*3.7987*	*4.1190*	*4.6177*	*5.4651*	*7.3067*
0.8	*4.4246*	*4.4298*	*4.4558*	*4.5308*	*4.6899*	*4.9768*	*5.4651*	*6.3321*	*8.2276*
0.9	*6.5788*	*6.5791*	*6.5791*	*6.6012*	*6.6723*	*6.8675*	*7.3067*	*8.2276*	*10.4272*

If we look at these results, we see that the number of incorrect results obtained by the Bisection method increases. Even in the last table we can see that there were no correct values.

2. *Conclusion*

As a result, based on the results of the table, in some cases (see bolded results in the table) we see that the calculation of the Bisection method is more profitable than the calculation via of the figure. In some values, you can see that the results of the Bisection calculations are incorrect (see normally and italic results in the table).As already mentioned above, the method[3] of determining the Box dimensions using imageof Sierpinski triangle, which was long and hard is absolutely correct. However, because the open set condition is sometimes not satisfied for the members of a set, the Bisection methodmay not always be efficient and correct.

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