IRSTI 27.15.21

G. Turgynbai[1]
[1]Suleyman Demirel University, Kaskelen, Kazakhstan

GENERALIZED CONTINUED FRACTION EXPANSION FOR EULER CONSTANT

Abstract. The theory of continued fractions is one of the oldest fields of number theory going back as far as Euclid 300 BCE. As opposed to generalized continued fractions, the theory of simple continued fractions is well-developed and finds many applications in various fields. In this article we consider a problem of elegant representations of famous mathematical constants with generalized continued fraction and prove that the Euler constant e satisfies the generalized continued fraction formula

$$\frac{e}{-2} = -1 + \frac{1}{-2+}\,\frac{4}{-4+}\,\frac{8}{-6+}\,\frac{12}{-8+}\,....$$

This is one of the conjectures generated in www.RamanujanMachine.com using machine learning techniques. Our method of proof uses only elementary techniques.

Keywords: Generalized Continued Fraction, Euler number e, Ramanujan Machine, Number theory.

***

Аңдатпа. Шектеусіз үздіксіз бөлшектер теориясы Евклид секілді б.з.д 300 жыл бұрын қолданысқа еңген сандар теориясының ескі тармақтарының бірі болып табылады. Жәй үздіксіз бөлшектер теориясы шектеусіз үздіксіз бөлшектерге қарағанда кеңірек дамыған және көптеген салаларда өзінің қолданысын тапқан. Бұл мақалада біз белгілі математикалық тұрақтылардың көрінісін шектеусіз үздіксіз бөлшектер арқылы қарастырамыз және Эйлер константасы е-нің төмендегі шектеусіз үздіксіз бөлшектер формуласына сәйкес келетіндігін дәлелдейміз.

$$\frac{e}{-2} = -1 + \frac{1}{-2+}\,\frac{4}{-4+}\,\frac{8}{-6+}\,\frac{12}{-8+}\,....$$

Бұл www.RamanujanMachine.com сайтында машиналық оқыту әдістерін қолданып жасалған болжамдардың бірі. Біздің дәлелдеу әдісі тек қарапайым әдістерді қолданады.

Түйін сөздер: Шектеусіз үздіксіз бөлшектер, е Эйлер саны, Рамануджан машинасы, Сандар теориясы

***

Аннотация. Теория непрерывных дробей - одна из старейших областей теории чисел, восходящая к Евклиду 300 г. до н. э. В отличие от обобщенных цепных дробей, теория простых непрерывных дробей хорошо развита и находит множество применений в различных областях. В этой статье мы рассматриваем проблему элегантного представления известных математических констант с помощью обобщенной цепной дроби и доказываем, что постоянная Эйлера e удовлетворяет формуле обобщенной цепной дроби

$$\frac{e}{-2} = -1 + \frac{1}{-2+}\frac{4}{-4+}\frac{8}{-6+}\frac{12}{-8+}\dots$$

Это одна из гипотез, созданных на сайте www.RamanujanMachine.com с использованием методов машинного обучения. Наш метод доказательства использует только элементарные приемы.

Ключевые слова: Обобщенные цепные дроби, Постоянная Эйлера e, Машина Раманджана, Теория чисел

1. *Introduction*

Infinite (generalized) continued fractions have the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}},$$

for the given sequences (a_n) and (b_n) of integers. For simplicity of notation we instead write

$$b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\dots \quad (1)$$

We say that a real number x has a *continued fraction representation* as in (1) if

$$x = \lim_{n\to\infty}\frac{A_n}{B_n},$$

where for any n we define the approximants A_n, B_n by

$$\frac{A_n}{B_n} = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\dots\frac{a_n}{+b_n}. \quad (2)$$

When $a_n = 1$ and $b_n \in N$ for all n, it is called the *simple continued fraction expansion,* one of the well-studied fields in number theory. Simple continued fractions yield unique representations for irrational numbers, with the approximants being the best rational approximations. However, when $a_n's$ are allowed to vary, the representations are not necessarily unique which provides opportunities for finding elegant generalized continued fraction representations of the given number. Euler himself obtained [4] simple continued fraction representation of his famous constant e. One of the well-known continued fraction representations for e is given by

$$e = 2 + \frac{1}{1+}\frac{2}{2+}\ \frac{3}{3+}\frac{4}{4+}\cdots$$

Recently, researchers have created a program [3] that independently generates representations of mathematical constants, such as e and π, in the form of infinite continued fractions, without proof. The generated expressions resemble formulas obtained by a mathematician of the beginning of the 20th century Ramanujan. The algorithm is called "Ramanujan Machine" and the method of finding expressions is a new approach, reminiscent of the intuition of mathematicians rather than the logic of formal proofs. From a technical point of view, the Ramanujan Machine is a distributed computing program that iteratively finds expressions with continued fractions, combining algorithms of meeting in the middle and gradient descent. Both algorithms work by gradually selecting an increasingly accurate numerical value, therefore the result is only unproven hypotheses, the truth of which must be strictly confirmed by other methods.

In this note, our goal is to prove one of the conjectures listed in Ramanujan site. To this end, our main result is the following.

Theorem 1. The Euler constant e satisfies the following continued fraction representation

$$\frac{e}{-2} = -1 + \cfrac{1}{-2 + \cfrac{4}{-4 + \cfrac{8}{-6 + \cfrac{12}{-8 + \ddots}}}}.$$

In the next section we prove Theorem 1 using elementary tools. The idea is based on transforming the problem into series representation of e.

2. *Proof of Main Result*

In this section we prove our main result, Theorem 1. It is easy to see that the approximants A_n and B_n given in (2) satisfies the following difference equations with non-constant coefficients

$$A_n = b_n A_{n-1} + a_n A_{n-2}\,, B_n = b_n B_{n-1} + a_n B_{n-2}\,, n \geq 1, \tag{2}$$

$$A_{-1} = 1, A_0 = b_0, B_{-1} = 0, B_0 = 1, \tag{3}$$

where the sequences (a_n) and (b_n) are coming from the continued fraction expansion. Thus, provided that A_n, B_n are found, one obtains x by taking the limit of the quotient A_n/B_n. However, computing closed form arrangements to distinction conditions as in (2) with variable coefficients a_n, b_n is troublesome. A basic approach is to figure out the closed form of A_n and B_n from the primary few terms and demonstrate it utilizing mathematical induction. Lately, this approach was utilized in [1] and in [2] to demonstrate one of the numerous conjectures on generalized proceeded division development for Euler number e listed in [3].

Using the similar approach, we prove one of the conjectures for e, Theorem 1. The first step is to recovery the a_n, b_n terms. Take the recursive formula of A_n and B_n implementing (2) and (3). Calculate the first few terms of A_n and B_n and use OEIS website to seek for possible closed-form representations of A_n and B_n. The next step is to prove the formula utilizing mathematical induction. Finally, take the limit of the quotient and show that it equals the desired conclusion.

Proof of Theorem 1.

Let a_n and b_n be given as in Theorem 1. We recall the initial conditions $A_{-1} = 1\,, A_0 = b_0\,, B_{-1} = 0, B_0 = 1$ and note that:

$$b_0 = -1, b_n = -2n,$$

$$a_1 = 1, a_n = 4(n-1).$$

Substituting into (2) and using (3) we see that

$$(A_n) = (3, -16, 120, -1152, 13440, \ldots).$$

The search from https://oeis.org/ website using the first 5 terms of A_n yields exact match with the terms of the sequence A187735 which has the following closed form representation

$$A_n = (-1)^{n-1} 2^{n-1} (n+2) n! \tag{3}$$

which still requires a proof. Next we see that B_n satisfies

$$B_n = (1, -2, 12, -88, 848, -9888, \dots .) \tag{4}$$

We see that B_n can be represented as a product of OEIS sequence A000255 with $(-2)^n$. More specifically, we have

$$B_n = (-2)^n \left[\frac{(n+2)n!}{e}\right], \tag{5}$$

where the symbol [ ] stands for the nearest integer. We still need to justify (5).

We now turn in proving (3) using mathematical induction. It clearly holds for the basis step. More specifically, the first two terms satisfy

$$A_0 = (-1)^{0-1}2^{0-1}(0+2)0! = -1,$$

$$A_1 = (-1)^{1-1}2^{1-1}(1+2)1! = 3.$$

Now, let us assume $A_k = (-1)^{k-1}2^{k-1}(k+2)k! \quad \forall k \leq n, n \geq 1$ and set $k = n+1$. Using the recursive relation, we get

$$\begin{aligned}
A_{n+1} &= b_{n+1}A_n + a_{n+1}A_{n-1} \\
&= -2(n+1)(-1)^{n-1}2^{n-1}(n+2)n! + 4n(-1)^{n-2}2^{n-2}(n+1)(n-1)! \\
&= (-1)^n 2^n (n+2)(n+1)n! + (-1)^{n-2}2^n \times n(n+1)(n-1)! \\
&= (-1)^n 2^n (n+2)! + (-1)^{n-2}2^n(n+1)! \\
&= (-1)^n 2^n (n+1)!\,(n+2+(-1)^{-2}) \\
&= (-1)^n 2^n (n+1)!\,(n+3),
\end{aligned}$$

which finishes the proof of (3) by mathematical induction.

We turn in justifying (5). We use a simple change of variable to show that our recursive relation (2) for B_n is equivalent to the recursive relation provided in OEIS site for the sequence A000255, namely

$$a(n) = n * a(n-1) + (n-1) * a(n-2), a(0) = 1, a(1) = 1. \tag{6}$$

To this end, we substitute $B_n = (-2)^n C_n,\ n \geq 1$ into (2) with $b_{n+1} = -2(n+1)$ and $a_{n+1} = 4n$ and obtain

$$(-2)^{n+1}C_{n+1} = -2(n+1)(-2)^n C_n + 4n(-2)^{n-1}C_{n-1},$$

Which simplifies to $C_{n+1} = (n+1)C_n + nC_{n-1},\ \ n \geq 1$ or

$$C_n = nC_{n-1} + (n-1)C_{n-2},\ \ n \geq 2$$

with initial conditions $C_0 = B_0 = 1\ , C_1 = \frac{B_1}{-2} = 1$. It follows that C_n is equivalent to (6), the recursive relation from OEIS website corresponding to the sequence A000255. On the other hand, OEIS also provides the formula for A000255 proposed by Simon Plouffe, March 1993:

$$C_n = \left[\frac{(n+2)n!}{e}\right] \quad \forall n \geq 1.$$

Thus, we conclude that

$$B_n = (-2)^n C_n = (-2)^n \left[\frac{(n+2)n!}{e}\right]. \tag{7}$$

It remains to show that A_n/B_n tends to the desired limit $-e/2$. Using the fact that [ ] is the nearest integer symbol, we may write B_n as

$$B_n = (-2)^n \left(\frac{(n+2)n!}{e} + \varepsilon(n)\right) \text{ where } -0{,}5 < \varepsilon(n) < 0{,}5. \tag{8}$$

Using (3) together with (8) we arrive at

$$\lim_{n\to\infty} \frac{A_n}{B_n} = \frac{(-1)^{n-1}2^{n-1}(n+2)n!}{(-2)^n\left(\frac{(n+2)n!}{e} + \varepsilon(n)\right)} = \frac{e}{(-2 - \varepsilon(n)/((n+2)n!)} = \frac{e}{-2}.$$

This finished the proof of Theorem 1.

3. *Conclusion*

In this paper, a special representation of numbers called continued fraction is studied. The continued fraction has a rich history and it is one of the most striking and powerful representations of numbers. For e, a continued fraction expansion often reveals beautiful number patterns which remain obscured in their decimal expansion. In our work we tried to prove some new continued fraction identities for Euler constant e which has taken from Ramanujan site. The proof here is direct and elementary, where we use the recursive formula to derive closed form formulas for the convergents of particular continued fractions.

References

1 Kadyrov, Sh., Mashurov, F. Generalized continued fraction expansions for pi and e. arXiv *preprint arXiv*:1912.03214, (2019): pp. 1-9.
2 Zhentao Lu. Elementary proofs of generalized continued fraction formulae for e. *arXiv preprint* arXiv:1907.05563, (2019): pp. 1-4.
3 Gal Raayoni, Shahar Gottlieb, George Pisha, Yoav Harris, Yahel Manor, Uri Mendlovic, Doron Haviv, Yaron Hadad, and Ido Kaminer. The ramanujan machine: automatically generated conjectures on fundamental constants. *arXiv preprint* arXiv:1907.00205, (2019): pp.1-28.
4 Sandifer, Ed. How Euler Did It: Who proved e is irrational. (2006): pp. 1-7.
5 Sloane, N.J. et al., The on-line encyclopedia of integer sequences, URL: https://oeis.org/ (2003).
6 William, B.J., Wolfgang, J.Th., Continued fractions: Analytic theory and applications, Cambridge University Press, 1984. – 459 p.
7 David H Bailey, Jonathan M Borwein, and Roland Girgensohn. Experimental evaluation of euler sums. Experimental Mathematics, 3 (1), (1994): pp. 17–30.
8 Bruce, C. Berndt. Ramanujan's notebooks. Springer Science & Business Media, 2012. – 357 p.
9 James Mc Laughlin, Nancy J. Wyshinski. Real numbers with polynomial continued fraction expansions. arxiv *preprint arXiv*:1912.06345v2, (2004): pp.1-18.