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# Hypoelliptic functional inequalities and applications

THESIS

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# Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Seitkan Meruyert

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I would like to express my deepest gratitude to Nurgissa Yessirkegenov for his invaluable guidance and support.

# Dedication

This thesis is dedicated to:  
My family and Madina Kalaman.

# Abstract

In this thesis we discuss cylindrical extensions of the improved Rellich inequalities on  $\mathbb{R}^k \times \mathbb{R}^{n-k}$  with the Euclidean norm  $|\cdot|_k$  on  $\mathbb{R}^k$ . Actually, we show sharp remainders of the Rellich type inequalities yielding the classical Rellich inequality. Moreover, Rellich type identities and inequalities with more general weights are established. In addition, we present horizontal extensions of these results on stratified Lie groups.

# Аңдатпа

Бұл диссертацияда  $\mathbb{R}^k \times \mathbb{R}^{n-k}$  кеңістігіндегі жақсартылған Релих теңсіздіктерінің цилиндрлік кеңейтулерін  $\mathbb{R}^k$  кеңістігіндегі евклидтік нормамен  $|\cdot|_k$  аламыз. Сондай-ақ, біз классикалық Релих теңсіздігіне келтірілетін Релих типті теңсіздіктердің нақты қалдықтарын талқылаймыз. Сонымен қатар, жалпы салмақтары бар Релих типті теңдіктер мен теңсіздіктер алынды. Біз осы нәтижелердің көлденең нұсқаларын стратификацияланған Ли топтарына кеңейтеміз.

# Аннотация

В данной диссертации мы обсуждаем цилиндрические расширения неравенств улучшенного Релиха на  $\mathbb{R}^k \times \mathbb{R}^{n-k}$  с евклидовой нормой  $|\cdot|_k$  на  $\mathbb{R}^k$ . Более того, мы покажем точные остатки неравенств типа Релиха приводящие к классическому неравенству Релиха. Мы устанавливаем тождества и неравенства типа Релиха с более общими весами. Кроме того, мы расширяем горизонтальные версии этих результатов на стратифицированные группы Ли.

# Table of Contents

<b>Declaration</b>	<b>i</b>
<b>Acknowledgements</b>	<b>ii</b>
<b>Dedication</b>	<b>iii</b>
<b>Abstract</b>	<b>iv</b>
<b>Аңдатпа</b>	<b>v</b>
<b>Аннотация</b>	<b>vi</b>
<b>1 Background and motivations</b>	<b>1</b>
1.1 Introduction . . . . .	1
<b>2 Preliminaries</b>	<b>5</b>
2.1 Stratified Lie groups . . . . .	5
<b>3 Rellich type identities and inequalities on <math>\mathbb{R}^k \times \mathbb{R}^{n-k}</math></b>	<b>8</b>
3.1 Rellich type identity and inequality on $\mathbb{R}^k \times \mathbb{R}^{n-k}$ . . . . .	8
<b>4 Rellich type identities and inequalities on stratified Lie groups</b>	<b>23</b>
4.1 Rellich type identity and inequality on stratified Lie groups . . . . .	23
<b>5 Conclusions and future work</b>	<b>37</b>
5.1 Conclusions . . . . .	37
<b>Bibliography</b>	<b>38</b>

# Chapter 1

## Background and motivations

### 1.1 Introduction

In 1954 at the International Congress of Mathematicians German mathematician Rellich for the first time introduced the following inequality for all  $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$  and  $n \in \mathbb{N} \setminus \{2\}$ :

$$\left(\frac{n(n-4)}{4}\right)^2 \left\| \frac{f}{\|x\|_E^2} \right\|_{L^2(\mathbb{R}^n)}^2 \leq \|\Delta f\|_{L^2(\mathbb{R}^n)}^2, \quad (1.1.1)$$

where  $\|\cdot\|_E$  denotes the Euclidean norm on  $\mathbb{R}^n$  and  $\Delta$  denotes the standard Laplacian on  $\mathbb{R}^n$ . The constant  $\left(\frac{n(n-4)}{4}\right)^2$  is sharp. In the case when  $n = 2$  this inequality is still true with the following additional condition:

$$\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\cos\theta d\theta = \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\sin\theta d\theta = 0.$$

The Rellich inequality plays a key role in the theory of partial differential equations and spectral theory. For example, such inequalities find diverse applications in exploring fourth-order elliptic and parabolic partial differential equations see, e.g. [1], [2] and [3]. Therefore, Rellich type inequalities have many important applications in various fields of mathematics and physics.

Define the operator  $T$  by

$$Tu = \Delta\Delta u - c|x|^{-4}u$$

on  $L^2(\mathbb{R}^n)$  with domain  $\mathcal{D}(T) = C_0^\infty(\mathbb{R}^n \setminus \{0\})$ . Then  $T$  is symmetric on  $\mathcal{D}(T)$ . As Rellich mentions in [4] his inequality can be viewed as a means of finding the largest positive real number  $c$  such that the eigenvalues for the Friedrich's extension of  $T$  are nonnegative. For the solution we turn to Rellich's inequality.

Hence, for the operator  $T$  given above,  $c = n^2(n-4)^2/16$  is the largest real

number such that the operator  $T$  is nonnegative.

Rellich inequalities become valuable when dealing with critical potentials. As a basic illustration, these inequalities provide insights into the existence of solutions and their asymptotic behavior for the equation  $u_t = -\Delta^2 + V$  for critical potentials  $V$ .

The  $L_p$  version of the Rellich inequality for  $n > 2p$  and  $1 < p < \infty$  was proved in [2] for all  $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ :

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{2p}} dx \leq \left( \frac{p^2}{n(p-1)(n-2p)} \right)^p \int_{\mathbb{R}^n} |\Delta f(x)|^p dx.$$

Evans and Lewis introduced Rellich inequality with remainder terms in [5]. Nowadays, there are many works concerning the Rellich inequality, see e.g. [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18] and [19].

In this thesis, our approach is mainly inspired by the works [20] and [21] of Japanese mathematicians H. Wadade, N. Bez, S. Machihara and T. Ozawa. There, they presented a new perspective on the Rellich inequality, namely by investigating the impact arising from radial derivatives and spherical derivatives. It is well-known that the Laplacian decomposes as  $\Delta = \Delta_r + \frac{1}{|x|^2} \Delta_{\mathbb{S}^{n-1}}$  where  $\Delta_r$  denotes the radial Laplacian and  $\Delta_{\mathbb{S}^{n-1}}$  denotes the Laplace-Beltrami operator. The radial Laplacian is given by

$$\Delta_r f = \partial_r^2 f + \frac{n-1}{|x|} \partial_r f,$$

where  $\partial_r$  is the radial derivative given by  $\partial_r = \frac{x}{|x|} \cdot \nabla$ . Also, for each  $j = 1, \dots, n$  these authors introduced the spherical-type derivative  $L_j = \partial_j - \frac{x_j}{|x|} \partial_r$  in which case there is the relation

$$\Delta_{\mathbb{S}^{n-1}} = |x|^2 \sum_{j=1}^n L_j^2.$$

These authors in [20] and [21] presented following Rellich type identity by encompassing the radial and spherical contributions on both sides of (1.1.1) for  $n \geq 2$  and  $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ :

$$\begin{aligned} \left( \frac{n(n-4)}{4} \right)^2 \left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)}^2 &= \left\| \partial_r^2 f + \frac{n-1}{|x|} \partial_r f \right\|_{L^2(\mathbb{R}^n)}^2 \\ &- \left\| \partial_r^2 f + \frac{n-1}{|x|} \partial_r f + \frac{n(n-4)}{4|x|^2} f \right\|_{L^2(\mathbb{R}^n)}^2 - \frac{n(n-4)}{2} \left\| \frac{1}{|x|} \partial_r f + \frac{n-4}{2|x|^2} f \right\|_{L^2(\mathbb{R}^n)}^2, \end{aligned} \quad (1.1.2)$$

which imply the Rellich type inequality for the radial part of the Laplacian by

dropping non-negative remainder terms:

$$\left(\frac{n(n-4)}{4}\right)^2 \left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)}^2 \leq \left\| \partial_r^2 f + \frac{n-1}{|x|} \partial_r f \right\|_{L^2(\mathbb{R}^n)}^2.$$

By [20, Corollary 1.3] for  $n \geq 3$  it follows that

$$\left(\frac{n(n-4)}{4}\right)^2 \left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)}^2 \leq \left\| \partial_r^2 f + \frac{n-1}{|x|} \partial_r f \right\|_{L^2(\mathbb{R}^n)}^2 \leq \|\Delta f\|_{L^2(\mathbb{R}^n)}^2, \quad (1.1.3)$$

which means that this inequality gives improvement of the classical Rellich inequality.

It was hypothesized by Badiale and Tarantello [22] that the best constant in the following extended Hardy inequality:

$$\left\| \frac{1}{|x'|} f \right\|_{L^p(\mathbb{R}^n)} \leq C_{n,k,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad (1.1.4)$$

on  $\mathbb{R}^k \times \mathbb{R}^{n-k}$  is  $\frac{p}{k-p}$ , where  $\nabla$  is the standard Euclidean gradient. This conjecture was proved by S. Secchi, D. Smets and M. Willen [23]. It is natural question to obtain such cylindrical version of the classical Rellich inequality.

Our aim is to investigate Rellich type inequalities on  $\mathbb{R}^k \times \mathbb{R}^{n-k}$  where we take  $|x'|$  to be the Euclidean norm on  $\mathbb{R}^k$  as a weight function. Let  $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$  and  $\alpha - 2 \leq k \leq n$ . Then we have the following Rellich identity:

$$\begin{aligned} \left(\frac{k(k-\alpha)}{4}\right)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 &= \left\| \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\ &- \left\| \frac{k(k-\alpha)}{4|x'|_k^{\frac{\alpha}{2}}} f + \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\ &- \frac{k(k-\alpha)}{2} \left\| \frac{k-\alpha}{2} \frac{f}{|x'|_k^{\frac{\alpha}{2}}} + \frac{(x' \cdot \nabla_k)}{|x'|_k^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2, \quad (1.1.5) \end{aligned}$$

for all  $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ , where  $\nabla_k$  is the Euclidean gradient on  $\mathbb{R}^k$ . Note that when  $k = n$  and  $\alpha = 4$  we obtain identity (1.1.2). Such results on  $\mathbb{R}^k \times \mathbb{R}^{n-k}$  will be discussed in more details in Chapter 3.

In the special case when  $\alpha = 4$  we get the following identity:

$$\begin{aligned} \left(\frac{k(k-4)}{4}\right)^2 \left\| \frac{f}{|x'|^2} \right\|_{L^2(\mathbb{R}^n)}^2 &= \left\| \left(\frac{x' \cdot \nabla_k}{|x'|}\right)^2 f + \frac{k-1}{|x'|} \left(\frac{x' \cdot \nabla_k}{|x'|}\right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\ &- \left\| \frac{k(k-4)}{4|x'|^2} f + \left(\frac{x' \cdot \nabla_k}{|x'|}\right)^2 f + \frac{k-1}{|x'|} \left(\frac{x' \cdot \nabla_k}{|x'|}\right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\ &- \frac{k(k-4)}{2} \left\| \frac{k-4}{2} \frac{f}{|x'|^2} + \frac{(x' \cdot \nabla_k)}{|x'|^2} f \right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \quad (1.1.6)$$

In the identity (1.1.6), by dropping the last two non-negative terms on the right-hand side, we obtain the following Rellich type inequality:

$$\left(\frac{k(k-4)}{4}\right)^2 \left\| \frac{f}{|x'|_k^2} \right\|_{L^2(\mathbb{R}^n)}^2 \leq \left\| \left(\frac{x' \cdot \nabla_k}{|x'|_k}\right)^2 f + \frac{k-1}{|x'|_k} \left(\frac{x' \cdot \nabla_k}{|x'|_k}\right) f \right\|_{L^2(\mathbb{R}^n)}^2,$$

for all  $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$  and  $k \geq 2$ . In the case when  $k = n$ , taking into account  $\frac{d}{d|x|_E} = \frac{x \cdot \nabla}{|x|_E}$  we get the improvement of the classical Rellich inequality (1.1.3)

Furthermore, in this thesis we also discuss extensions of these identities and inequalities on stratified Lie groups in Chapter 4. Necessary notations and definitions needed for our analysis will be discussed in Chapter 2. In Chapter 5, we give a conclusion on the obtained results and state possible future directions.

Results of this thesis partly were announced in [24] and [25].

# Chapter 2

## Preliminaries

In this chapter, we provide necessary definitions and notations needed for our analysis from [26], [27] and [28].

### 2.1 Stratified Lie groups

In this section the essential notations and definitions concerning the setting of stratified Lie groups are shortly recalled.

A group  $\mathbb{G} = (\mathbb{R}^n, \circ)$  which satisfies the following two conditions:

- For some natural numbers  $N + N_2 + \dots + N_r = n$ , that is  $N = N_1$ , the decomposition  $\mathbb{R}^n = \mathbb{R}^N \times \dots \times \mathbb{R}^{N_r}$  is valid, and for every  $\lambda > 0$  the dilation  $\delta_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by

$$\delta_\lambda(x) = \delta_\lambda(x', x^{(2)}, \dots, x^{(r)}) := (\lambda x', \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)})$$

is an automorphism of the group  $\mathbb{G}$ . Here  $x' \equiv x^{(1)} \in \mathbb{R}^N$  and  $x^{(k)} \in \mathbb{R}^{N_k}$  for  $k = 2, \dots, r$ ;

- Let  $N$  be as in above and let  $X_1, \dots, X_N$  be the left invariant vector fields on  $\mathbb{G}$  such that  $X_k(0) = \frac{\partial}{\partial x_k} \Big|_0$  for  $k = 1, \dots, N$ . Then

$$\text{rank}(\text{Lie}\{X_1, \dots, X_N\}) = n,$$

for every  $x \in \mathbb{R}^n$ , i.e. the iterated commutators of  $X_1, \dots, X_N$  span the Lie algebra of  $\mathbb{G}$ ;

is called a stratified Lie group (or a homogeneous Carnot group).

That's why the triple  $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$  is a stratified Lie group. Folland have investigated such groups in detail [29]. A more basic technique without identifying them with  $\mathbb{R}^n$  is possible, but it can be demonstrated to reduce to the above definition, so we may use it from the start. For more thoroughly discussions from the Lie algebra point of view we refer to e.g. [30].

The (Jacobian) generators of  $\mathbb{G}$  is the left invariant vector fields  $X_1, \dots, X_N$  and  $r$  is a step of  $\mathbb{G}$

$$Q = \sum_{k=1}^r k N_k, \quad N_1 = N,$$

is called the homogeneous dimension of  $\mathbb{G}$ . We also recall that the standard Lebesgue measure  $dx$  on  $\mathbb{R}^n$  is the Haar measure for  $\mathbb{G}$  (see, e.g. [31, Proposition 1.6.6]). For more details on stratified Lie groups we refer to [32] or [31].

The left invariant vector fields  $X_j$  have an explicit form and satisfy the divergence theorem, see e.g. [27] and [26] for the derivation of the exact formula: more precisely, we can write

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{\ell=2}^r \sum_{m=1}^{N_1} a_{k,m}^{(\ell)}(x', \dots, x^{\ell-1}) \frac{\partial}{\partial x_m^{(\ell)}}, \quad (2.1.1)$$

see also [30, Section 3.1.5] for a general presentation. Throughout this paper, we will also use the following notations:

$$\nabla_H := (X_1, \dots, X_N)$$

for the horizontal gradient,

$$\operatorname{div}_H v := \nabla_H \cdot v$$

for the horizontal divergence, and

$$|x'| = \sqrt{x_1'^2 + \dots + x_N'^2}$$

for the Euclidean norm on  $\mathbb{R}^N$ .

The explicit representation of the left invariant vector fields  $X_j$  from (2.1.1) allows us to verify the identities

$$|\nabla_H |x'|^\gamma| = \gamma |x'|^{\gamma-1},$$

and

$$\operatorname{div}_H \left( \frac{x'}{|x'|^\gamma} \right) = \frac{\sum_{j=1}^N |x'|^\gamma X_j x'_j - \sum_{j=1}^N x'_j \gamma |x'|^{\gamma-1} X_j |x'|}{|x'|^{2\gamma}} = \frac{N - \gamma}{|x'|^\gamma}$$

for any  $\gamma \in \mathbb{R}, |x'| \neq 0$ .

We will also use the following lemma based on the lammata from [20, Lemma 1.4] and [33, Lemma 1.1] to prove our main results of the thesis:

**Lemma 1.** *Let  $X$  be a scalar product space with scalar product  $\langle \cdot, \cdot \rangle$ . Let  $c > 0$ .*

Then the following statements are equivalent:

- The equality

$$\|u\|^2 = -2c \operatorname{Re}\langle u, v \rangle \quad (2.1.2)$$

holds for all  $u, v \in X$ .

- The equality

$$\|u\|^2 = 4c^2 \|v\|^2 - \|u + 2cv\|^2 \quad (2.1.3)$$

holds for all  $u, v \in X$ .

Moreover, the next two statements are equivalent:

- The equality

$$\|u\|^2 = -c \operatorname{Re} \langle u, v \rangle + b \quad (2.1.4)$$

holds for all  $u, v \in X$ .

- The equality

$$\frac{1}{c^2} \|u\|^2 = \|v\|^2 - \left\| v + \frac{1}{c}u \right\|^2 + \frac{2b}{c} \quad (2.1.5)$$

holds for all  $u, v \in X$ .

In this thesis, we use the scalar product in the  $L^2$  setting, that is, the scalar product of the two functions  $f(x)$  and  $g(x)$  is computed as an integral of the type

$$\langle f(x), g(x) \rangle = \int f(x) \cdot \overline{g(x)} dx.$$

# Chapter 3

## Rellich type identities and inequalities on $\mathbb{R}^k \times \mathbb{R}^{n-k}$

In this chapter, we discuss cylindrical extensions of the improved Rellich inequalities on  $\mathbb{R}^k \times \mathbb{R}^{n-k}$  with the Euclidean norm  $|\cdot|_k$  on  $\mathbb{R}^k$ . Importance of such cylindrical versions of the classical functional inequalities due to applications in investigating cylindrical solutions of partial differential equations arising in astrophysics. Moreover, we establish weighted versions of the improved Rellich type identities.

### 3.1 Rellich type identity and inequality on $\mathbb{R}^k \times \mathbb{R}^{n-k}$

Here we establish a sharp remainder formula for the weighted improved Rellich inequality. In the proof we adopt the following notations:

$$H_k = \frac{k - \alpha + 2}{2} \quad \text{and} \quad R_k = \frac{k(k - \alpha)}{4}.$$

**Theorem 1.** *Let  $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$  and  $\alpha - 2 \leq k \leq n$ . Then we have the Rellich identity for all functions  $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ :*

$$\begin{aligned} \left(\frac{k(k - \alpha)}{4}\right)^2 \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 &= \left\| \left(\frac{x' \cdot \nabla_k}{|x'|_k}\right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left(\frac{x' \cdot \nabla_k}{|x'|_k}\right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\ &- \left\| \frac{k(k - \alpha)}{4|x'|_k^{\frac{\alpha}{2}}} f + \left(\frac{x' \cdot \nabla_k}{|x'|_k}\right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left(\frac{x' \cdot \nabla_k}{|x'|_k}\right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\ &- \frac{k(k - \alpha)}{2} \left\| \frac{k - \alpha}{2} \frac{f}{|x'|_k^{\frac{\alpha}{2}}} + \frac{(x' \cdot \nabla_k)}{|x'|_k^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2, \end{aligned} \quad (3.1.1)$$

where  $|\cdot|$  is the Euclidean norm on  $\mathbb{R}^k$  and  $\nabla_k$  is the Euclidean gradient on  $\mathbb{R}^k$ .

**Remark 1.** In the case when  $\alpha = 4$  and by dropping two non-negative terms of the right-hand side, we obtain the following Rellich type inequality:

$$\begin{aligned} & \left(\frac{k(k-4)}{4}\right)^2 \left\| \frac{f}{|x'|_k^2} \right\|_{L^2(\mathbb{R}^n)}^2 \\ & \leq \left\| \left(\frac{x' \cdot \nabla_k}{|x'|_k}\right)^2 f + \frac{k-1}{|x'|_k} \left(\frac{x' \cdot \nabla_k}{|x'|_k}\right) f \right\|_{L^2(\mathbb{R}^n)}^2 \end{aligned}$$

for all functions  $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$  and  $k \geq 2$ . Note that when  $k = n$ , it implies the following improvement of the classical Rellich inequality if we take into account  $\frac{d}{d|x|_E} = \frac{x \cdot \nabla_k}{|x|_E}$  and [20, Corollary 1.3]:

$$\begin{aligned} & \left(\frac{n(n-4)}{4}\right)^2 \left\| \frac{f}{|x|_E^2} \right\|_{L^2(\mathbb{R}^n)}^2 \\ & \leq \left\| \frac{d^2}{d|x|_E^2} f + \frac{n-1}{|x|_E} \frac{d}{d|x|_E} f \right\|_{L^2(\mathbb{R}^n)}^2 \\ & \leq \|\Delta f\|_{L^2(\mathbb{R}^n)}^2, \quad (3.1.2) \end{aligned}$$

for all functions  $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$  and  $n \geq 3$ .

**Remark 2.** In the case when  $k = n$  and  $\alpha = 4$ , taking into account  $\frac{d}{d|x|_E} = \frac{x \cdot \nabla_k}{|x|_E}$ , we derive from (3.1.1) the following result by N. Bez, S. Machihara and T. Ozawa in [21]:

$$\begin{aligned} & \left(\frac{n(n-4)}{4}\right)^2 \left\| \frac{f}{|x|_E^2} \right\|_{L^2(\mathbb{R}^n)}^2 \\ & = \left\| \frac{d^2 f}{d|x|_E^2} + \frac{n-1}{|x|} \frac{df}{d|x|_E} \right\|_{L^2(\mathbb{R}^n)}^2 \\ & - \left\| \frac{d^2 f}{d|x|_E^2} + \frac{n-1}{|x|_E} \frac{df}{d|x|_E} + \frac{n(n-4)}{4|x|_E^2} f \right\|_{L^2(\mathbb{R}^n)}^2 \\ & - \frac{n(n-4)}{2} \left\| \frac{1}{|x|_E} \frac{df}{d|x|_E} + \frac{n-4}{2|x|_E^2} f \right\|_{L^2(\mathbb{R}^n)}^2, \quad (3.1.3) \end{aligned}$$

for all functions  $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$  and  $n \geq 2$ .

**Proof of Theorem 1.** Let us first prove the identity

$$\begin{aligned} \left(\frac{k-\beta}{2}\right)^2 \left\| \frac{f}{|x'|_k^{\frac{\beta}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 &= \left\| \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\beta}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad - \left\| \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\beta}{2}}} f + \frac{k-\beta}{2|x'|_k^{\frac{\beta}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \quad (3.1.4)$$

For this, we first compute the following equality:

$$\int_{\mathbb{R}^n} \frac{|f|^2}{|x'|_k^\beta} dx = \frac{1}{k-\beta} \int_{\mathbb{R}^n} |f|^2 \operatorname{div}_k \left( \frac{x'}{|x'|_k^\beta} \right) dx.$$

Using integration by parts, it follows that

$$\int_{\mathbb{R}^n} \frac{|f|^2}{|x'|_k^\beta} dx = -\frac{2}{k-\beta} \operatorname{Re} \int_{\mathbb{R}^n} \frac{\overline{x' \cdot \nabla_k f}}{|x'|_k^\beta} f dx. \quad (3.1.5)$$

Recall from (2.1.2) and (2.1.3) that the following two statements are equivalent:

$$\|u\|^2 = -2c \operatorname{Re} \langle u, v \rangle$$

and

$$\|u\|^2 = 4c^2 \|v\|^2 - \|u + 2cv\|^2.$$

Then, using Lemma 1 with

$$\begin{aligned} X &= L^2(\mathbb{R}^n), \\ u &= \frac{f}{|x'|_k^{\frac{\beta}{2}}}, \\ v &= \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\beta}{2}}} \right) f \end{aligned}$$

and

$$c = \frac{1}{k-\beta}$$

it follows that the identity (3.1.5) is equivalent to

$$\begin{aligned} \left\| \frac{f}{|x'|_k^{\frac{\beta}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 &= \left( \frac{2}{k-\beta} \right)^2 \left\| \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\beta}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad - \left\| \frac{f}{|x'|_k^{\frac{\beta}{2}}} + \frac{2}{k-\beta} \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\beta}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2 \end{aligned}$$

yielding the following:

$$\left(\frac{k-\beta}{2}\right)^2 \left\| \frac{f}{|x'|_k^{\frac{\beta}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 = \left\| \frac{x' \cdot \nabla_k f}{|x'|_k^{\frac{\beta}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 - \left\| \frac{x' \cdot \nabla_k f}{|x'|_k^{\frac{\beta}{2}}} + \frac{k-\beta}{2|x'|_k^{\frac{\beta}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2.$$

Now we split rest of the proof into two cases:  $k \neq \alpha - 1$  with  $k \geq \alpha - 2$  and  $k = \alpha - 1$ . So, in the case  $k \neq \alpha - 1$  we obtain

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x'|_k^\alpha} dx = \frac{1}{k-\alpha} \int_{\mathbb{R}^n} |f|^2 \operatorname{div}_k \left( \frac{x'}{|x'|_k^\alpha} \right) dx.$$

If we apply integration by parts, we get

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x'|_k^\alpha} dx = -\frac{2}{k-\alpha} \operatorname{Re} \int_{\mathbb{R}^n} \frac{\overline{x' \cdot \nabla_k f}}{|x'|_k^\alpha} f dx.$$

It implies that

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x'|_k^\alpha} dx = -\frac{2}{k-\alpha} \operatorname{Re} \int_{\mathbb{R}^n} \frac{\overline{x' \cdot \nabla_k f}}{|x'|_k} f \frac{1}{(k-\alpha+1)} \operatorname{div}_k \left( \frac{x'}{|x'|_k^{\alpha-1}} \right) dx.$$

We now apply integration by parts to get

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x'|_k^\alpha} dx \\ &= \frac{2}{(k-\alpha)(k-\alpha+1)} \left( \operatorname{Re} \int_{\mathbb{R}^n} \frac{(x' \cdot \nabla_k f) \overline{(x' \cdot \nabla_k f)}}{|x'|_k^\alpha} dx + \operatorname{Re} \int_{\mathbb{R}^n} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f \cdot \frac{f}{|x'|_k^{\alpha-2}} dx \right) \\ &= \frac{2}{(k-\alpha)(k-\alpha+1)} \left( \left\| \frac{(x' \cdot \nabla_k)}{|x'|_k^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2 + \operatorname{Re} \int_{\mathbb{R}^n} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f dx \right). \end{aligned} \tag{3.1.6}$$

The first term on the right-hand side of the last equality can be represented as

$$\begin{aligned}
\left\| \frac{(x' \cdot \nabla_k)}{|x'|_k^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2 &= \left\| \frac{(x' \cdot \nabla_k)}{|x'|_k^{\frac{\alpha}{2}}} f + f(-1)|x'|^{-\frac{\alpha}{2}} + \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \left\| \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right) + \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 = \left\| \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right) \right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + 2 \operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right), \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\rangle + \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2. \quad (3.1.7)
\end{aligned}$$

Let us rewrite (3.1.4) with  $f$  replaced by  $\frac{f}{|x'|}$  and with  $\beta$  replaced by  $\alpha - 2$  as follows

$$\begin{aligned}
\left\| \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right) \right\|_{L^2(\mathbb{R}^n)}^2 &= \left( \frac{k - \alpha + 2}{2} \right)^2 \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + \left\| |x'|_k^{-\frac{k+\alpha}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2. \quad (3.1.8)
\end{aligned}$$

The aim is now to calculate scalar product

$$\begin{aligned}
2 \operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right), \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\rangle \\
= 2 \operatorname{Re} \int_{\mathbb{R}^n} \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right) \cdot \overline{\frac{f}{|x'|_k^{\frac{\alpha}{2}}}} dx = \int_{\mathbb{R}^n} \frac{x'}{|x'|_k^{\alpha-2}} \nabla_k \left( \frac{|f|^2}{|x'|_k^2} \right) dx.
\end{aligned}$$

Integrating by parts yields that

$$\begin{aligned}
2 \operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right), \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\rangle \\
= - \int_{\mathbb{R}^n} \operatorname{div}_k \left( \frac{x'}{|x'|_k^{\alpha-2}} \right) \frac{|f|^2}{|x'|_k^2} dx \\
= - \int_{\mathbb{R}^n} \sum_{j=1}^k \left( \frac{1}{|x'|_k^{\alpha-2}} - (\alpha-2) \frac{x'_j}{|x'|_k^{\alpha-1}} \cdot \frac{x'_j}{|x'|_k} \right) \frac{|f|^2}{|x'|_k^2} dx \quad (3.1.9) \\
= - \int_{\mathbb{R}^n} \left( \frac{k}{|x'|_k^{\alpha-2}} - \frac{\alpha-2}{|x'|_k^{\alpha-2}} \right) \frac{|f|^2}{|x'|_k^2} dx = -(k - \alpha + 2) \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}$$

A combination of (3.1.8) and (3.1.9) with (3.1.7) gives that

$$\begin{aligned}
& \left\| \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}}} \right) f \right\|_{L^2(\mathbb{R}^n)}^2 = \left( \frac{k - \alpha + 2}{2} \right)^2 \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
& + \left\| |x'|_k^{\frac{-k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2 - (k - \alpha + 2) \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
& + \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 = \left( \frac{(k - \alpha + 2)^2}{4} - (k - \alpha + 2) + 1 \right) \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
& + \left\| |x'|_k^{\frac{-k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2 = \left( \frac{k - \alpha}{2} \right)^2 \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
& + \left\| |x'|_k^{\frac{-k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2. \quad (3.1.10)
\end{aligned}$$

The second term on the right-hand side of (3.1.6) can be represented as

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-2}} \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2} f(x) dx \\
& = \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-2}} \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2} f(x) dx \\
& + \left( k - \frac{\alpha}{2} + 1 \right) \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-1}} \cdot \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)} f(x) dx \\
& - \left( k - \frac{\alpha}{2} + 1 \right) \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-1}} \cdot \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)} f(x) dx, \quad (3.1.11)
\end{aligned}$$

that is,

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-2}} \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2} f(x) dx \\
& = \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-2}} \overline{\left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x) + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x) \right)} dx \\
& - \left( k - \frac{\alpha}{2} + 1 \right) \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)} f(x) dx,
\end{aligned}$$

where the last integral is given by

$$\operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)} f(x) dx = \frac{1}{2} \operatorname{Re} \int_{\mathbb{R}^n} \frac{x'}{|x'|_k^\alpha} \cdot \nabla_k (|f|^2) dx.$$

Using integration by parts we obtain

$$\begin{aligned}
\operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)} f dx &= -\frac{1}{2} \int_{\mathbb{R}^n} \operatorname{div}_k \left( \frac{x'}{|x'|_k^\alpha} \right) (|f|^2) dx \\
&= -\frac{1}{2} \int_{\mathbb{R}^n} \sum_{j=1}^k \left( \frac{1}{|x'|_k^\alpha} - \frac{\alpha x'_j}{|x'|_k^{\alpha+1}} \cdot \frac{x'_j}{|x'|_k} \right) (|f|^2) dx \\
&= -\frac{1}{2} \int_{\mathbb{R}^n} \left( \frac{k}{|x'|_k^\alpha} - \frac{\alpha}{|x'|_k^\alpha} \right) |f|^2 dx \\
&= -\frac{k-\alpha}{2} \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned} \tag{3.1.12}$$

We substitute (3.1.10), (3.1.11) and (3.1.12) into (3.1.6) to get

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x'|_k^\alpha} dx &= \frac{2}{(k-\alpha)(k-\alpha+1)} \left( \left( \frac{k-\alpha}{2} \right)^2 \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \right. \\
&\quad \left. \left\| |x'|_k^{-\frac{k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2 \right) \\
&\quad + \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-2}} \overline{\left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x) + \frac{k-\frac{\alpha}{2}+1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x) \right)} dx \\
&\quad + (k-\frac{\alpha}{2}+1) \left( \frac{k-\alpha}{2} \right) \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \frac{2}{(k-\alpha)(k-\alpha+1)} \left( \left( \frac{(k-\alpha)^2}{4} + \frac{(k-\frac{\alpha}{2}+1)(k-\alpha)}{2} \right) \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \right. \\
&\quad \left. + \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-2}} \overline{\left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x) + \frac{k-\frac{\alpha}{2}+1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x) \right)} dx \right. \\
&\quad \left. + \left\| |x'|_k^{-\frac{k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2 \right),
\end{aligned} \tag{3.1.13}$$

then we find

$$\begin{aligned}
& \frac{k(k-4)}{4} \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= -\operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-2}} \overline{\left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x) + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x) \right)} dx \\
&\quad - \left\| |x'|_k^{\frac{-k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2. \quad (3.1.14)
\end{aligned}$$

Recall that we have the following equivalence from (2.1.4) and (2.1.5):

$$\|u\|^2 = -c \operatorname{Re} \langle u, v \rangle + b \Leftrightarrow \frac{1}{c^2} \|u\|^2 = \|v\|^2 - \left\| v + \frac{1}{c} u \right\|^2 + \frac{2b}{c}.$$

Using Lemma 1 with

$$\begin{aligned}
X &= L^2(\mathbb{R}^n), \\
u &= \frac{f}{|x'|_k^{\frac{\alpha}{2}}}, \\
v &= \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f, \\
c &= \frac{4}{k(k-\alpha)}
\end{aligned}$$

and

$$a = -c \left\| |x'|_k^{\frac{-k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2,$$

we observe that the identity (3.1.14) is equivalent to the following identity

$$\begin{aligned}
& \left( \frac{4}{k(k-\alpha)} \right)^2 \left\| \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} + \frac{4}{k(k-\alpha)} \left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f \right) \right\|_{L^2(\mathbb{R}^n)}^2 \\
&+ \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 + 2 \cdot \frac{4}{k(k-\alpha)} \left\| |x'|_k^{\frac{-k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}$$

Let us rewrite it as follows

$$\begin{aligned}
& \left\| \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \left\| \frac{k(k - \alpha)}{4} \left[ \frac{f}{|x'|_k^{\frac{\alpha}{2}}} + \frac{4}{k(k - \alpha)} \left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f \right) \right] \right\|_{L^2(\mathbb{R}^n)}^2 \\
&+ \left( \frac{k(k - \alpha)}{4} \right)^2 \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 + 2 \cdot \frac{k(k - \alpha)}{4} \left\| |x'|_k^{-\frac{k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}$$

It means that

$$\begin{aligned}
\left( \frac{k(k - \alpha)}{4} \right)^2 \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 &= \left\| \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\
&- \left\| \frac{k(k - \alpha)}{4|x'|_k^{\frac{\alpha}{2}}} f + \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\
&- \frac{k(k - \alpha)}{2} \left\| |x'|_k^{-\frac{k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2,
\end{aligned}$$

where the last term is given by

$$\begin{aligned}
& \left\| |x'|_k^{-\frac{k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \left\| |x'|_k^{-\frac{k+2}{2}-1} \sum_{j=1}^k \left( x'_j \left( \partial_{x'_j} \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right) \right) \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \left\| |x'|_k^{-\frac{k+2}{2}-1} \sum_{j=1}^k x'_j \left( \left( \frac{k - \alpha + 2}{2} - 1 \right) |x'|_k^{\frac{k-\alpha+2}{2}-2} \frac{x'_j}{|x'|_k} f + \partial_{x'_j} f |x'|_k^{\frac{k-\alpha+2}{2}-1} \right) \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \left\| |x'|_k^{-\frac{k+2}{2}} \sum_{j=1}^k \frac{x'_j{}^2}{|x'|_k} \left( \frac{k - \alpha + 2}{2} - 1 \right) |x'|_k^{\frac{k-\alpha+2}{2}-2} f + |x'|_k^{-\frac{k+2}{2}} \sum_{j=1}^k x'_j \partial_{x'_j} f |x'|_k^{\frac{k-\alpha+2}{2}-1} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \left\| |x'|_k^{-\frac{k+2}{2}} \left( \frac{k - \alpha}{2} \right) |x'|_k^{\frac{k-\alpha+2}{2}-1} f + |x'|_k^{\frac{k-\alpha+2}{2}} \frac{(x' \cdot \nabla_k)}{|x'|_k} f \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \left\| \frac{k - \alpha}{2} \frac{f}{|x'|_k^{\frac{\alpha}{2}}} + \frac{(x' \cdot \nabla_k)}{|x'|_k^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}$$

Then we conclude that

$$\begin{aligned} \left(\frac{k(k-\alpha)}{4}\right)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 &= \left\| \left(\frac{x' \cdot \nabla_k}{|x'|^k}\right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|^k} \left(\frac{x' \cdot \nabla_k}{|x'|^k}\right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\ &- \left\| \frac{k(k-\alpha)}{4|x'|^{\frac{\alpha}{2}}} f + \left(\frac{x' \cdot \nabla_k}{|x'|^k}\right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|^k} \left(\frac{x' \cdot \nabla_k}{|x'|^k}\right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\ &- \frac{k(k-\alpha)}{2} \left\| \frac{k-\alpha}{2} \frac{f}{|x'|^{\frac{\alpha}{2}}} + \frac{(x' \cdot \nabla_k)}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

Now, consider the case  $k = \alpha - 1$ . We write (3.1.6) as follows

$$\begin{aligned} \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 &= \frac{2}{(k-\alpha)(k-\alpha+1)} \left( \left\| \frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2 \right. \\ &\quad \left. + \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left(\frac{x' \cdot \nabla_k}{|x'|}\right)^2 f \right\rangle \right). \end{aligned}$$

In the case when  $k = \alpha - 1$ , this identity has the following form:

$$\left\| \frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2 = -\operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left(\frac{x' \cdot \nabla_k}{|x'|}\right)^2 f \right\rangle. \quad (3.1.15)$$

Let us rewrite the equality (3.1.4) as follows

$$H_{\alpha-1}^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 = \left\| \frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2 - \left\| \frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}}} f + H_{\alpha-1} \frac{f}{|x'|} \right\|_{L^2(\mathbb{R}^n)}^2. \quad (3.1.16)$$

Now we calculate the first term on the right-hand side of (3.1.16) in the following way:

$$\begin{aligned} \left\| \frac{(x' \cdot \nabla_k)}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{R}^n)}^2 &= \left\| \frac{(x' \cdot \nabla_k)}{|x'|^{\frac{\alpha}{2}}} f + f(-1)|x'|^{-\frac{\alpha}{2}} + \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\ &= \left\| \left(\frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}-1}}\right) \left(\frac{f}{|x'|^k}\right) + \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 = \left\| \left(\frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}-1}}\right) \left(\frac{f}{|x'|^k}\right) \right\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad + 2 \operatorname{Re} \left\langle \left(\frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}-1}}\right) \left(\frac{f}{|x'|^k}\right), \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\rangle + \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2. \quad (3.1.17) \end{aligned}$$

We replaced  $f$  by  $\frac{f}{|x'|}$  and  $\beta$  by  $\alpha - 2$  in (3.1.16) to obtain

$$\begin{aligned} \left\| \left( \frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right) \right\|_{L^2(\mathbb{R}^n)}^2 &= H_{\alpha-1}^2 \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\ &+ \left\| |x'|_k^{-\frac{k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \quad (3.1.18)$$

Also, calculating the scalar product

$$\begin{aligned} 2 \operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right), \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\rangle &= 2 \operatorname{Re} \int_{\mathbb{R}^n} \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right) \cdot \overline{\frac{f}{|x'|_k^{\frac{\alpha}{2}}}} dx \\ &= \int_{\mathbb{R}^n} \frac{x'}{|x'|_k^{\alpha-2}} \nabla_k \left( \frac{|f|^2}{|x'|_k^2} \right) dx \end{aligned}$$

and integration by parts, we get

$$\begin{aligned} 2 \operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right), \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\rangle &= - \int_{\mathbb{R}^n} \operatorname{div}_k \left( \frac{x'}{|x'|_k^{\alpha-2}} \right) \frac{|f|^2}{|x'|_k^2} dx \\ &= - \int_{\mathbb{R}^n} \sum_{j=1}^k \left( \frac{1}{|x'|_k^{\alpha-2}} - (\alpha-2) \frac{x'_j}{|x'|_k^{\alpha-1}} \cdot \frac{x'_j}{|x'|_k} \right) \frac{|f|^2}{|x'|_k^2} dx \\ &= - \int_{\mathbb{R}^n} \left( \frac{k}{|x'|_k^{\alpha-2}} - \frac{\alpha-2}{|x'|_k^{\alpha-2}} \right) \frac{|f|^2}{|x'|_k^2} dx = -(k-\alpha+2) \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

Dividing by 2 both sides of the above we get

$$\operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|_k} \right), \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\rangle = -H_{\alpha-1} \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2. \quad (3.1.19)$$

We put (3.1.18) and (3.1.19) into (3.1.17) to derive

$$\begin{aligned}
\left\| \left( \frac{x' \cdot \nabla_k}{|x'|_k^{\frac{\alpha}{2}}} \right) f \right\|_{L^2(\mathbb{R}^n)}^2 &= H_{\alpha-1}^2 \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&+ \left\| |x'|_k^{\frac{-k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2 \\
&+ 2H_{\alpha-1} \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= \left( \frac{(k-\alpha+2)^2}{4} - (k-\alpha+2) + 1 \right) \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \quad (3.1.20) \\
&+ \left\| |x'|_k^{\frac{-k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2 \\
&= (H_{\alpha-1} - 1)^2 \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&+ \left\| |x'|_k^{\frac{-k+2}{2}} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) \left( |x'|_k^{\frac{k-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}$$

However, the right-hand side of (3.1.15) gives

$$\begin{aligned}
\operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \frac{x' \cdot \nabla_k}{|x'|} \right)^2 f \right\rangle &= \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-2}} \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x)} dx \\
&= \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-2}} \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x)} dx \\
&+ \left( k - \frac{\alpha}{2} + 1 \right) \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-1}} \cdot \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x)} dx \\
&- \left( k - \frac{\alpha}{2} + 1 \right) \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-1}} \cdot \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x)} dx,
\end{aligned}$$

then

$$\begin{aligned}
\operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \frac{x' \cdot \nabla_k}{|x'|} \right)^2 f \right\rangle &= \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-2}} \overline{\left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x) + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x) \right)} dx \\
&- \left( k - \frac{\alpha}{2} + 1 \right) \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x)} dx. \quad (3.1.21)
\end{aligned}$$

Here, the last term can be simplified as follows

$$\operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)} f dx = \frac{1}{2} \operatorname{Re} \int_{\mathbb{R}^n} \frac{x'}{|x'|_k^\alpha} \cdot \nabla_k (|f|^2) dx.$$

It follows after applying integration by parts that

$$\begin{aligned} \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)} f dx &= -\frac{1}{2} \int_{\mathbb{R}^n} \operatorname{div}_k \left( \frac{x'}{|x'|_k^\alpha} \right) (|f|^2) dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^n} \sum_{j=1}^k \left( \frac{1}{|x'|_k^\alpha} - \frac{\alpha x'_j}{|x'|_k^{\alpha+1}} \cdot \frac{x'_j}{|x'|_k} \right) (|f|^2) dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^n} \left( \frac{k}{|x'|_k^\alpha} - \frac{\alpha}{|x'|_k^\alpha} \right) |f|^2 dx = -\frac{k-\alpha}{2} \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \quad (3.1.22)$$

If we put (3.1.22) into (3.1.21) to get

$$\begin{aligned} \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \frac{x' \cdot \nabla_k}{|x'|} \right)^2 f \right\rangle &= \\ \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k^{\alpha-2}} \overline{\left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x) + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x) \right)} dx &+ \frac{(k-1)(k-\alpha)}{2} \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\ = \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x) + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x) \right) \right\rangle &+ \frac{\alpha}{2} (H_{\alpha-1} - 1) \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \quad (3.1.23)$$

We substitute (3.1.20) and (3.1.23) into (3.1.15) to obtain

$$\begin{aligned} (H_{\alpha-1} - 1)^2 \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| \frac{x' \cdot \nabla_k f}{|x'|_k^{\frac{\alpha}{2}-1} |x'|} + H_{\alpha-1} \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 &= \\ - \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x) + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x) \right) \right\rangle &- \frac{\alpha}{2} (H_{\alpha-1} - 1) \left\| \frac{f}{|x'|_k^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

Combining similar terms implies that

$$\begin{aligned} & \left( (H_{\alpha-1} - 1)^2 + \frac{\alpha}{2}(H_{\alpha-1} - 1) \right) \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 = \\ & - \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x) + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x) \right) \right\rangle \\ & - \left\| \frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}-1}} \frac{f}{|x'|} + H_{\alpha-1} \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2, \end{aligned}$$

that is,

$$\begin{aligned} & \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\ & = - \frac{1}{R_{\alpha-1}} \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f(x) + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f(x) \right) \right\rangle \\ & - \frac{1}{R_{\alpha-1}} \left\| \frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}-1}} \frac{f}{|x'|} + H_{\alpha-1} \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2. \quad (3.1.24) \end{aligned}$$

We have the following easily verified equivalence from (2.1.4) and (2.1.5):

$$\|u\|^2 = -c \operatorname{Re} \langle u, v \rangle + b \Leftrightarrow \frac{1}{c^2} \|u\|^2 = \|v\|^2 - \left\| v + \frac{1}{c} u \right\|^2 + \frac{2b}{c}.$$

By Lemma 1 with

$$\begin{aligned} X &= L_2(\mathbb{R}^n), \\ u &= \frac{f}{|x'|_k^{\frac{\alpha}{2}}}, \\ v &= \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|_k} \left( \frac{x' \cdot \nabla_k}{|x'|_k} \right) f, \\ c &= \frac{1}{R_{\alpha-1}} \end{aligned}$$

and

$$b = - \left\| \frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}-1}} \frac{f}{|x'|} + H_{\alpha-1} \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2,$$

the identity (3.1.24) is equivalent to the identity

$$\begin{aligned}
R_k^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 &= \left\| \left( \frac{x' \cdot \nabla_k}{|x'|} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_k}{|x'|} \right) f \right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad - \left\| \left( \frac{x' \cdot \nabla_k}{|x'|} \right)^2 f + \frac{k - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_k}{|x'|} \right) f + R_k \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad - 2R_k \left\| \frac{x' \cdot \nabla_k}{|x'|^{\frac{\alpha}{2}}} f + \frac{k - \alpha}{2} \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^n)}^2
\end{aligned}$$

concluding the proof for the case  $k = \alpha - 1$ .

Thus, proof of Theorem 1 is completed.

# Chapter 4

## Rellich type identities and inequalities on stratified Lie groups

In this chapter, we discuss hypoelliptic versions of the Rellich identities from the previous Chapter 3 on stratified Lie groups.

### 4.1 Rellich type identity and inequality on stratified Lie groups

Here we obtain a sharp remainder formula for the weighted improved Rellich inequality on stratified Lie groups, which implies the improved Rellich inequality when  $\mathbb{G} = (\mathbb{R}^n; +)$  is the Euclidean space and  $N = n \geq 3$ .

**Theorem 2.** *Let  $\mathbb{G}$  be a stratified Lie group with  $N$  being the dimension of the first stratum and  $N \geq \alpha - 2$ . Then we have for all functions  $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$  the following identity:*

$$\begin{aligned} \left(\frac{N(N-\alpha)}{4}\right)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 &= \left\| \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f \right\|_{L^2(\mathbb{G})}^2 \\ &- \left\| \frac{N(N-\alpha)}{4|x'|^{\frac{\alpha}{2}}} f + \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f \right\|_{L^2(\mathbb{G})}^2 \\ &- \frac{N(N-\alpha)}{2} \left\| \frac{N-\alpha}{2} \frac{f}{|x'|^{\frac{\alpha}{2}}} + \frac{(x' \cdot \nabla_H)}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{G})}^2, \quad (4.1.1) \end{aligned}$$

where  $|\cdot|$  is the Euclidean norm on  $\mathbb{R}^N$ . We denote by  $x'$  the variables from the first stratum of  $\mathbb{G}$ .

**Remark 3.** *By dropping two non-negative terms on the right-hand side we obtain*

the following inequality

$$\begin{aligned} \left(\frac{N(N-\alpha)}{4}\right)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\ \leq \left\| \left(\frac{x' \cdot \nabla_H}{|x'|}\right)^2 f + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left(\frac{x' \cdot \nabla_H}{|x'|}\right) f \right\|_{L^2(\mathbb{G})}^2 \end{aligned}$$

for all functions  $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$  and  $N \geq \alpha - 2$ .

**Remark 4.** In the unweighted case  $\alpha = 4$  and by dropping non-negative terms we get

$$\begin{aligned} \left(\frac{N(N-4)}{4}\right)^2 \left\| \frac{f}{|x'|^2} \right\|_{L^2(\mathbb{G})}^2 \\ \leq \left\| \left(\frac{x' \cdot \nabla_H}{|x'|}\right)^2 f + \frac{N-1}{|x'|} \left(\frac{x' \cdot \nabla_H}{|x'|}\right) f \right\|_{L^2(\mathbb{G})}^2 \end{aligned} \quad , \quad (4.1.2)$$

for all functions  $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$  and  $N \geq 2$ . In the Abelian case  $\mathbb{G} = (\mathbb{R}^n; +)$ , hence  $N = n$  and  $\nabla_H = \nabla = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})$ , then the inequality (4.1.2) gives an improvement of the classical Rellich inequality as discussed in (1.1.3).

**Remark 5.** We refer to [34] and [35] for the Rellich type inequalities on stratified Lie groups.

**Proof of Theorem 2.** Firstly, we prove the identity

$$\begin{aligned} \left(\frac{N-\beta}{2}\right)^2 \left\| \frac{f}{|x'|^{\frac{\beta}{2}}} \right\|_{L^2(\mathbb{G})}^2 &= \left\| \frac{x' \cdot \nabla_H}{|x'|^{\frac{\beta}{2}}} f \right\|_{L^2(\mathbb{G})}^2 \\ &- \left\| \frac{x' \cdot \nabla_H}{|x'|^{\frac{\beta}{2}}} f + \frac{N-\beta}{2|x'|^{\frac{\beta}{2}}} f \right\|_{L^2(\mathbb{G})}^2 \end{aligned} \quad . \quad (4.1.3)$$

For this, we first compute the following equality

$$\int_{\mathbb{G}} \frac{|f|^2}{|x'|^\beta} dx = \frac{1}{N-\beta} \int_{\mathbb{G}} |f|^2 \operatorname{div}_H \left( \frac{x'}{|x'|^\beta} \right) dx.$$

Applying integration by parts, we have

$$\int_{\mathbb{G}} \frac{|f|^2}{|x'|^\beta} dx = -\frac{2}{N-\beta} \operatorname{Re} \int_{\mathbb{G}} \frac{\overline{x' \cdot \nabla_H f}}{|x'|^\beta} f dx. \quad (4.1.4)$$

Recall from (2.1.2) and (2.1.3) that the following two statements are equivalent:

$$\|u\|^2 = -2c \operatorname{Re} \langle u, v \rangle$$

and

$$\|u\|^2 = 4c^2 \|v\|^2 - \|u + 2cv\|^2.$$

Then, using Lemma 1 with

$$\begin{aligned} X &= L^2(\mathbb{G}), \\ u &= \frac{f}{|x'|^{\frac{\beta}{2}}}, \\ v &= \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\beta}{2}}} \right) f \end{aligned}$$

and

$$c = \frac{1}{N - \beta},$$

it follows that the identity (4.1.4) is equivalent to

$$\begin{aligned} \left\| \frac{f}{|x'|^{\frac{\beta}{2}}} \right\|_{L^2(\mathbb{G})}^2 &= \left( \frac{2}{N - \beta} \right)^2 \left\| \frac{x' \cdot \nabla_H}{|x'|^{\frac{\beta}{2}}} f \right\|_{L^2(\mathbb{G})}^2 \\ &\quad - \left\| \frac{f}{|x'|^{\frac{\beta}{2}}} + \frac{2}{N - \beta} \frac{x' \cdot \nabla_H}{|x'|^{\frac{\beta}{2}}} f \right\|_{L^2(\mathbb{G})}^2. \end{aligned}$$

yielding (4.1.3).

Now we split rest of the proof into two cases:  $N \neq \alpha - 1$  with  $N \geq \alpha - 2$  and  $N = \alpha - 1$ . So, in the case  $N \neq \alpha - 1$  we obtain

$$\int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^\alpha} dx = \frac{1}{N - \alpha} \int_{\mathbb{R}^n} |f|^2 \operatorname{div}_H \left( \frac{x'}{|x'|^\alpha} \right) dx.$$

Integrating by parts, we have

$$\int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^\alpha} dx = -\frac{2}{N - \alpha} \operatorname{Re} \int_{\mathbb{G}} \frac{\overline{x' \cdot \nabla_H f}}{|x'|^\alpha} f dx.$$

It implies that

$$\int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^\alpha} dx = -\frac{2}{N - \alpha} \operatorname{Re} \int_{\mathbb{G}} \frac{\overline{x' \cdot \nabla_H f}}{|x'|} f \frac{1}{(N - \alpha + 1)} \operatorname{div}_H \left( \frac{x'}{|x'|^{\alpha-1}} \right) dx.$$

Using integration by parts to get

$$\begin{aligned}
& \int_{\mathbf{G}} \frac{|f(x)|^2}{|x'|^\alpha} dx \\
&= \frac{2}{(N-\alpha)(N-\alpha+1)} \left( \operatorname{Re} \int_{\mathbf{G}} \frac{(x' \cdot \nabla_H f) \overline{(x' \cdot \nabla_H f)}}{|x'|^\alpha} dx + \operatorname{Re} \int_{\mathbf{G}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2} f \cdot \frac{f}{|x'|^{\alpha-2}} dx \right) \\
&= \frac{2}{(N-\alpha)(N-\alpha+1)} \left( \left\| \frac{(x' \cdot \nabla_H)}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbf{G})}^2 + \operatorname{Re} \int_{\mathbf{G}} \frac{f(x)}{|x'|^{\alpha-2}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2} f dx \right). \tag{4.1.5}
\end{aligned}$$

The first term on the right-hand side of (4.1.5) can be rewritten as

$$\begin{aligned}
\left\| \frac{(x' \cdot \nabla_H)}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbf{G})}^2 &= \left\| \frac{(x' \cdot \nabla_H)}{|x'|^{\frac{\alpha}{2}}} f + f(-1)|x'|^{-\frac{\alpha}{2}} + \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbf{G})}^2 \\
&= \left\| \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|} \right) + \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbf{G})}^2 = \left\| \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|} \right) \right\|_{L^2(\mathbf{G})}^2 \\
&\quad + 2 \operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|} \right), \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\rangle + \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbf{G})}^2. \tag{4.1.6}
\end{aligned}$$

Let us represent (4.1.3) with  $f$  replaced by  $\frac{f}{|x'|}$  and with  $\beta$  replaced by  $\alpha - 2$  as follows

$$\begin{aligned}
\left\| \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|} \right) \right\|_{L^2(\mathbf{G})}^2 &= \left( \frac{N-\alpha+2}{2} \right)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbf{G})}^2 \\
&\quad + \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbf{G})}^2. \tag{4.1.7}
\end{aligned}$$

Now we calculate the scalar product

$$\begin{aligned}
& 2 \operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|} \right), \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\rangle \\
&= 2 \operatorname{Re} \int_{\mathbf{G}} \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|} \right) \cdot \overline{\frac{f}{|x'|^{\frac{\alpha}{2}}}} dx = \int_{\mathbf{G}} \frac{x'}{|x'|^{\alpha-2}} \nabla_H \left( \frac{|f|^2}{|x'|^2} \right) dx.
\end{aligned}$$

Applying integration by parts that implies

$$\begin{aligned}
& 2 \operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|} \right), \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\rangle \\
&= - \int_{\mathbb{G}} \operatorname{div}_H \left( \frac{x'}{|x'|^{\alpha-2}} \right) \frac{|f|^2}{|x'|^2} dx \\
&= - \int_{\mathbb{G}} \sum_{j=1}^N \left( \frac{1}{|x'|^{\alpha-2}} - (\alpha-2) \frac{x'_j}{|x'|^{\alpha-1}} \cdot \frac{x'_j}{|x'|} \right) \frac{|f|^2}{|x'|^2} dx \\
&= - \int_{\mathbb{G}} \left( \frac{N}{|x'|^{\alpha-2}} - \frac{\alpha-2}{|x'|^{\alpha-2}} \right) \frac{|f|^2}{|x'|^2} dx = -(N-\alpha+2) \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2.
\end{aligned} \tag{4.1.8}$$

A combining of (4.1.7) and (4.1.8) with (4.1.6) gives that

$$\begin{aligned}
& \left\| \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}}} \right) f \right\|_{L^2(\mathbb{G})}^2 = \left( \frac{N-\alpha+2}{2} \right)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\
&+ \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2 - (N-\alpha+2) \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\
&+ \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 = \left( \frac{(N-\alpha+2)^2}{4} - (N-\alpha+2) + 1 \right) \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\
&+ \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2 = \left( \frac{N-\alpha}{2} \right)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\
&+ \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2.
\end{aligned} \tag{4.1.9}$$

The second term on the right-hand side of (4.1.5) can be represented as

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-2}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x)} dx \\
&= \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-2}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x)} dx \\
&+ \left( N - \frac{\alpha}{2} + 1 \right) \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-1}} \cdot \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x)} dx \\
&- \left( N - \frac{\alpha}{2} + 1 \right) \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-1}} \cdot \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x)} dx,
\end{aligned} \tag{4.1.10}$$

that is,

$$\begin{aligned}
& \operatorname{Re} \int_{\mathbf{G}} \frac{f(x)}{|x'|^{\alpha-2}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x)} dx \\
&= \operatorname{Re} \int_{\mathbf{G}} \frac{f(x)}{|x'|^{\alpha-2}} \left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x) + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x) \right) dx \\
&\quad - \left( N - \frac{\alpha}{2} + 1 \right) \operatorname{Re} \int_{\mathbf{G}} \frac{f(x)}{|x'|^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x)} dx,
\end{aligned}$$

where the last integral is given by

$$\operatorname{Re} \int_{\mathbf{G}} \frac{f(x)}{|x'|^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x)} dx = \frac{1}{2} \operatorname{Re} \int_{\mathbf{G}} \frac{x'}{|x'|^\alpha} \cdot \nabla_H (|f|^2) dx.$$

Integration by parts we obtain

$$\begin{aligned}
\operatorname{Re} \int_{\mathbf{G}} \frac{f(x)}{|x'|^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x)} dx &= -\frac{1}{2} \int_{\mathbf{G}} \operatorname{div}_H \left( \frac{x'}{|x'|^\alpha} \right) (|f|^2) dx \\
&= -\frac{1}{2} \int_{\mathbf{G}} \sum_{j=1}^N \left( \frac{1}{|x'|^\alpha} - \frac{\alpha x'_j}{|x'|^{\alpha+1}} \cdot \frac{x'_j}{|x'|} \right) (|f|^2) dx \\
&= -\frac{1}{2} \int_{\mathbf{G}} \left( \frac{N}{|x'|^\alpha} - \frac{\alpha}{|x'|^\alpha} \right) |f|^2 dx \\
&= -\frac{N - \alpha}{2} \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbf{G})}^2.
\end{aligned} \tag{4.1.11}$$

We substitute (4.1.9), (4.1.10) and (4.1.11) into (4.1.5) to derive

$$\begin{aligned}
\int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^\alpha} dx &= \frac{2}{(N-\alpha)(N-\alpha+1)} \left( \left( \frac{N-\alpha}{2} \right)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \right. \\
&\quad \left. \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) (|x'|^{\frac{N-\alpha+2}{2}-1} f) \right\|_{L^2(\mathbb{G})}^2 \right. \\
&\quad \left. + \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-2}} \left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x) + \frac{N-\frac{\alpha}{2}+1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x) \right) dx \right. \\
&\quad \left. + (N-\frac{\alpha}{2}+1) \left( \frac{N-\alpha}{2} \right) \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \right) \\
&= \frac{2}{(N-\alpha)(N-\alpha+1)} \left( \left( \frac{(N-\alpha)^2}{4} + \frac{(N-\frac{\alpha}{2}+1)(N-\alpha)}{2} \right) \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \right. \\
&\quad \left. + \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-2}} \left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x) + \frac{N-\frac{\alpha}{2}+1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x) \right) dx \right. \\
&\quad \left. + \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) (|x'|^{\frac{N-\alpha+2}{2}-1} f) \right\|_{L^2(\mathbb{G})}^2 \right), \tag{4.1.12}
\end{aligned}$$

then we find

$$\begin{aligned}
&\frac{N(N-4)}{4} \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\
&= -\operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-2}} \left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x) + \frac{N-\frac{\alpha}{2}+1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x) \right) dx \\
&\quad - \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) (|x'|^{\frac{N-\alpha+2}{2}-1} f) \right\|_{L^2(\mathbb{G})}^2. \tag{4.1.13}
\end{aligned}$$

Recall that we have the following equivalence from (2.1.4) and (2.1.5):

$$\|u\|^2 = -c \operatorname{Re} \langle u, v \rangle + b \Leftrightarrow \frac{1}{c^2} \|u\|^2 = \|v\|^2 - \left\| v + \frac{1}{c} u \right\|^2 + \frac{2b}{c}$$

Using Lemma 1 with

$$\begin{aligned}
X &= L^2(\mathbb{G}), \\
u &= \frac{f}{|x'|^{\frac{\alpha}{2}}}, \\
v &= \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N-\frac{\alpha}{2}+1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f,
\end{aligned}$$

$$c = \frac{4}{N(N - \alpha)}$$

and

$$a = -c \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2.$$

we observe that the identity (4.1.13) is equivalent to the following identity

$$\begin{aligned} & \left( \frac{4}{N(N - \alpha)} \right)^2 \left\| \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f \right\|_{L^2(\mathbb{G})}^2 \\ &= \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} + \frac{4}{N(N - \alpha)} \left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f \right) \right\|_{L^2(\mathbb{G})}^2 \\ &+ \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 + 2 \cdot \frac{4}{N(N - \alpha)} \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2. \end{aligned}$$

Let us represent it as follows

$$\begin{aligned} & \left\| \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f \right\|_{L^2(\mathbb{G})}^2 \\ &= \left\| \frac{N(N - \alpha)}{4} \left[ \frac{f}{|x'|^{\frac{\alpha}{2}}} + \frac{4}{N(N - \alpha)} \left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f \right) \right] \right\|_{L^2(\mathbb{G})}^2 \\ &+ \left( \frac{N(N - \alpha)}{4} \right)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 + 2 \cdot \frac{N(N - \alpha)}{4} \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2. \end{aligned}$$

It means that

$$\begin{aligned} & \left( \frac{N(N - \alpha)}{4} \right)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 = \left\| \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f \right\|_{L^2(\mathbb{G})}^2 \\ &- \left\| \frac{N(N - \alpha)}{4|x'|^{\frac{\alpha}{2}}} f + \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f \right\|_{L^2(\mathbb{G})}^2 \\ &- \frac{N(N - \alpha)}{2} \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2, \end{aligned}$$

where the last term is given by

$$\begin{aligned}
& \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2 \\
&= \left\| |x'|^{-\frac{N+2}{2}-1} \sum_{j=1}^N \left( x'_j \left( X_j \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right) \right) \right\|_{L^2(\mathbb{G})}^2 \\
&= \left\| |x'|^{-\frac{N+2}{2}-1} \sum_{j=1}^N x'_j \left( \left( \frac{N-\alpha+2}{2} - 1 \right) |x'|^{\frac{N-\alpha+2}{2}-2} \frac{x'_j}{|x'|} f + X_j f |x'|^{\frac{N-\alpha+2}{2}-1} \right) \right\|_{L^2(\mathbb{G})}^2 \\
&= \left\| |x'|^{-\frac{N+2}{2}} \sum_{j=1}^N \frac{x'_j{}^2}{|x'|} \left( \frac{N-\alpha+2}{2} - 1 \right) |x'|^{\frac{N-\alpha+2}{2}-2} f + |x'|^{-\frac{N+2}{2}} \sum_{j=1}^N x'_j X_j f |x'|^{\frac{N-\alpha+2}{2}-1} \right\|_{L^2(\mathbb{G})}^2 \\
&= \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{N-\alpha}{2} \right) |x'|^{\frac{N-\alpha+2}{2}-1} f + |x'|^{-\frac{N+2}{2}} \frac{(x' \cdot \nabla_H)}{|x'|} f \right\|_{L^2(\mathbb{G})}^2 \\
&= \left\| \frac{N-\alpha}{2} \frac{f}{|x'|^{\frac{\alpha}{2}}} + \frac{(x' \cdot \nabla_H)}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{G})}^2.
\end{aligned}$$

Then we conclude that

$$\begin{aligned}
& \left( \frac{N(N-\alpha)}{4} \right)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 = \left\| \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N-\frac{\alpha}{2}+1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f \right\|_{L^2(\mathbb{G})}^2 \\
& - \left\| \frac{N(N-\alpha)}{4|x'|^{\frac{\alpha}{2}}} f + \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N-\frac{\alpha}{2}+1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f \right\|_{L^2(\mathbb{G})}^2 \\
& - \frac{N(N-\alpha)}{2} \left\| \frac{N-\alpha}{2} \frac{f}{|x'|^{\frac{\alpha}{2}}} + \frac{(x' \cdot \nabla_H)}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{G})}^2.
\end{aligned}$$

Now, consider the case  $N = \alpha - 1$ . We adopt the following notations:

$$H_N = \frac{N-\alpha+2}{2}$$

and

$$R_N = \frac{N(N-\alpha)}{4}.$$

We write (4.1.5) in the following way:

$$\begin{aligned}
\left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 &= \frac{2}{(N-\alpha)(N-\alpha+1)} \left( \left\| \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{G})}^2 \right. \\
& \quad \left. + \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f \right\rangle \right).
\end{aligned}$$

In the case when  $N = \alpha - 1$ , this identity has the following form:

$$\left\| \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{G})}^2 = -\operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f \right\rangle. \quad (4.1.14)$$

Let us represent the equality (4.1.3) in the way

$$H_{\alpha-1}^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 = \left\| \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{G})}^2 - \left\| \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}}} f + H_{\alpha-1} \frac{f}{|x'|} \right\|_{L^2(\mathbb{G})}^2. \quad (4.1.15)$$

Now we calculate the first term on the right-hand side of (4.1.15) in the following way:

$$\begin{aligned} \left\| \frac{(x' \cdot \nabla_H)}{|x'|^{\frac{\alpha}{2}}} f \right\|_{L^2(\mathbb{G})}^2 &= \left\| \frac{(x' \cdot \nabla_H)}{|x'|^{\frac{\alpha}{2}}} f + f(-1)|x'|^{-\frac{\alpha}{2}} + \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\ &= \left\| \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1} } \right) \left( \frac{f}{|x'|} \right) + \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 = \left\| \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1} } \right) \left( \frac{f}{|x'|} \right) \right\|_{L^2(\mathbb{G})}^2 \\ &\quad + 2 \operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1} } \right) \left( \frac{f}{|x'|} \right), \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\rangle + \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (4.1.16)$$

We replaced  $f$  by  $\frac{f}{|x'|}$  and  $\beta$  replaced by  $\alpha - 2$  in (4.1.15) to obtain

$$\begin{aligned} \left\| \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1} } \right) \left( \frac{f}{|x'|} \right) \right\|_{L^2(\mathbb{G})}^2 &= H_{\alpha-1}^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\ &\quad + \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (4.1.17)$$

Also, calculating the scalar product

$$\begin{aligned} 2 \operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1} } \right) \left( \frac{f}{|x'|} \right), \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\rangle &= 2 \operatorname{Re} \int_{\mathbb{G}} \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1} } \right) \left( \frac{f}{|x'|} \right) \cdot \overline{\frac{f}{|x'|^{\frac{\alpha}{2}}}} dx \\ &= \int_{\mathbb{G}} \frac{x'}{|x'|^{\alpha-2}} \nabla_H \left( \frac{|f|^2}{|x'|^2} \right) dx \end{aligned}$$

and apply integration by parts we derive

$$\begin{aligned} 2 \operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1} } \right) \left( \frac{f}{|x'|} \right), \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\rangle &= - \int_{\mathbb{G}} \operatorname{div}_H \left( \frac{x'}{|x'|^{\alpha-2}} \right) \frac{|f|^2}{|x'|^2} dx \\ &= - \int_{\mathbb{G}} \sum_{j=1}^N \left( \frac{1}{|x'|^{\alpha-2}} - (\alpha-2) \frac{x'_j}{|x'|^{\alpha-1}} \cdot \frac{x'_j}{|x'|} \right) \frac{|f|^2}{|x'|^2} dx \\ &= - \int_{\mathbb{G}} \left( \frac{N}{|x'|^{\alpha-2}} - \frac{\alpha-2}{|x'|^{\alpha-2}} \right) \frac{|f|^2}{|x'|^2} dx = -(N - \alpha + 2) \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2. \end{aligned}$$

Dividing by 2 both sides of the above we get

$$\operatorname{Re} \left\langle \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}-1}} \right) \left( \frac{f}{|x'|} \right), \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\rangle = -H_{\alpha-1} \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2. \quad (4.1.18)$$

We put (4.1.17),(4.1.18) into (4.1.16) to obtain

$$\begin{aligned} \left\| \left( \frac{x' \cdot \nabla_H}{|x'|^{\frac{\alpha}{2}}} \right) f \right\|_{L^2(\mathbb{G})}^2 &= H_{\alpha-1}^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\ &+ \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2 \\ &+ 2H_{\alpha-1} \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 + \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\ &= \left( \frac{(N-\alpha+2)^2}{4} - (N-\alpha+2) + 1 \right) \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\ &+ \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2 \\ &= (H_{\alpha-1} - 1)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\ &+ \left\| |x'|^{-\frac{N+2}{2}} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) \left( |x'|^{\frac{N-\alpha+2}{2}-1} f \right) \right\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (4.1.19)$$

However, the right-hand side of (4.1.14) gives

$$\begin{aligned} \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f \right\rangle &= \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-2}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x)} dx \\ &= \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-2}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x)} dx \\ &+ (N - \frac{\alpha}{2} + 1) \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-1}} \cdot \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x)} dx \\ &- (N - \frac{\alpha}{2} + 1) \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-1}} \cdot \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x)} dx, \end{aligned}$$

then

$$\begin{aligned}
& \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f \right\rangle \\
&= \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-2}} \overline{\left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x) + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x) \right)} dx \\
&\quad - (N - \frac{\alpha}{2} + 1) \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x)} dx. \quad (4.1.20)
\end{aligned}$$

Here, the second term of (4.1.20) can be simplified as follows

$$\operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right) f} dx = \frac{1}{2} \operatorname{Re} \int_{\mathbb{G}} \frac{x'}{|x'|^\alpha} \cdot \nabla_H (|f|^2) dx.$$

It follows after applying integration by parts that

$$\begin{aligned}
\operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-1}} \overline{\left( \frac{x' \cdot \nabla_H}{|x'|} \right) f} dx &= -\frac{1}{2} \int_{\mathbb{G}} \operatorname{div}_H \left( \frac{x'}{|x'|^\alpha} \right) (|f|^2) dx \\
&= -\frac{1}{2} \int_{\mathbb{G}} \sum_{j=1}^N \left( \frac{1}{|x'|^\alpha} - \frac{\alpha x'_j}{|x'|^{\alpha+1}} \cdot \frac{x'_j}{|x'|} \right) (|f|^2) dx \\
&= -\frac{1}{2} \int_{\mathbb{G}} \left( \frac{N}{|x'|^\alpha} - \frac{\alpha}{|x'|^\alpha} \right) |f|^2 dx = -\frac{N - \alpha}{2} \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2. \quad (4.1.21)
\end{aligned}$$

If we substitute (4.1.21) into (4.1.20) to get

$$\begin{aligned}
& \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f \right\rangle = \\
& \operatorname{Re} \int_{\mathbb{G}} \frac{f(x)}{|x'|^{\alpha-2}} \overline{\left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x) + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x) \right)} dx \\
&\quad + \frac{(N-1)(N-\alpha)}{2} \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\
&= \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x) + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x) \right) \right\rangle \\
&\quad + \frac{\alpha}{2} (H_{\alpha-1} - 1) \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2. \quad (4.1.22)
\end{aligned}$$

We put (4.1.19) and (4.1.22) into (4.1.14) to obtain

$$\begin{aligned} & (H_{\alpha-1} - 1)^2 \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 + \left\| \frac{x' \cdot \nabla_H f}{|x'|^{\frac{\alpha}{2}-1} |x'|} + H_{\alpha-1} \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 = \\ & - \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x) + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x) \right) \right\rangle \\ & \quad - \frac{\alpha}{2} (H_{\alpha-1} - 1) \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2. \end{aligned}$$

Combining similar terms yields that

$$\begin{aligned} & \left( (H_{\alpha-1} - 1)^2 + \frac{\alpha}{2} (H_{\alpha-1} - 1) \right) \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 = \\ & - \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x) + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x) \right) \right\rangle \\ & \quad - \left\| \frac{x' \cdot \nabla_H f}{|x'|^{\frac{\alpha}{2}-1} |x'|} + H_{\alpha-1} \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2, \quad (4.1.23) \end{aligned}$$

that is,

$$\begin{aligned} & \left\| \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 \\ & = - \frac{1}{R_{\alpha-1}} \operatorname{Re} \left\langle \frac{f}{|x'|^{\alpha-2}}, \left( \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f(x) + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f(x) \right) \right\rangle \\ & \quad - \frac{1}{R_{\alpha-1}} \left\| \frac{x' \cdot \nabla_H f}{|x'|^{\frac{\alpha}{2}-1} |x'|} + H_{\alpha-1} \frac{f}{|x'|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2. \end{aligned}$$

We have the following easily verified equivalence from (2.1.4) and (2.1.5):

$$\|u\|^2 = -c \operatorname{Re} \langle u, v \rangle + b$$

and

$$\frac{1}{c^2} \|u\|^2 = \|v\|^2 - \left\| v + \frac{1}{c} u \right\|^2 + \frac{2b}{c}.$$

By Lemma 1 with

$$\begin{aligned} X &= L_2(\mathbb{G}), \\ u &= \frac{f}{|x'|^{\frac{\alpha}{2}}}, \\ v &= \left( \frac{x' \cdot \nabla_H}{|x'|} \right)^2 f + \frac{N - \frac{\alpha}{2} + 1}{|x'|} \left( \frac{x' \cdot \nabla_H}{|x'|} \right) f, \end{aligned}$$

concluding the proof for the case  $N = \alpha - 1$ .

$$\begin{aligned}
 & \left\| \frac{f}{|x|^{\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{G})}^2 = \left\| \left( \frac{|x|}{H} \cdot \Delta_H \right)^2 f + \frac{|x|}{N - \frac{\alpha}{2} + 1} \left( \frac{|x|}{H} \cdot \Delta_H \right) f + f \right\|_{L^2(\mathbb{G})}^2 \\
 & - \left\| \left( \frac{|x|}{H} \cdot \Delta_H \right)^2 \left( f + R_N \frac{|x|^{\frac{\alpha}{2}}}{f} \right) + \frac{|x|}{N - \frac{\alpha}{2} + 1} \left( \frac{|x|}{H} \cdot \Delta_H \right) \left( f + R_N \frac{|x|^{\frac{\alpha}{2}}}{f} \right) + \frac{2}{N - \alpha} \frac{|x|^{\frac{\alpha}{2}}}{f} \right\|_{L^2(\mathbb{G})}^2 \\
 & - 2R_N \left\| \frac{|x|^{\frac{\alpha}{2}}}{f} \right\|_{L^2(\mathbb{G})}^2
 \end{aligned}$$

the identity (4.1.23) is equivalent to the identity

$$b = - \left\| \frac{|x|^{\frac{\alpha}{2}}}{f} \cdot \Delta_H \left( \frac{|x|}{H} \cdot \Delta_H \right)^2 f + H^{\alpha-1} \frac{|x|^{\frac{\alpha}{2}}}{f} \right\|_{L^2(\mathbb{G})}^2,$$

and

$$c = \frac{R^{\alpha-1}}{1}$$

# Chapter 5

## Conclusions and future work

### 5.1 Conclusions

The Rellich inequality has many important applications in various fields of mathematics and physics. This inequality holds significant value in the theory of partial differential equations and spectral theory.

Recently, Japanese mathematicians N. Bez, S. Machihara and T. Ozawa in [20] and [21] introduced improvements of the classical Rellich inequality for the radial part of the Laplacian. In this thesis we discuss cylindrical extensions of these improved Rellich inequalities on  $\mathbb{R}^k \times \mathbb{R}^{n-k}$  with the Euclidean norm  $|\cdot|_k$  on  $\mathbb{R}^k$ . Importance of such cylindrical versions of the classical functional inequalities due to applications in investigating cylindrical solutions of partial differential equations arising in astrophysics. For example, in [22] Badiale and Tarantello used the cylindrical Hardy inequality in studying a model describing the dynamics of galaxies.

Moreover, we discussed sharp remainders of the Rellich type inequalities, namely Rellich type identities yielding Rellich type inequalities.

Furthermore, we established Rellich type identities and inequalities with more general weights. In the unweighted case  $\alpha = 4$  and when  $k = n$ , our results imply the results of N. Bez, S. Machihara and T. Ozawa in [20] and [21]. In addition, we obtained horizontal versions of these results on stratified Lie groups.

Further research could be to investigate applications of the obtained results in studying the well-posedness of partial differential equations. Another possible direction is to study the critical case  $n = 4$  of Rellich type inequalities and identities. Finally, to extend the results to more general Lie groups.

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