

Omer Cakir
Suleyman Demirel University

About existense of solution of third-order derivative of $-y''' + q(x)y = f$ like equations

Let's consider following equation in range $L_2 \llbracket \pi; \pi^- \rrbracket$:

$$-y''' + q(x)y = f \in L_2 \llbracket \pi; \pi^- \rrbracket \quad (0.1)$$

$$y^{(i)} \llbracket \pi \rrbracket = y^{(i)}(\pi), \quad i = 0,1,2. \quad (0.2)$$

or

$$Ly = -y''' + q(x)y$$

by adding this notation:

$$Ly = f \in L_2 \llbracket \pi; \pi^- \rrbracket \quad (1)$$

$$y^{(i)} \llbracket \pi \rrbracket = y^{(i)}(\pi), \quad i = 0,1,2. \quad (2)$$

(1) we get this operator. Let's prove this operator (1) by principles of conditions (2).

Theorem 1. Assume that the function $q(x) \geq 1$ is continuous on $\llbracket \pi; \pi^- \rrbracket$, then for $\forall f \in L_2$ there is only one solution for problems (2.1)-(2.2).

Before proving this theorem we will prove some inequalities and lemmas.

Lemma 1. Assume that $y \in C^3_\pi \llbracket \pi; \pi^- \rrbracket$, then

$$\int_{-\pi}^{\pi} y'''y dx = 0 \quad (3)$$

equation is true.

Proof: To prove lemma we use integration by part.

$$\begin{aligned} \int_{-\pi}^{\pi} y'''y dx &= \int_{-\pi}^{\pi} y dy'' = yy'' \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} y''y' dx = - \int_{-\pi}^{\pi} y' dy' = -y'y' \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} y'y'' dx = \int_{-\pi}^{\pi} y'' dy = y''y \Big|_{-\pi}^{\pi} - \\ &- \int_{-\pi}^{\pi} yy''' dx = - \int_{-\pi}^{\pi} yy''' dx \end{aligned}$$

As a result we get

$$\int_{-\pi}^{\pi} y'''y dx = - \int_{-\pi}^{\pi} yy''' dx.$$

It means $2 \int_{-\pi}^{\pi} yy''' dx = 0$. Proof is complete.

Lemma 2. For every $y \in C^3_\pi \llbracket \pi; \pi^- \rrbracket$, the following inequality holds:

$$\|Ly\|_2 \geq \|y\|_2$$

Proof: To prove lemma we consider scalar product and equation (3),

$$\begin{aligned} \langle Ly, y \rangle &= \int_{-\pi}^{\pi} \llbracket y''' + q(x)y \rrbracket dx = - \int_{-\pi}^{\pi} y'''y dx + \int_{-\pi}^{\pi} q(x)y^2 dx = \\ &= \int_{-\pi}^{\pi} q(x)y^2 dx \geq \int_{-\pi}^{\pi} y^2 dx = \int_{-\pi}^{\pi} y^2 dx = \|y\|_2^2. \end{aligned}$$

Using the Cauchy-Bunyakovskiy-Schwarz inequality we get inequality:

$$|\langle Ly, y \rangle| \leq \|Ly\|_2 \|y\|_2$$

It means,

$$\|y\|_2^2 \leq |\langle Ly, y \rangle| \leq \|Ly\|_2 \|y\|_2.$$

From this

$$\|Ly\|_2 \geq \|y\|_2$$

inequality is true. Proof is complete.

Lemma3. Assume that theorem 1 holds, and then $KerL^* = \mathbb{C}$ is true.

Proof: Let's prove this lemma by contradiction. Suppose that the set of values of operator L doesn't coincide with space L_2 , or shortly $R(L) \neq L_2$. It means, $\forall v \in L_2, v \perp R(L)$, so, $\forall y \in D(L)$.

By Riezs theorem there exists an operator conjugated to operator L , that satisfies the equation below

$$\langle Ly, v \rangle = \langle y, L^*v \rangle = 0.$$

So,

$$\langle y, L^*v \rangle = \int_{-\pi}^{\pi} y(v''' + q(x)v) dx = 0.$$

This equation holds for $\forall y \in D(L)$ and since $D(L)$ is a dense we get $v''' + q(x)v = 0$. From $v''' = -q(x)v$, where $\forall v \in L_2$, and $q(x)$ is continuous function. So $v''' \in L_2$, it means that we have a third-order differential. Because of existence of 1,2,3-order differentials we can integrate by parting.

$$\langle y, L^*v \rangle = \int_{-\pi}^{\pi} y(v''' + q(x)v) dx = \int_{-\pi}^{\pi} yv''' dx + \int_{-\pi}^{\pi} q(x)yv dx = 0$$

Let's integrate each part one by one. Then:

$$\int_{-\pi}^{\pi} yv''' dx = \int_{-\pi}^{\pi} ydv'' = yv'' \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} v''y' dx = yv'' \Big|_{-\pi}^{\pi} - y'v' \Big|_{-\pi}^{\pi} + y''v \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} yv''' dx$$

y is periodic function, so we'll look for v, v', v'' . $y = \sin x \in D(L)$, then, for $y = \sin x$ we will check the equation above. Then

$$\int_{-\pi}^{\pi} yv''' dx = yv'' \Big|_{-\pi}^{\pi} - y'v' \Big|_{-\pi}^{\pi} + y''v \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} yv''' dx$$

where $y yv'' \Big|_{-\pi}^{\pi} = 0$ and $y''v \Big|_{-\pi}^{\pi} = 0$. Then

$$\int_{-\pi}^{\pi} y(v''' + q(x)v) dx = \cos(\pi)v'(\pi) - \cos(-\pi)v'(-\pi) + \int_{-\pi}^{\pi} (-y''' + q(x)y)v dx = -v'(\pi) + v'(-\pi) = 0$$

Now it is easy to prove that $v'(\pi) = v'(-\pi)$.

If $y = 1$, then $y'v' \Big|_{-\pi}^{\pi} = 0$, so $y''v \Big|_{-\pi}^{\pi} = 0$ is true. We can also see that $v''(\pi) = v''(-\pi)$ is true.

Similarly if $y = \cos x$, then $y'v' \Big|_{-\pi}^{\pi} = 0$. If we look at $v''(\pi) = v''(-\pi)$ we get that $yv'' \Big|_{-\pi}^{\pi} = 0$. Then from

$$-\cos(\pi)v(\pi) + \cos(-\pi)v(-\pi) = 0$$

ratio we get equation $v(\pi) = v(-\pi)$.

By this we show that functions v, v', v'' are periodic. And we also get $v \in C_{\pi}^3 \llbracket \pi; \pi \rrbracket$.

From Lemma 3.2.2 the following inequality holds

$$\|L^*v\|_2 \geq \|v\|_2.$$

$v''' + q(x)v = 0$ but it is also equal to L^*v .

Then we get

$$\|0\|_2 \geq \|v\|_2,$$

inequality, it means, $v = 0$. So $\text{Ker}L = \{0\}$. We proved that the kernel is free.

Lemma 4. Assume that the theorem holds, then for operator L determined at all L_2 , there exist an inverse operator L^{-1} .

Proof: Let's prove by contradiction. Assume that for elements $y_1, y_2 \in D(L)$ there is only one suitable f , it means $Ly_1 = f$ and $Ly_2 = f$. If we consider that operator L is linear $L(y_1 - y_2) = 0$. From this we get inequality

$$\|L(y_1 - y_2)\|_2 \geq \|y_1 - y_2\|_2.$$

From this inequality we get $y_1 = y_2$. So L is self-compatible

If in inequality

$$\|Ly\|_2 \geq \|y\|_2$$

We consider that $Ly = f$ then we get inequality:

$$\|f\|_2 \geq \|L^{-1}f\|_2.$$

So, for operator L there exists a limited inverse operator L^{-1} . Lemma is proved.

Proof of theorem: To prove theorem we use 3 and 4 lemmas.

USED MATERIALS (REFERENCES)

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Түйіндеме

Бұл жұмыста Гильберт кеңістігінде үшінші ретті дифференциалдық теңдеудің шешімінің бар болуы зерттелді.

Rezume

In this work we research existence of a solution of third ordered differential equations in Hilbert Space.

Özet

Biz Hilbert uzayda üçüncü dereceden diferansiyel denklemlerin çözülebilen araştırmak.