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Identities in mutations of bicommutative algebras

THESIS

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Topic of the thesis:

Identities in mutations of bicommutative algebras

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Declaration

I affirm that this work is my own and that all material sourced from other references has been correctly and fully acknowledged.

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Dedication

This thesis is dedicated to my parents and many other for their support, help, sense of humour and useful comments for improving this project.

Abstract

An algebra with identities $a \cdot (b \cdot c) = b \cdot (a \cdot c)$, $(a \cdot b) \cdot c = (a \cdot c) \cdot b$ is called bicommutative. In this work, we study bicommutative algebras under mutation product and prove that any bicommutative algebra under mutation product satisfies a Lie-admissible identity, which follows from two independent identities of the third degree, and we obtain all identities of the fourth degree.

Аңдатпа

$a{\cdot}(b{\cdot}c) = b{\cdot}(a{\cdot}c),\ (a{\cdot}b){\cdot}c = (a{\cdot}c){\cdot}b$ тепе-теңдіктері бар алгебра бикоммутативті деп аталады. Біз бұл жұмыста, бикоммутативті алгебраны мутация көбейтіндісі бойынша зерттейміз және мутация көбейтіндісі бойынша кез келген бикоммутативті алгебра үшінші дәрежелі екі тәуелсіз тепе-теңдіктен туындайтын Ли-адмиссибл тепе-теңдігін қанағаттандыратынын дәлелдейміз және біз барлық төртінші дәрежелі тепе-теңдіктерді аламыз.

Аннотация

Алгебра с тождествами $a{\cdot}(b{\cdot}c) = b{\cdot}(a{\cdot}c)$, $(a{\cdot}b){\cdot}c = (a{\cdot}c){\cdot}b$ называется бикоммутативной. В этой работе мы изучаем бикоммутативные алгебры при произведении мутаций и доказываем, что любая бикоммутативная алгебра при произведении мутаций удовлетворяет допустимому тождеству Ли, которое следует из двух независимых тождеств третьей степени, и получаем все тождества четвертой степени.

Contents

1. Introduction

An algebra $(A, \cdot)$ is called bicommutative if any $a, b, c \in A$ are satisfied the following identities:

$$a \cdot (b \cdot c) = b \cdot (a \cdot c)$$

$$(a \cdot b) \cdot c = (a \cdot c) \cdot b.$$

[1],[2]. Bicommutative algebra is also called LR - (left and right) algebra [3]. The base of free bicommutative algebra was obtained in [1].

Let $\mathcal{B}$ be the free bicommutative algebra with generators a, b, p, q with multiplication $(a, b) \longmapsto ab$. We assume that all polynomials and vector fields are defined over the field of complex numbers $\mathbf{C}$, then we define the mutation product on $\mathcal{B}$ by

$$\langle a, b \rangle = (ap)b - b(qa).$$

Let $\mathcal{B}_{p,q}$ be an algebra consisting of a vector space $\mathcal{B}$ and mutation product $\langle \cdot, \cdot \rangle$.

Let $\mathcal{B}icom$ be a variety of bicommutative algebras. Let $\mathcal{B}icom_{p,q}$ for fixed $p, q \in \mathcal{B}$ be the class of mutation of bicommutative algebras. Put simply, the elements of $\mathcal{B}icom_{p,q}$ are algebras represented as $\mathcal{B}_{p,q}$, where $\mathcal{B} \in \mathcal{B}icom$.

There are publications in which the properties of mutation and bicommutative algebras have been studied. We present some of them:

Bicommutative algebras have been a topic of study in mathematics for many years. The concept of bicommutative algebras was first introduced as LR algebra by the mathematicians D.Burde, K.Dekimpe and S.Deschamps as a way to study algebraic structures that are bicommutative with respect to both multiplication and addition [3].

An important area of research in bicommutative algebras is the study of mutations, which are operations that transform an algebraic structure into a new one. Mutation algebras are an important for understanding the relationships between different types of algebras and for studying the properties of these structures.

Mutation algebras were first introduced by theoretical physicists around 1980; see [4, 5]. For a survey of early work by mathematicians on this topic, see [6]. For a detailed exposition of the structure theory of mutation algebras, see [7, 8, 9, 10].

The concepts of mutation algebras in nonassociative algebras is considered in [11]. It discusses the Lie-admissibility for mutation algebras, in particular, special attention is given to the conditions under which the algebra of mutations $A(p, q)$ is Lie-admissible.

In this paper authors investigate the properties of the nonassociative algebra $A(r, s)$, and the relation between $A(r, s)$ and A, where, $A(r, s)$ denote the elements of A under the new product $x * y = xry - ysx, x, y$ element of A [12].

Montaner examined identities of polynomials for mutation algebras for the first time in [13], using the classical techniques of nonassociative algebra [14]. And this Montaner's article devoted to the study of mutations of associative artinian (left or right) algebras.

In [15], the authors develop the necessary and sufficient conditions for the existence of a unit element in a mutation algebra.

In 1983, Myung published a paper where A is considered as an alternative algebra [16]. Then, in 1985, Gonzalez examined the mutation algebras $A(p, q)$, where p and q are elements of the nucleus of the algebra A [17]. This particular choice of p and q made it easier to obtain good results in the corresponding mutation algebras. One of Gonzalez's theorems states that, if p and q belong to the nucleus, and A is a nonassociative flexible Lie-admissible algebra, then $A(p, q)$ is Lie-admissible.

Authors [18] studied polynomial identities that are satisfied by the mutation product $xpy - yqx$ on the basic vector space of an associative algebra A, where fixed elements A are represented by p and q. They provide results for identities in degree $4, 5$ and in degree 6.

The purpose of this work is to identiy the identities for $\mathcal{B}icom_{p,q}$. A. Dzhumadil'daev and N. Ismailov considered the case, when $p = \emptyset$, $q = \emptyset$ in [2]. They obtained the exact expressions of left-normed commutator bracket elements in a free bicommutative algebra. Using this result, they proved that any identity satisfied by the commutator in every bicommutative algebra is a consequence of anti-commutativity, Jacobi and the metabelian identities. In this note, we generalize this result for mutation products. Moreover, we demonstrate that any bicommutative algebra with a mutation product satisfies an identity of degree three and all identities of degree four.

2. Preliminaries

In this section, we will introduce several definitions that are essential for stating a theorem. References are provided alongside each definition.

Definition 2.0.1. ([19]) Let A be a vector space over a field F with given bilinear mapping $\cdot$ (usually called multiplication)

$$\cdot : A \times A \to A,$$

such that for any $x, y, z \in A$ and any $\lambda \in F$, with the following conditions:

$$(x + y) \cdot z = x \cdot z + y \cdot z,$$
$$x \cdot (y + z) = x \cdot y + x \cdot z,$$
$$\lambda(x \cdot y) = (\lambda x) \cdot y = x \cdot (\lambda y).$$

Then A is called an algebra over field F.

Definition 2.0.2. ([20]) Let $f = f(t_1, t_2, \ldots, t_k)$ be a nonassociative, noncommutative polynomial. Let A = (A, $\cdot$) be an algebra with vector space A and multiplication $\cdot$.

We say that A satisfies an identity $f = 0$ if $f(a_1, a_2, \ldots, a_k) = 0$ for any substitutions $t_1 := a_1, \ldots, t_k := a_k$ by elements of A.

Example 2.0.1. Let, for example,

$$f(a, b, c) = a(bc) - (ab)c = 0$$

Then, any algebra that satisfies the identity $f = 0$ is associative.

Definition 2.0.3. ([13]) Let A be an arbitrary algebra over a field F and let p, q be two fixed elements of A. Then a new algebra is derived from A by defining on the same vector space as A a new multiplication

$$\langle x, y \rangle = (xp)y - y(qx)$$

for $x, y \in A$. The resulting algebra is called the (p, q) -mutation of the algebra A and it is denoted by $A(p, q)$.

3. Main Result. Bicommutative algebras under mutation product

3.1 Statements of the main theorem

3.1.1 Identities

$$a{\cdot}(b{\cdot}c) = b{\cdot}(a{\cdot}c),$$
$$(a{\cdot}b){\cdot}c = (a{\cdot}c){\cdot}b.$$
$$\langle a, b\rangle = (ap)b - b(qa)$$

Theorem 3.1.1. *Every identity of degree 3 in mutations of free bicommutative algebra follows from these identities:*

$$\langle\langle a, b\rangle, c\rangle - \langle\langle a, c\rangle, b\rangle + \langle b, \langle c, a\rangle\rangle - \langle c, \langle b, a\rangle\rangle = 0$$

$$\langle a, \langle b, c\rangle\rangle - \langle a, \langle c, b\rangle\rangle - \langle b, \langle a, c\rangle\rangle + \langle b, \langle c, a\rangle\rangle + \langle c, \langle a, b\rangle\rangle - \langle c, \langle b, a\rangle\rangle = 0$$

Corollary 3.1.2. *Any bicommutative algebra under the mutation product is Lie-admissible algebra.*

$$L(x_1, x_2, x_3) = \sum_{\sigma\in S_3} sgn(\sigma)(\langle\langle x_{\sigma(1)}, x_{\sigma(2)}\rangle, x_{\sigma(3)}\rangle - \langle x_{\sigma(1)}, \langle x_{\sigma(2)}, x_{\sigma(3)}\rangle\rangle) = 0$$

Theorem 3.1.3. *Every identity of degree no more than 4 satisfied by the mutation products in every bicommutative algebra is a consequence of identities:*

$$\langle\langle a, \langle d, c\rangle\rangle, b\rangle - \langle a, \langle\langle d, c\rangle, b\rangle\rangle = 0.$$

$$\langle\langle\langle b, c\rangle, a\rangle, d\rangle - \langle\langle\langle b, d\rangle, a\rangle, c\rangle + \langle c, \langle\langle d, b\rangle, a\rangle\rangle - \langle d, \langle\langle c, b\rangle, a\rangle\rangle = 0,$$
$$\langle\langle b, \langle a, c\rangle\rangle, d\rangle - \langle b, \langle\langle a, d\rangle, c\rangle\rangle + \langle b, \langle c, \langle d, a\rangle\rangle\rangle - \langle b, \langle d, \langle c, a\rangle\rangle\rangle = 0,$$

$$\langle\langle\langle d,b\rangle,a\rangle,c\rangle-\langle\langle\langle d,c\rangle,a\rangle,b\rangle+\langle b,\langle\langle d,c\rangle,a\rangle\rangle$$
$$-\langle c,\langle\langle d,b\rangle,a\rangle\rangle-\langle d,\langle\langle b,c\rangle,a\rangle\rangle+\langle d,\langle\langle c,b\rangle,a\rangle\rangle=0,$$
$$\langle\langle b,\langle d,a\rangle\rangle,c\rangle-\langle b,\langle\langle d,c\rangle,a\rangle\rangle+\langle a,\langle b,\langle d,c\rangle\rangle\rangle$$
$$-\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle b,\langle d,\langle c,a\rangle\rangle\rangle=0,$$
$$\langle\langle\langle a,b\rangle,c\rangle,d\rangle+\langle b,\langle\langle d,c\rangle,a\rangle\rangle-\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle$$
$$-\langle\langle\langle a,d\rangle,b\rangle,c\rangle+\langle a,\langle d,\langle b,c\rangle\rangle\rangle+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle d,\langle\langle b,c\rangle,a\rangle\rangle=0.$$

3.2 Proof of the Main Theorem

3.2.1 Identities in degree 3

Let

$$f_1(a,b,c)=\langle\langle a,b\rangle,c\rangle-\langle\langle a,c\rangle,b\rangle+\langle b,\langle c,a\rangle\rangle-\langle c,\langle b,a\rangle\rangle$$
$$f_2(a,b,c)=\langle\langle b,a\rangle,c\rangle-\langle\langle b,c\rangle,a\rangle+\langle a,\langle c,b\rangle\rangle-\langle c,\langle a,b\rangle\rangle$$
$$f_3(a,b,c)=\langle\langle c,a\rangle,b\rangle-\langle\langle c,b\rangle,a\rangle+\langle a,\langle c,b\rangle\rangle-\langle b,\langle c,a\rangle\rangle-\langle c,\langle a,b\rangle\rangle+\langle c,\langle b,a\rangle\rangle$$
$$f_4(a,b,c)=\langle a,\langle b,c\rangle\rangle-\langle a,\langle c,b\rangle\rangle-\langle b,\langle a,c\rangle\rangle+\langle b,\langle c,a\rangle\rangle+\langle c,\langle a,b\rangle\rangle-\langle c,\langle b,a\rangle\rangle$$

where

$$\langle a,b\rangle=(ap)b-b(qa).$$

Lemma 3.2.1. *Let $(\mathcal{B},\cdot)$ be a bicommutative algebra. Then $(\mathcal{B},\langle\cdot,\cdot\rangle)$ satisfies the identity*

$$f_1(a,b,c)=\langle\langle a,b\rangle,c\rangle-\langle\langle a,c\rangle,b\rangle+\langle b,\langle c,a\rangle\rangle-\langle c,\langle b,a\rangle\rangle=0,$$

where $\langle a,b\rangle=(ap)b-b(qa)$.

Proof. By direct calculations, we have

$$\langle\langle a,b\rangle,c\rangle=(((ab)c)p)p-q(((ba)c)p)-q(c((ab)p))+q(q(c(ba))),$$
$$-\langle\langle a,c\rangle,b\rangle=-(((ab)c)p)p+q(((ca)b)p)+q(b((ac)p))-q(q(c(ba))),$$
$$\langle b,\langle c,a\rangle\rangle=c(((ba)p)p)-q(b((ac)p))-q(((ca)b)p)+q(q((ab)c)),$$
$$-\langle c,\langle b,a\rangle\rangle=-c(((ba)p)p)+q(c((ab)p))+q(((ba)c)p)-q(q((ab)c)).$$

The sum of the above elements gives us the desired result.

□

Lemma 3.2.2. *Let $(\mathcal{B},\cdot)$ be a bicommutative algebra. Then $(\mathcal{B},\langle\ ,\ \rangle)$ satisfies the identities $f_2(a,b,c)=0$ and $f_3(a,b,c)=0$, where $\langle a,b\rangle=(ap)b-b(qa)$.*

Proof. Firstly, we will prove that

$$f_2(a,b,c) = \langle\langle c,a\rangle,b\rangle - \langle\langle c,b\rangle,a\rangle + \langle a,\langle c,b\rangle\rangle - \langle b,\langle c,a\rangle\rangle - \langle c,\langle a,b\rangle\rangle + \langle c,\langle b,a\rangle\rangle = 0.$$

then

$$f_3(a,b,c) = \langle a,\langle b,c\rangle\rangle - \langle a,\langle c,b\rangle\rangle - \langle b,\langle a,c\rangle\rangle + \langle b,\langle c,a\rangle\rangle + \langle c,\langle a,b\rangle\rangle - \langle c,\langle b,a\rangle\rangle = 0$$

By direct calculations for the first identity, we have

$$\begin{aligned}
\langle\langle c,a\rangle,b\rangle &= (((ca)b)p)p - q(((ab)c)p) - q(c((ba)p)) + q(q(b(ac)))\\
-\langle\langle c,b\rangle,a\rangle &= -(((ca)b)p)p + q(((ba)c)p) + q(c((ab)p)) - q(q(b(ac)))\\
\langle a,\langle c,b\rangle\rangle &= c(((ab)p)p) - q(b((ac)p)) - q(((ca)b)p) + q(q((ba)c))\\
-\langle b,\langle c,a\rangle\rangle &= -c(((ba)p)p) + q(b((ac)p)) + q(((ca)b)p) - q(q((ab)c))\\
-\langle c,\langle a,b\rangle\rangle &= -c(((ab)p)p) + q(c((ba)p)) + q(((ab)c)p) - q(q((ba)c))\\
\langle c,\langle b,a\rangle\rangle &= c(((ba)p)p) - q(c((ab)p)) - q(((ba)c)p) + q(q((ab)c))
\end{aligned}$$

for the second identity, we have

$$\begin{aligned}
\langle a,\langle b,c\rangle\rangle &= b(((ac)p)p) - q(c((ab)p)) - q(((ba)c)p) + q(q((ca)b))\\
-\langle\langle c,b\rangle,a\rangle &= -(((ca)b)p)p + q(((ba)c)p) + q(c((ab)p)) - q(q(b(ac)))\\
-\langle a,\langle c,b\rangle\rangle &= -c(((ab)p)p) + q(b((ac)p)) + q(((ca)b)p) - q(q((ba)c))\\
\langle b,\langle c,a\rangle\rangle &= c(((ba)p)p) - q(b((ac)p)) - q(((ca)b)p) + q(q((ab)c))\\
\langle c,\langle a,b\rangle\rangle &= c(((ab)p)p) - q(c((ba)p)) - q(((ab)c)p) + q(q((ba)c))\\
-\langle c,\langle b,a\rangle\rangle &= -c(((ba)p)p) + q(c((ab)p)) + q(((ba)c)p) - q(q((ab)c))
\end{aligned}$$

The sum of the above elements gives us the desired results.

□

Lemma 3.2.3. *The identity $f_2(a,b,c)$ and $f_4(a,b,c)$ follows from the identities $f_1(a,b,c)$ and $f_3(a,b,c)$.*

Proof. There are 12 nonassociative monomials of degree 3, and we present them in the following order:

$$\{\langle\langle a,b\rangle,c\rangle, \langle\langle a,c\rangle,b\rangle, \langle\langle b,a\rangle,c\rangle, \langle\langle b,c\rangle,a\rangle, \langle\langle c,a\rangle,b\rangle, \langle\langle c,b\rangle,a\rangle,$$
$$\langle a,\langle b,c\rangle\rangle, \langle a,\langle c,b\rangle\rangle, \langle b,\langle a,c\rangle\rangle, \langle b,\langle c,a\rangle\rangle, \langle c,\langle a,b\rangle\rangle, \langle c,\langle b,a\rangle\rangle\}.$$

We select the coefficients of the monomials relative to the order of the above monomials. In other words, the columns correspond to the monomials and the rows represent each polynomial with all possible permutations of f_1 and f_3 in the variables a, b, c.

Then we have the following matrix, the first 6 rows of which are permutations of f_1, and the next 6 rows are f_3:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we see that $rank(A_1) = 4$. Now, we create a matrix of size 18×12 without changing columns, as in the previous matrix. Here, columns represent monomials of degree three, while the rows represent every polynomial resulting from all possible permutations of f_1, f_2 and f_3.

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we see the $rank(A_2) = 4$. Therefore, the identity f_2 follows from f_1 and f_3. We form the matrix whose rows are the polynomial with all possible permutations of f_i, where $i \in \{1, 2, 3, 4\}$ Also, write the columns as a previous matrices.

$$A_3 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\
1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\
-1 & 1 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
1 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1
\end{pmatrix}$$

The $rank(A_3) = 4$. Hence, the identities $f_2(a,b,c)$ and $f_4(a,b,c)$ follows from the identities $f_1(a,b,c)$ and $f_3(a,b,c)$. □

Let

$$L(x_1,x_2,x_3) = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)(\langle\langle x_{\sigma(1)}, x_{\sigma(2)}\rangle, x_{\sigma(3)}\rangle - \langle x_{\sigma(1)}, \langle x_{\sigma(2)}, x_{\sigma(3)}\rangle\rangle) = 0$$

Corollary 3.2.4. *Any bicommutative algebra under the mutation product is Lie-admissible algebra.*

Theorem 3.2.5. *Every identity of degree 3 in mutations of free bicommutative algebra follows from these identities:*

$$f_1(a,b,c) = \langle\langle a,b\rangle,c\rangle - \langle\langle a,c\rangle,b\rangle + \langle b,\langle c,a\rangle\rangle - \langle c,\langle b,a\rangle\rangle = 0 \tag{3.2.1}$$

$$f_4(a,b,c) = \langle a,\langle b,c\rangle\rangle - \langle a,\langle c,b\rangle\rangle - \langle b,\langle a,c\rangle\rangle + \langle b,\langle c,a\rangle\rangle + \langle c,\langle a,b\rangle\rangle - \langle c,\langle b,a\rangle\rangle = 0 \tag{3.2.2}$$

Proof. Let $F(a,b,c) =$

$$\lambda_1\langle\langle a,b\rangle,c\rangle+\lambda_2\langle\langle a,c\rangle,b\rangle+\lambda_3\langle\langle b,a\rangle,c\rangle+\lambda_4\langle\langle b,c\rangle,a\rangle+\lambda_5\langle\langle c,a\rangle,b\rangle+\lambda_6\langle\langle c,b\rangle,a\rangle$$
$$+\lambda_7\langle a,\langle b,c\rangle\rangle+\lambda_8\langle a,\langle c,b\rangle\rangle+\lambda_9\langle b,\langle a,c\rangle\rangle+\lambda_{10}\langle b,\langle c,a\rangle\rangle+\lambda_{11}\langle c,\langle a,b\rangle\rangle+\lambda_{12}\langle c,\langle b,a\rangle\rangle$$

be a nonassociative polynomial, which is an element of the free nonassociative algebra of degree 3.

Let $\mathcal{B}$ be the free bicommutative algebra generated by a,b,c,p,q with multiplication defined by $(a,b)\longmapsto ab$. We compute $F(a,b,c)\in\mathcal{B}$ using the mutation product $\langle a,b\rangle = (ap)b - b(qa)$. Then, we have

$F(a,b,c) =$

$$\lambda_1\langle\langle a,b\rangle,c\rangle+\lambda_2\langle\langle a,c\rangle,b\rangle+\lambda_3\langle\langle b,a\rangle,c\rangle+\lambda_4\langle\langle b,c\rangle,a\rangle+\lambda_5\langle\langle c,a\rangle,b\rangle+\lambda_6\langle\langle c,b\rangle,a\rangle$$
$$+\lambda_7\langle a,\langle b,c\rangle\rangle+\lambda_8\langle a,\langle c,b\rangle\rangle+\lambda_9\langle b,\langle a,c\rangle\rangle+\lambda_{10}\langle b,\langle c,a\rangle\rangle+\lambda_{11}\langle c,\langle a,b\rangle\rangle+$$
$$\lambda_{12}\langle c,\langle b,a\rangle\rangle$$
$$= (((ab)c)p)p(\lambda_1+\lambda_2)$$
$$+(((ba)c)p)p(\lambda_3+\lambda_4)$$
$$+(((ca)b)p)p(\lambda_5+\lambda_6)$$
$$+b(((ac)p)p)(\lambda_7+\lambda_9)$$
$$+c(((ab)p)p)(\lambda_8+\lambda_{11})$$
$$+c(((ba)p)p)(\lambda_{10}+\lambda_{12})$$
$$+q(q(c(ba)))(\lambda_1+\lambda_2)$$
$$+q(q(c(ab)))(\lambda_3+\lambda_4)$$
$$+q(q(b(ac)))(\lambda_5+\lambda_6)$$
$$+q(q((ca)b))(\lambda_7+\lambda_9)$$
$$+q(q((ba)c))(\lambda_8+\lambda_{11})$$
$$+q(q((ab)c))(\lambda_{10}+\lambda_{12})$$
$$+q(((ba)c)p)(-\lambda_1-\lambda_6-\lambda_7-\lambda_{12})$$
$$+q(((ab)c)p)(-\lambda_3-\lambda_5-\lambda_9-\lambda_{11})$$
$$+q(b((ac)p))(-\lambda_2-\lambda_4-\lambda_8-\lambda_{10})$$
$$+q(c((ab)p))(-\lambda_1-\lambda_6-\lambda_7-\lambda_{12})$$
$$+q(c((ba)p))(-\lambda_3-\lambda_5-\lambda_9-\lambda_{11})$$
$$-q(((ca)b)p)(-\lambda_2-\lambda_4-\lambda_8-\lambda_{10})$$

Since the mutation product of elements $\{a,b,c\}$ in bicommutative algebra can be expressed with the elements

$$\{(((ab)c)p)p, (((ba)c)p)p, (((ca)b)p)p, b(((ac)p)p), c(((ab)p)p), c(((ba)p)p),$$
$$q(((ba)c)p), q(((ab)c)p), q(c((ab)p)), q(c((ba)p)), q(((ca)b)p), q(b((ac)p)),$$
$$q(q(c(ba))), q(q(c(ab))), q(q(b(ac))), q(q((ca)b)), q(q((ba)c)), q(q((ab)c))\}$$

in the free bicommutative algebra of degree 5, we find that $F(a,b,c)=0$ leads to a system of 18 linear equations with 12 unknowns λ_i for $i=1,\ldots,12$.. This system has a rank of 8, which allows us to select $\lambda_1,\lambda_3,\lambda_5$, and λ_7 as free parameters and express the remaining parameters as follows:

$$\lambda_2 = -\lambda_1,$$
$$\lambda_4 = -\lambda_3,$$
$$\lambda_6 = -\lambda_5,$$
$$\lambda_8 = \lambda_3 + \lambda_5 - \lambda_7,$$
$$\lambda_9 = -\lambda_7,$$
$$\lambda_{10} = \lambda_1 - \lambda_5 + \lambda_7,$$
$$\lambda_{11} = -\lambda_3 - \lambda_5 + \lambda_7,$$
$$\lambda_{12} = -\lambda_1 + \lambda_5 - \lambda_7,$$

Therefore,

$$f = \lambda_1 h_1 + \lambda_3 h_2 + \lambda_5 h_3 + \lambda_7 h_4 = 0$$

where

$$h_1 = \langle\langle a, b\rangle, c\rangle - \langle\langle a, c\rangle, b\rangle + \langle b, \langle c, a\rangle\rangle - \langle c, \langle b, a\rangle\rangle,$$
$$h_2 = \langle\langle b, a\rangle, c\rangle - \langle\langle b, c\rangle, a\rangle + \langle a, \langle c, b\rangle\rangle - \langle c, \langle a, b\rangle\rangle,$$
$$h_3 = \langle\langle c, a\rangle, b\rangle - \langle\langle c, b\rangle, a\rangle + \langle a, \langle c, b\rangle\rangle - \langle b, \langle c, a\rangle\rangle - \langle c, \langle a, b\rangle\rangle + \langle c, \langle b, a\rangle\rangle,$$
$$h_4 = \langle a, \langle b, c\rangle\rangle - \langle a, \langle c, b\rangle\rangle - \langle b, \langle a, c\rangle\rangle + \langle b, \langle c, a\rangle\rangle + \langle c, \langle a, b\rangle\rangle - \langle c, \langle b, a\rangle\rangle,$$

We see that the following equalities hold:

$$h_1(a, b, c) = \langle\langle a, b\rangle, c\rangle - \langle\langle a, c\rangle, b\rangle + \langle b, \langle c, a\rangle\rangle - \langle c, \langle b, a\rangle\rangle = f_1(a, b, c),$$

$$h_2(a, b, c) = \langle\langle b, a\rangle, c\rangle - \langle\langle b, c\rangle, a\rangle + \langle a, \langle c, b\rangle\rangle - \langle c, \langle a, b\rangle\rangle = f_1(b, a, c),$$

$h_3(a, b, c) = \langle\langle c, a\rangle, b\rangle - \langle\langle c, b\rangle, a\rangle + \langle a, \langle c, b\rangle\rangle - \langle b, \langle c, a\rangle\rangle - \langle c, \langle a, b\rangle\rangle + \langle c, \langle b, a\rangle\rangle = f_2(a, b, c),$
$h_4(a, b, c) = \langle a, \langle b, c\rangle\rangle - \langle a, \langle c, b\rangle\rangle - \langle b, \langle a, c\rangle\rangle + \langle b, \langle c, a\rangle\rangle + \langle c, \langle a, b\rangle\rangle - \langle c, \langle b, a\rangle\rangle = f_3(a, b, c).$
This means that any identity of degree three in the class $\mathcal{B}icom_{pq}$ is derived from the identities $\{h_1, h_2, h_3, h_4\}$. And by Lemma 3.2.3, the proof is complete. □

Now, we prove that a bicommutative algebra, under mutation product, satisfies identities of degree four that do not follow from $f_1(a, b, c)$ and $f_3(a, b, c)$.

3.2.2 Young diagrams and basis element construction

Assume that X is an ordered set. Refer to a basis of the free bicommutative algebra generated by X as described in X in [2]. We examine Young diagrams of the forms (n) and $(n-k, 1^k)$ where $k = 1, \ldots, n-2$, to identify basis elements in degree n. Young diagrams are filled with elements of X such that $a_1 \leq a_2 \leq \cdots \leq a_k, b_1 \leq b_2 \leq \cdots \leq b_k$ and $k, l > 0$ for $a_1, \ldots, a_k, b_1, \ldots, b_l \in X$ and these can be corresponded to monomials of bicommutative base elements in the following way.

$$\begin{array}{|c|c|c|c|c|}\hline a_1 & b_1 & b_2 & \dots & b_l \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} \end{array} \longmapsto a_k(\dots(a_2((\dots((a_1b_1)b_2)\dots)b_l))\dots).$$

We provide the construction of the multilinear base elements of $Bicom(\{a,b,c\})$ as an example and assume that $a<b<c$.

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline \end{array} \longmapsto (ab)c, \quad \begin{array}{|c|c|c|}\hline b & a & c \\ \hline \end{array} \longmapsto (ba)c, \quad \begin{array}{|c|c|c|}\hline c & a & b \\ \hline \end{array} \longmapsto (ca)b$$

$$\begin{array}{|c|c|}\hline a & c \\ \hline b \\ \cline{1-1} \end{array} \longmapsto b(ac), \quad \begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1} \end{array} \longmapsto c(ab), \quad \begin{array}{|c|c|}\hline b & a \\ \hline c \\ \cline{1-1} \end{array} \longmapsto c(ba).$$

Let us define the following diagrams based on the basic elements

$$\begin{array}{|c|c|c|c|c|c|}\hline a_1 & b_1 & b_2 & \dots & b_l & p^l \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} q^{k-1} \\ \cline{1-1} \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|c|c|c|c|c|}\hline a_1 & b_1 & b_2 & \dots & b_l & p & \dots & p \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} q \\ \cline{1-1} \dots \\ \cline{1-1} q \\ \cline{1-1} \end{array}$$

where elements from p occur l times, and elements from q occur m times, with $m=k-1$.

Let us define

$$\begin{array}{|c|c|c|c|c|}\hline a_1 & b_1 & b_2 & \dots & b_l \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} \end{array}^{(-)} = \begin{array}{|c|c|c|c|c|}\hline a_1 & b_1 & b_2 & \dots & b_l \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} \end{array} - \begin{array}{|c|c|c|c|c|}\hline b_1 & a_1 & a_2 & \dots & a_k \\ \hline b_2 \\ \cline{1-1} \vdots \\ \cline{1-1} b_l \\ \cline{1-1} \end{array},$$

and

$$\begin{array}{|c|c|c|c|c|}\hline a_1 & b_1 & b_2 & \dots & b_l \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} \end{array}^{(+)} = \begin{array}{|c|c|c|c|c|}\hline a_1 & b_1 & b_2 & \dots & b_l \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} \end{array} + \begin{array}{|c|c|c|c|c|}\hline b_1 & a_1 & a_2 & \dots & a_k \\ \hline b_2 \\ \cline{1-1} \vdots \\ \cline{1-1} b_l \\ \cline{1-1} \end{array}.$$

We need the following two lemmas to prove the main result.

Lemma 3.2.6. *Let f be an odd element of $Bicom(X)$. Then, any identity of degree three in $Bicom(X)$ satisfies the following identity.*

$$\left\langle \begin{array}{|c|c|c|c|c|c|} \hline a_1 & b_1 & b_2 & \dots & b_l & p^l \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} q^{k-1} \\ \cline{1-1} \end{array}^{(+)} , \ \begin{array}{|c|} \hline c \\ \hline \end{array} \right\rangle =$$

$$= \begin{array}{|c|c|c|c|c|c|c|} \hline a_1 & b_1 & b_2 & \dots & b_l & c & p^{l+1} \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} q^{k-1} \\ \cline{1-1} \end{array}^{(-)} - \begin{array}{|c|c|c|c|c|c|} \hline a_1 & b_1 & b_2 & \dots & b_l & p^l \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} c \\ \cline{1-1} q^k \\ \cline{1-1} \end{array}^{(-)}$$

Proof. Here, the number of elements of this type in the multilinear case is k, l. And to verify the symmetry of function f, assume that this statement holds for elements of degree less than n. We can observe:

$$\left(\begin{array}{|c|c|c|c|c|c|} \hline a_1 & b_1 & b_2 & \dots & b_l & p^l \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} q^{k-1} \\ \cline{1-1} \end{array} \ p \right) c - c \left(q \ \begin{array}{|c|c|c|c|c|c|} \hline a_1 & b_1 & b_2 & \dots & b_l & p^l \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} q^{k-1} \\ \cline{1-1} \end{array} \right)$$

$$+ \left(\begin{array}{|c|c|c|c|c|c|} \hline b_1 & a_1 & a_2 & \dots & a_k & p^{k-1} \\ \hline b_2 \\ \cline{1-1} \vdots \\ \cline{1-1} b_l \\ \cline{1-1} q^l \\ \cline{1-1} \end{array} \ p \right) c - c \left(q \ \begin{array}{|c|c|c|c|c|c|} \hline b_1 & a_1 & a_2 & \dots & a_k & p^{k-1} \\ \hline b_2 \\ \cline{1-1} \vdots \\ \cline{1-1} b_l \\ \cline{1-1} q^l \\ \cline{1-1} \end{array} \right)$$

$$
= \begin{array}{|c|c|c|c|c|c|c|}
\hline
a_1 & b_1 & b_2 & \dots & b_l & c & p^{l+1} \\
\hline
a_2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
a_k \\
\cline{1-1}
q^{k-1} \\
\cline{1-1}
\end{array}
\; - \;
\begin{array}{|c|c|c|c|c|c|}
\hline
a_1 & b_1 & b_2 & \dots & b_l & p^{l} \\
\hline
a_2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
a_k \\
\cline{1-1}
c \\
\cline{1-1}
q^{k} \\
\cline{1-1}
\end{array}
\; +
$$

$$
+ \begin{array}{|c|c|c|c|c|c|c|}
\hline
b_1 & a_1 & a_2 & \dots & a_k & c & p^{k} \\
\hline
b_2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
b_l \\
\cline{1-1}
q^{l} \\
\cline{1-1}
\end{array}
\; - \;
\begin{array}{|c|c|c|c|c|c|}
\hline
b_1 & a_1 & a_2 & \dots & a_k & p^{k-1} \\
\hline
b_2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
b_l \\
\cline{1-1}
c \\
\cline{1-1}
q^{l+1} \\
\cline{1-1}
\end{array}
$$

where $k + l = n$.

$\square$

Lemma 3.2.7. *Let f be an even element of $Bicom(X)$. Then, any identity of degree three in $Bicom(X)$ satisfies the following identity.*

$$
\left\langle
\begin{array}{|c|c|c|c|c|c|}
\hline
a_1 & b_1 & b_2 & \dots & b_l & p^{l} \\
\hline
a_2 \\
\cline{1-1}
a_3 \\
\cline{1-1}
\vdots \\
\cline{1-1}
a_k \\
\cline{1-1}
q^{k-1} \\
\cline{1-1}
\end{array}^{(-)}
,\;
\begin{array}{|c|}
\hline
c \\
\hline
\end{array}
\right\rangle =
$$

$$
= \begin{array}{|c|c|c|c|c|c|c|}
\hline
a_1 & b_1 & b_2 & \dots & b_l & c & p^{l+1} \\
\hline
a_2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
a_k \\
\cline{1-1}
q^{k-1} \\
\cline{1-1}
\end{array}^{(+)}
\; - \;
\begin{array}{|c|c|c|c|c|c|}
\hline
a_1 & b_1 & b_2 & \dots & b_l & p^{l} \\
\hline
a_2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
a_k \\
\cline{1-1}
c \\
\cline{1-1}
q^{k} \\
\cline{1-1}
\end{array}^{(+)}
$$

Proof. It is easy to see that

$$\left(\begin{array}{cccccc} a_1 & b_1 & b_2 & \dots & b_l & p^l \\ a_2 & & & & & \\ \vdots & & & & & \\ a_k & & & & & \\ q^{k-1} & & & & & \end{array} \; p\right) c - c \left(q \; \begin{array}{cccccc} a_1 & b_1 & b_2 & \dots & b_l & p^l \\ a_2 & & & & & \\ \vdots & & & & & \\ a_k & & & & & \\ q^{k-1} & & & & & \end{array}\right)$$

$$- \left(\begin{array}{cccccc} b_1 & a_1 & a_2 & \dots & a_k & p^{k-1} \\ b_2 & & & & & \\ \vdots & & & & & \\ b_l & & & & & \\ q^l & & & & & \end{array} \; p\right) c + c \left(q \; \begin{array}{cccccc} b_1 & a_1 & a_2 & \dots & a_k & p^{k-1} \\ b_2 & & & & & \\ \vdots & & & & & \\ b_l & & & & & \\ q^l & & & & & \end{array}\right)$$

$$=$$

$$\begin{array}{ccccccc} a_1 & b_1 & b_2 & \dots & b_l & c & p^{l+1} \\ a_2 & & & & & & \\ \vdots & & & & & & \\ a_k & & & & & & \\ q^{k-1} & & & & & & \end{array} - \begin{array}{cccccc} a_1 & b_1 & b_2 & \dots & b_l & p^l \\ a_2 & & & & & \\ \vdots & & & & & \\ a_k & & & & & \\ c & & & & & \\ q^k & & & & & \end{array}$$

$$- \begin{array}{ccccccc} b_1 & a_1 & a_2 & \dots & a_k & c & p^k \\ b_2 & & & & & & \\ \vdots & & & & & & \\ b_l & & & & & & \\ q^l & & & & & & \end{array} + \begin{array}{cccccc} b_1 & a_1 & a_2 & \dots & a_k & p^{k-1} \\ b_2 & & & & & \\ \vdots & & & & & \\ b_l & & & & & \\ c & & & & & \\ q^{l+1} & & & & & \end{array}$$

where $k + l = n$.

□

Now, we prove the main result of this note.

Theorem 3.2.8. *If n is odd, then*

$$\langle\langle \dots \langle\langle a_1, a_2\rangle, a_3\rangle \dots \rangle, a_n\rangle =$$

$$\begin{array}{|c|c|c|c|c|}\hline a_1 & a_2 & \dots & a_n & p^{n-1} \\ \hline \end{array}^{(+)} - \sum_{i=3}^{n} \begin{array}{|c|c|c|c|c|}\hline a_1 & a_2 & \dots & a_n & p^{n-2} \\ \hline a_i \\ \cline{1-1} q \\ \cline{1-1} \end{array}^{(+)} + \dots$$

$$\dots + \sum_{i=3}^{n} \begin{array}{|c|c|c|c|}\hline a_1 & a_2 & a_i & p \\ \hline a_3 \\ \cline{1-1} \vdots \\ \cline{1-1} a_n \\ \cline{1-1} q^{n-3} \\ \cline{1-1} \end{array}^{(+)} - \begin{array}{|c|c|c|}\hline a_1 & a_2 & p \\ \hline a_3 \\ \cline{1-1} \vdots \\ \cline{1-1} a_n \\ \cline{1-1} q^{n-2} \\ \cline{1-1} \end{array}^{(+)} .$$

If n is even, then

$$\langle\langle \dots \langle\langle a_1, a_2\rangle, a_3\rangle \dots \rangle, a_n\rangle =$$

$$\begin{array}{|c|c|c|c|c|}\hline a_1 & a_2 & \dots & a_n & p^{n-1} \\ \hline \end{array}^{(-)} - \sum_{i=3}^{n} \begin{array}{|c|c|c|c|c|}\hline a_1 & a_2 & \dots & a_n & p^{n-2} \\ \hline a_i \\ \cline{1-1} q \\ \cline{1-1} \end{array}^{(-)} + \dots$$

$$\dots - \sum_{i=3}^{n} \begin{array}{|c|c|c|c|c|}\hline a_1 & a_2 & a_i & p & p \\ \hline a_3 \\ \cline{1-1} \vdots \\ \cline{1-1} a_n \\ \cline{1-1} q^{n-3} \\ \cline{1-1} \end{array}^{(-)} + \begin{array}{|c|c|c|}\hline a_1 & a_2 & p \\ \hline a_3 \\ \cline{1-1} \vdots \\ \cline{1-1} a_n \\ \cline{1-1} q^{n-2} \\ \cline{1-1} \end{array}^{(-)} .$$

Proof. Assume that the above formulas are true for $k < n$. So, $k = n - 1$. Let $n - 1$ be even, then n is odd and

$$\langle\langle \dots \langle\langle a_1, a_2\rangle, a_3\rangle, \dots, a_{n-1}\rangle, a_n\rangle =$$

$$\left\langle \begin{array}{|c|c|c|c|c|}\hline a_1 & a_2 & \dots & a_{n-1} & p^{n-2} \\ \hline \end{array}^{(-)} - \sum_{i=3}^{n-1} \begin{array}{|c|c|c|c|c|}\hline a_1 & a_2 & \dots & a_{n-1} & p^{n-3} \\ \hline \multicolumn{1}{|c|}{a_i} \\ \cline{1-1} \multicolumn{1}{|c|}{q} \\ \cline{1-1} \end{array}^{(-)} + \dots \right.$$

$$\left. \dots - \sum_{i=3}^{n} \begin{array}{|c|c|c|c|c|}\hline a_1 & a_2 & a_i & p & p \\ \hline \multicolumn{1}{|c|}{a_3} \\ \cline{1-1} \multicolumn{1}{|c|}{\vdots} \\ \cline{1-1} \multicolumn{1}{|c|}{a_{n-1}} \\ \cline{1-1} \multicolumn{1}{|c|}{q^{n-4}} \\ \cline{1-1} \end{array}^{(-)} + \begin{array}{|c|c|c|}\hline a_1 & a_2 & p \\ \hline \multicolumn{1}{|c|}{a_3} \\ \cline{1-1} \multicolumn{1}{|c|}{\vdots} \\ \cline{1-1} \multicolumn{1}{|c|}{a_{n-1}} \\ \cline{1-1} \multicolumn{1}{|c|}{q^{n-3}} \\ \cline{1-1} \end{array}^{(-)} , a_n \right\rangle$$

By Lemma 3.2.6 and 3.2.7:

$$\begin{array}{|c|c|c|c|c|c|}\hline a_1 & a_2 & \dots & a_{n-1} & a_n & p^{n-1} \\ \hline \end{array}^{(+)} - \begin{array}{|c|c|c|c|}\hline a_1 & \dots & a_{n-1} & p^{n-2} \\ \hline \multicolumn{1}{|c|}{a_n} \\ \cline{1-1} \multicolumn{1}{|c|}{q} \\ \cline{1-1} \end{array}^{(+)} -$$

$$- \sum_{i=3}^{n-1} \begin{array}{|c|c|c|c|c|c|}\hline a_1 & a_2 & \dots & a_{n-1} & a_n & p^{n-2} \\ \hline \multicolumn{1}{|c|}{a_i} \\ \cline{1-1} \multicolumn{1}{|c|}{q} \\ \cline{1-1} \end{array}^{(+)} + \sum_{i=3}^{n-1} \begin{array}{|c|c|c|c|c|}\hline a_1 & a_2 & \dots & a_{n-1} & p^{n-3} \\ \hline \multicolumn{1}{|c|}{a_i} \\ \cline{1-1} \multicolumn{1}{|c|}{a_n} \\ \cline{1-1} \multicolumn{1}{|c|}{q} \\ \cline{1-1} \multicolumn{1}{|c|}{q} \\ \cline{1-1} \end{array}^{(+)} + \dots$$

$$
\cdots - \sum_{i=3}^{n-1} \left(\begin{array}{|c|c|c|c|c|c|c|}
\hline
a_1 & a_2 & a_i & a_n & p & p & p \\
\hline
a_3 & \multicolumn{6}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{6}{c}{} \\
\cline{1-1}
a_{n-1} & \multicolumn{6}{c}{} \\
\cline{1-1}
q^{n-4} & \multicolumn{6}{c}{} \\
\cline{1-1}
\end{array}\right)^{(+)}
+ \sum_{i=3}^{n-1} \left(\begin{array}{|c|c|c|c|c|}
\hline
a_1 & a_2 & a_i & p & p \\
\hline
a_3 & \multicolumn{4}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{4}{c}{} \\
\cline{1-1}
a_{n-1} & \multicolumn{4}{c}{} \\
\cline{1-1}
a_n & \multicolumn{4}{c}{} \\
\cline{1-1}
q^{n-3} & \multicolumn{4}{c}{} \\
\cline{1-1}
\end{array}\right)^{(+)} +
$$

$$
+ \left(\begin{array}{|c|c|c|c|c|}
\hline
a_1 & a_2 & a_n & p & p \\
\hline
a_3 & \multicolumn{4}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{4}{c}{} \\
\cline{1-1}
a_{n-1} & \multicolumn{4}{c}{} \\
\cline{1-1}
q^{n-3} & \multicolumn{4}{c}{} \\
\cline{1-1}
\end{array}\right)^{(+)}
- \left(\begin{array}{|c|c|c|}
\hline
a_1 & a_2 & p \\
\hline
a_3 & \multicolumn{2}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{2}{c}{} \\
\cline{1-1}
a_{n-1} & \multicolumn{2}{c}{} \\
\cline{1-1}
a_n & \multicolumn{2}{c}{} \\
\cline{1-1}
q^{n-2} & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array}\right)^{(+)} =
$$

$$
\left(\begin{array}{|c|c|c|c|c|}
\hline
a_1 & a_2 & \ldots & a_n & p^{n-1} \\
\hline
\end{array}\right)^{(+)}
- \sum_{i=3}^{n} \left(\begin{array}{|c|c|c|c|c|}
\hline
a_1 & a_2 & \ldots & a_n & p^{n-2} \\
\hline
a_i & \multicolumn{4}{c}{} \\
\cline{1-1}
q & \multicolumn{4}{c}{} \\
\cline{1-1}
\end{array}\right)^{(+)} + \cdots
$$

$$
+ \cdots + \sum_{i=3}^{n} \left(\begin{array}{|c|c|c|c|c|}
\hline
a_1 & a_2 & a_i & p & p \\
\hline
a_3 & \multicolumn{4}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{4}{c}{} \\
\cline{1-1}
a_n & \multicolumn{4}{c}{} \\
\cline{1-1}
q^{n-3} & \multicolumn{4}{c}{} \\
\cline{1-1}
\end{array}\right)^{(+)}
- \left(\begin{array}{|c|c|c|}
\hline
a_1 & a_2 & p \\
\hline
a_3 & \multicolumn{2}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{2}{c}{} \\
\cline{1-1}
a_n & \multicolumn{2}{c}{} \\
\cline{1-1}
q^{n-2} & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array}\right)^{(+)} .
$$

$\square$

3.2.3 Identities in degree 4

Let

$$g_1(a,b,c,d) = \langle\langle a, \langle d, c\rangle\rangle, b\rangle - \langle a, \langle\langle d, c\rangle, b\rangle\rangle,$$

$$\begin{aligned}
g_2(a,b,c,d) &= \langle\langle\langle b, c\rangle, a\rangle, d\rangle - \langle\langle\langle b, d\rangle, a\rangle, c\rangle + \langle c, \langle\langle d, b\rangle, a\rangle\rangle - \langle d, \langle\langle c, b\rangle, a\rangle\rangle,\\
g_3(a,b,c,d) &= \langle\langle b, \langle a, c\rangle\rangle, d\rangle - \langle b, \langle\langle a, d\rangle, c\rangle\rangle + \langle b, \langle c, \langle d, a\rangle\rangle\rangle - \langle b, \langle d, \langle c, a\rangle\rangle\rangle,\\
g_4(a,b,c,d) &= \langle\langle\langle d, b\rangle, a\rangle, c\rangle - \langle\langle\langle d, c\rangle, a\rangle, b\rangle + \langle b, \langle\langle d, c\rangle, a\rangle\rangle - \langle c, \langle\langle d, b\rangle, a\rangle\rangle\\
&\quad - \langle d, \langle\langle b, c\rangle, a\rangle\rangle + \langle d, \langle\langle c, b\rangle, a\rangle\rangle,\\
g_5(a,b,c,d) &= \langle\langle b, \langle d, a\rangle\rangle, c\rangle - \langle b, \langle\langle d, c\rangle, a\rangle\rangle + \langle a, \langle b, \langle d, c\rangle\rangle\rangle - \langle b, \langle c, \langle d, a\rangle\rangle\rangle\\
&\quad - \langle b, \langle d, \langle a, c\rangle\rangle\rangle + \langle b, \langle d, \langle c, a\rangle\rangle\rangle,\\
g_6(a,b,c,d) &= \langle\langle\langle a, b\rangle, c\rangle, d\rangle + \langle b, \langle\langle d, c\rangle, a\rangle\rangle - \langle a, \langle b, \langle d, c\rangle\rangle\rangle - \langle c, \langle d, \langle b, a\rangle\rangle\rangle\\
&\quad - \langle\langle\langle a, d\rangle, b\rangle, c\rangle + \langle a, \langle d, \langle b, c\rangle\rangle\rangle + \langle b, \langle c, \langle d, a\rangle\rangle\rangle - \langle d, \langle\langle b, c\rangle, a\rangle\rangle
\end{aligned}$$

where

$$\langle a, b\rangle = (ap)b - b(qa).$$

Lemma 3.2.9. *Let $(\mathcal{B}, \cdot)$ be a bicommutative algebra. Then $(\mathcal{B}, \langle\ ,\ \rangle)$ satisfies the identity $f_i(a,b,c,d) = 0$, where $i \in \{1,2,3,4,5,6\}$.*

Proof. Firstly, we will prove that

$$f_1(a,b,c,d) = \langle\langle a, \langle d, c\rangle\rangle, b\rangle - \langle a, \langle\langle d, c\rangle, b\rangle\rangle = 0.$$

By direct calculations for the first identity, we have

$$\begin{aligned}
\langle\langle a, \langle d, c\rangle\rangle, b\rangle &= d(((((ab)c)p)p)p) - q(c((((ab)d)p)p)) - q(((((da)b)c)p)p)\\
&\quad + q(q((((ca)b)d)p)) - q(d(b(((ac)p)p))) + q(q(c(b((ad)p))))\\
&\quad + q(q(d(((ba)c)p))) - q(q(q(c((ba)d))))
\end{aligned}$$

$$\begin{aligned}
-\langle a, \langle\langle d, c\rangle, b\rangle\rangle &= -d(((((ab)c)p)p)p) + q(c((((ab)d)p)p)) + q(d(b(((ac)p)p)))\\
&\quad - q(q(c(b((ad)p)))) + q(((((da)b)c)p)p) - q(q((((ca)b)d)p))\\
&\quad - q(q(d(((ba)c)p))) + q(q(q(c((ba)d))))
\end{aligned}$$

The sum of the above elements gives us the desired results. □

The same calculations 3.2.9 are also used to prove the following five identities $g_i(a,b,c,d) = 0$, where $i \in \{2,3,4,5,6\}$.

$$f_2(a,b,c,d) = \langle\langle\langle b, c\rangle, a\rangle, d\rangle - \langle\langle\langle b, d\rangle, a\rangle, c\rangle + \langle c, \langle\langle d, b\rangle, a\rangle\rangle - \langle d, \langle\langle c, b\rangle, a\rangle\rangle = 0,$$

$$f_3(a,b,c,d) = \langle\langle b,\langle a,c\rangle\rangle,d\rangle - \langle b,\langle\langle a,d\rangle,c\rangle\rangle + \langle b,\langle c,\langle d,a\rangle\rangle\rangle - \langle b,\langle d,\langle c,a\rangle\rangle\rangle = 0,$$

$$\begin{aligned} f_4(a,b,c,d) &= \langle\langle\langle d,b\rangle,a\rangle,c\rangle - \langle\langle\langle d,c\rangle,a\rangle,b\rangle + \langle b,\langle\langle d,c\rangle,a\rangle\rangle \\ &-\langle c,\langle\langle d,b\rangle,a\rangle\rangle - \langle d,\langle\langle b,c\rangle,a\rangle\rangle + \langle d,\langle\langle c,b\rangle,a\rangle\rangle = 0, \end{aligned}$$

$$\begin{aligned} f_5(a,b,c,d) &= \langle\langle b,\langle d,a\rangle\rangle,c\rangle - \langle b,\langle\langle d,c\rangle,a\rangle\rangle + \langle a,\langle b,\langle d,c\rangle\rangle\rangle \\ &-\langle b,\langle c,\langle d,a\rangle\rangle\rangle - \langle b,\langle d,\langle a,c\rangle\rangle\rangle + \langle b,\langle d,\langle c,a\rangle\rangle\rangle = 0, \end{aligned}$$

$$\begin{aligned} f_6(a,b,c,d) &= \langle\langle\langle a,b\rangle,c\rangle,d\rangle + \langle b,\langle\langle d,c\rangle,a\rangle\rangle - \langle a,\langle b,\langle d,c\rangle\rangle\rangle - \langle c,\langle d,\langle b,a\rangle\rangle\rangle \\ &-\langle\langle\langle a,d\rangle,b\rangle,c\rangle + \langle a,\langle d,\langle b,c\rangle\rangle\rangle + \langle b,\langle c,\langle d,a\rangle\rangle\rangle - \langle d,\langle\langle b,c\rangle,a\rangle\rangle = 0. \end{aligned}$$

Now, we proved that the bicommutative algebra under mutation product satisfies these identities $g_i(a,b,c,d)$, where $i \in \{1,2,3,4,5,6\}$.

Proposition 3.2.10. *Let $(\mathcal{B},\circ)$ be a bicommutative algebra. Then $(\mathcal{B},\langle\ ,\ \rangle)$ satisfies the identity $f(a,b,c,d) = 0$, where $\langle a,b\rangle = (ap)b - b(qa)$.*

$$\langle\langle\langle a,b\rangle,c\rangle,d\rangle = \langle\langle\langle a,b\rangle,d\rangle,c\rangle \tag{3.2.3}$$

$$\langle\langle a,b\rangle,\langle c,d\rangle\rangle = \langle\langle c,d\rangle,\langle a,b\rangle\rangle \tag{3.2.4}$$

$$\langle a,\langle b,\langle c,d\rangle\rangle\rangle = \langle b,\langle a,\langle c,d\rangle\rangle\rangle \tag{3.2.5}$$

By direct calculations for the first identity, we have

$$\begin{aligned} \langle\langle\langle a,b\rangle,c\rangle,d\rangle &= (((((ab)c)d)p)p)p - q(((((ba)c)d)p)p) - q(c((((ab)d)p)p)) \\ &+ q(q(c(((ba)d)p))) - q(d((((ab)c)p)p)) + q(q(d(((ba)c)p))) \\ &+ q(q(d(c((ab)p)))) - q(q(q(d(c(ba))))) \end{aligned}$$

$$\begin{aligned} \langle\langle\langle a,b\rangle,d\rangle,c\rangle &= (((((ab)c)d)p)p)p - q(((((ba)c)d)p)p) - q(d((((ab)c)p)p)) \\ &+ q(q(d(((ba)c)p))) - q(c((((ab)d)p)p)) + q(q(c(((ba)d)p))) \\ &+ q(q(d(c((ab)p)))) - q(q(q(d(c(ba))))) \end{aligned}$$

for the second identity, we have

$$\begin{aligned} \langle\langle a,b\rangle,\langle c,d\rangle\rangle &= c(((((ab)d)p)p)p) - q(d((((ab)c)p)p)) - q(c((((ba)d)p)p)) \\ &+ q(q(d(((ba)c)p))) - q(c((((ab)d)p)p)) + q(q(c(((ba)d)p))) \\ &+ q(q(d(((ab)c)p))) - q(q(q(d((ba)c)))) \end{aligned}$$

$$\begin{aligned} \langle\langle c,d\rangle,\langle a,b\rangle\rangle &= c(((((ab)d)p)p)p) - q(c((((ba)d)p)p)) - q(d((((ab)c)p)p)) \\ &+ q(q(d(((ba)c)p))) - q(c((((ab)d)p)p)) + q(q(d(((ab)c)p))) \\ &+ q(q(c(((ba)d)p))) - q(q(q(d((ba)c)))) \end{aligned}$$

for the third identity, we have

$$\begin{aligned}\langle a,\langle b,\langle c,d\rangle\rangle\rangle = {} & c(b((((ad)p)p)p)) - q(d(b(((ac)p)p))) - q(c((((ab)d)p)p)) \\ & + q(q(d(((ab)c)p))) - q(c((((ba)d)p)p)) + q(q(d(((ba)c)p))) \\ & + q(q((((ca)b)d)p)) - q(q(q(((da)b)c)))\end{aligned}$$

$$\begin{aligned}\langle b,\langle a,\langle c,d\rangle\rangle\rangle = {} & c(b((((ad)p)p)p)) - q(d(b(((ac)p)p))) - q(c((((ba)d)p)p)) \\ & + q(q(d(((ba)c)p))) - q(c((((ab)d)p)p)) + q(q(d(((ab)c)p))) \\ & + q(q((((ca)b)d)p)) - q(q(q(((da)b)c)))\end{aligned}$$

Now, we have proved that the above identities are equal to each other.

Theorem 3.2.11. *Every identity of degree no more than 4 satisfied by the mutation products in every bicommutative algebra is a consequence of identities:*

$$f_1(a,b,c,d) = \langle\langle a,\langle d,c\rangle\rangle,b\rangle - \langle a,\langle\langle d,c\rangle,b\rangle\rangle = 0 \tag{3.2.6}$$

$$f_2(a,b,c,d) = \langle\langle\langle b,c\rangle,a\rangle,d\rangle - \langle\langle\langle b,d\rangle,a\rangle,c\rangle + \langle c,\langle\langle d,b\rangle,a\rangle\rangle - \langle d,\langle\langle c,b\rangle,a\rangle\rangle = 0 \tag{3.2.7}$$

$$f_3(a,b,c,d) = \langle\langle b,\langle a,c\rangle\rangle,d\rangle - \langle b,\langle\langle a,d\rangle,c\rangle\rangle + \langle b,\langle c,\langle d,a\rangle\rangle\rangle - \langle b,\langle d,\langle c,a\rangle\rangle\rangle = 0 \tag{3.2.8}$$

$$\begin{aligned}f_4(a,b,c,d) = {} & \langle\langle\langle d,b\rangle,a\rangle,c\rangle - \langle\langle\langle d,c\rangle,a\rangle,b\rangle + \langle b,\langle\langle d,c\rangle,a\rangle\rangle - \\ & - \langle c,\langle\langle d,b\rangle,a\rangle\rangle - \langle d,\langle\langle b,c\rangle,a\rangle\rangle + \langle d,\langle\langle c,b\rangle,a\rangle\rangle = 0\end{aligned} \tag{3.2.9}$$

$$\begin{aligned}f_5(a,b,c,d) = {} & \langle\langle b,\langle d,a\rangle\rangle,c\rangle - \langle b,\langle\langle d,c\rangle,a\rangle\rangle + \langle a,\langle b,\langle d,c\rangle\rangle\rangle - \\ & - \langle b,\langle c,\langle d,a\rangle\rangle\rangle - \langle b,\langle d,\langle a,c\rangle\rangle\rangle + \langle b,\langle d,\langle c,a\rangle\rangle\rangle = 0\end{aligned} \tag{3.2.10}$$

$$\begin{aligned}f_6(a,b,c,d) = {} & \langle\langle\langle a,b\rangle,c\rangle,d\rangle + \langle b,\langle\langle d,c\rangle,a\rangle\rangle - \langle a,\langle b,\langle d,c\rangle\rangle\rangle - \\ & - \langle c,\langle d,\langle b,a\rangle\rangle\rangle - \langle\langle\langle a,d\rangle,b\rangle,c\rangle + \langle a,\langle d,\langle b,c\rangle\rangle\rangle + \\ & + \langle b,\langle c,\langle d,a\rangle\rangle\rangle - \langle d,\langle\langle b,c\rangle,a\rangle\rangle = 0\end{aligned} \tag{3.2.11}$$

Proof. By using the program Albert, we obtain a basis of degree 4 of algebra defined by identities $g_1(a,b,c) = 0$ and $g_3(a,b,c) = 0$. The basis of this algebra contains 84 nonassociative monomials of degree 4, and we present them in the following order:

$$\{\langle\langle\langle a,b\rangle,c\rangle,d\rangle,\langle\langle\langle b,c\rangle,a\rangle,d\rangle,\langle\langle\langle a,c\rangle,b\rangle,d\rangle,\langle\langle\langle a,d\rangle,b\rangle,c\rangle,\langle\langle\langle b,a\rangle,c\rangle,d\rangle,$$
$$\langle\langle\langle b,d\rangle,a\rangle,c\rangle,\langle\langle\langle c,a\rangle,b\rangle,d\rangle,\langle\langle\langle c,b\rangle,a\rangle,d\rangle,\langle\langle\langle c,d\rangle,a\rangle,b\rangle,\langle\langle\langle d,a\rangle,b\rangle,c\rangle,$$
$$\langle\langle\langle d,b\rangle,a\rangle,c\rangle,\langle\langle\langle d,c\rangle,a\rangle,b\rangle,\langle\langle a,b\rangle,\langle c,d\rangle\rangle,\langle\langle a,c\rangle,\langle b,d\rangle\rangle,\langle\langle a,d\rangle,\langle b,c\rangle\rangle,$$
$$\langle\langle b,a\rangle,\langle c,d\rangle\rangle,\langle\langle c,a\rangle,\langle b,d\rangle\rangle,\langle\langle c,b\rangle,\langle a,d\rangle\rangle,\langle\langle d,a\rangle,\langle b,c\rangle\rangle,\langle\langle d,b\rangle,\langle a,c\rangle\rangle,$$
$$\langle\langle d,c\rangle,\langle a,b\rangle\rangle,\langle\langle b,a\rangle,\langle d,c\rangle\rangle,\langle\langle b,d\rangle,\langle a,c\rangle\rangle,\langle\langle c,a\rangle,\langle d,b\rangle\rangle,\langle\langle a,\langle b,c\rangle\rangle,d\rangle,$$
$$\langle\langle a,\langle b,d\rangle\rangle,c\rangle,\langle\langle a,\langle c,b\rangle\rangle,d\rangle,\langle\langle a,\langle c,d\rangle\rangle,b\rangle,\langle\langle a,\langle d,b\rangle\rangle,c\rangle,\langle\langle a,\langle d,c\rangle\rangle,b\rangle,$$
$$\langle\langle b,\langle a,c\rangle\rangle,d\rangle,\langle\langle b,\langle a,d\rangle\rangle,c\rangle,\langle\langle b,\langle d,a\rangle\rangle,c\rangle,\langle\langle b,\langle d,c\rangle\rangle,a\rangle,\langle\langle b,\langle c,a\rangle\rangle,d\rangle,$$
$$\langle\langle b,\langle c,d\rangle\rangle,a\rangle,\langle\langle c,\langle a,b\rangle\rangle,d\rangle,\langle\langle c,\langle a,d\rangle\rangle,b\rangle,\langle\langle c,\langle b,a\rangle\rangle,b\rangle,\langle\langle c,\langle b,d\rangle\rangle,a\rangle,$$
$$\langle\langle c,\langle d,a\rangle\rangle,b\rangle,\langle\langle c,\langle d,b\rangle\rangle,a\rangle,\langle\langle d,\langle a,b\rangle\rangle,c\rangle,\langle\langle d,\langle a,c\rangle\rangle,b\rangle,\langle\langle d,\langle b,a\rangle\rangle,c\rangle,$$
$$\langle\langle d,\langle b,c\rangle\rangle,a\rangle,\langle\langle d,\langle c,a\rangle\rangle,b\rangle,\langle\langle d,\langle c,b\rangle\rangle,a\rangle,\langle a,\langle\langle b,c\rangle,d\rangle\rangle,\langle a,\langle\langle b,d\rangle,c\rangle\rangle,$$
$$\langle a,\langle\langle c,b\rangle,d\rangle\rangle,\langle a,\langle\langle c,d\rangle,b\rangle\rangle,\langle a,\langle\langle d,b\rangle,c\rangle\rangle,\langle a,\langle\langle d,c\rangle,b\rangle\rangle,\langle b,\langle\langle a,c\rangle,d\rangle\rangle,$$
$$\langle b,\langle\langle a,d\rangle,c\rangle\rangle,\langle b,\langle\langle c,a\rangle,d\rangle\rangle,\langle b,\langle\langle c,d\rangle,a\rangle\rangle,\langle b,\langle\langle d,a\rangle,c\rangle\rangle,\langle b,\langle\langle d,c\rangle,a\rangle\rangle,$$
$$\langle c,\langle\langle a,b\rangle,d\rangle\rangle,\langle c,\langle\langle a,d\rangle,b\rangle\rangle,\langle c,\langle\langle b,a\rangle,d\rangle\rangle,\langle c,\langle\langle b,d\rangle,a\rangle\rangle,\langle c,\langle\langle d,a\rangle,b\rangle\rangle,$$
$$\langle c,\langle\langle d,b\rangle,a\rangle\rangle,\langle d,\langle\langle a,b\rangle,c\rangle\rangle,\langle d,\langle\langle a,c\rangle,b\rangle\rangle,\langle d,\langle\langle b,a\rangle,c\rangle\rangle,\langle d,\langle\langle b,c\rangle,a\rangle\rangle,$$
$$\langle d,\langle\langle c,a\rangle,b\rangle\rangle,\langle d,\langle\langle c,b\rangle,a\rangle\rangle,\langle a,\langle b,\langle c,d\rangle\rangle\rangle,\langle a,\langle b,\langle d,c\rangle\rangle\rangle,\langle a,\langle c,\langle b,d\rangle\rangle\rangle,$$
$$\langle a,\langle c,\langle d,b\rangle\rangle\rangle,\langle a,\langle d,\langle b,c\rangle\rangle\rangle,\langle a,\langle d,\langle c,b\rangle\rangle\rangle,\langle b,\langle c,\langle a,d\rangle\rangle\rangle,\langle b,\langle c,\langle d,a\rangle\rangle\rangle,$$
$$\langle b,\langle d,\langle a,c\rangle\rangle\rangle,\langle b,\langle d,\langle c,a\rangle\rangle\rangle,\langle c,\langle d,\langle a,b\rangle\rangle\rangle,\langle c,\langle d,\langle b,a\rangle\rangle\rangle\} \tag{3.2.12}$$

Let

$$\begin{aligned}
&F(a,b,c,d)\\
&\quad=\lambda_1\langle\langle\langle a,b\rangle,c\rangle,d\rangle+\lambda_2\langle\langle\langle a,c\rangle,b\rangle,d\rangle+\lambda_3\langle\langle\langle a,d\rangle,b\rangle,c\rangle+\lambda_4\langle\langle\langle b,a\rangle,c\rangle,d\rangle\\
&\quad+\lambda_5\langle\langle\langle b,c\rangle,a\rangle,d\rangle+\lambda_6\langle\langle\langle b,d\rangle,a\rangle,c\rangle+\lambda_7\langle\langle\langle c,a\rangle,b\rangle,d\rangle+\lambda_8\langle\langle\langle c,b\rangle,a\rangle,d\rangle\\
&\quad+\lambda_9\langle\langle\langle c,d\rangle,a\rangle,b\rangle+\lambda_{10}\langle\langle\langle d,a\rangle,b\rangle,c\rangle+\lambda_{11}\langle\langle\langle d,b\rangle,a\rangle,c\rangle+\lambda_{12}\langle\langle\langle d,c\rangle,a\rangle,b\rangle\\
&\quad+\lambda_{13}\langle\langle a,b\rangle,\langle c,d\rangle\rangle+\lambda_{14}\langle\langle a,b\rangle,\langle d,c\rangle\rangle+\lambda_{15}\langle\langle a,c\rangle,\langle b,d\rangle\rangle+\lambda_{16}\langle\langle a,c\rangle,\langle d,b\rangle\rangle\\
&\quad+\lambda_{17}\langle\langle a,d\rangle,\langle b,c\rangle\rangle+\lambda_{18}\langle\langle a,d\rangle,\langle c,b\rangle\rangle+\lambda_{19}\langle\langle b,a\rangle,\langle c,d\rangle\rangle+\lambda_{20}\langle\langle b,a\rangle,\langle d,c\rangle\rangle\\
&\quad+\lambda_{21}\langle\langle b,c\rangle,\langle d,a\rangle\rangle+\lambda_{22}\langle\langle b,d\rangle,\langle c,a\rangle\rangle+\lambda_{23}\langle\langle c,a\rangle,\langle d,b\rangle\rangle+\lambda_{24}\langle\langle c,b\rangle,\langle d,a\rangle\rangle\\
&\quad+\lambda_{25}\langle\langle a,\langle b,c\rangle\rangle,d\rangle+\lambda_{26}\langle\langle a,\langle b,d\rangle\rangle,c\rangle+\lambda_{27}\langle\langle a,\langle c,b\rangle\rangle,d\rangle+\lambda_{28}\langle\langle a,\langle c,d\rangle\rangle,b\rangle\\
&\quad+\lambda_{29}\langle\langle a,\langle d,b\rangle\rangle,c\rangle+\lambda_{30}\langle\langle a,\langle d,c\rangle\rangle,b\rangle+\lambda_{31}\langle\langle b,\langle a,c\rangle\rangle,d\rangle+\lambda_{32}\langle\langle b,\langle a,d\rangle\rangle,c\rangle\\
&\quad+\lambda_{33}\langle\langle b,\langle c,a\rangle\rangle,d\rangle+\lambda_{34}\langle\langle b,\langle c,d\rangle\rangle,a\rangle+\lambda_{35}\langle\langle b,\langle d,a\rangle\rangle,c\rangle+\lambda_{36}\langle\langle b,\langle d,c\rangle\rangle,a\rangle\\
&\quad+\lambda_{37}\langle\langle c,\langle a,b\rangle\rangle,d\rangle+\lambda_{38}\langle\langle c,\langle a,d\rangle\rangle,b\rangle+\lambda_{39}\langle\langle c,\langle b,a\rangle\rangle,d\rangle+\lambda_{40}\langle\langle c,\langle b,d\rangle\rangle,a\rangle\\
&\quad+\lambda_{41}\langle\langle c,\langle d,a\rangle\rangle,b\rangle+\lambda_{42}\langle\langle c,\langle d,b\rangle\rangle,a\rangle+\lambda_{43}\langle\langle d,\langle a,b\rangle\rangle,c\rangle+\lambda_{44}\langle\langle d,\langle a,c\rangle\rangle,b\rangle\\
&\quad+\lambda_{45}\langle\langle d,\langle b,a\rangle\rangle,c\rangle+\lambda_{46}\langle\langle d,\langle b,c\rangle\rangle,a\rangle+\lambda_{47}\langle\langle d,\langle c,a\rangle\rangle,b\rangle+\lambda_{48}\langle\langle d,\langle c,b\rangle\rangle,a\rangle\\
&\quad+\lambda_{49}\langle a,\langle\langle b,c\rangle,d\rangle\rangle+\lambda_{50}\langle a,\langle\langle b,d\rangle,c\rangle\rangle+\lambda_{51}\langle a,\langle\langle c,b\rangle,d\rangle\rangle+\lambda_{52}\langle a,\langle\langle c,d\rangle,b\rangle\rangle\\
&\quad+\lambda_{53}\langle a,\langle\langle d,b\rangle,c\rangle\rangle+\lambda_{54}\langle a,\langle\langle d,c\rangle,b\rangle\rangle+\lambda_{55}\langle b,\langle\langle a,c\rangle,d\rangle\rangle+\lambda_{56}\langle b,\langle\langle a,d\rangle,c\rangle\rangle\\
&\quad+\lambda_{57}\langle b,\langle\langle c,a\rangle,d\rangle\rangle+\lambda_{58}\langle b,\langle\langle c,d\rangle,a\rangle\rangle+\lambda_{59}\langle b,\langle\langle d,a\rangle,c\rangle\rangle+\lambda_{60}\langle b,\langle\langle d,c\rangle,a\rangle\rangle
\end{aligned}$$

$$+\lambda_{61}\langle c,\langle\langle a,b\rangle,d\rangle\rangle+\lambda_{62}\langle c,\langle\langle a,d\rangle,b\rangle\rangle+\lambda_{63}\langle c,\langle\langle b,a\rangle,d\rangle\rangle+\lambda_{64}\langle c,\langle\langle b,d\rangle,a\rangle\rangle$$
$$+\lambda_{65}\langle c,\langle\langle d,a\rangle,b\rangle\rangle+\lambda_{66}\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\lambda_{67}\langle d,\langle\langle a,b\rangle,c\rangle\rangle+\lambda_{68}\langle d,\langle\langle a,c\rangle,b\rangle\rangle$$
$$+\lambda_{69}\langle d,\langle\langle b,a\rangle,c\rangle\rangle+\lambda_{70}\langle d,\langle\langle b,c\rangle,a\rangle\rangle+\lambda_{71}\langle d,\langle\langle c,a\rangle,b\rangle\rangle+\lambda_{72}\langle d,\langle\langle c,b\rangle,a\rangle\rangle$$
$$+\lambda_{73}\langle a,\langle b,\langle c,d\rangle\rangle\rangle+\lambda_{74}\langle a,\langle b,\langle d,c\rangle\rangle\rangle+\lambda_{75}\langle a,\langle c,\langle b,d\rangle\rangle\rangle+\lambda_{76}\langle a,\langle c,\langle d,b\rangle\rangle\rangle$$
$$+\lambda_{77}\langle a,\langle d,\langle c,b\rangle\rangle\rangle+\lambda_{78}\langle a,\langle d,\langle b,c\rangle\rangle\rangle+\lambda_{79}\langle b,\langle c,\langle a,d\rangle\rangle\rangle+\lambda_{80}\langle b,\langle c,\langle d,a\rangle\rangle\rangle$$
$$+\lambda_{81}\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\lambda_{82}\langle b,\langle d,\langle c,a\rangle\rangle\rangle+\lambda_{83}\langle c,\langle d,\langle a,b\rangle\rangle\rangle+\lambda_{84}\langle c,\langle d,\langle b,a\rangle\rangle\rangle$$

be a nonassociative polynomial, i.e., an element of the free nonassociative algebra of degree 4.
Let $\mathcal{B}$ be the free bicommutative algebra with multiplication $(a,b)\longmapsto ab$. We calculate $F(a,b,c,d)\in\mathcal{B}$ in terms of the mutation product $\langle a,b\rangle=(ap)b-b(qa)$.

We have

$$F(a,b,c,d)=$$
$$(((((ab)c)d)p)p)p(\lambda_1+\lambda_2+\lambda_3)$$
$$+(((((ba)c)d)p)p)p(\lambda_4+\lambda_5+\lambda_6)$$
$$+(((((ca)b)d)p)p)p(\lambda_7+\lambda_8+\lambda_9)$$
$$+(((((da)b)c)p)p)p(\lambda_{10}+\lambda_{11}+\lambda_{12})$$
$$+c(((((ab)d)p)p)p)(\lambda_{13}+\lambda_{18}+\lambda_{27}+\lambda_{28}+\lambda_{37}+\lambda_{38}+\lambda_{51}+\lambda_{52}+\lambda_{61}+\lambda_{62})$$
$$+d(((((ab)c)p)p)p)(\lambda_{14}+\lambda_{16}+\lambda_{29}+\lambda_{30}+\lambda_{43}+\lambda_{44}+\lambda_{53}+\lambda_{54}+\lambda_{67}+\lambda_{68})$$
$$+b(((((ac)d)p)p)p)(\lambda_{15}+\lambda_{17}+\lambda_{25}+\lambda_{26}+\lambda_{31}+\lambda_{32}+\lambda_{49}+\lambda_{50}+\lambda_{55}+\lambda_{56})$$
$$+c(((((ba)d)p)p)p)(\lambda_{19}+\lambda_{22}+\lambda_{33}+\lambda_{34}+\lambda_{39}+\lambda_{40}+\lambda_{57}+\lambda_{58}+\lambda_{63}+\lambda_{64})$$
$$+d(((((ba)c)p)p)p)(\lambda_{20}+\lambda_{21}+\lambda_{35}+\lambda_{36}+\lambda_{45}+\lambda_{46}+\lambda_{59}+\lambda_{60}+\lambda_{69}+\lambda_{70})$$
$$+d(((((ca)b)p)p)p)(\lambda_{23}+\lambda_{24}+\lambda_{41}+\lambda_{42}+\lambda_{47}+\lambda_{48}+\lambda_{65}+\lambda_{66}+\lambda_{71}+\lambda_{72})$$
$$+c(b((((ad)p)p)p))(\lambda_{73}+\lambda_{75}+\lambda_{79})$$
$$+d(b((((ac)p)p)p))(\lambda_{74}+\lambda_{78}+\lambda_{81})$$
$$+d(c((((ab)p)p)p))(\lambda_{76}+\lambda_{77}+\lambda_{83})$$
$$+d(c((((ba)p)p)p))(\lambda_{80}+\lambda_{82}+\lambda_{84})$$
$$+q(((((ba)c)d)p)p)(-\lambda_1-\lambda_8-\lambda_{11}-\lambda_{25}-\lambda_{26}-\lambda_{39}-\lambda_{40}-\lambda_{45}-\lambda_{46}-\lambda_{49}-$$
$$-\lambda_{50}-\lambda_{63}-\lambda_{64}-\lambda_{69}-\lambda_{70})$$
$$+q(((((ca)b)d)p)p)(-\lambda_2-\lambda_5-\lambda_{12}-\lambda_{27}-\lambda_{28}-\lambda_{33}-\lambda_{34}-\lambda_{47}-\lambda_{48}-\lambda_{51}-$$
$$-\lambda_{52}-\lambda_{57}-\lambda_{58}-\lambda_{71}-\lambda_{72})$$
$$+q(((((da)b)c)p)p)(-\lambda_3-\lambda_6-\lambda_9-\lambda_{29}-\lambda_{30}-\lambda_{35}-\lambda_{36}-\lambda_{41}-\lambda_{42}-\lambda_{53}-$$
$$-\lambda_{54}-\lambda_{59}-\lambda_{60}-\lambda_{65}-\lambda_{66})$$
$$+q(((((ab)c)d)p)p)(-\lambda_4-\lambda_7-\lambda_{10}-\lambda_{31}-\lambda_{32}-\lambda_{37}-\lambda_{38}-\lambda_{43}-\lambda_{44}-\lambda_{55}-$$
$$-\lambda_{56}-\lambda_{61}-\lambda_{62}-\lambda_{67}-\lambda_{68})$$
$$+q(d((((ab)c)p)p))(-\lambda_1-\lambda_2-\lambda_{11}-\lambda_{12}-\lambda_{13}-\lambda_{14}-\lambda_{15}-\lambda_{16}-\lambda_{20}-\lambda_{23}-$$
$$-\lambda_{26}-\lambda_{28}-\lambda_{45}-\lambda_{47}-\lambda_{50}-\lambda_{52}-\lambda_{69}-\lambda_{71}-\lambda_{74}-$$
$$-\lambda_{76}-\lambda_{81}-\lambda_{83})$$
$$+q(c((((ab)d)p)p))(-\lambda_1-\lambda_3-\lambda_8-\lambda_9-\lambda_{13}-\lambda_{14}-\lambda_{17}-\lambda_{18}-\lambda_{19}-\lambda_{24}-$$
$$-\lambda_{25}-\lambda_{30}-\lambda_{39}-\lambda_{41}-\lambda_{49}-\lambda_{54}-\lambda_{63}-\lambda_{65}-\lambda_{73}-$$
$$-\lambda_{77}-\lambda_{79}-\lambda_{83})$$

$$+q(b((((ac)d)p)p))(-\lambda_2-\lambda_3-\lambda_5-\lambda_6-\lambda_{15}-\lambda_{16}-\lambda_{17}-\lambda_{18}-\lambda_{21}-\lambda_{22}-$$
$$-\lambda_{27}-\lambda_{29}-\lambda_{33}-\lambda_{35}-\lambda_{51}-\lambda_{53}-\lambda_{57}-\lambda_{59}-\lambda_{75}-$$
$$-\lambda_{78}-\lambda_{79}-\lambda_{81})$$
$$+q(d((((ba)c)p)p))(-\lambda_4-\lambda_5-\lambda_{10}-\lambda_{12}-\lambda_{14}-\lambda_{17}-\lambda_{19}-\lambda_{20}-\lambda_{21}-\lambda_{24}-$$
$$-\lambda_{32}-\lambda_{34}-\lambda_{43}-\lambda_{48}-\lambda_{56}-\lambda_{58}-\lambda_{67}-\lambda_{72}-\lambda_{74}-$$
$$-\lambda_{78}-\lambda_{84})$$
$$+q(c((((ba)d)p)p))(-\lambda_4-\lambda_6-\lambda_7-\lambda_9-\lambda_{13}-\lambda_{15}-\lambda_{19}-\lambda_{20}-\lambda_{22}-\lambda_{23}-$$
$$-\lambda_{31}-\lambda_{36}-\lambda_{37}-\lambda_{42}-\lambda_{55}-\lambda_{60}-\lambda_{61}-\lambda_{65}-\lambda_{73}-$$
$$-\lambda_{75}-\lambda_{82}-\lambda_{84})$$
$$+q(d((((ca)b)p)p))(-\lambda_7-\lambda_8-\lambda_{10}-\lambda_{11}-\lambda_{16}-\lambda_{18}-\lambda_{21}-\lambda_{22}-\lambda_{23}-\lambda_{24}-$$
$$-\lambda_{38}-\lambda_{40}-\lambda_{44}-\lambda_{46}-\lambda_{62}-\lambda_{64}-\lambda_{68}-\lambda_{70}-$$
$$-\lambda_{76}-\lambda_{77}-\lambda_{80}-\lambda_{82})$$
$$+q(d(b(((ac)p)p)))(-\lambda_{25}-\lambda_{30}-\lambda_{31}-\lambda_{36}-\lambda_{44}-\lambda_{46}-\lambda_{49}-\lambda_{54}-\lambda_{55}-$$
$$-\lambda_{60}-\lambda_{68}-\lambda_{70}-\lambda_{73}-\lambda_{77}-\lambda_{82})$$
$$+q(c(b(((ad)p)p)))(-\lambda_{26}-\lambda_{28}-\lambda_{32}-\lambda_{34}-\lambda_{38}-\lambda_{40}-\lambda_{50}-\lambda_{52}-\lambda_{56}-$$
$$-\lambda_{58}-\lambda_{62}-\lambda_{64}-\lambda_{74}-\lambda_{76}-\lambda_{80})$$
$$+q(d(c(((ab)p)p)))(-\lambda_{27}-\lambda_{29}-\lambda_{37}-\lambda_{42}-\lambda_{43}-\lambda_{48}-\lambda_{51}-\lambda_{53}-\lambda_{61}-$$
$$-\lambda_{66}-\lambda_{67}-\lambda_{72}-\lambda_{75}-\lambda_{78}-\lambda_{84})$$
$$+q(d(c(((ba)p)p)))(-\lambda_{33}-\lambda_{35}-\lambda_{39}-\lambda_{41}-\lambda_{45}-\lambda_{47}-\lambda_{57}-\lambda_{59}-\lambda_{63}-$$
$$-\lambda_{65}-\lambda_{69}-\lambda_{71}-\lambda_{79}-\lambda_{81}-\lambda_{83})$$
$$+q(q(c(((ba)d)p)))(\lambda_1+\lambda_2+\lambda_{11}+\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{15}+\lambda_{16}+\lambda_{20}+\lambda_{23}+$$
$$+\lambda_{26}+\lambda_{28}+\lambda_{45}+\lambda_{47}+\lambda_{50}+\lambda_{52}+\lambda_{69}+\lambda_{71}+\lambda_{74}+$$
$$+\lambda_{76}+\lambda_{81}+\lambda_{83})$$
$$+q(q(d(((ba)c)p)))(\lambda_1+\lambda_3+\lambda_8+\lambda_9+\lambda_{13}+\lambda_{14}+\lambda_{17}+\lambda_{18}+\lambda_{19}+\lambda_{24}+$$
$$+\lambda_{25}+\lambda_{30}+\lambda_{39}+\lambda_{41}+\lambda_{49}+\lambda_{54}+\lambda_{63}+\lambda_{65}+\lambda_{73}+$$
$$+\lambda_{77}+\lambda_{79}+\lambda_{83})$$
$$+q(q(c(((ab)d)p)))(\lambda_4+\lambda_5+\lambda_6+\lambda_{10}+\lambda_{12}+\lambda_{14}+\lambda_{17}+\lambda_{19}+\lambda_{20}+\lambda_{21}+$$
$$+\lambda_{24}+\lambda_{32}+\lambda_{34}+\lambda_{43}+\lambda_{48}+\lambda_{56}+\lambda_{58}+\lambda_{67}+\lambda_{72}+$$
$$+\lambda_{74}+\lambda_{78}+\lambda_{80}+\lambda_{84})$$
$$+q(q(d(((ab)c)p)))(\lambda_4+\lambda_6+\lambda_7+\lambda_9+\lambda_{13}+\lambda_{15}+\lambda_{19}+\lambda_{20}+\lambda_{22}+\lambda_{23}+$$
$$+\lambda_{31}+\lambda_{36}+\lambda_{37}+\lambda_{42}+\lambda_{55}+\lambda_{60}+\lambda_{61}+\lambda_{66}+\lambda_{73}+$$
$$+\lambda_{75}+\lambda_{82}+\lambda_{84})$$
$$+q(q(b(((ac)d)p)))(\lambda_7+\lambda_8+\lambda_{10}+\lambda_{11}+\lambda_{16}+\lambda_{18}+\lambda_{21}+\lambda_{22}+\lambda_{23}+\lambda_{24}+$$
$$+\lambda_{38}+\lambda_{40}+\lambda_{44}+\lambda_{46}+\lambda_{62}+\lambda_{64}+\lambda_{68}+\lambda_{70}+\lambda_{76}+$$
$$+\lambda_{77}+\lambda_{80}+\lambda_{82})$$
$$+q(q(d(((ca)b)p)))(\lambda_2+\lambda_3+\lambda_5+\lambda_{15}+\lambda_{16}+\lambda_{17}+\lambda_{18}+\lambda_{21}+\lambda_{22}+\lambda_{27}+$$
$$+\lambda_{29}+\lambda_{33}+\lambda_{35}+\lambda_{51}+\lambda_{53}+\lambda_{57}+\lambda_{59}+\lambda_{75}+\lambda_{78}+$$
$$+\lambda_{79}+\lambda_{81})$$

$$
\begin{aligned}
&+q(q(((((ca)b)d)p))(\lambda_{25}+\lambda_{30}+\lambda_{31}+\lambda_{36}+\lambda_{44}+\lambda_{46}+\lambda_{49}+\lambda_{54}+\lambda_{55}+\lambda_{60}+\\
&\qquad +\lambda_{68}+\lambda_{70}+\lambda_{73}+\lambda_{77}+\lambda_{82})\\
&+q(q(((((da)b)c)p))(\lambda_{26}+\lambda_{28}+\lambda_{32}+\lambda_{34}+\lambda_{38}+\lambda_{40}+\lambda_{50}+\lambda_{52}+\lambda_{56}+\lambda_{58}+\\
&\qquad +\lambda_{62}+\lambda_{64}+\lambda_{74}+\lambda_{76}+\lambda_{80})\\
&+q(q(((((ba)c)d)p))(\lambda_{27}+\lambda_{29}+\lambda_{37}+\lambda_{42}+\lambda_{43}+\lambda_{48}+\lambda_{51}+\lambda_{53}+\lambda_{61}+\lambda_{66}+\\
&\qquad +\lambda_{67}+\lambda_{72}+\lambda_{75}+\lambda_{78}+\lambda_{84})\\
&+q(q(((((ab)c)d)p))(\lambda_{33}+\lambda_{35}+\lambda_{39}+\lambda_{41}+\lambda_{45}+\lambda_{47}+\lambda_{57}+\lambda_{59}+\lambda_{63}+\lambda_{65}+\\
&\qquad +\lambda_{69}+\lambda_{71}+\lambda_{79}+\lambda_{81}+\lambda_{83})\\
&+q(q(d(c((ab)p))))(\lambda_{1}+\lambda_{8}+\lambda_{11}+\lambda_{25}+\lambda_{26}+\lambda_{39}+\lambda_{40}+\lambda_{45}+\lambda_{46}+\lambda_{49}+\\
&\qquad +\lambda_{50}+\lambda_{63}+\lambda_{64}+\lambda_{69}+\lambda_{70})\\
&+q(q(d(b((ac)p))))(\lambda_{2}+\lambda_{5}+\lambda_{12}+\lambda_{27}+\lambda_{28}+\lambda_{33}+\lambda_{34}+\lambda_{47}+\lambda_{48}+\lambda_{51}+\\
&\qquad +\lambda_{52}+\lambda_{57}+\lambda_{58}+\lambda_{71}+\lambda_{72})\\
&+q(q(c(b((ad)p))))(\lambda_{3}+\lambda_{6}+\lambda_{9}+\lambda_{29}+\lambda_{30}+\lambda_{35}+\lambda_{36}+\lambda_{41}+\lambda_{42}+\lambda_{53}+\\
&\qquad +\lambda_{54}+\lambda_{59}+\lambda_{60}+\lambda_{65}+\lambda_{66})\\
&+q(q(d(c((ba)p))))(\lambda_{4}+\lambda_{7}+\lambda_{10}+\lambda_{31}+\lambda_{32}+\lambda_{37}+\lambda_{38}+\lambda_{43}+\lambda_{44}+\lambda_{55}+\\
&\qquad +\lambda_{56}+\lambda_{61}+\lambda_{62}+\lambda_{67}+\lambda_{68})\\
&+q(q(q(d(c(ba)))))(-\lambda_{1}-\lambda_{2}-\lambda_{3})\\
&+q(q(q(d(c(ab)))))(-\lambda_{4}-\lambda_{5}-\lambda_{6})\\
&+q(q(q(d(b(ac)))))(-\lambda_{7}-\lambda_{8}-\lambda_{9})\\
&+q(q(q(c(b(ad)))))(-\lambda_{10}-\lambda_{11}-\lambda_{12})\\
&+q(q(q(d((ba)c))))(-\lambda_{13}-\lambda_{18}-\lambda_{27}-\lambda_{28}-\lambda_{37}-\lambda_{38}-\lambda_{51}-\lambda_{52}-\lambda_{61}-\lambda_{62})\\
&+q(q(q(c((ba)d))))(-\lambda_{14}-\lambda_{16}-\lambda_{29}-\lambda_{30}-\lambda_{43}-\lambda_{44}-\lambda_{53}-\lambda_{54}-\lambda_{67}-\lambda_{68})\\
&+q(q(q(d((ca)b))))(-\lambda_{15}-\lambda_{17}-\lambda_{25}-\lambda_{26}-\lambda_{31}-\lambda_{32}-\lambda_{49}-\lambda_{50}-\lambda_{55}-\lambda_{56})\\
&+q(q(q(d((ab)c))))(-\lambda_{19}-\lambda_{22}-\lambda_{33}-\lambda_{34}-\lambda_{39}-\lambda_{40}-\lambda_{57}-\lambda_{58}-\lambda_{63}-\lambda_{64})\\
&+q(q(q(c((ab)d))))(-\lambda_{20}-\lambda_{21}-\lambda_{35}-\lambda_{36}-\lambda_{45}-\lambda_{46}-\lambda_{59}-\lambda_{60}-\lambda_{69}-\lambda_{70})\\
&+q(q(q(b((ac)d))))(-\lambda_{23}-\lambda_{24}-\lambda_{41}-\lambda_{42}-\lambda_{47}-\lambda_{48}-\lambda_{65}-\lambda_{66}-\lambda_{71}-\lambda_{72})\\
&+q(q(q(((da)b)c)))(-\lambda_{73}-\lambda_{75}-\lambda_{79})\\
&+q(q(q(((ca)b)d)))(-\lambda_{74}-\lambda_{78}-\lambda_{81})\\
&+q(q(q(((ba)c)d)))(-\lambda_{76}-\lambda_{77}-\lambda_{83})\\
&+q(q(q(((ab)c)d)))(-\lambda_{80}-\lambda_{82}-\lambda_{84})
\end{aligned}
$$

Since, the mutation products of $\{a,b,c,d\}$ are expressed with the set of elements

$$
\begin{aligned}
&\{(((((ab)c)d)p)p)p, (((((ba)c)d)p)p)p, (((((ca)b)d)p)p)p, (((((da)b)c)p)p)p,\\
&c(((((ab)d)p)p)p), d(((((ab)c)p)p)p), b(((((ac)d)p)p)p), c(((((ba)d)p)p)p),\\
&d(((((ba)c)p)p)p), d(((((ca)b)p)p)p), c(b((((ad)p)p)p)), d(b((((ac)p)p)p)),\\
&d(c((((ab)p)p)p)), d(c((((ba)p)p)p)), q(((((ba)c)d)p)p), q(((((ca)b)d)p)p),\\
&q(((((da)b)c)p)p), q(((((ab)c)d)p)p), q(d((((ab)c)p)p)), q(c((((ab)d)p)p)),\\
&q(b((((ac)d)p)p)), q(d((((ba)c)p)p)), q(c((((ba)d)p)p)), q(d((((ca)b)p)p)),\\
&q(d(b(((ac)p)p))), q(c(b(((ad)p)p))), q(d(c(((ab)p)p))), q(d(c(((ba)p)p))),\\
&q(q(c(((ba)d)p))), q(q(d(((ba)c)p))), q(q(c(((ab)d)p))), q(q(d(((ab)c)p))),
\end{aligned}
$$

$$q(q(b(((ac)d)p))), q(q(d(((ca)b)p))), q(q((((ca)b)d)p)), q(q((((da)b)c)p)),$$
$$q(q((((ba)c)d)p)), q(q((((ab)c)d)p)), q(q(d(c((ab)p)))), q(q(d(b((ac)p)))),$$
$$q(q(c(b((ad)p)))), q(q(d(c((ba)p)))), q(q(q(d(c(ba))))), q(q(q(d(c(ab))))),$$
$$q(q(q(d(b(ac))))), q(q(q(c(b(ad))))), q(q(q(d((ba)c)))), q(q(q(c((ba)d)))),$$
$$q(q(q(d((ca)b)))), q(q(q(d((ab)c)))), q(q(q(c((ab)d)))), q(q(q(b((ac)d)))),$$
$$q(q(q(((da)b)c))), q(q(q(((ca)b)d))), q(q(q(((ba)c)d))), q(q(q(((ab)c)d)))\}$$

from a basis of the free bicommutative algebra in degree 7 on the set $\{a, b, c, d, p, p, p, q, q, q\}$, we see that $F(a, b, c, d) = 0$ leads to a system of 61 linear equations involving 84 unknowns, denoted by λ_i, where $i = 1, \ldots, 84$. This system has a rank of 23. Therefore, we can consider $\lambda_3, \lambda_6, \lambda_9, \lambda_{12}, \lambda_{24}, \lambda_{54}, \lambda_{56}, \lambda_{60}, \lambda_{62}, \lambda_{64}, \lambda_{66}, \lambda_{68}, \lambda_{70}, \lambda_{72}, \lambda_{74}, \lambda_{76}, \lambda_{78}, \lambda_{78}, \lambda_{80}, \lambda_{81}, \lambda_{82}, \lambda_{83}$ and λ_{84} as free parameters, the remaining parameters can be expressed in terms of these free parameters:

$\lambda_3 = -\lambda_1 - \lambda_2,$

$\lambda_6 = -\lambda_4 - \lambda_5,$

$\lambda_9 = -\lambda_7 - \lambda_8,$

$\lambda_{12} = -\lambda_{10} - \lambda_{11},$

$\lambda_{24} = -\lambda_{13} - \lambda_{14} - \lambda_{15} - \lambda_{16} - \lambda_{17} - \lambda_{18} - \lambda_{19} - \lambda_{20} - \lambda_{21} - \lambda_{22} - \lambda_{23},$

$$\begin{aligned}\lambda_{54} =& \lambda_{10} - \lambda_{13} - \lambda_{14} - \lambda_{15} - \lambda_{16} - \lambda_{17} - \lambda_{18} - \lambda_{25} - \lambda_{26} - \lambda_{27} - \lambda_{28} - \lambda_{29} - \\ & - \lambda_{30} + \lambda_4 - \lambda_{49} - \lambda_{50} - \lambda_{51} - \lambda_{52} - \lambda_{53} + \lambda_7,\end{aligned}$$

$\lambda_{56} = -\lambda_{15} - \lambda_{17} - \lambda_{25} - \lambda_{26} - \lambda_{31} - \lambda_{32} - \lambda_{49} - \lambda_{50} - \lambda_{55},$

$$\begin{aligned}\lambda_{60} =& \lambda_1 + \lambda_{11} - \lambda_{19} - \lambda_{20} - \lambda_{21} - \lambda_{22} + \lambda_{25} + \lambda_{26} - \lambda_{33} - \lambda_{34} - \lambda_{35} - \lambda_{36} + \\ & + \lambda_{49} + \lambda_{50} - \lambda_{57} - \lambda_{58} - \lambda_{59} + \lambda_8,\end{aligned}$$

$\lambda_{62} = -\lambda_{13} - \lambda_{18} - \lambda_{27} - \lambda_{28} - \lambda_{37} - \lambda_{38} - \lambda_{51} - \lambda_{52} - \lambda_{61},$

$\lambda_{64} = -\lambda_{19} - \lambda_{22} - \lambda_{33} - \lambda_{34} - \lambda_{39} - \lambda_{40} - \lambda_{57} - \lambda_{58} - \lambda_{63},$

$$\begin{aligned}\lambda_{66} =& - \lambda_{10} - \lambda_{11} + \lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{16} + \lambda_{17} + \lambda_{18} + \lambda_{19} + \lambda_2 + \lambda_{20} + \lambda_{21} + \\ & + \lambda_{22} + \lambda_{27} + \lambda_{28} + \lambda_{33} + \lambda_{34} - \lambda_{41} - \lambda_{42} + \lambda_5 + \lambda_{51} + \lambda_{52} + \lambda_{57} + \lambda_{58} - \\ & - \lambda_{65},\end{aligned}$$

$$\begin{aligned}\lambda_{68} =& \lambda_{10} + \lambda_{13} + \lambda_{15} + \lambda_{17} + \lambda_{18} + \lambda_{25} + \lambda_{26} + \lambda_{27} + \lambda_{28} - \lambda_4 - \lambda_{43} - \lambda_{44} + \\ & + \lambda_{49} + \lambda_{50} + \lambda_{51} + \lambda_{52} - \lambda_{67} - \lambda_7,\end{aligned}$$

$$\begin{aligned}\lambda_{70} =& - \lambda_1 - \lambda_{11} + \lambda_{19} + \lambda_{22} - \lambda_{25} - \lambda_{26} + \lambda_{33} + \lambda_{34} - \lambda_{45} - \lambda_{46} - \lambda_{49} - \lambda_{50} + \\ & + \lambda_{57} + \lambda_{58} - \lambda_{69} - \lambda_8,\end{aligned}$$

$$\begin{aligned}\lambda_{72} =& \lambda_{10} + \lambda_{11} - \lambda_2 - \lambda_{27} - \lambda_{28} - \lambda_{33} - \lambda_{34} - \lambda_{47} - \lambda_{48} - \lambda_5 - \lambda_{51} - \lambda_{52} - \\ & - \lambda_{57} - \lambda_{58} - \lambda_{71},\end{aligned}$$

$$\begin{aligned}\lambda_{74} =& - \lambda_1 + \lambda_{15} + \lambda_{16} + \lambda_{17} + \lambda_{18} + \lambda_{21} + \lambda_{22} + \lambda_{27} + \lambda_{29} + \lambda_{33} + \lambda_{35} - \lambda_4 + \\ & + \lambda_{51} + \lambda_{53} + \lambda_{57} + \lambda_{59} - \lambda_{73},\end{aligned}$$

$$\begin{aligned}\lambda_{76} =& \lambda_{10} - \lambda_{13} - \lambda_{14} - 2\lambda_{15} - 2\lambda_{16} - \lambda_{17} - \lambda_{18} - \lambda_2 - \lambda_{20} - \lambda_{21} - \lambda_{22} - \lambda_{23} - \\ & - \lambda_{26} - \lambda_{27} - \lambda_{28} - \lambda_{29} + \lambda_{39} + \lambda_4 + \lambda_{41} - \lambda_{50} - \lambda_{51} - \lambda_{52} - \lambda_{53} + \lambda_{63} + \\ & + \lambda_{65} - \lambda_{75},\end{aligned}$$

$$\begin{aligned}\lambda_{78} =& \lambda_1 - \lambda_{10} + \lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{16} + \lambda_2 + \lambda_{20} + \lambda_{23} + \lambda_{26} + \lambda_{28} + \lambda_{45} + \\ & + \lambda_{47} + \lambda_{50} + \lambda_{52} + \lambda_{69} + \lambda_{71} - \lambda_{77},\end{aligned}$$

$\lambda_{79} = -\lambda_{73} - \lambda_{75},$
$\lambda_{80} = \lambda_1 - \lambda_{10} + 2\lambda_{13} + \lambda_{14} + 2\lambda_{15} + \lambda_{16} + \lambda_{17} + \lambda_{18} + \lambda_{19} + \lambda_2 + \lambda_{20} + \lambda_{22} +$
$+ \lambda_{23} + \lambda_{25} + \lambda_{26} + \lambda_{27} + \lambda_{28} + \lambda_{31} - \lambda_{35} + \lambda_{37} - \lambda_{41} + \lambda_{49} + \lambda_{50} + \lambda_{51} +$
$+ \lambda_{52} + \lambda_{55} - \lambda_{59} + \lambda_{61} - \lambda_{65} + \lambda_{73} + \lambda_{75},$
$\lambda_{81} = \lambda_{10} - \lambda_{13} - \lambda_{14} - 2\lambda_{15} - 2\lambda_{16} - \lambda_{17} - \lambda_{18} - \lambda_2 - \lambda_{20} - \lambda_{21} - \lambda_{22} - \lambda_{23} -$
$- \lambda_{26} - \lambda_{27} - \lambda_{28} - \lambda_{29} - \lambda_{33} - \lambda_{35} + \lambda_4 - \lambda_{45} - \lambda_{47} - \lambda_{50} - \lambda_{51} - \lambda_{52} -$
$- \lambda_{53} - \lambda_{57} - \lambda_{59} - \lambda_{69} - \lambda_{71} + \lambda_{73} + \lambda_{77},$
$\lambda_{82} = \lambda_{14} + \lambda_{16} + \lambda_{20} + \lambda_{21} - \lambda_{25} + \lambda_{29} - \lambda_{31} + \lambda_{35} + \lambda_{43} + \lambda_{45} - \lambda_{49} + \lambda_{53} -$
$- \lambda_{55} + \lambda_{59} + \lambda_{67} + \lambda_{69} - \lambda_{73} - \lambda_{77},$
$\lambda_{83} = - \lambda_{10} + \lambda_{13} + \lambda_{14} + 2\lambda_{15} + 2\lambda_{16} + \lambda_{17} + \lambda_{18} + \lambda_2 + \lambda_{20} + \lambda_{21} + \lambda_{22} +$
$+ \lambda_{23} + \lambda_{26} + \lambda_{27} + \lambda_{28} + \lambda_{29} - \lambda_{39} - \lambda_4 - \lambda_{41} + \lambda_{50} + \lambda_{51} + \lambda_{52} + \lambda_{53} -$
$- \lambda_{63} - \lambda_{65} + \lambda_{75} - \lambda_{77},$
$\lambda_{84} = - \lambda_1 + \lambda_{10} - 2\lambda_{13} - 2\lambda_{14} - 2\lambda_{15} - 2\lambda_{16} - \lambda_{17} - \lambda_{18} - \lambda_{19} - \lambda_2 - 2\lambda_{20} -$
$- \lambda_{21} - \lambda_{22} - \lambda_{23} - \lambda_{26} - \lambda_{27} - \lambda_{28} - \lambda_{29} - \lambda_{37} + \lambda_{41} - \lambda_{43} - \lambda_{45} -$
$- \lambda_{50} - \lambda_{51} - \lambda_{52} - \lambda_{53} - \lambda_{61} + \lambda_{65} - \lambda_{67} - \lambda_{69} - \lambda_{75} + \lambda_{77}.$

From these equations, we get 61 free parameters. We apply these parameters to nonassociative monomials 3.2.12.
$\lambda_1\langle\langle\langle a, b\rangle, c\rangle, d\rangle + \lambda_2\langle\langle\langle a, c\rangle, b\rangle, d\rangle + (-\lambda_1 - \lambda_2)\langle\langle\langle a, d\rangle, b\rangle, c\rangle + \lambda_4\langle\langle\langle b, a\rangle, c\rangle, d\rangle +$
$\lambda_5\langle\langle\langle b, c\rangle, a\rangle, d\rangle + (-\lambda_4 - \lambda_5)\langle\langle\langle b, d\rangle, a\rangle, c\rangle + \lambda_7\langle\langle\langle c, a\rangle, b\rangle, d\rangle + \lambda_8\langle\langle\langle c, b\rangle, a\rangle, d\rangle +$
$(-\lambda_7 - \lambda_8)\langle\langle\langle c, d\rangle, a\rangle, b\rangle + \lambda_{10}\langle\langle\langle d, a\rangle, b\rangle, c\rangle + \lambda_{11}\langle\langle\langle d, b\rangle, a\rangle, c\rangle +$
$(-\lambda_{10} - \lambda_{11})\langle\langle\langle d, c\rangle, a\rangle, b\rangle + \lambda_{13}\langle\langle a, b\rangle, \langle c, d\rangle\rangle + \lambda_{14}\langle\langle a, b\rangle, \langle d, c\rangle\rangle + \lambda_{15}\langle\langle a, \rangle b, \langle b, d\rangle\rangle +$
$\lambda_{16}\langle\langle a, \rangle b, \langle d, b\rangle\rangle + \lambda_{17}\langle\langle a, d\rangle, \langle b, c\rangle\rangle + \lambda_{18}\langle\langle a, d\rangle, \langle c, b\rangle\rangle + \lambda_{19}\langle\langle b, a\rangle, \langle c, d\rangle\rangle +$
$\lambda_{20}\langle\langle b, a\rangle, \langle d, c\rangle\rangle + \lambda_{21}\langle\langle b, c\rangle, \langle d, a\rangle\rangle + \lambda_{22}\langle\langle b, d\rangle, \langle c, a\rangle\rangle + \lambda_{23}\langle\langle c, a\rangle, \langle d, b\rangle\rangle + (-\lambda_{13} -$
$\lambda_{14} - \lambda_{15} - \lambda_{16} - \lambda_{17} - \lambda_{18} - \lambda_{19} - \lambda_{20} - \lambda_{21} - \lambda_{22} - \lambda_{23})\langle\langle c, b\rangle, \langle d, a\rangle\rangle + \lambda_{25}\langle\langle a, \langle b, c\rangle\rangle, d\rangle +$
$\lambda_{26}\langle\langle a, \langle b, d\rangle\rangle, c\rangle + \lambda_{27}\langle\langle a, \langle c, b\rangle\rangle, d\rangle + \lambda_{28}\langle\langle a, \langle c, d\rangle\rangle, b\rangle + \lambda_{29}\langle\langle a, \langle d, b\rangle\rangle, c\rangle +$
$\lambda_{30}\langle\langle a, \langle d, c\rangle\rangle, b\rangle + \lambda_{31}\langle\langle b, \langle a, c\rangle\rangle, d\rangle + \lambda_{32}\langle\langle b, \langle a, d\rangle\rangle, c\rangle + \lambda_{33}\langle\langle b, \langle c, a\rangle\rangle, d\rangle +$
$\lambda_{34}\langle\langle b, \langle c, d\rangle\rangle, a\rangle + \lambda_{35}\langle\langle b, \langle d, a\rangle\rangle, c\rangle + \lambda_{36}\langle\langle b, \langle d, c\rangle\rangle, a\rangle + \lambda_{37}\langle\langle c, \langle a, b\rangle\rangle, d\rangle +$
$\lambda_{38}\langle\langle c, \langle a, d\rangle\rangle, b\rangle + \lambda_{39}\langle\langle c, \langle b, a\rangle\rangle, d\rangle + \lambda_{40}\langle\langle c, \langle b, d\rangle\rangle, a\rangle + \lambda_{41}\langle\langle c, \langle d, a\rangle\rangle, b\rangle +$
$\lambda_{42}\langle\langle c, \langle d, b\rangle\rangle, a\rangle + \lambda_{43}\langle\langle d, \langle a, b\rangle\rangle, c\rangle + \lambda_{44}\langle\langle d, \langle a, c\rangle\rangle, b\rangle + \lambda_{45}\langle\langle d, \langle b, a\rangle\rangle, c\rangle +$
$\lambda_{46}\langle\langle d, \langle b, c\rangle\rangle, a\rangle + \lambda_{47}\langle\langle d, \langle c, a\rangle\rangle, b\rangle + \lambda_{48}\langle\langle d, \langle c, b\rangle\rangle, a\rangle + \lambda_{49}\langle a, \langle\langle b, c\rangle, d\rangle\rangle +$
$\lambda_{50}\langle a, \langle\langle b, d\rangle, c\rangle\rangle + \lambda_{51}\langle a, \langle\langle c, b\rangle, d\rangle\rangle + \lambda_{52}\langle a, \langle\langle c, d\rangle, b\rangle\rangle + \lambda_{53}\langle a, \langle\langle d, b\rangle, c\rangle\rangle +$
$(\lambda_{10} - \lambda_{13} - \lambda_{14} - \lambda_{15} - \lambda_{16} - \lambda_{17} - \lambda_{18} - \lambda_{25} - \lambda_{26} - \lambda_{27} - \lambda_{28} - \lambda_{29} - \lambda_{30} + \lambda_4 -$
$\lambda_{49} - \lambda_{50} - \lambda_{51} - \lambda_{52} - \lambda_{53} + \lambda_7)\langle a, \langle\langle d, c\rangle, b\rangle\rangle + \lambda_{55}\langle b, \langle\langle a, c\rangle, d\rangle\rangle + (-\lambda_{15} - \lambda_{17} - \lambda_{25} -$
$\lambda_{26} - \lambda_{31} - \lambda_{32} - \lambda_{49} - \lambda_{50} - \lambda_{55})\langle b, \langle\langle a, d\rangle, c\rangle\rangle + \lambda_{57}\langle b, \langle\langle c, a\rangle, d\rangle\rangle + \lambda_{58}\langle b, \langle\langle c, d\rangle, a\rangle\rangle +$
$\lambda_{59}\langle b, \langle\langle d, a\rangle, c\rangle\rangle + (\lambda_1 + \lambda_{11} - \lambda_{19} - \lambda_{20} - \lambda_{21} - \lambda_{22} + \lambda_{25} + \lambda_{26} - \lambda_{33} - \lambda_{34} - \lambda_{35} - \lambda_{36} +$
$\lambda_{49} + \lambda_{50} - \lambda_{57} - \lambda_{58} - \lambda_{59} + \lambda_8)\langle b, \langle\langle d, c\rangle, a\rangle\rangle + \lambda_{61}\langle c, \langle\langle a, b\rangle, d\rangle\rangle + (-\lambda_{13} - \lambda_{18} - \lambda_{27} -$
$\lambda_{28} - \lambda_{37} - \lambda_{38} - \lambda_{51} - \lambda_{52} - \lambda_{61})\langle c, \langle\langle a, d\rangle, b\rangle\rangle + \lambda_{63}\langle c, \langle\langle b, a\rangle, d\rangle\rangle + (-\lambda_{19} - \lambda_{22} -$
$\lambda_{33} - \lambda_{34} - \lambda_{39} - \lambda_{40} - \lambda_{57} - \lambda_{58} - \lambda_{63})\langle c, \langle\langle b, d\rangle, a\rangle\rangle + \lambda_{65}\langle c, \langle\langle d, a\rangle, b\rangle\rangle + (-\lambda_{10} -$
$\lambda_{11} + \lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{16} + \lambda_{17} + \lambda_{18} + \lambda_{19} + \lambda_2 + \lambda_{20} + \lambda_{21} + \lambda_{22} + \lambda_{27} + \lambda_{28} + \lambda_{33} + \lambda_{34} -$
$\lambda_{41} - \lambda_{42} + \lambda_5 + \lambda_{51} + \lambda_{52} + \lambda_{57} + \lambda_{58} - \lambda_{65})\langle c, \langle\langle d, b\rangle, a\rangle\rangle + \lambda_{67}\langle d, \langle\langle a, b\rangle, c\rangle\rangle + (-\lambda_{10} +$
$\lambda_{13} + \lambda_{15} + \lambda_{17} + \lambda_{18} + \lambda_{25} + \lambda_{26} + \lambda_{27} + \lambda_{28} - \lambda_4 - \lambda_{43} - \lambda_{44} + \lambda_{49} + \lambda_{50} + \lambda_{51} + \lambda_{52} -$

$\lambda_{67}-\lambda_7)\langle d,\langle\langle a,c\rangle,b\rangle\rangle+\lambda_{69}\langle d,\langle\langle b,a\rangle,c\rangle\rangle+(-\lambda_1-\lambda_{11}+\lambda_{19}+\lambda_{22}-\lambda_{25}-\lambda_{26}+\lambda_{33}+\lambda_{34}-\lambda_{45}-\lambda_{46}-\lambda_{49}-\lambda_{50}+\lambda_{57}+\lambda_{58}-\lambda_{69}-\lambda_8)\langle d,\langle\langle b,c\rangle,a\rangle\rangle+\lambda_{71}\langle d,\langle\langle c,a\rangle,b\rangle\rangle+(\lambda_{10}+\lambda_{11}-\lambda_2-\lambda_{27}-\lambda_{28}-\lambda_{33}-\lambda_{34}-\lambda_{47}-\lambda_{48}-\lambda_5-\lambda_{51}-\lambda_{52}-\lambda_{57}-\lambda_{58}-\lambda_{71})\langle d,\langle\langle c,b\rangle,a\rangle\rangle+\lambda_{73}\langle a,\langle b,\langle c,d\rangle\rangle\rangle+(-\lambda_1+\lambda_{15}+\lambda_{16}+\lambda_{17}+\lambda_{18}+\lambda_{21}+\lambda_{22}+\lambda_{27}+\lambda_{29}+\lambda_{33}+\lambda_{35}-\lambda_4+\lambda_{51}+\lambda_{53}+\lambda_{57}+\lambda_{59}-\lambda_{73})\langle a,\langle b,\langle d,c\rangle\rangle\rangle+\lambda_{75}\langle a,\langle c,\langle b,d\rangle\rangle\rangle+(\lambda_{10}-\lambda_{13}-\lambda_{14}-2\lambda_{15}-2\lambda_{16}-\lambda_{17}-\lambda_{18}-\lambda_2-\lambda_{20}-\lambda_{21}-\lambda_{22}-\lambda_{23}-\lambda_{26}-\lambda_{27}-\lambda_{28}-\lambda_{29}+\lambda_{39}+\lambda_4+\lambda_{41}-\lambda_{50}-\lambda_{51}-\lambda_{52}-\lambda_{53}+\lambda_{63}+\lambda_{65}-\lambda_{75})\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\lambda_{77}\langle a,\langle d,\langle c,b\rangle\rangle\rangle+(\lambda_1-\lambda_{10}+\lambda_{13}+\lambda_{14}+\lambda_{15}+\lambda_{16}+\lambda_2+\lambda_{20}+\lambda_{23}+\lambda_{26}+\lambda_{28}+\lambda_{45}+\lambda_{47}+\lambda_{50}+\lambda_{52}+\lambda_{69}+\lambda_{71}-\lambda_{77})\langle a,\langle d,\langle b,c\rangle\rangle\rangle+(-\lambda_{73}-\lambda_{75})\langle b,\langle c,\langle a,d\rangle\rangle\rangle+(\lambda_1-\lambda_{10}+2\lambda_{13}+\lambda_{14}+2\lambda_{15}+\lambda_{16}+\lambda_{17}+\lambda_{18}+\lambda_{19}+\lambda_2+\lambda_{20}+\lambda_{22}+\lambda_{23}+\lambda_{25}+\lambda_{26}+\lambda_{27}+\lambda_{28}+\lambda_{31}-\lambda_{35}+\lambda_{37}-\lambda_{41}+\lambda_{49}+\lambda_{50}+\lambda_{51}+\lambda_{52}+\lambda_{55}-\lambda_{59}+\lambda_{61}-\lambda_{65}+\lambda_{73}+\lambda_{75})\langle b,\langle c,\langle d,a\rangle\rangle\rangle+(\lambda_{10}-\lambda_{13}-\lambda_{14}-2\lambda_{15}-2\lambda_{16}-\lambda_{17}-\lambda_{18}-\lambda_2-\lambda_{20}-\lambda_{21}-\lambda_{22}-\lambda_{23}-\lambda_{26}-\lambda_{27}-\lambda_{28}-\lambda_{29}-\lambda_{33}-\lambda_{35}+\lambda_4-\lambda_{45}-\lambda_{47}-\lambda_{50}-\lambda_{51}-\lambda_{52}-\lambda_{53}-\lambda_{57}-\lambda_{59}-\lambda_{69}-\lambda_{71}+\lambda_{73}+\lambda_{77})\langle b,\langle d,\langle a,c\rangle\rangle\rangle+(\lambda_{14}+\lambda_{16}+\lambda_{20}+\lambda_{21}-\lambda_{25}+\lambda_{29}-\lambda_{31}+\lambda_{35}+\lambda_{43}+\lambda_{45}-\lambda_{49}+\lambda_{53}-\lambda_{55}+\lambda_{59}+\lambda_{67}+\lambda_{69}-\lambda_{73}-\lambda_{77})\langle b,\langle d,\langle c,a\rangle\rangle\rangle+(-\lambda_{10}+\lambda_{13}+\lambda_{14}+2\lambda_{15}+2\lambda_{16}+\lambda_{17}+\lambda_{18}+\lambda_2+\lambda_{20}+\lambda_{21}+\lambda_{22}+\lambda_{23}+\lambda_{26}+\lambda_{27}+\lambda_{28}+\lambda_{29}-\lambda_{39}-\lambda_4-\lambda_{41}+\lambda_{50}+\lambda_{51}+\lambda_{52}+\lambda_{53}-\lambda_{63}-\lambda_{65}+\lambda_{75}-\lambda_{77})\langle c,\langle d,\langle a,b\rangle\rangle\rangle+(-\lambda_1+\lambda_{10}-2\lambda_{13}-2\lambda_{14}-2\lambda_{15}-2\lambda_{16}-\lambda_{17}-\lambda_{18}-\lambda_{19}-\lambda_2-2\lambda_{20}-\lambda_{21}-\lambda_{22}-\lambda_{23}-\lambda_{26}-\lambda_{27}-\lambda_{28}-\lambda_{29}-\lambda_{37}+\lambda_{41}-\lambda_{43}-\lambda_{45}-\lambda_{50}-\lambda_{51}-\lambda_{52}-\lambda_{53}-\lambda_{61}+\lambda_{65}-\lambda_{67}-\lambda_{69}-\lambda_{75}+\lambda_{77})\langle c,\langle d,\langle b,a\rangle\rangle\rangle.$

From here we collect the lambdas. To do this, use monomials inside the $F(a,b,c,d)$ from 1 to 84 in terms of the mutation product $\langle a,b\rangle=(ap)b-b(qa)$.

$F(a,b,c,d)=$

$$\lambda_{30}(\langle\langle a,\langle d,c\rangle\rangle,b\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle)+\lambda_5(\langle\langle\langle b,c\rangle,a\rangle,d\rangle-\langle\langle\langle b,d\rangle,a\rangle,c\rangle+$$
$$+\langle c,\langle\langle d,b\rangle,a\rangle\rangle-\langle d,\langle\langle c,b\rangle,a\rangle\rangle)+\lambda_7(\langle\langle\langle c,a\rangle,b\rangle,d\rangle-\langle\langle\langle c,d\rangle,a\rangle,b\rangle+$$
$$+\langle a,\langle\langle d,c\rangle,b\rangle\rangle-\langle d,\langle\langle a,c\rangle,b\rangle\rangle)+\lambda_{31}(\langle\langle b,\langle a,c\rangle\rangle,d\rangle-\langle b,\langle\langle a,d\rangle,c\rangle\rangle+$$
$$+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle c,a\rangle\rangle\rangle)+\lambda_{37}(\langle\langle c,\langle a,b\rangle\rangle,d\rangle-\langle c,\langle\langle a,d\rangle,b\rangle\rangle+$$
$$+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{43}(\langle\langle d,\langle a,b\rangle\rangle,c\rangle-\langle d,\langle\langle a,c\rangle,b\rangle\rangle+$$
$$+\langle b,\langle d,\langle c,a\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{55}(\langle b,\langle\langle a,c\rangle,d\rangle\rangle-\langle b,\langle\langle a,d\rangle,c\rangle\rangle+$$
$$+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle c,a\rangle\rangle\rangle)+\lambda_{61}(\langle c,\langle\langle a,b\rangle,d\rangle\rangle-\langle c,\langle\langle a,d\rangle,b\rangle\rangle+$$
$$+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{67}(\langle d,\langle\langle a,b\rangle,c\rangle\rangle-\langle d,\langle\langle a,c\rangle,b\rangle\rangle+$$
$$+\langle b,\langle d,\langle c,a\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{11}(\langle\langle\langle d,b\rangle,a\rangle,c\rangle-\langle\langle\langle d,c\rangle,a\rangle,b\rangle+$$
$$+\langle b,\langle\langle d,c\rangle,a\rangle\rangle-\langle c,\langle\langle d,b\rangle,a\rangle\rangle-\langle d,\langle\langle b,c\rangle,a\rangle\rangle+\langle d,\langle\langle c,b\rangle,a\rangle\rangle)+$$
$$+\lambda_{34}(\langle\langle b,\langle c,d\rangle\rangle,a\rangle-\langle b,\langle\langle d,c\rangle,a\rangle\rangle-\langle c,\langle\langle b,d\rangle,a\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+$$
$$+\langle d,\langle\langle b,c\rangle,a\rangle\rangle-\langle d,\langle\langle c,b\rangle,a\rangle\rangle)+\lambda_{35}(\langle\langle b,\langle d,a\rangle\rangle,c\rangle-\langle b,\langle\langle d,c\rangle,a\rangle\rangle+$$
$$+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle b,\langle d,\langle c,a\rangle\rangle\rangle)+$$
$$+\lambda_{41}(\langle\langle c,\langle d,a\rangle\rangle,b\rangle-\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\langle a,\langle c,\langle d,b\rangle\rangle\rangle-\langle b,\langle c,\langle d,a\rangle\rangle\rangle-$$
$$-\langle c,\langle d,\langle a,b\rangle\rangle\rangle+\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{45}(\langle\langle d,\langle b,a\rangle\rangle,c\rangle-\langle d,\langle\langle b,c\rangle,a\rangle\rangle+$$
$$+\langle a,\langle d,\langle b,c\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle b,\langle d,\langle c,a\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+$$
$$+\lambda_{58}(\langle b,\langle\langle c,d\rangle,a\rangle\rangle-\langle b,\langle\langle d,c\rangle,a\rangle\rangle-\langle c,\langle\langle b,d\rangle,a\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+$$

$$+\langle d,\langle\langle b,c\rangle,a\rangle\rangle-\langle d,\langle\langle c,b\rangle,a\rangle\rangle+)\lambda_{59}(\langle b,\langle\langle d,a\rangle,c\rangle\rangle-\langle b,\langle\langle d,c\rangle,a\rangle\rangle+$$
$$+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle b,\langle d,\langle c,a\rangle\rangle\rangle)+$$
$$+\lambda_{65}(\langle c,\langle\langle d,a\rangle,b\rangle\rangle-\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\langle a,\langle c,\langle d,b\rangle\rangle\rangle-\langle b,\langle c,\langle d,a\rangle\rangle\rangle-$$
$$-\langle c,\langle d,\langle a,b\rangle\rangle\rangle+\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{69}(\langle d,\langle\langle b,a\rangle,c\rangle\rangle-\langle d,\langle\langle b,c\rangle,a\rangle\rangle+$$
$$+\langle a,\langle d,\langle b,c\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle b,\langle d,\langle c,a\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+$$
$$+\lambda_{73}(\langle a,\langle b,\langle c,d\rangle\rangle\rangle-\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle b,\langle c,\langle a,d\rangle\rangle\rangle+\langle b,\langle c,\langle d,a\rangle\rangle\rangle+$$
$$+\langle b,\langle d,\langle a,c\rangle\rangle\rangle-\langle b,\langle d,\langle c,a\rangle\rangle\rangle)+\lambda_{75}(\langle a,\langle c,\langle b,d\rangle\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle-$$
$$-\langle b,\langle c,\langle a,d\rangle\rangle\rangle+\langle b,\langle c,\langle d,a\rangle\rangle\rangle+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+$$
$$+\lambda_{77}(\langle a,\langle d,\langle c,b\rangle\rangle\rangle-\langle a,\langle d,\langle b,c\rangle\rangle\rangle+\langle b,\langle d,\langle a,c\rangle\rangle\rangle-\langle b,\langle d,\langle c,a\rangle\rangle\rangle-$$
$$-\langle c,\langle d,\langle a,b\rangle\rangle\rangle+\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{1}(\langle\langle\langle a,b\rangle,c\rangle,d\rangle+\langle b,\langle\langle d,c\rangle,a\rangle\rangle-$$
$$-\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle-\langle\langle\langle a,d\rangle,b\rangle,c\rangle+\langle a,\langle d,\langle b,c\rangle\rangle\rangle+$$
$$+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle d,\langle\langle b,c\rangle,a\rangle\rangle)+\lambda_{4}(\langle\langle\langle b,a\rangle,c\rangle,d\rangle-\langle\langle\langle b,d\rangle,a\rangle,c\rangle)+$$
$$+\langle a,\langle\langle d,c\rangle,b\rangle\rangle-\langle d,\langle\langle a,c\rangle,b\rangle\rangle-\langle a,\langle b,\langle d,c\rangle\rangle\rangle+\langle a,\langle c,\langle d,b\rangle\rangle\rangle+$$
$$+\langle b,\langle d,\langle a,c\rangle\rangle\rangle-\langle c,\langle d,\langle a,b\rangle\rangle\rangle)+\lambda_{19}(\langle\langle b,a\rangle,\langle c,d\rangle\rangle-\langle\langle c,b\rangle,\langle d,a\rangle\rangle-$$
$$-\langle b,\langle\langle d,c\rangle,a\rangle\rangle-\langle c,\langle\langle b,d\rangle,a\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\langle d,\langle\langle b,c\rangle,a\rangle\rangle+$$
$$+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{23}(\langle\langle c,a\rangle,\langle b,d\rangle\rangle-\langle\langle c,b\rangle,\langle d,a\rangle\rangle-$$
$$-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\langle a,\langle d,\langle b,c\rangle\rangle\rangle+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+$$
$$+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{25}(\langle\langle a,\langle b,c\rangle\rangle,d\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle-$$
$$-\langle b,\langle\langle a,d\rangle,c\rangle\rangle+\langle b,\langle\langle d,c\rangle,a\rangle\rangle+\langle d,\langle\langle a,c\rangle,b\rangle\rangle-\langle d,\langle\langle b,c\rangle,a\rangle\rangle+$$
$$+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle c,a\rangle\rangle\rangle)+\lambda_{29}(\langle\langle a,\langle d,b\rangle\rangle,c\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle+$$
$$+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle b,\langle d,\langle c,a\rangle\rangle\rangle+$$
$$+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{33}(\langle\langle b,\langle c,a\rangle\rangle,d\rangle-\langle b,\langle\langle d,c\rangle,a\rangle\rangle-$$
$$-\langle c,\langle\langle b,d\rangle,a\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\langle d,\langle\langle b,c\rangle,a\rangle\rangle-\langle d,\langle\langle c,b\rangle,a\rangle\rangle+$$
$$+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle)+\lambda_{49}(\langle a,\langle\langle b,c\rangle,d\rangle\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle-$$
$$-\langle b,\langle\langle a,d\rangle,c\rangle\rangle+\langle b,\langle\langle d,c\rangle,a\rangle\rangle+\langle d,\langle\langle a,c\rangle,b\rangle\rangle-\langle d,\langle\langle b,c\rangle,a\rangle\rangle+$$
$$+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle c,a\rangle\rangle\rangle)+\lambda_{53}(\langle a,\langle\langle d,b\rangle,c\rangle\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle+$$
$$+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle b,\langle d,\langle c,a\rangle\rangle\rangle+$$
$$+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{57}(\langle b,\langle\langle c,a\rangle,d\rangle\rangle-\langle b,\langle\langle d,c\rangle,a\rangle\rangle-$$
$$-\langle c,\langle\langle b,d\rangle,a\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\langle d,\langle\langle b,c\rangle,a\rangle\rangle-\langle d,\langle\langle c,b\rangle,a\rangle\rangle+$$
$$+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle)+\lambda_{2}(\langle\langle\langle a,c\rangle,b\rangle,d\rangle-\langle\langle\langle a,d\rangle,b\rangle,c\rangle+$$
$$+\langle c,\langle\langle d,b\rangle,a\rangle\rangle-\langle d,\langle\langle c,b\rangle,a\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\langle a,\langle d,\langle b,c\rangle\rangle\rangle+$$
$$+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+$$
$$+\lambda_{21}(\langle\langle b,c\rangle,\langle d,a\rangle\rangle-\langle\langle c,b\rangle,\langle d,a\rangle\rangle-\langle b,\langle\langle d,c\rangle,a\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+$$
$$+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle b,\langle d,\langle c,a\rangle\rangle\rangle+$$
$$+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{13}(\langle\langle a,b\rangle,\langle c,d\rangle\rangle-\langle\langle c,b\rangle,\langle d,a\rangle\rangle-$$
$$-\langle a,\langle\langle d,c\rangle,b\rangle\rangle-\langle c,\langle\langle a,d\rangle,b\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\langle d,\langle\langle a,c\rangle,b\rangle\rangle-$$

$$
\begin{aligned}
&-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\langle a,\langle d,\langle b,c\rangle\rangle\rangle+2\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\\
&+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-2\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{20}(\langle\langle b,a\rangle,\langle d,c\rangle\rangle-\langle\langle c,b\rangle,\langle d,a\rangle\rangle-\\
&-\langle b,\langle\langle d,c\rangle,a\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\langle a,\langle d,\langle b,c\rangle\rangle\rangle+\\
&+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle b,\langle d,\langle c,a\rangle\rangle\rangle+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\\
&-2\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{10}(\langle\langle\langle d,a\rangle,b\rangle,c\rangle-\langle\langle\langle d,c\rangle,a\rangle,b\rangle+\langle a,\langle\langle d,c\rangle,b\rangle\rangle-\\
&-\langle c,\langle\langle d,b\rangle,a\rangle\rangle-\langle d,\langle\langle a,c\rangle,b\rangle\rangle+\langle d,\langle\langle c,b\rangle,a\rangle\rangle+\langle a,\langle c,\langle d,b\rangle\rangle\rangle-\\
&-\langle a,\langle d,\langle b,c\rangle\rangle\rangle-\langle b,\langle c,\langle d,a\rangle\rangle\rangle+\langle b,\langle d,\langle a,c\rangle\rangle\rangle-\langle c,\langle d,\langle a,b\rangle\rangle\rangle+\\
&+\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\lambda_{14}(\langle\langle a,b\rangle,\langle d,c\rangle\rangle-\langle\langle c,b\rangle,\langle d,a\rangle\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle+\\
&+\langle c,\langle\langle d,b\rangle,a\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\langle a,\langle d,\langle b,c\rangle\rangle\rangle+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\\
&-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle b,\langle d,\langle c,a\rangle\rangle\rangle+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-2\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\\
&+\lambda_{22}(\langle\langle b,d\rangle,\langle c,a\rangle\rangle-\langle\langle c,b\rangle,\langle d,a\rangle\rangle-\langle b,\langle\langle d,c\rangle,a\rangle\rangle-\langle c,\langle\langle b,d\rangle,a\rangle\rangle+\\
&+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\langle d,\langle\langle b,c\rangle,a\rangle\rangle+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\\
&+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\\
&+\lambda_{26}(\langle\langle a,\langle b,d\rangle\rangle,c\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle-\langle b,\langle\langle a,d\rangle,c\rangle\rangle+\langle b,\langle\langle d,c\rangle,a\rangle\rangle+\\
&+\langle d,\langle\langle a,c\rangle,b\rangle\rangle-\langle d,\langle\langle b,c\rangle,a\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\langle a,\langle d,\langle b,c\rangle\rangle\rangle+\\
&+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\\
&+\lambda_{27}(\langle\langle a,\langle c,b\rangle\rangle,d\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle-\langle c,\langle\langle a,d\rangle,b\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\\
&+\langle d,\langle\langle a,c\rangle,b\rangle\rangle-\langle d,\langle\langle c,b\rangle,a\rangle\rangle+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\\
&-\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\\
&+\lambda_{28}(\langle\langle a,\langle c,d\rangle\rangle,b\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle-\langle c,\langle\langle a,d\rangle,b\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\\
&+\langle d,\langle\langle a,c\rangle,b\rangle\rangle-\langle d,\langle\langle c,b\rangle,a\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\langle a,\langle d,\langle b,c\rangle\rangle\rangle+\\
&+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\\
&+\lambda_{16}(\langle\langle a,c\rangle,\langle d,b\rangle\rangle-\langle\langle c,b\rangle,\langle d,a\rangle\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\\
&+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-2\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\langle a,\langle d,\langle b,c\rangle\rangle\rangle+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\\
&-2\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle b,\langle d,\langle c,a\rangle\rangle\rangle+2\langle c,\langle d,\langle a,b\rangle\rangle\rangle-2\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\\
&+\lambda_{17}(\langle\langle a,d\rangle,\langle b,c\rangle\rangle-\langle\langle c,b\rangle,\langle d,a\rangle\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle-\langle b,\langle\langle a,d\rangle,c\rangle\rangle+\\
&+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\langle d,\langle\langle a,c\rangle,b\rangle\rangle+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\\
&+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\\
&+\lambda_{18}(\langle\langle a,d\rangle,\langle c,b\rangle\rangle-\langle\langle c,b\rangle,\langle d,a\rangle\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle-\langle c,\langle\langle a,d\rangle,b\rangle\rangle+\\
&+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\langle d,\langle\langle a,c\rangle,b\rangle\rangle+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\\
&+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\\
&+\lambda_{50}(\langle a,\langle\langle b,d\rangle,c\rangle\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle-\langle b,\langle\langle a,d\rangle,c\rangle\rangle+\langle b,\langle\langle d,c\rangle,a\rangle\rangle+\\
&+\langle d,\langle\langle a,c\rangle,b\rangle\rangle-\langle d,\langle\langle b,c\rangle,a\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+\langle a,\langle d,\langle b,c\rangle\rangle\rangle+\\
&+\langle b,\langle c,\langle d,a\rangle\rangle\rangle-\langle b,\langle d,\langle a,c\rangle\rangle\rangle+\langle c,\langle d,\langle a,b\rangle\rangle\rangle-\langle c,\langle d,\langle b,a\rangle\rangle\rangle)+\\
&+\lambda_{51}(\langle a,\langle\langle c,b\rangle,d\rangle\rangle-\langle a,\langle\langle d,c\rangle,b\rangle\rangle-\langle c,\langle\langle a,d\rangle,b\rangle\rangle+\langle c,\langle\langle d,b\rangle,a\rangle\rangle+\\
&+\langle d,\langle\langle a,c\rangle,b\rangle\rangle-\langle d,\langle\langle c,b\rangle,a\rangle\rangle+\langle a,\langle b,\langle d,c\rangle\rangle\rangle-\langle a,\langle c,\langle d,b\rangle\rangle\rangle+
\end{aligned}
$$

$$+ \langle b, \langle c, \langle d, a\rangle\rangle\rangle - \langle b, \langle d, \langle a, c\rangle\rangle\rangle + \langle c, \langle d, \langle a, b\rangle\rangle\rangle - \langle c, \langle d, \langle b, a\rangle\rangle\rangle) +$$
$$+ \lambda_{52}(\langle a, \langle\langle c, d\rangle, b\rangle\rangle - \langle a, \langle\langle d, c\rangle, b\rangle\rangle - \langle c, \langle\langle a, d\rangle, b\rangle\rangle + \langle c, \langle\langle d, b\rangle, a\rangle\rangle +$$
$$+ \langle d, \langle\langle a, c\rangle, b\rangle\rangle - \langle d, \langle\langle c, b\rangle, a\rangle\rangle - \langle a, \langle c, \langle d, b\rangle\rangle\rangle + \langle a, \langle d, \langle b, c\rangle\rangle\rangle +$$
$$+ \langle b, \langle c, \langle d, a\rangle\rangle\rangle - \langle b, \langle d, \langle a, c\rangle\rangle\rangle + \langle c, \langle d, \langle a, b\rangle\rangle\rangle - \langle c, \langle d, \langle b, a\rangle\rangle\rangle) +$$
$$+ \lambda_{15}(\langle\langle a, c\rangle, \langle b, d\rangle\rangle - \langle\langle c, b\rangle, \langle d, a\rangle\rangle - \langle a, \langle\langle d, c\rangle, b\rangle\rangle - \langle b, \langle\langle a, d\rangle, c\rangle\rangle +$$
$$+ \langle c, \langle\langle d, b\rangle, a\rangle\rangle + \langle d, \langle\langle a, c\rangle, b\rangle\rangle + \langle a, \langle b, \langle d, c\rangle\rangle\rangle - 2\langle a, \langle c, \langle d, b\rangle\rangle\rangle +$$
$$+ \langle a, \langle d, \langle b, c\rangle\rangle\rangle + 2\langle b, \langle c, \langle d, a\rangle\rangle\rangle - 2\langle b, \langle d, \langle a, c\rangle\rangle\rangle + 2\langle c, \langle d, \langle a, b\rangle\rangle\rangle -$$
$$- 2\langle c, \langle d, \langle b, a\rangle\rangle\rangle).$$

From there we take that the monomials are interdependent. As a result, we obtain 6 identities.
Therefore,

$$g = \lambda_{30}k_1 + \lambda_5 k_2 + \lambda_{31}k_3 + \lambda_{11}k_4 + \lambda_{35}k_5 + \lambda_1 k_6 = 0$$

where

$$k_1(a, b, c, d) = \langle\langle a, \langle d, c\rangle\rangle, b\rangle - \langle a, \langle\langle d, c\rangle, b\rangle\rangle$$

$$k_2(a, b, c, d) = \langle\langle\langle b, c\rangle, a\rangle, d\rangle - \langle\langle\langle b, d\rangle, a\rangle, c\rangle + \langle c, \langle\langle d, b\rangle, a\rangle\rangle - \langle d, \langle\langle c, b\rangle, a\rangle\rangle$$

$$k_3(a, b, c, d) = \langle\langle b, \langle a, c\rangle\rangle, d\rangle - \langle b, \langle\langle a, d\rangle, c\rangle\rangle + \langle b, \langle c, \langle d, a\rangle\rangle\rangle - \langle b, \langle d, \langle c, a\rangle\rangle\rangle$$

$$\begin{aligned} k_4(a, b, c, d) = {} & \langle\langle\langle d, b\rangle, a\rangle, c\rangle - \langle\langle\langle d, c\rangle, a\rangle, b\rangle + \langle b, \langle\langle d, c\rangle, a\rangle\rangle - \langle c, \langle\langle d, b\rangle, a\rangle\rangle \\ & - \langle d, \langle\langle b, c\rangle, a\rangle\rangle + \langle d, \langle\langle c, b\rangle, a\rangle\rangle \end{aligned}$$

$$\begin{aligned} k_5(a, b, c, d) = {} & \langle\langle b, \langle d, a\rangle\rangle, c\rangle - \langle b, \langle\langle d, c\rangle, a\rangle\rangle + \langle a, \langle b, \langle d, c\rangle\rangle\rangle - \langle b, \langle c, \langle d, a\rangle\rangle\rangle \\ & - \langle b, \langle d, \langle a, c\rangle\rangle\rangle + \langle b, \langle d, \langle c, a\rangle\rangle\rangle \end{aligned}$$

$$\begin{aligned} k_6(a, b, c, d) = {} & \langle\langle\langle a, b\rangle, c\rangle, d\rangle + \langle b, \langle\langle d, c\rangle, a\rangle\rangle - \langle a, \langle b, \langle d, c\rangle\rangle\rangle - \langle c, \langle d, \langle b, a\rangle\rangle\rangle \\ & - \langle\langle\langle a, d\rangle, b\rangle, c\rangle + \langle a, \langle d, \langle b, c\rangle\rangle\rangle + \langle b, \langle c, \langle d, a\rangle\rangle\rangle - \langle d, \langle\langle b, c\rangle, a\rangle\rangle \end{aligned}$$

This implies that any identity of degree 4 of the class $\mathcal{B}_{pq}$ can be derived from the identities $\{k_1, k_2, k_3, k_4, k_5, k_6.\}$

So, we have obtained 61 identities. Now, we need to show that all of them follows from the Theorem 3.2.11.

We select the coefficients of the monomials based on their order in the preceding list 3.2.12. In other words, the columns of a matrix correspond to the monomials and the rows of a matrix represent each polynomial with all possible permutations of f_i where $i \in \{1, 2, 3, 4, 5, 6\}$ in the variables a, b, c, d. The resulting matrix achieves a rank of 106.

After that, we construct the matrix without changing its columns; however, the rows represent each polynomial resulting from all possible permutations of

$g_i(a, b, c, d)$ where $i \in \{1, 2, 3, 4, 5, 6\}$ and the other 61 identities. Hence, the 61 identities follows from the identities in the Theorem 3.2.11.

$\square$

4. Conclusion

The main task of this thesis was to prove that any bicommutative algebra under the mutation product satisfies an identity of degree three and all identities of degree four.

We have built bases by constructing theorems and proved theorems by using the methods of linear algebra.

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