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“Integral power balance method in heat problems with free boundary”

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INTRODUCTION

The dissertation is devoted to investigation and development the method that consider models describing the processes of phase transformations in the presence of several free moving boundaries, which could degenerate at the initial time. Such problems arise, for example, the mathematical modeling of the dynamics of the temperature field of liquid-metal electrical contact bridge generated between the electrodes when they are in the opening stages of the arc. To solve these problems will be developed approximate method.

Problems of heat conduction in domain with a free boundary are among the most difficult problems of mathematical physics, the solution of which is necessary for progress in many areas of practice, such as metallurgy, technologies using low-temperature plasma, aircraft engineering, oil production and a number of others. From a mathematical point of view, these problems are among the most difficult nonlinear problems in the theory of parabolic equations, in which, along with the desired solutions of the equations, it is also necessary to find unknown moving boundaries.

One of the important areas of application of the free boundary problems is the mathematical modelling of phenomena in the low-temperature plasma of an electric arc and in contacts of electrical devices. Analysis of solutions makes it possible to verify the obtained theoretical results, to test the effectiveness of the developed algorithms for specific evolutionary processes in electrical apparatuses, and to interpret the available experimental data. The evolution of contact bridge and arcing processes is so fast (nano- and microsecond range) that their experimental study is very difficult. In some cases, only mathematical modeling can give an idea of their dynamics. Thus, the need for modeling is required not only for optimization of the experiment, but also due to the impossibility of using a some different approach.

One of the most effective methods of solving heat problem is the method of heat potentials, which reduces the initial boundary value problems to integral equations. However, in the case of regions degenerating at the initial time, additional difficulties arise releted to the singularity of these integral equations. These difficulties are compounded in the case when an unknown function appears not only in the boundary condition, but also in the coefficients of the equation. This method enables us to obtain an approximate solution with desirable degree of accuracy and to evaluate the approximation error, using the maximum principle.

Analytical methods for solution of heat and mass transfer problems have recently received a new stimulus to their further development due to the growing need to solve multicriteria problems for which numerical methods are unable to estimate the influence of a large number of input parameters on the behaviour of the solution and especially on its dynamics. In particular, an integral thermal balance method, a perturbation method, and a number of other methods are widely used to solve problems of the Stefan type with a free boundary, describing heat transfer with phase transitions. The main problem with the use of this method is the estimation of the approximation error, which, as a rule, is replaced for applied problems by comparison of the analytical solution with the experimental data.

The results of the performed work will make it possible a significant contribution to the development of the theory and methods for solution of boundary value problems for the heat equation with phase transformations in domains with moving boundaries, degenerating at the initial time.

1 INTEGRAL POWER BALANCE METHOD

The method easily applicable to one dimensional, time dependent boundary value problems of heat conduction and Stefan problems. The results are approximate, but several solutions obtained with this method when compared with the exact solutions have confirmed that the accuracy is generally acceptable for many engineering applications.

Idea of the method is integrate the heat equation over some distance, that called the thermal layer and choose suitable profile for temperature function. When the temperature profile thus constructed is introduced into the energy integral equation and the indicated operations are performed, an ordinary differential equation is obtained for the thermal layer thickness $\delta(t)$ with time as the independent variable. To illustrate the mathematical models discussed above we will start with simplest problem in a semi infinite domain.

Suppose we have a rod that is heated by heat flow $P(t)$. The temperature along the rod distributes by the curve till $\alpha(t)$. Here we consider the zone of temperature shifting (*Figure 3.5.1*).

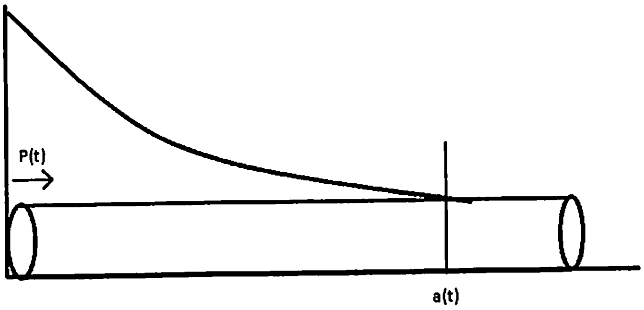


Figure 3.5.1 Heat flux

The mathematical formulation of problem:

$$\frac{\partial\theta}{\partial t} = a^2\frac{\partial^2\theta}{\partial x^2}, \qquad 0 < x < \infty \tag{1.1}$$

$$\theta|_{t=0} = 0 \tag{1.2}$$

$$\theta|_{\infty} = 0 \tag{1.3}$$

$$-\lambda\frac{\partial\theta}{\partial x}|_{x=0} = P(t) \tag{1.4}$$

We now choose temperature profile $\theta(x,t)$ in the form:

$$\theta(x,t)=\begin{cases} Q(t)\left[1-\dfrac{x}{\alpha(t)}\right]^2, & 0<x<\alpha(t) \\ 0, & x>\alpha(t) \end{cases} \qquad (1.5)$$

$0<x<\alpha(t)$ - the zone of temperature shifting. After heating, the zone $\int_0^{\alpha(t)}\frac{\partial}{\partial t}\left\{\frac{\alpha(t)}{2\lambda}P(t)\left[1-\frac{x}{\alpha(t)}\right]^2\right\}dx=a^2\frac{\partial\theta}{\partial x}\Big]_0^{\alpha(t)}=\frac{1}{c\gamma}P(t)$ of heating starts the shifting:

$$\frac{\partial\theta}{\partial x}\Big|_{x=\alpha(t)}=0 \qquad (1.6)$$

The main equation is finding the $Q(t)$ and $\alpha(t)$. Firstly, let's find the link between the boundary conditions and $Q(t)$. By the conditions (1.4) and (1.5) we get:

$$\frac{\partial\theta}{\partial x}=2Q(t)\left[1-\frac{x}{\alpha(t)}\right]\left(-\frac{1}{\alpha(t)}\right)$$

$$P(t)=-\lambda\frac{\partial\theta}{\partial x}\Big|_{x=0}=-2\lambda Q(t)\left(-\frac{1}{\alpha(t)}\right)=\frac{2\lambda Q(t)}{\alpha(t)}$$

$$Q(t)=\frac{\alpha(t)P(t)}{2\lambda}$$

Now to find the $\alpha(t)$. We substitute the differential equation with the equation of integral power balance. However as it is not available on every point, we choose the middle as $0<x<\alpha(t)$.

Firstly, let's begin the solution as follows:

$$\theta(x,t)=\begin{cases} \dfrac{\alpha(t)}{2\lambda}P(t)\left[1-\dfrac{x}{\alpha(t)}\right]^2, & 0<x<\alpha(t) \\ 0, & x>\alpha(t) \end{cases}$$

$$\frac{d}{dt}\int_0^{\alpha(t)} P(t)\alpha(t)\left[1-\frac{x}{\alpha(t)}\right]^2 dx = 2a^2P(t)$$

Therefore, we get next expression:

$$\frac{d}{dt}\left\{P(t)\alpha(t)\left[1-\frac{x}{\alpha(t)}\right]^3\frac{1}{3}[-\alpha(t)]\right]_0^{\alpha(t)}\right\} = 2a^2P(t)$$

$$\frac{d}{dt}\left\{P(t)\alpha^2(t)\right\} = 6a^2P(t)$$

$$P(t)\alpha^2(t) = 6a^2\int_0^t P(\tau)d\tau$$

$$\alpha(t) = \left[\frac{6a^2}{P(t)}\int_0^t P(\tau)d\tau\right]^{\frac{1}{2}}$$

Now let us consider the case when we have a constant flow:

$$P(t) = P_0 = const, \qquad \alpha(t) = a\sqrt{6t}, \qquad \theta(0,t) = \frac{P_0 a\sqrt{6t}}{2\lambda}$$

At $x=0$ the solution coincides to: $\theta(0,t) = \bar{\theta}(0,t) = \theta_0$, but at $x = a\sqrt{6t}$ $\theta(\alpha(t),t) = \theta_0 erfc(\sqrt{6}/2) = 0.08\theta_0$, as 8% from initial condition at $x=0$, while $\bar{\theta}(\alpha(t),t) = 0$. Similarly, if instead of boundary condition (1.2) given a general boundary condition:

$$\theta(0,t) = f(t),$$

Approximate solution has the same solution as (1.4):

$$\alpha(t) = \frac{a\sqrt{6}}{f(t)}\left[\int_0^t f^2(\tau)d\tau\right]^{1/2}$$

Exact solution:

$$\theta(x,t) = \frac{2aP_0}{\lambda}\sqrt{t}\cdot ierfc\frac{x}{2a\sqrt{t}}$$

$$\theta(0,t) = \frac{2aP_0}{\lambda\sqrt{\pi}}\sqrt{t}$$

2 APPROXIMATE SOLUTION OF TWO PHASE INVERSE STEFAN PROBLEM

Exact solutions of heat conduction problems are rather cumbersome and time-consuming. In addition, they are practically absent in the problems of the radial heat flux in spherical coordinates with the change in the aggregate state [1, 2]. Therefore, to solve practical problems are commonly used charts, obtained by numerical or approximate methods [3]. One of the approximate analytical methods is an integral method of heat balance, which primarily attracted to its physical clarity, simplicity and precision of the results is high enough that demonstrates T. Goodman [4] on numerous examples. The main difficulty to be faced when using the integral method of heat balance is the setting right temperature profile, which according to T. Goodman, significantly affects the accuracy of the results.

There are several approaches in selecting the temperature profiles. In [5] A.I. Veinik offers to use temperature profiles in the form of ordinary polynomials for problems of any geometry, which should simplify the solution of the problem.

Problem Formulation

In the following two sections we will deal with the approximate solution of heat equations

$$\frac{\partial u_1}{\partial t} = a_1^2\frac{\partial^2 u_1}{\partial x^2}, \quad 0 < x < \alpha(t), \quad t > 0 \tag{2.1}$$

$$\frac{\partial u_2}{\partial t} = a_2^2 \frac{\partial^2 u_2}{\partial x^2}, \quad \alpha(t) < x < \infty, \quad t > 0 \tag{2.2}$$

Subjected to the following conditions:

$$u_1(0,0) = 0 \tag{2.3}$$

$$u_2(x,0) = f(x) \quad \alpha(t) < x < \infty \tag{2.4}$$

$$-\lambda_1 \frac{\partial u_1}{\partial x} |_{x=0} = P(t), \quad t > 0 \tag{2.5}$$

$$u_1(\alpha(t),t) = u_2(\alpha(t),t) = u_m \tag{2.6}$$

The Stefan's condition:

$$-\lambda_1 \frac{\partial u_1}{\partial x} |_{x=\alpha(t)} = -\lambda_2 \frac{\partial u_2}{\partial x} |_{x=\alpha(t)} + L\gamma \frac{d(\alpha(t))}{dt} \tag{2.7}$$

$$u_2 |_{x=\infty} = 0 \tag{2.8}$$

It is necessary to find temperature distribution $u_1(x,t)$ and $u_2(x,t)$ also it is required to reconstruct the boundary function $P(t)$ if the free boundary $\alpha(t)$ is given. Such problem is called two phase inverse Stefan Problem. The heat spread in the solid is negligible because of the physical properties of contact material. This condition is valid for refractory metals like wolfram.

Problem Solution

Let us consider this problem at the suggestion that $\alpha(t)=\sum_{n=1}^{\infty}\alpha_n t^{n/2}$ is given.

To find the unknown $u_2(x,t)$, we identify the temperature profile corresponding following conditions:

$$u_2(\alpha(t),t)=u_m \tag{2.9}$$

$$u_2\big|_{\beta(t)}=0 \tag{2.10}$$

$$\left.\frac{\partial u_2}{\partial x}\right|_{x=\beta(t)}=0 \tag{2.11}$$

$$u_2\big|_{x=\infty}=0 \tag{2.12}$$

Therefore the temperature profile will be:

$$u_2(x,t)=\begin{cases} u_m\left[\dfrac{x-\beta(t)}{\alpha(t)-\beta(t)}\right]^2, & \alpha(t)<x<\beta(t) \\ 0, & x>\beta(t)\end{cases} \tag{2.13}$$

Here we need to find the $\beta(t)$, where $\beta(0)=0$ by using the equation (2.10) to use the integral of power balance:

$$\int_{\alpha(t)}^{\beta(t)}\frac{\partial u_2}{\partial t}dx=a_2^2\left.\frac{\partial u_2}{\partial x}\right]_{\alpha(t)}^{\beta(t)} \tag{2.14}$$

$$\int_{\alpha(t)}^{\beta(t)}\frac{\partial u_2}{\partial t}dx=a_2^2\left[\left.\frac{\partial u_2}{\partial x}\right|_{x=\beta(t)}-\left.\frac{\partial u_2}{\partial x}\right|_{x=\alpha(t)}\right]$$

$$\int_{\alpha(t)}^{\beta(t)}\frac{\partial u_2}{\partial t}dx=a_2^2\left[0-2u_m\left[\frac{x-\beta(t)}{\alpha(t)-\beta(t)}\right]\left(\frac{1}{\alpha(t)-\beta(t)}\right)\Bigg]_{x=\alpha(t)}\right]$$

$$\int_{\alpha(t)}^{\beta(t)} \frac{\partial u_2}{\partial t} dx = -a_2^2 \frac{2u_m}{\alpha(t)-\beta(t)}$$

For the right side we use the Leibniz integral rule, also differentiation under the integral sign:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x))\cdot b'(x) - f(x,a(x))\cdot a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt$$

Therefore,

$$\int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t} f(x,t)dx = \frac{d}{dt}\left(\int_{\alpha(x)}^{\beta(x)} f(x,t)dx\right) - f(\beta(t),t)\cdot\beta'(t) + f(\alpha(t),t)\cdot\alpha'(t) \quad (2.15)$$

By using Leibniz's method we get:

$$\left[\frac{d}{dt}\left(\int_{\alpha(t)}^{\beta(t)} u_2 dx\right) - u_2\big|_{x=\beta(t)}\frac{d\beta(t)}{dt} + u_2\big|_{x=\alpha(t)}\frac{d\alpha(t)}{dt}\right] = -a_2^2 \frac{2u_m}{\alpha(t)-\beta(t)}$$

Here we can calculate them separately:

$$u_2\big|_{x=\beta(t)}\frac{d\beta(t)}{dt} = 0$$

$$u_2\big|_{x=\alpha(t)}\frac{d\alpha(t)}{dt} = u_m\frac{d\alpha(t)}{dt}$$

After substituting them we get integral of power balance:

$$u_m \frac{d}{dt}\left(\int_{\alpha(t)}^{\beta(t)}\left[\frac{x-\beta(t)}{\alpha(t)-\beta(t)}\right]^2 dx\right) = -u_m\left(\frac{d\alpha(t)}{dt} + \frac{2a_2^2}{\alpha(t)-\beta(t)}\right)$$

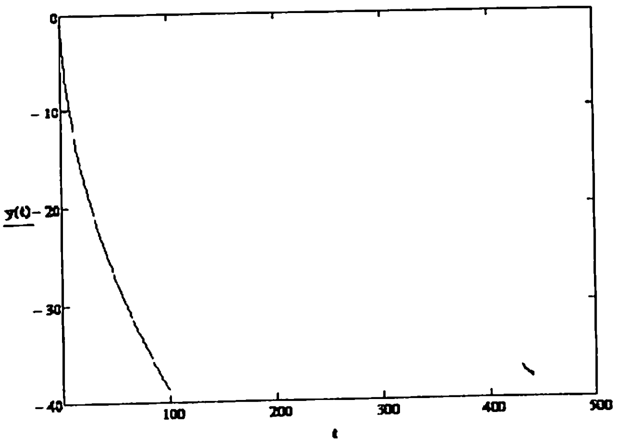
$$\frac{d}{dt}\left[\frac{1}{3}\left[\frac{x-\beta(t)}{\alpha(t)-\beta(t)}\right]^3\left(\alpha(t)-\beta(t)\right)\right]_{\alpha(t)}^{\beta(t)}=-\frac{d\alpha(t)}{dt}-\frac{2a_2^2}{\alpha(t)-\beta(t)}$$

$$\frac{d}{dt}\left[\left(\alpha(t)-\beta(t)\right)\right]=\frac{3d\alpha(t)}{dt}+\frac{6a_2^2}{\alpha(t)-\beta(t)}$$

$$\frac{d\beta(t)}{dt}\left(\beta(t)-\alpha(t)\right)+2\frac{d\alpha(t)}{dt}\left(\beta(t)-\alpha(t)\right)-6a_2^2 \qquad (2.16)$$

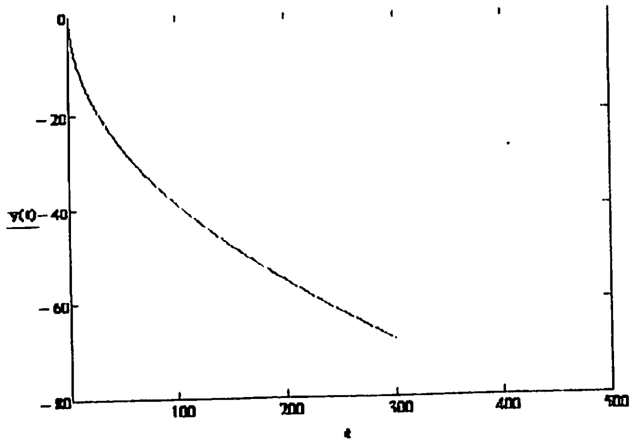
To solve the equation (2.18) we have used the mathematical program Mathcad. To find the solution we considered three particular cases and got following graphs:

$$\frac{d}{dt}y(t)\left(y(t)-\alpha(t)\right)+2\frac{d\alpha(t)}{dt}\left(y(t)-\alpha(t)\right)-6a_2^2=0 \qquad (2.17)$$



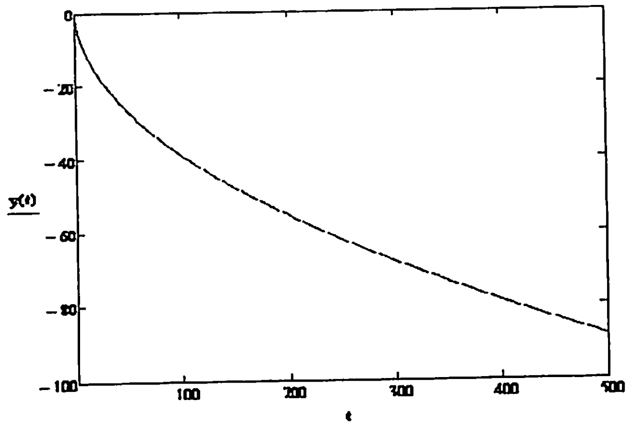
Equation (2.17) have been plotted in the following way for the first particular case when $t=100$:

Figure 3.6.3.1 $\beta(t)$ at $t=100$



Equation (2.17) have been plotted in the following way for the second particular case when $t=300$:

Figure 3.6.3.2 $\beta(t)$ at $t=300$



Equation (2.17) have been plotted in the following way for the third particular case when $t = 500$

Figure 3.6.3.3 $\beta(t)$ at $t = 500$

This given plots approximately show the value of $\beta(t)$. Therefore, we can consider that we know the value of $\beta(t)$ and consider other cases.

For $u_1(x,t)$ we have the temperature profile $A(t)x^2 + B(t)x + C(t)$ as it is linear. We use the conditions (2.13), (2.14) for $u_1(x,t)$, (2.17) and (2.9):

$$u_1(x,t) = A(t)x^2 + B(t)x + C(t), \qquad 0 < x < \alpha(t)$$

$$\frac{\partial u_1}{\partial x} = 2A(t)x + B(t)$$

By using the condition (2.13) we get:

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=0} = B(t) = -\frac{P(t)}{\lambda_1} \tag{2.18}$$

By using the condition (2.14) for $u_1(x,t)$ we have:

$$u_1(\alpha(t),t) = A(t)\alpha^2(t) + B(t)\alpha(t) + C(t) = u_m \tag{2.19}$$

By using the Stefan condition (2.15) we get:

$$-\lambda_1(2A(t)\alpha(t) + B(t)) = -\lambda_2\left(\frac{2u_m}{\alpha(t) - \beta(t)}\right) + L\gamma\frac{d\alpha(t)}{dt}$$

If we equalize the right side of the equation above to $Q(t)$, we will have following expression:

$$2A(t)\alpha(t)+B(t)=-\frac{Q(t)}{\lambda_1} \qquad (2.20)$$

Let us use the last equation (2.9) and get an integral of power balance:

$$\int_0^{\alpha(t)}\frac{\partial^2 u_1}{\partial x^2}dx=\left.\frac{\partial u_1}{\partial x}\right]_0^{\alpha(t)}$$

$$a_1^2\left[\left.\frac{\partial u_1}{\partial x}\right|_{x=\alpha(t)}-\left.\frac{\partial u_1}{\partial x}\right|_{x=0}\right]=\int_0^{\alpha(t)}\frac{\partial u_1}{\partial t}dx$$

$$a_1^2\left[2A(t)\cdot\alpha(t)+B(t)-B(t)\right]=a_1^2\left[2A(t)\cdot\alpha(t)\right]=\int_0^{\alpha(t)}\frac{\partial u_1}{\partial t}dx$$

For the right side we use the Leibniz integral rule, also differentiation under the integral sign.

Therefore,

$$\int_0^{\alpha(t)}\frac{\partial}{\partial t}u_1 dx=\frac{d}{dt}\left(\int_0^{\alpha(t)}u_1 dx\right)-u_1\big|_{x=\alpha(t)}\cdot\frac{d\alpha(t)}{dt}+u_1\big|_{x=0}\cdot\frac{d(0)}{dt}$$

After substituting we get total integral power of balance:

$$\frac{d}{dt}\left[\int_0^{\alpha(t)}\left(A(t)x^2+B(t)x+C(t)\right)dx-u_m\frac{d\alpha(t)}{dt}\right]=a_1^2\left[2A(t)\cdot\alpha(t)\right]$$

$$\frac{d}{dt}\left[A(t)\frac{\alpha^3(t)}{3}+B(t)\frac{\alpha^2(t)}{2}+C(t)\alpha(t)\right]=u_m\alpha'(t)+2a_1^2\left[A(t)\cdot\alpha(t)\right]$$

$$\frac{d}{dt}\left[2A(t)\alpha^2(t)+3B(t)\alpha(t)+6C(t)\right]=u_m\frac{\alpha'(t)}{\alpha(t)}+12a_1^2A(t)$$

As $B(t)=-\left(Q(t)/\lambda_1\right)-2A(t)\cdot\alpha(t)$ and $C(t)=u_m+A(t)\cdot\alpha^2(t)+\left(Q(t)\cdot\alpha(t)/\lambda_1\right)$:

$$\frac{d}{dt}\left[2A(t)\alpha^2(t)+6u_m+\frac{3Q(t)\alpha(t)}{\lambda_1}\right]=u_m\frac{\alpha'(t)}{\alpha(t)}+12a_1^2A(t)$$

$$A'(t)\left[2\alpha^2(t)+4\alpha'(t)\cdot\alpha(t)\right]+A(t)\left[4\alpha'(t)+4\left(\alpha''(t)\cdot\alpha(t)+\left(\alpha'(t)\right)^2\right)-12a_1^2\right]+$$

$$+\left[\frac{d}{dt}\left[\frac{3\alpha'(t)Q(t)}{\lambda_1}+\frac{3\alpha(t)Q'(t)}{\lambda_1}\right]-u_m\frac{\alpha'(t)}{\alpha(t)}\right]=0 \qquad (2.21)$$

If we denote the functions that is multiplied to $A'(t)$, $A(t)$ and some function by $Z_1(t)$, $Z_2(t)$ and $Z_3(t)$ respectively, then we get:

$$A'(t)\cdot Z_1(t)+A(t)\cdot Z_2(t)+Z_3(t)=0$$

We can easily rewrite our differential equation in the following form:

$$y'(x)+y(x)\cdot S(x)+T(x)=0$$

By solving our differential equation by Integrating Factor method, we should firstly find Integrating factor in the following form:.

$$IF=e^{\int S(t)dt}=\gamma(t)$$

$$A'(t)\gamma(t)+A(t)\cdot\gamma(t)\cdot S(t)+T(t)\gamma(t)=0$$

$$\frac{d}{dt}\left[\gamma(t)\cdot y(t)\right]=-T(t)\gamma(t)$$

By integrating both parts we can find $A(t)$:

$$A(t) = -e^{-\int S(t)dt} \cdot \left(\int T(t) \cdot e^{\int S(t)dt} dt \right) + c \cdot e^{-\int S(t)dt} \tag{2.22}$$

where $S(t) = Z_2(t) / Z_1(t)$ and $T(t) = Z_3(t) / Z_1(t)$. Let us denote the right side of above expression as $K(t)$, therefore we can show the temperature profile for $u_1(x,t)$ and reconstruct the boundary function $P(t)$:

$$u_1(x,t) = K(t)x^2 - \left[\frac{Q(t)}{\lambda_1} + 2K(t)\alpha(t) \right] x + \left[u_m + K(t)\alpha^2(t) + \frac{Q(t)\alpha(t)}{\lambda_1} \right] \tag{2.23}$$

$$P(t) = Q(t) + \frac{2K(t)\alpha(t)}{\lambda_1} \tag{2.24}$$

3 STEFAN PROBLEM IN ELLIPSOIDAL COORDINATES

Consider the quasi-stationary Stefan problem in symmetric electrical contacts. The analytical solution of three-phase stationary Stefan problem is found. Based on that decision was constructed the temperature profile to the approximate solution of the three-phase Stefan problem with Joule heating in ellipsoidal coordinates.

Stationary temperature and electromagnetic fields in symmetric electrical contacts have been described in [6]. Working with the scale of a mile second range, we think that every time the stationary state manages to instantly achieve stationary. And therefore this solution is suitable for constructing a temperature profile of the quasi-stationary problem.

3.1 Quasi-stationary nonlinear mathematical model of melting in ellipsoidal coordinates

The system of equations for the temperature $T_i(r,z)$ and electrical potential $\Phi_i(r,z)$ in this case can be written in the form

$$div(\lambda_1 \operatorname{grad} T_1) + \frac{1}{\rho_1} grad^2 \Phi_1 = 0$$

$$div(\frac{1}{\rho_1} grad\Phi_1) = 0$$

$$div(\lambda_2 \operatorname{grad} T_2) + \frac{1}{\rho_2} grad^2 \Phi_2 = 0$$

$$div(\frac{1}{\rho_2} grad\Phi_2) = 0$$

where Φ_i, λ_i, ρ_i are electrical potential, heat conductance and electrical resistivity respectively. In cylindrical co-ordinates these equations can be written as

$$\rho_i \frac{d\lambda_i}{dT_i}\left[\left(\frac{\partial T_i}{\partial r}\right)^2 + \left(\frac{\partial T_i}{\partial z}\right)^2\right] + \rho_i \lambda_i \Delta T_i + \left(\frac{\partial \Phi_i}{\partial r}\right)^2 + \left(\frac{\partial \Phi_i}{\partial z}\right)^2 = 0 \tag{3.1}$$

$$\frac{1}{\rho_i}\Delta\Phi_i - \frac{d\rho_i}{dT}\frac{1}{\rho_i^2}\left(\frac{\partial T_i}{\partial r}\frac{\partial \Phi_i}{\partial r} + \frac{\partial T_i}{\partial z}\frac{\partial \Phi_i}{\partial z}\right) = 0 \tag{3.2}$$

The index $i=1$ relates to the melted zone occupying the domain $D_1(0<z<\infty,\ r_0<r<r_m(t)$, and $i=2$ corresponds to the solid zone in the domain $D_2(0<z<\infty,\ r_m(t)<r<\infty)$.

It has to be mentioned that this problem is essentially non-linear due to temperature dependence of thermal conductivity $\lambda_i=\lambda_i(T_i)$ and electrical conductivity $\rho_i=\rho_i(T_i)$. The method of the solution can be obtained from the suggestion that the identity of equipotential and isothermal surfaces in contacts, which is correct for stationary fields in linear case, keeps safe for non-linear case as well. In linear case these surfaces are ellipsoids of revolution.

Equations (3.1) and (3.2) can be transformed into ellipsoidal coordinates and using well known relations among cylindrical and elliptical coordinates, if we suggest similarly like above that

$$\Phi_i=\Phi_i(\xi)\,,\quad T_i=T_i(\xi)\,, \tag{3.3}$$

where

$$\xi=\sqrt{s+\sqrt{s^2+4r_0^{\,2}z^2}}\,,\qquad s=r^2+z^2-r_0^{\,2}$$

then the equations (3.4) and (3.5) should be replaced by the equation

$$\rho_i\frac{d\lambda_i}{d\mathrm{T}_i}\cdot\left(\frac{d\mathrm{T}_i}{d\xi}\right)^2+\rho_i\lambda_i\frac{d^2\mathrm{T}_i}{d\xi^2}+\rho_i\lambda_i\frac{d\mathrm{T}_i}{d\xi}\cdot\frac{2\xi}{r_0^{\,2}+\xi^2}+\left(\frac{d\Phi_i}{d\xi}\right)^2=0 \tag{3.4}$$

$$\frac{d^2\Phi_i}{d\xi^2}+\frac{2\xi}{r_0^{\,2}+\xi^2}\cdot\frac{d\Phi_i}{d\xi}-\frac{1}{\rho_i}\frac{d\rho_i}{d\mathrm{T}_i}\cdot\frac{d\mathrm{T}_i}{d\xi}\frac{d\Phi_i}{d\xi}=0 \tag{3.5}$$

$$D:0<r<\infty,\ 0<z<\infty,\ z=0,\cup\ \ 0\le r<r_0\ ,\ 0<\xi<\infty\,,\ 0\le\eta<r_0 \tag{3.6}$$

The boundary conditions are

$z=0\ (\xi=0)$ $\qquad$ $\dfrac{dT_1}{d\xi}=0$ (3.7) $\qquad$ $\Phi_1\big|_{0\le r\le r_0}=0$ (3.8) $\qquad$ $\dfrac{\partial\Phi_1}{\partial z}\bigg|_{r<r_m(t)}=0$ (3.9)

$$T_1 = T_2 = T_m \quad (3.10) \qquad \Phi_1 = \Phi \quad (3.11)$$

$$z = \sigma(\mathrm{r},\mathrm{t})\ (\xi = \xi_m(\mathrm{t})) \qquad \lambda_1 \frac{dT_1}{d\xi} = \lambda_2 \frac{dT_2}{d\xi} \quad (3.12) \qquad \frac{1}{\rho_1}\frac{d\Phi_1}{d\xi} = \frac{1}{\rho_2}\frac{d\Phi_2}{d\xi} \quad (3.13)$$

$$z = \infty\ or\ r = \infty\ (\xi = \infty) \qquad T_2 = 0 \quad (3.14) \qquad \Phi_2 = \frac{U_c}{2} \quad (3.15)$$

while the solution for electric potentials

$$\Phi_1'(\xi) = \frac{I^2 \rho_1(\mathrm{T}_1)}{2\pi(\mathrm{r}_0^2+\xi^2)}, \qquad \Phi_2'(\xi) = \frac{I^2 \rho_2(\mathrm{T}_2)}{2\pi(\mathrm{r}_0^2+\xi^2)} \tag{3.16}$$

Putting (2.16) into (2.4) we get

$$\frac{1}{\lambda_i}\frac{d\lambda_i}{d\mathrm{T}_i}\cdot\left(\frac{d\mathrm{T}_i}{d\xi}\right)^2 + \frac{d^2\mathrm{T}_i}{d\xi^2} + \frac{d\mathrm{T}_i}{d\xi}\cdot\frac{2\xi}{r_0^2+\xi^2} + \frac{I^2\rho_i}{4\pi^2(\mathrm{r}_0^2+\xi^2)} = 0 \tag{3.17}$$

Let us introduce the new independent variable ζ using formula and consider the case when thermal conductivity doesn't depend on temperature, $\frac{d\lambda}{dT} = 0$

$$\zeta = \arctan\frac{\xi}{r_0} \tag{3.18}$$

Taking into account that $\rho_1 = \rho_{10}(1+\alpha_{10}(T_1 - T_m))$, $\rho_2 = \rho_{20}(1+\alpha_{20}T_2)$

And using [7] $\omega_i^2 = \frac{I^2\rho_{i0}\alpha_{i0}}{4\pi^2 r_0^2 \lambda_i}$

then the equation (3.17) for melted zone can be reduced to the form

$$\frac{d^2T_1}{d\zeta^2} + \frac{\omega_1^2}{\alpha_{10}}[1+\alpha_{10}(\mathrm{T}_1 - \mathrm{T}_m)] = 0 \tag{3.19}$$

The general solution of this equation is

$$T_1 = \frac{A_1}{\alpha_{10}} \cos \omega_1 \zeta + \frac{B_1}{\alpha_{10}} \sin \omega_1 \zeta + T_m - \frac{1}{\alpha_{10}} \qquad (3.20)$$

and A_1, B_1 are arbitrary constants, which can be found from the boundary conditions (3.7) and (3.8)

From (3.7) and (3.10)

$$B_1 = 0 \qquad A_1 = \frac{1}{\cos \omega_1 \frac{\pi}{2}}$$

Finally,

$$T_1 = \frac{1}{\alpha_{10}} \left(\frac{\cos \omega_1 \zeta}{\cos \omega_1 \zeta_m} + \alpha_{10} T_m - 1 \right)$$

The equation (3.17) for solid zone can be reduced to the form

$$\frac{d^2 T_2}{d\zeta^2} + \frac{\omega_2^2}{\alpha_{20}} [1 + \alpha_{20} \mathrm{T}_2] = 0 \qquad (3.21)$$

the general solution can be represented

$$T_2 = \frac{1}{\alpha_{20}} \left(A_2 \frac{\cos \omega_2 \zeta}{\cos \omega_2 \frac{\pi}{2}} + B_2 \frac{\sin \omega_2 \zeta}{\sin \omega_2 \frac{\pi}{2}} - 1 \right) \qquad (3.22)$$

From (3.14) and (3.10) can be found A_2, B_2 and temperature will be in the form

$$T_2 = \frac{1}{\alpha_{20} \sin \omega_2 (\frac{\pi}{2} - \zeta_m)} \left\{ (1 + \alpha_{20} T_m) \sin \omega_2 (\frac{\pi}{2} - \zeta) - \sin \omega_2 (\zeta_m - \zeta) - \sin \omega_2 (\frac{\pi}{2} - \zeta_m) \right\}$$

Noting first that

$$\left.\frac{dT_1}{d\xi}\right|_{\xi=\xi_m(t)}=\left.\frac{dT_1}{d\zeta}\cdot\frac{d\zeta}{d\xi}\right|_{\xi=\varsigma_m(t)}=-\frac{\omega_1\sin\omega_1\zeta_m}{\alpha_{10}\cos\omega_1\zeta_m}\cdot\frac{r_0}{r_0^2+\xi_m^2(t)}$$

$$\left.\frac{dT_2}{d\xi}\right|_{\xi=\xi_m(t)}=\left.\frac{dT_2}{d\zeta}\cdot\frac{d\zeta}{d\xi}\right|_{\xi=\xi_m(t)}=-\frac{\omega_2\left[(1+\alpha_{20}T_m)\cos\left(\frac{\pi}{2}-\zeta_m\right)-1\right]}{\alpha_{20}\sin\omega_2\left(\frac{\pi}{2}-\zeta_m\right)}\cdot\frac{r_0}{r_0^2+\xi_m^2(t)}$$

From (3.12),

$$\frac{\lambda_1\omega_1\sin\omega_1\zeta_m}{\alpha_{10}\cos\omega_1\zeta_m}=\frac{\lambda_2\omega_2\left[(1+\alpha_{20}T_m)\cos\omega_2\left(\frac{\pi}{2}-\zeta_m\right)-1\right]}{\alpha_{20}\sin\omega_2\left(\frac{\pi}{2}-\zeta_m\right)}$$

Finally we get

$$\zeta_m=\frac{1}{\omega_1}\arctan\frac{\lambda_2\omega_2\alpha_{10}}{\lambda_1\omega_1\alpha_{20}}\left[(1+\alpha_{20}T_m)\cot\omega_2\left(\frac{\pi}{2}-\zeta_m\right)-\operatorname{cosec}\omega_2\left(\frac{\pi}{2}-\zeta_m\right)\right]$$

3.2 Approximate solution of three phase Stefan problem

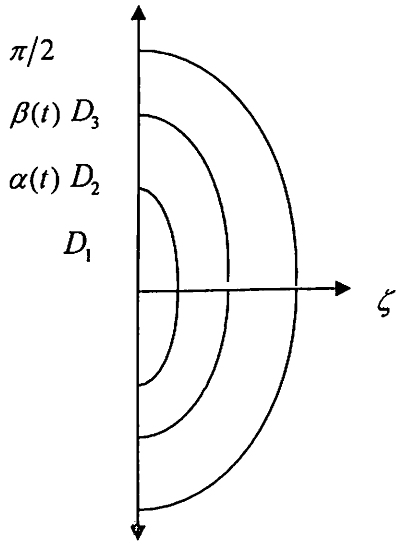


Fig. 1 |Ellipsoidal model of electrical contact. -

D_1 - (liquid), D_2 - (soft), D_3 -(solid)

The process of contact heating can be divided in a chain of consecutive stages.

The first stage $T_1(0 < t < t_s)$ corresponds to the initial period of contact heating by arc flux P . The temperature of electrode $\theta_1(\zeta, t)$ during this stage increases from the initial value $\theta_1(\zeta, 0)$ to the softening temperature θ_s , thus the termination time t_s of the first stage is defined from the equation

$$\theta_1(0, t_s) = \theta_s \tag{3.0.1}$$

The electrode at this stage consists only of one zone $D_3(0 < \zeta < \pi/2)$

The second stage $T_2(t_s < t < t_m)$ lasts from t_s to t_m ,when the electrode begins to melt. The electrode now consists of two zones: soft $D_3(0 < \zeta < \beta(t))$ and solid $D_2(\beta(t) < \zeta < \pi/2)$. Here $\zeta = \beta(t)$ is a free moving boundary interface of phase transformation. The time t_m can be found from the equation

$$\theta_2(0, t_m) = \theta_m \tag{3.0.2}$$

where $\theta_2(\zeta, t)$ is the temperature distribution inside the zone D_2, and θ_m is the melting of electrode.

The third stage $T_3(t_m < t < t_A)$ is characterized by appearance of new region $D_1(0 < \zeta < \alpha(t))$ occupied by melted region of electrode. The region $D_2(\alpha(t) < \zeta < \beta(t))$ with two moving boundaries is occupied by softened material, while the region $D_3(\beta(t) < \zeta < \pi/2)$ remains solid. It is the last stage of arcing of total duration t_A .

Using well known relations among cylindrical and elliptical coordinates, we get

$$\frac{\partial T}{\partial t} = \frac{a^2}{r_0^2 - \eta^2 + \xi^2}\left[\left(r_0^2 - \eta^2\right)\frac{\partial^2 T}{\partial \eta} + \frac{r_0^2 - 2\eta^2}{\eta^2}\frac{\partial T}{\partial \eta} + \left(r_0^2 + \xi^2\right)\frac{\partial^2 T}{\partial \xi^2} + 2\xi\frac{\partial T}{\partial \xi} + \frac{I^2 \rho}{4\pi^2 c\gamma\left(r_0^2 + \xi^2\right)}\right] \tag{3.3}$$

$$\frac{\nabla^2 \xi}{(\nabla \xi)^2} = \frac{2\xi}{r_0^2 + \xi^2}$$

The last relation allows us to conclude that the stationary temperature T depends only on ξ and does not depend on η. $T = T(\xi)$. Ellipsoids $\frac{r^2}{\xi^2 + r_0^2} + \frac{z^2}{\xi^2} = 1$ are isotherms and equipotential, and hyperboloids $\frac{r^2}{\eta^2} - \frac{z^2}{r_0^2 - \eta^2} = 1$ are surfaces of heat flow of electric current.

Then the equation (3.3) should be replaced by the equation

$$\frac{\partial T}{\partial t} = \frac{a^2}{r_0^2 + \xi^2}\left[\left(r_0^2 + \xi^2\right)\frac{\partial^2 T}{\partial \xi^2} + 2\xi\frac{\partial T}{\partial \xi} + \frac{I^2 \rho}{4\pi^2 c\gamma\left(r_0^2 + \xi^2\right)}\right]$$

or

$$\frac{\partial T}{\partial t} = a^2\left[\frac{\partial^2 T}{\partial \xi^2} + \frac{2\xi}{r_0^2 + \xi^2}\frac{\partial T}{\partial \xi} + \frac{I^2 \rho}{4\pi^2 c\gamma\left(r_0^2 + \xi^2\right)^2}\right]$$

After substitution $\zeta = \arctan\frac{\xi}{r_0}$, we get

$$\frac{\partial T}{\partial t} = \frac{a^2}{r_0^2}\cos^4(\zeta)\left[\frac{\partial^2 T}{\partial \zeta^2} + \frac{I^2 \rho}{4\pi^2 c\gamma r_0^4}\right]$$

The temperature fields $\theta_1(\zeta,t)$, $\theta_2(\zeta,t)$, $\theta_3(\zeta,t)$ inside corresponding zones D_1, D_2, D_3 are described by heat equations in ellipsoidal symmetry:

$$\frac{\partial \theta_i}{\partial t} = \frac{a_i^2}{r_0^2}\cos^4(\zeta)\left[\frac{\partial^2 \theta_i}{\partial \zeta^2} + \frac{I^2 \rho_i}{4\pi^2 r_0^4 c_i \gamma_i}\right], \quad i = 1, 2, 3 \qquad (3.0.4)$$

where a_i^2 is the thermal diffusivity of the zone D_i.

θ_{ij} can be determined in three regions where i=1,2,3; j=1,2 correspond to heat transfer regions and stages respectively.

The stage 1

In the first stage we have only one region D_3, where contact material is solid and temperature attains softening point. In this case we consider the heat equation

$$\frac{\partial\theta_1}{\partial t}=\frac{a_1^2}{r_0^2}\cos^4(\zeta)\left[\frac{\partial^2\theta_1}{\partial\zeta^2}+\omega_1^2\left(\theta_1+\frac{1}{\alpha_1}\right)\right] \qquad 0<\zeta<\pi/2 \qquad (3.1.1)$$

subjected to boundary conditions

$$\zeta=0: \qquad \frac{\partial\theta_1(0,t)}{\partial\zeta}=0 \qquad (3.1.2)$$

$$\zeta=\pi/2: \qquad \theta_1(\pi/2,t)=0 \qquad (3.1.3)$$

and initial condition

$$t=0: \qquad \theta_1(\zeta,0)=0 \qquad (3.1.4)$$

where $\omega_1=\frac{I}{2\pi r_0^2}\sqrt{\frac{\rho_{10}\alpha_1}{c_1\gamma_1}}$

For the temperature distribution $\theta_1(\zeta,t)$, let us assume that the temperature profile as given in the form

$$\theta_1(\zeta,t)=A_1(t)\cos(\omega_1\zeta)+B_1(t)\sin(\omega_1\zeta)+C_1(t) \qquad \text{in} \qquad 0\le\zeta\le\frac{\pi}{2} \qquad (3.1.5)$$

where the coefficients are in general functions of time.

Using conditions (3.1.2) and (3.1.3) we get

$$\begin{cases} B_1(t)=0 \\ A_1(t)\cos(\omega_1\frac{\pi}{2})+C_1(t)=0 \end{cases} \qquad (3.1.6)$$

Integration equation (3.1.1) with respect to the space variable form $\zeta=0$ to $\zeta=\pi/2$, noting first that

$$\int_0^{\pi/2}\cos^4(\zeta)\left[\frac{\partial^2\theta_1}{\partial\zeta^2}+\omega_1^2\left(\theta_1+\frac{1}{\alpha_1}\right)\right]d\zeta=\cos^4(\zeta)\frac{\partial\theta_1}{\partial\zeta}\bigg|_0^{\pi/2}+4\theta_1\cos^3(\zeta)\sin(\zeta)\Big|_0^{\pi/2}+$$

$$+\frac{3\pi\omega_1^2}{16\alpha_1}+\int_0^{\pi/2}\left[12\cos^2(\zeta)+\left(\omega_1^2-16\right)\cos^4(\zeta)\right]\theta_1 d\zeta$$

then we have

$$\frac{r_0^2}{a_1^2}\int_0^{\pi/2}\frac{\partial\theta_1}{\partial t}d\zeta=\frac{3\pi\omega_1^2}{16\alpha_1}+\int_0^{\pi/2}\left[12\cos^2(\zeta)+\left(\omega_1^2-16\right)\cos^4(\zeta)\right]\theta_1 d\zeta$$

When the integral on the left-hand side is performed using Leibniz's integral formula, we obtain

$$\frac{r_0^2}{a_1^2}\frac{d}{dt}\left[\int_0^{\pi/2}\theta_1 d\zeta\right]=\frac{3\pi\omega_1^2}{16\alpha_1}+\int_0^{\pi/2}\left[12\cos^2(\zeta)+\left(\omega_1^2-16\right)\cos^4(\zeta)\right]\theta_1 d\zeta \qquad (3.1.7)$$

(3.1.7) is called the energy integral equation for the problem considered here.

Substituting (3.1.5) and (3.1.6) the above into the energy integral equation (3.1.7) we obtain the following ordinary for $C_1(t)$

$$\frac{r_0^2}{a_1^2}\left(\frac{\pi}{2}-\frac{\tan(\omega_1\pi/2)}{\omega_1}\right)\frac{dC_1(\mathrm{t})}{dt}=\frac{3\pi\omega_1^2}{16\alpha_1}\left(\frac{1}{\alpha_1}+C_1(\mathrm{t})\right)$$

$$\begin{cases}C_1(0)=0\\ A_1(t)=-C_1(t)\sec\left(\omega_1\dfrac{\pi}{2}\right)\end{cases}$$

The solution of equation

$$C_1(\mathrm{t})=\exp\left(\left(\frac{\pi}{2}-\frac{\tan(\omega_1\pi/2)}{\omega_1}\right)^{-1}\frac{3\pi\omega_1^2a_1^2}{16\alpha_1r_0^2}t-\ln(\alpha_1)\right)-\frac{1}{\alpha_1}$$

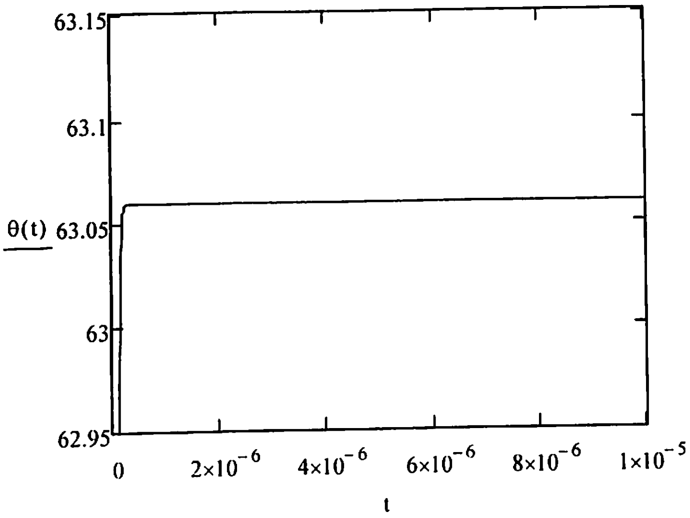
And

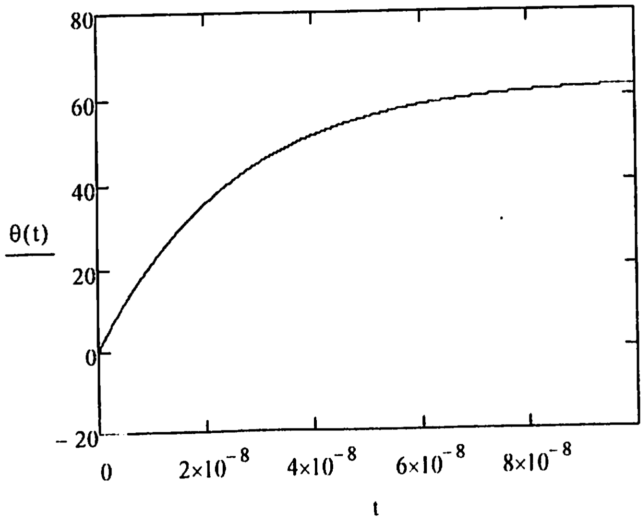
$$A_1(\mathrm{t}) = \left[\frac{1}{\alpha_1} - \exp\left(\left(\frac{\pi}{2} - \frac{\tan(\omega_1\,\pi/2)}{\omega_1}\right)^{-1} \frac{3\pi\omega_1^2 a_1^2}{16\alpha_1 r_0^2} t - \ln(\alpha_1)\right)\right] \sec\left(\omega_1 \frac{\pi}{2}\right)$$

Finally temperature profile

$$\theta_1(\zeta,t) = \left[\frac{1}{\alpha_1} - \exp\left(\left(\frac{\pi}{2} - \frac{\tan(\omega_1\,\pi/2)}{\omega_1}\right)^{-1} \frac{3\pi\omega_1^2 a_1^2}{16\alpha_1 r_0^2} t - \ln(\alpha_1)\right)\right] \sec\left(\omega_1 \frac{\pi}{2}\right)\cos(\omega_1\zeta) +$$

$$+ \exp\left(\left(\frac{\pi}{2} - \frac{\tan(\omega_1\,\pi/2)}{\omega_1}\right)^{-1} \frac{3\pi\omega_1^2 a_1^2}{16\alpha_1 r_0^2} t - \ln(\alpha_1)\right) - \frac{1}{\alpha_1}$$





The stage 2

In the second stage a new region D_2 occurs and now we have two regions D_2 and D_3. We determine t_s (softening time) from $\theta_{21}(0,t_s)=\theta_s$. Mathematical formulation for these zone are given as

$$\frac{\partial\theta_{21}}{\partial t}=\frac{a_{21}{}^2}{r_0^2}\cos^4(\zeta)\left[\frac{\partial^2\theta_{21}}{\partial\zeta^2}+\omega_{21}^2\left(\theta_{21}+\frac{1}{\alpha_{21}}\right)\right] \qquad 0<\zeta<\beta(t) \qquad (3.2.1)$$

$$\frac{\partial\theta_{22}}{\partial t}=\frac{a_{22}{}^2}{r_0^2}\cos^4(\zeta)\left[\frac{\partial^2\theta_{22}}{\partial\zeta^2}+\omega_{22}^2\left(\theta_{22}+\frac{1}{\alpha_{22}}\right)\right] \qquad \beta(t)<\zeta<\pi/2 \qquad (3.2.2)$$

$$\theta_{22}(0,0)=\theta_s \qquad (3.2.3)$$

$$\theta_{21}(\zeta,0)=f(\zeta) \qquad f(0)=\theta_s \qquad \beta(0)=0 \qquad f\left(\frac{\pi}{2}\right)=0 \qquad (3.2.4)$$

$$\zeta=0: \qquad \left.\frac{\partial\theta_{21}}{\partial\zeta}\right|_{\zeta=0}=0 \qquad (3.2.5)$$

$$\zeta=\beta(t): \qquad \theta_{21}\big|_{\zeta=\beta(t)}=\theta_{22}\big|_{\zeta=\beta(t)}=\theta_s \qquad (3.2.6)$$

$$\zeta=\frac{\pi}{2}: \qquad \theta_{22}\big|_{\zeta=\frac{\pi}{2}}=0 \qquad (3.2.7)$$

Stefan's condition

$$-\lambda_{21}\left.\frac{\partial\theta_{21}}{\partial\zeta}\right|_{\zeta=\beta(t)}=-\lambda_{22}\left.\frac{\partial\theta_{22}}{\partial\zeta}\right|_{\zeta=\beta(t)} \qquad (3.2.8)$$

where

$$f(\zeta)=\left[\frac{1}{\alpha_1}-\exp\left(\left(\frac{\pi}{2}-\frac{\tan(\omega_1\,\pi/2)}{\omega_1}\right)^{-1}\frac{3\pi\omega_1^2a_1^2}{16\alpha_1r_0^2}t_s-\ln(\alpha_1)\right)\right]\sec\left(\omega_1\frac{\pi}{2}\right)\cos(\omega_1\zeta)+$$

$$+\exp\left(\left(\frac{\pi}{2}-\frac{\tan(\omega_1\,\pi/2)}{\omega_1}\right)^{-1}\frac{3\pi\omega_1^2a_1^2}{16\alpha_1r_0^2}t_s-\ln(\alpha_1)\right)-\frac{1}{\alpha_1}$$

$$\omega_{21} = \frac{I}{2\pi r_0^2}\sqrt{\frac{\rho_{11}\alpha_{21}}{c_{11}\gamma_{11}}}, \qquad \omega_{22} = \frac{I}{2\pi r_0^2}\sqrt{\frac{\rho_{12}\alpha_{22}}{c_{12}\gamma_{12}}}$$

We note that the location of the softening interface $\zeta = \beta(t)$ is identical to the definition of the thermal layer. Hence, we choose the region $0 \le \zeta \le \beta(t)$ as the thermal layer appropriate for this problem and integrate the heat equation (3.2.1) form $\zeta = 0$ to $\zeta = \beta(t)$ noting that

$$\int_0^{\beta(t)} \cos^4(\zeta)\left[\frac{\partial^2\theta_{21}}{\partial\zeta^2} + \omega_{21}^2\left(\theta_{21} + \frac{1}{\alpha_{21}}\right)\right]d\zeta = \cos^4(\zeta)\frac{\partial\theta_{21}}{\partial\zeta}\bigg|_0^{\beta(t)} + 4\theta_{21}\cos^3(\zeta)\sin(\zeta)\Big|_0^{\beta(t)} +$$

$$+\frac{\omega_{21}^2}{32\alpha_{21}}\left(12\zeta + 8\sin(2\zeta) + \sin(4\zeta)\right)\Big|_0^{\beta(t)} + \int_0^{\beta(t)}\left[12\cos^2(\zeta)\sin^2(\zeta) + \left(\omega_{21}^2 - 4\right)\cos^4(\zeta)\right]\theta_{21}d\zeta$$

then we have

$$\left.\begin{aligned}
&\frac{r_0^2}{a_{21}^2}\left(\frac{d}{dt}\left[\int_0^{\beta(t)}\theta_{21}d\zeta\right] - \frac{d\beta(t)}{dt}\theta_s\right) = \frac{\lambda_{22}}{\lambda_{21}}\cos^4\left(\beta(t)\right)\frac{\partial\theta_{22}}{\partial\zeta}\bigg|_{\zeta=\beta(t)} + \\
&+\int_0^{\beta(t)}\left[12\cos^2(\zeta)\sin^2(\zeta) + \left(\omega_{21}^2 - 4\right)\cos^4(\zeta)\right]\theta_{21}d\zeta + \\
&+\frac{\omega_{21}^2}{32\alpha_{21}}\left(12\beta(t) + 8\sin[2\beta(t)] + \sin[4\beta(t)]\right) + 4\theta_s\cos^3(\beta(t))\sin(\beta(t))
\end{aligned}\right\} \qquad (3.2.9)$$

where (3.2.9) is the energy integral equation.

We now choose temperature profile for the equation $\theta_{21}(\zeta,t)$, in the form

$$\theta_{21}(\zeta,t) = A_{21}(t)\cos(\omega_{21}\zeta) + B_{21}(t)\sin(\omega_{21}\zeta) + C_{21}(t) \qquad \text{in} \qquad 0 \le \zeta \le \beta(t) \qquad (3.2.10)$$

Applying conditions (3.2.5), (3.2.8), (3.2.6) we get systems of equations

$$\begin{cases} B_{21}(t)=0 \\ A_{21}(t)\sin(\omega_{21}\beta(t))-B_{21}(t)\cos(\omega_{21}\beta(t))=-\dfrac{\lambda_{22}}{\omega_{21}\lambda_{21}}\dfrac{\partial\theta_{22}(\beta(t),t)}{\partial\zeta} \\ A_{21}(t)\cos(\omega_{21}\beta(t))+B_{21}(t)\sin(\omega_{21}\beta(t))+C_{21}(t)=\theta_s \end{cases}$$

$$\begin{cases} A_{21}(t)\sin(\omega_{21}\beta(t))=-\dfrac{\lambda_{22}}{\omega_{21}\lambda_{21}}\dfrac{\partial\theta_{22}(\beta(t),t)}{\partial\zeta} \\ A_{21}(t)\cos(\omega_{21}\beta(t))+C_{21}(t)=\theta_s \end{cases}$$

which solution is

$$A_{21}(t)=-\frac{\lambda_{22}}{\omega_{21}\lambda_{21}}\frac{1}{\sin(\omega_{21}\beta(t))}\frac{\partial\theta_{22}(\beta(t),t)}{\partial\zeta},$$
$$C_{21}(t)=\theta_s+\frac{\lambda_{22}}{\omega_{21}\lambda_{21}}\tan(\omega_{21}\beta(t))\frac{\partial\theta_{22}(\beta(t),t)}{\partial\zeta}$$

And temperature profile becomes

$$\theta_{21}(\zeta,t)=\left(\tan(\omega_{21}\beta(t))-\frac{\cos(\omega_{21}\zeta)}{\sin(\omega_{21}\beta(t))}\right)\frac{\lambda_{22}}{\omega_{21}\lambda_{21}}\frac{\partial\theta_{22}(\beta(t),t)}{\partial\zeta}+\theta_s$$

To solve this problem, we again choose temperature profile for $\theta_{22}(\zeta,t)$, in the form

$$\theta_{22}(\zeta,t)=A_{22}(t)\cos(\omega_{22}\zeta)+B_{22}(t)\sin(\omega_{22}\zeta)+C_{22}(t) \qquad \text{in} \qquad \beta(t)\le\zeta\le\pi/2 \qquad (3.2.13)$$

The coefficients are determined by utilizing conditions, as follow

(1) $\theta_{22}\big|_{\zeta=\beta(t)}=\theta_s$ (3.2.14$_1$)

(2) $\theta_{22}\big|_{\zeta=\frac{\pi}{2}}=0$ (3.2.14$_2$)

From conditions (3.2.14) we get system of equations

$$\begin{cases} A_{22}(t)\cos(\omega_{22}\beta(t)) + B_{22}(t)\sin(\omega_{22}\beta(t)) = \theta_s - C_{22}(t) \\ A_{22}(t)\cos(\omega_{22}\frac{\pi}{2}) + B_{22}(t)\sin(\omega_{22}\frac{\pi}{2}) = -C_{22}(t) \end{cases}$$

which solution is

$$A_{22} = \frac{\theta_s \sin\left(\frac{\pi w_{22}}{2}\right) + \left(\sin\left(\beta w_{22}\right) - \sin\left(\frac{\pi w_{22}}{2}\right)\right) C_{22}}{\sin\left(\frac{\pi w_{22}}{2} - \beta w_{22}\right)}$$

$$B_{22} = \frac{\left(\cos\left(\frac{\pi w_{22}}{2}\right) - \cos\left(\beta w_{22}\right)\right) C_{22} - \theta_s \cos\left(\frac{\pi w_{22}}{2}\right)}{\sin\left(\frac{\pi w_{22}}{2} - \beta w_{22}\right)}$$

Temperature profile became

$$\theta_{22}(\zeta,t) = \frac{\theta_s \sin\left(\frac{\pi w_{22}}{2}\right) + \left(\sin\left(\beta w_{22}\right) - \sin\left(\frac{\pi w_{22}}{2}\right)\right) C_{22}}{\sin\left(\frac{\pi w_{22}}{2} - \beta w_{22}\right)} \cos(\omega_{22}\zeta) + C_{22}(t) +$$

$$+ \frac{\left(\cos\left(\frac{\pi w_{22}}{2}\right) - \cos\left(\beta w_{22}\right)\right) C_{22} - \theta_s \cos\left(\frac{\pi w_{22}}{2}\right)}{\sin\left(\frac{\pi w_{22}}{2} - \beta w_{22}\right)} \sin(\omega_{22}\zeta)$$

The integration of the differential equation (3.2.2) over the thermal layer $\zeta = \pi/2$ gives

$$\left.\begin{aligned}&\frac{r_0^2}{a_{22}^2}\left(\frac{d}{dt}\left[\int_{\beta(t)}^{\pi/2}\theta_{22}d\zeta\right]+\frac{d\beta(t)}{dt}\theta_s\right)=\int_{\beta(t)}^{\pi/2}\left(12\cos^2(\zeta)+(\omega_{22}^2-16)\cos^4(\zeta)\right)\theta_{22}d\zeta+\\&+\frac{\omega_{22}^2}{\alpha_{22}}\left[\frac{3\pi}{16}-\frac{3}{8}\beta(t)-\frac{1}{4}\sin(2\beta(t))-\frac{1}{32}\sin(4\beta(t))\right]-4\theta_s\cos^3(\beta(t))\sin(\beta(t))-\\&-\cos^4(\beta(t))\frac{\partial\theta_{22}(\beta(t),t)}{\partial\zeta}\end{aligned}\right\}\qquad(3.2.12)$$

$\beta(t)$, $C_{22}(t)$ Can be found from system of ordinary differential equations (3.2.9), (3.2.12).

The stage 3

The third stage characterized by formation of third and last region D_1 where electrode begins to melt. Following model proposed to describe the phenomenon

$$\frac{\partial\theta_{31}}{\partial t}=\frac{a_{31}{}^2}{r_0^2}\cos^4(\zeta)\left[\frac{\partial^2\theta_{31}}{\partial\zeta^2}+\omega_{31}^2\left(\theta_{31}+\frac{1}{\alpha_{31}}\right)\right]\qquad 0<\zeta<\alpha(t)\qquad(3.3.1)$$

$$\frac{\partial\theta_{32}}{\partial t}=\frac{a_{32}{}^2}{r_0^2}\cos^4(\zeta)\left[\frac{\partial^2\theta_{32}}{\partial\zeta^2}+\omega_{32}^2\left(\theta_{32}+\frac{1}{\alpha_{32}}\right)\right]\qquad \alpha(t)<\zeta<\pi/2\qquad(3.3.2)$$

$$\theta_{32}(0,0)=\theta_m\qquad(3.3.3)$$

$$\theta_{31}(\zeta,0)=g(\zeta)\qquad g(0)=\theta_m\qquad \alpha(0)=0\qquad g\left(\frac{\pi}{2}\right)=0\qquad(3.3.4)$$

$$\zeta=0:\qquad \left.\frac{\partial\theta_{31}}{\partial\zeta}\right|_{\zeta=0}=0\qquad(3.3.5)$$

$$\zeta=\alpha(t):\qquad \theta_{31}\big|_{\zeta=\alpha(t)}=\theta_{32}\big|_{\zeta=\alpha(t)}=\theta_m\qquad(3.3.6)$$

$$\zeta=\frac{\pi}{2}:\qquad \theta_{32}\big|_{\zeta=\frac{\pi}{2}}=0\qquad(3.3.7)$$

Stefan's condition

$$-\lambda_{31}\frac{\partial\theta_{31}}{\partial\zeta}\bigg|_{\zeta=\alpha(t)} = -\lambda_{32}\frac{\partial\theta_{32}}{\partial\zeta}\bigg|_{\zeta=\alpha(t)} + L\gamma\frac{d\alpha(t)}{dt} \qquad (3.3.8)$$

where $g(\zeta) = A_{21}(t_m)\cos(\omega_{21}\zeta) + B_{21}(t_m)\sin(\omega_{21}\zeta) + C_{21}(t_m)$

$$\omega_{31} = \frac{I}{2\pi r_0^2}\sqrt{\frac{\rho_{21}\alpha_{31}}{c_{21}\gamma_{21}}}, \qquad \omega_{32} = \frac{I}{2\pi r_0^2}\sqrt{\frac{\rho_{22}\alpha_{32}}{c_{22}\gamma_{22}}}$$

Again if we note that the location of the solid-liquid interface $\zeta = \alpha(t)$ is identical to the definition of the thermal layer. Hence, we choose the region $0 \le \zeta \le \alpha(t)$ as the thermal layer appropriate for this problem and integrate the heat equation (3.3.1) form $\zeta = 0$ to $\zeta = \alpha(t)$, to obtain

$$\left.\begin{array}{l} \dfrac{r_0^2}{a_{31}^2}\left(\dfrac{d}{dt}\left[\displaystyle\int_0^{\alpha(t)}\theta_{31}d\zeta\right] - \dfrac{d\alpha(t)}{dt}\theta_m\right) = \cos^4(\alpha(t))\left(\dfrac{\lambda_{32}}{\lambda_{31}}\dfrac{\partial\theta_{32}}{\partial\zeta}\bigg|_{\zeta=\alpha(t)} - \dfrac{L\gamma}{\lambda_{31}}\dfrac{d\alpha(t)}{dt}\right) + 4\theta_m\cos^3(\alpha(t))\sin^2(\alpha(t)) + \\ + \displaystyle\int_0^{\alpha(t)}\left(12\sin^2(\zeta) + (\omega_{31}^2 - 16)\cos^4(\zeta)\right)\theta_{31}d\zeta + \dfrac{\omega_{31}^2}{32\alpha_{31}}\left(12\alpha(t) + 8\sin[2\alpha(t)] + \sin[4\alpha(t)]\right) \end{array}\right\}$$

(3.3.9)

Temperature profile for the equation $\theta_{31}(\zeta, t)$, in the form

$$\theta_{31}(\zeta, t) = A_{31}(t)\cos(\omega_{31}\zeta) + B_{31}(t)\sin(\omega_{31}\zeta) + C_{31}(t) \quad \text{in} \quad 0 \le \zeta \le \alpha(t) \qquad (3.3.10)$$

Three conditions are needed to determine these three coefficients. Equations (3.3.5) and (3.3.6) provide two conditions; however, the relation given by equation (3.3.8) is not suitable for this purpose because if it is used, the resulting temperature profile will involve derivative of moving bounday. When such a profile is substituted into the energy integral equation, a second – order ordinary differential equation will result for α(t), instead of the expected first – order equation. The boundary condition (3.3.6) is differentiated, which yields

$$d\theta_{31}=\left[\frac{\partial\theta_{31}}{\partial\zeta}d\zeta+\frac{\partial\theta_{31}}{\partial t}dt\right]_{\zeta=\alpha(t)}=0 \qquad (3.3.11_1)$$

or

$$\frac{\partial\theta_{31}}{\partial\zeta}\frac{d\alpha(t)}{dt}+\frac{\partial\theta_{31}}{\partial t}=0 \qquad (3.3.11_2)$$

The term $d\alpha(t)/dt$ is eliminated between equations (3.8) and (3.11$_2$) to yield

$$\frac{\partial\theta_{31}}{\partial\zeta}\left[\lambda_{32}\frac{\partial\theta_{32}}{\partial\zeta}-\lambda_{31}\frac{\partial\theta_{31}}{\partial\zeta}\right]\frac{1}{L\gamma}+\frac{\partial\theta_{31}}{\partial t}=0 \qquad \text{at} \qquad \zeta=\alpha(t) \qquad (3.3.12)$$

Now eliminating $\partial\theta_{31}/\partial t$ between equations (3.1) and (3.12), we obtain

$$\lambda_{31}\left(\frac{\partial\theta_{31}}{\partial\zeta}\right)^2-\lambda_{32}\frac{\partial\theta_{31}}{\partial\zeta}\frac{\partial\theta_{32}}{\partial\zeta}=L\gamma\frac{a_{31}^2}{r_0^2}\cos^4(\zeta)\left[\frac{\partial^2\theta_{31}}{\partial\zeta^2}+\omega_{31}^2\left(\theta_{31}+\frac{1}{\alpha_{31}}\right)\right] \text{ at } \zeta=\alpha(t) \qquad (3.3.13)$$

Applying conditions (3.3.5), (3.3.6) and (3.3.13) we get system of equations

$$B_{31}(t)=0$$

$$\theta_{31}(\zeta,t)=A_{31}(t)\cos(\omega_{31}\zeta)+C_{31}(t)$$

$$\left\{\begin{aligned}
&A_{31}(t)\cos(\omega_{31}\alpha(t))+C_{31}(t)=\theta_m\\
&\left[\lambda_{32}\sin(\omega_{31}\alpha(t))\frac{\partial\theta_{32}(\alpha(t),t)}{\partial\zeta}+L\gamma\frac{a_{31}^2\omega_{31}^2}{r_0^2}\cos^4(\alpha(t))\cos(\omega_{31}\alpha(t))\right]A_{31}(t)-\\
&-\left[\theta_m+\frac{1}{\alpha_{31}}\right]L\gamma\frac{a_{31}^2\omega_{31}^2}{r_0^2}\cos^4(\alpha(t))+\omega_{31}\lambda_{31}\sin^2(\omega_{31}\alpha(t))A_{31}^2(t)=0
\end{aligned}\right.$$

We can determine other coefficients if determinant D grater or equal to zero

The integration of the differential equation (3.3.2) over the thermal layer $\zeta=\pi/2$ gives

$$\frac{r_0^2}{a_{32}^2}\left(\frac{d}{dt}\left[\int_{\alpha(t)}^{\pi/2}\theta_{32}d\zeta\right]+\frac{d\alpha(t)}{dt}\theta_{32}\big|_{\zeta=\alpha(t)}\right)=-\cos^4(\zeta)\left.\frac{\partial\theta_{32}}{\partial\zeta}\right|_{\zeta=\alpha(t)}-4\theta_{32}\cos^3(\zeta)\sin(\zeta)\big|_{\zeta=\alpha(t)}+$$

$$+\int_{\alpha(t)}^{\pi/2}\big(12\cos^2(\zeta)+(\omega_{32}^2-16)\cos^4(\zeta)\big)\theta_{32}d\zeta+ +\frac{\omega_{32}^2}{\alpha_{32}}\left[\frac{3\pi}{16}-\frac{3\alpha(t)}{8}-\frac{\sin(2\alpha(t))}{4}-\frac{\sin(4\alpha(t))}{32}\right]$$

In view of the conditions (3.3.6) and (3.3.7)
it becomes to the energy integral equation (3.15) for the considered problem

$$\left.\begin{aligned}&\frac{r_0^2}{a_{32}^2}\left(\frac{d}{dt}\left[\int_{\alpha(t)}^{\pi/2}\theta_{32}d\zeta\right]+\frac{d\alpha(t)}{dt}\theta_m\right)=-\cos^4(\alpha(t))\left.\frac{\partial\theta_{32}}{\partial\zeta}\right|_{\zeta=\alpha(t)}-4\theta_m\cos^3(\alpha(t))\sin(\alpha(t))+\\&+\int_{\alpha(t)}^{\pi/2}\big(12\cos^2(\zeta)+(\omega_{32}^2-16)\cos^4(\zeta)\big)\theta_{32}d\zeta+\frac{\omega_{32}^2}{\alpha_{32}}\left[\frac{3\pi}{16}-\frac{3\alpha(t)}{8}-\frac{\sin(2\alpha(t))}{4}-\frac{\sin(4\alpha(t))}{32}\right]\end{aligned}\right\}$$

(3.3.15)

To solve this problem, we again choose temperature profile like in previous case for $\theta_{32}(\zeta,t)$, in the form

$$\theta_{32}(\zeta,t)=A_{32}(t)\cos(\omega_{32}\zeta)+B_{32}(t)\sin(\omega_{32}\zeta)+C_{32}(t) \qquad \text{in} \qquad \alpha(t)\le\zeta\le\pi/2 \qquad (3.3.16)$$

The coefficients are determined by utilizing conditions, as follow

(1) $\theta_{32}(\alpha(t),t)=\theta_m$ (3.3.17$_1$)

(2) $\theta_{32}\big|_{\zeta=\frac{\pi}{2}}=0$ (3.3.17$_2$)

From conditions (3.17) we get system of equations

$$\begin{cases}A_{32}(t)\cos(\omega_{32}\alpha(t))+B_{32}(t)\sin(\omega_{32}\alpha(t))=\theta_m-C_{32}(t)\\A_{32}(t)\cos(\omega_{32}\frac{\pi}{2})+B_{32}(t)\sin(\omega_{32}\frac{\pi}{2})=-C_{32}(t)\end{cases}$$

which solution is

$$A_{32}=\frac{\theta_m \sin\left(\frac{\pi w_{32}}{2}\right)+\left(\sin\left(\alpha w_{32}\right)-\sin\left(\frac{\pi w_{32}}{2}\right)\right)C_{32}}{\sin\left(\frac{\pi w_{32}}{2}-\alpha w_{32}\right)}$$

$$B_{32}=\frac{\left(\cos\left(\frac{\pi w_{32}}{2}\right)-\cos\left(\alpha w_{32}\right)\right)C_{32}-\theta_m \cos\left(\frac{\pi w_{32}}{2}\right)}{\sin\left(\frac{\pi w_{32}}{2}-\alpha w_{32}\right)}$$

Temperature profile became

$$\theta_{32}(\zeta,t)=\frac{\theta_m \sin\left(\frac{\pi w_{32}}{2}\right)+\left(\sin\left(\alpha(t) w_{32}\right)-\sin\left(\frac{\pi w_{32}}{2}\right)\right)C_{32}(t)}{\sin\left(\frac{\pi w_{32}}{2}-\alpha(t) w_{32}\right)}\cos(\omega_{32}\zeta)+$$

$$+\frac{\left(\cos\left(\frac{\pi w_{32}}{2}\right)-\cos\left(\alpha(t) w_{32}\right)\right)C_{32}(t)-\theta_m \cos\left(\frac{\pi w_{32}}{2}\right)}{\sin\left(\frac{\pi w_{32}}{2}-\alpha(t) w_{32}\right)}\sin(\omega_{32}\zeta)+C_{32}(t)$$

$$\left.\begin{aligned}&\frac{r_0^2}{a_{32}^2}\left(\frac{d}{dt}\left[\int_{\alpha(t)}^{\pi/2}\theta_{32}d\zeta\right]+\frac{d\alpha(t)}{dt}\theta_m\right)=-\cos^4(\alpha(t))\left.\frac{\partial\theta_{32}}{\partial\zeta}\right|_{\zeta=\alpha(t)}-4\theta_m\cos^3(\alpha(t))\sin(\alpha(t))+\\&+\int_{\alpha(t)}^{\pi/2}\left(12\cos^2(\zeta)+(\omega_{32}^2-16)\cos^4(\zeta)\right)\theta_{32}d\zeta+\frac{\omega_{32}^2}{\alpha_{32}}\left[\frac{3\pi}{16}-\frac{3\alpha(t)}{8}-\frac{\sin(2\alpha(t))}{4}-\frac{\sin(4\alpha(t))}{32}\right]\end{aligned}\right\}$$

$$\left.\frac{\partial\theta_{32}(\zeta,t)}{\partial\zeta}\right|_{\zeta=\alpha(t)}=-w_{32}\frac{\theta_m\sin\left(\frac{\pi w_{32}}{2}\right)+\left(\sin\left(\alpha(t)w_{32}\right)-\sin\left(\frac{\pi w_{32}}{2}\right)\right)C_{32}(t)}{\sin\left(\frac{\pi w_{32}}{2}-\alpha(t)w_{32}\right)}\sin(w_{32}\alpha(t))+$$

$$+w_{32}\frac{\left(\cos\left(\frac{\pi w_{32}}{2}\right)-\cos\left(\alpha(t)w_{32}\right)\right)C_{32}(t)-\theta_m\cos\left(\frac{\pi w_{32}}{2}\right)}{\sin\left(\frac{\pi w_{32}}{2}-\alpha(t)w_{32}\right)}\cos(w_{32}\alpha(t))$$

4 BRIDGING. GENERAL AND SPHERICAL MODELS

4.1 General Model

A molten liquid drop, that appeared between contacts, when the temperature is equal to the melting point, is extended into a molten bridge in the course of further contact opening. This liquid metallic bridge consists of the variable part of the length $l(t)$ (the domain D_i) and the domain D_2 inside electrodes (*Figure 3.2.1*).

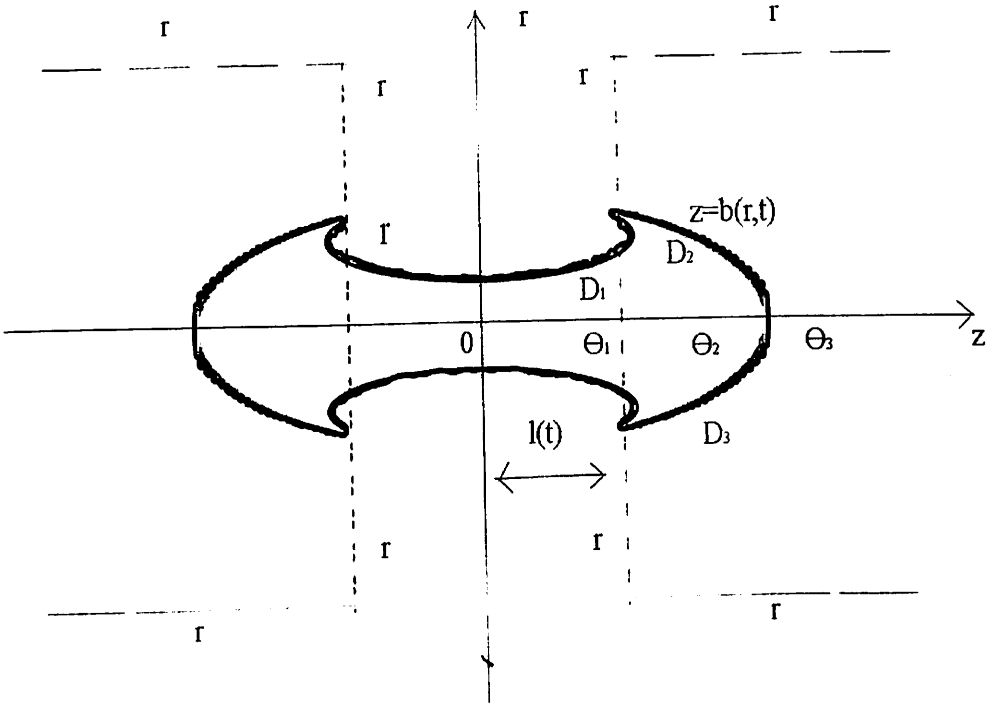


Figure 3.2.1 General Model of Bridging

The interface surface $z=\delta(r,t)$ between liquid zone D_2 and solid zone D_3 moves during contact heating. The temperature field $\theta_i(r,z,t)$ in each zone $D_i(1,2,3)$ can be described by the heat equation:

$$c_i\gamma_i\frac{\partial\theta_i}{\partial t}=div(\lambda_i grad\theta_i)+\rho_i j^2 \qquad (4.1)$$

With initial conditions:

$$t=0;\ l(0)=0; \qquad \delta(r,0)=\text{the circle}; \qquad z=0; \qquad 0\le r\le r_0$$

$$\theta_1(r,0,0)=\theta_m; \qquad \theta_2(r,z,0)=f(r,z); \tag{4.2}$$

where θ_m is the melting temperature, $f(r,t)$ is the initial temperature distribution at the time $t=t_m$ when temperature in solid opening contacts becomes equal to θ_m. The boundary conditions in a symmetric bridge is:

$$z=0; \qquad \frac{\partial\theta_i}{\partial z}=0; \tag{4.3}$$

$$z=\delta(r,t); \qquad \theta_1=\theta_2=\theta_m; \tag{4.4}$$

$$-\lambda_1\frac{\partial\theta_2}{\partial n}=-\lambda_2\frac{\partial\theta_3}{\partial n}+L\gamma\frac{\partial\delta}{\partial t}; \tag{4.5}$$

$$z=r \text{ (boundary with ambient air)}; \qquad \frac{\partial\theta_i}{\partial n}=0, \quad i=1,2,3 \tag{4.6}$$

$$z=\pm\infty; \qquad \text{or} \qquad r=\infty \qquad \theta_3=0; \tag{4.7}$$

This model is rather complicated and can applied for numerical solution mostly. Thus more simple model should be considered for analytical solution.

4.2 Spherical model

In spherical model the zones D_2 and D_3 is considered as spheres with the boundary interface $r=r_m(t)$, while the bridge is represented by a rod (*Figure 3.2.2*).

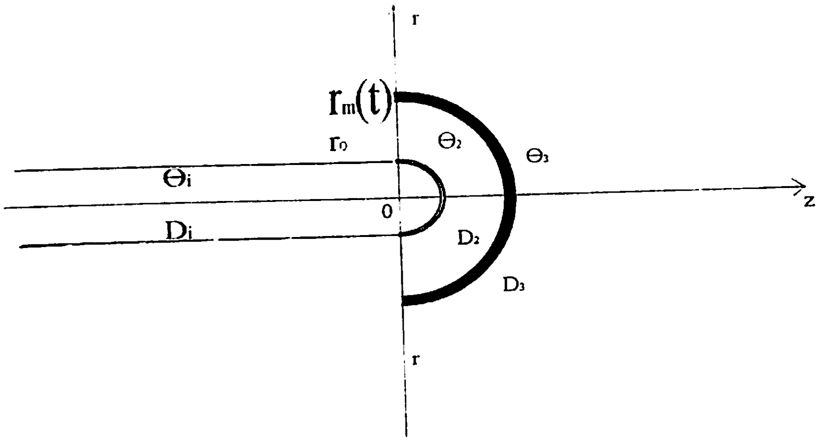


Figure 3.2.2 Spherical Model of Bridging

The heat equation for the bridge (1) transforms to the following

$$c_i\gamma_i \frac{\partial\theta_i}{\partial t} = \lambda_1 \frac{\partial^2\theta_1}{\partial z^2} + q_1 \qquad -l(t) < z < 0 \qquad (4.8)$$

where $q_1 = \left(I^2\rho_1\right)/\left(\pi^2 r_0^2\right)$.

The origin $z = 0$ is shifted to the contact of bridge with elected by means of sphere of ideal conductivity D_0. The centre of bridge symmetry now is $z = -l(t)$. The heat equation for the liquid zone D_2 can be written in spherical model as:

$$c_1\gamma_1 \frac{\partial\theta_2}{\partial t} = \lambda_1\left(\frac{\partial^2\theta_2}{\partial r^2} + \frac{2}{r}\frac{\partial\theta_2}{\partial r} + \frac{q_2}{r^4}\right) \qquad r_0 < r < r_m(t) \qquad q_2 = \frac{I^2\rho_1}{\pi^2\lambda_1} \qquad (4.9)$$

And for the solid zone D_3:

$$c_2\gamma_2 \frac{\partial\theta_3}{\partial t} = \lambda_2\left(\frac{\partial^2\theta_3}{\partial r^2} + \frac{2}{r}\frac{\partial\theta_3}{\partial r} + \frac{q_3}{r^4}\right) \qquad r_m(t) < r < \infty \qquad q_2 = \frac{I^2\rho_2}{\pi^2\lambda_2} \qquad (4.10)$$

It should be noted that for both liquid zones D_1 and D_2 the temperature properties are the same. The initial conditions (4.2) can be written in the form:

$$t=0, \qquad l(0)=0, \qquad r_m(0)=r_0,$$

$$\theta_1(z,0)=\theta_1(0,0)=\theta_m,$$

$$\theta_2(r,0)=\theta_2(r_0,0)=\theta_m, \qquad (4.11)$$

$$\theta_3(r,0)=f(r)\approx 0$$

Let us use the notation:

$$\theta_1(0,t)=\theta_0(t), \qquad \theta_1(-l(t),t)=\theta_2(t)$$

$$-\lambda_1\frac{\partial\theta_1(z,t)}{\partial z}\Big|_{z=0}=P_1(t), \quad -\lambda_1\frac{\partial\theta_2}{\partial r}\Big|_{r=r_m(t)}=P_2(t), \quad (4.12)$$

$$-\lambda_2\frac{\partial\theta_3}{\partial r}\Big|_{r=r_m(t)}=P_3(t)$$

All these functions are still not known and should be defined later on. The boundary conditions for the equations (4.8) - (4.10) can be written in the form:

$$z=-l(t) \qquad \frac{\partial\theta_1}{\partial z}\Big|_{z=-l(t)}=0 \qquad \text{(Due to thermal symmetry in the bridge)} \qquad (4.13)$$

$$z=0, \qquad r=r_0, \qquad \theta_1(0,t)=\theta_2(r_0,t)=\theta_0(t) \qquad (4.14)$$

$$-\lambda_1\frac{\partial\theta_1}{\partial z}\Big|_{z=0}=-2\lambda_1\frac{\partial\theta_2}{\partial r}\Big|_{r=r_0}=P_1(t) \qquad (4.15)$$

$$r=r_m(t) \qquad \theta_2=\theta_3=\theta_m \qquad (4.16)$$

$$-\lambda_1\frac{\partial\theta_2}{\partial r}\Big|_{r=r_m(t)}=-\lambda_2\frac{\partial\theta_3}{\partial r}\Big|_{r=r_m(t)}+L\gamma\frac{dr_m}{dt} \qquad (4.17)$$

$$r = \infty \qquad \theta_3(\infty, t) = 0 \tag{4.18}$$

Problem solution

The Stefan problem (4.8) – (4.18) is very complicated for the exact solution. Therefore we try to find an approximate solution by profile assignment of the temperature with change of differential equation (4.8) – (4.10) by heat balance equations, which are obtained by integration with respect to z and r.

Temperature profiles for each zone D_i can be chosen using corresponding stationary solutions with unknown coefficients depending on time. Thus we put:

$$\theta_1(z,t) = A_1(t)z^2 + B_1(t)z + C_1(t), \qquad -l(t) < z < 0 \tag{4.19}$$

$$\theta_2(r,t) = A_2(t) + \frac{B_2(t)}{r} + \frac{C_2(t)}{r^2}, \qquad r_0 < r < r_m(t) \tag{4.20}$$

$$\theta_3(r,t) = A_3(t) + \frac{B_3(t)}{r} + \frac{C_3(t)}{r^2}, \qquad r_m(t) < r < \infty \tag{4.21}$$

These sketchy profiles are represented as follows (*Figure 3.3.1*):

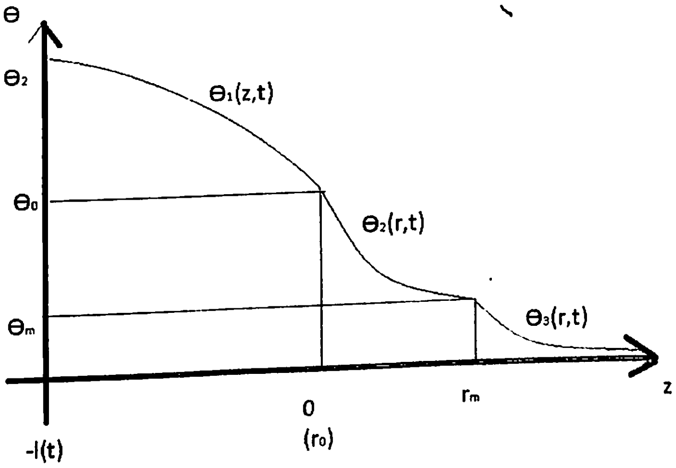


Figure 3.3.1 The distribution of temperature profiles

Let us consider (4.19). From the notation $\theta_1(0,t) = \theta_0(t)$ and $-\lambda_1(\partial\theta_1/\partial z)|_{z=0} = P_1(t)$

$$C_1(t)=\theta_0(t), \qquad B_1(t)=-\frac{1}{\lambda_1}P_1(t)$$

From the condition (4.13) we find

$$A_1(t)=\frac{1}{2l(t)}B_1(t)=-\frac{P_1(t)}{2\lambda_1 l(t)}$$

Thus

$$\theta_1(z,t)=\frac{P_1(t)}{2\lambda_1 l(t)}z^2-\frac{P_1(t)}{\lambda_1}z+\theta_0$$

That can be rewritten in the form

$$\theta_1(z,t)=\frac{P_1 l}{2\lambda_1}+\theta_0-\frac{P_1}{\lambda_1 l}(z+l)^2 \tag{4.22}$$

Let us consider (4.20). From the condition (4.12) we get the system of equations:

$$\begin{aligned}
&\lambda_1\frac{\partial\theta_2}{\partial r}|_{r=r_0}=P_1 &&\rightarrow &&\lambda_1\left(\frac{B_2}{r_0^2}+\frac{2C_2}{r_0^3}\right)=P_1\\
&\theta_2|_{r=r_0}=\theta_0 &&\rightarrow &&A_2+\frac{B_2}{r_0}+\frac{C_2}{r_0^2}=\theta_0\\
&\theta_2|_{r=r_m}=\theta_m &&\rightarrow &&A_2+\frac{B_2}{r_m}+\frac{C_2}{r_m^2}=\theta_m
\end{aligned}$$

Which solution is

$$A_2=I_1+\theta_0, \qquad B_2=-2r_0I_1-I_2, \qquad C_2=r_0I_2+r_0^2I_1 \tag{4.23}$$

$$I_1 = (\theta_m - \theta_0)\left(1 - \frac{r_0}{r_m}\right)^2 + \frac{P_1 r_0^2}{\lambda_1 r_m}\left(1 - \frac{r}{r_m}\right)^{-1}, \quad I_2 = \frac{P_1 r_0^2}{\lambda_1} \tag{4.24}$$

The profile (4.20) is defined now in terms of required values. Similarly for the profile (4.21) we get:

$$\theta_3(\infty, t) = 0 \quad \rightarrow \quad A_3(t) = 0$$

$$\theta_3(r_m, t) = \theta_m \quad \rightarrow \quad \frac{B_3}{r_m} + \frac{C_3}{r_m^2} = \theta_m$$

$$-\lambda_1 \frac{\partial \theta_3(r_m, t)}{\partial r} = P_3 \quad \rightarrow \quad \lambda_2\left(\frac{B_3}{r_m^2} + \frac{2C_2}{r_m^3}\right) = P_3$$

Thus

$$C_3 = \frac{P_3 r_m^3}{\lambda_2} - \theta_m r_m^2, \qquad B_3 = 2\theta_m r_m - \frac{P_3 r_m^2}{\lambda_2}$$

$$\theta_3(r, t) = \left(2\theta_m - \frac{P_3 r_m}{\lambda_2}\right)\frac{r_m}{r} + \left(\frac{P_3 r_m}{\lambda_2} - \theta_m\right)\frac{r_m^2}{r} \tag{4.25}$$

Equations for $\theta_0(t), P_i(t), r_m(t), \theta_l(t)$

From the condition $\theta_1 |_{z=-l(t)} = \theta_2(t)$ we get:

$$\theta_l = \theta_0 + \frac{P_1 l}{2\lambda_1} \tag{4.26}$$

From the condition $-\lambda_1 \left(\partial\theta_2 / \partial r\right)|_{r=r_m} = P_2$ we get the equation:

$$B_2 + \frac{2C_2}{r_m} = \frac{P_2 r_m^2}{\lambda_1} \tag{4.27}$$

where B_2 and C_2 are defined by the expressions (4.23) and (4.24).

Using Stefan condition (4.17), we can write:

$$P_2(t) = P_3(t) + L\gamma\frac{dr_m}{dt} \tag{4.28}$$

Integration of the equation (4.8) with respect to z gives:

$$c_1\gamma_1\int_{-l(t)}^{0}\frac{\partial\theta_1}{\partial t}dz = \int_{-l(t)}^{0}\left[-\lambda_1\frac{\partial^2\theta_1}{\partial z^2}+q_1\right]dz$$

$$c_1\gamma_1\frac{d}{dt}\int_{-l(t)}^{0}\theta_1(z,t)dz - c_1\gamma_1\frac{dl}{dt}\theta_1\big|_{z=-l(t)} = -\lambda_1\frac{\partial\theta_1}{\partial z}\Big|_{-l(t)}^{0} + q_1 l(t)$$

Using profile (4.22) we get

$$c_1\gamma_1\frac{d}{dt}\int_{-l(t)}^{0}\left[\frac{P_1 l}{2\lambda_1}+\theta_0-\frac{P_1}{2\lambda_1 l}(z+l)^2\right]dz = c_1\gamma_1\theta_l\frac{dl}{dt} - P_1 + q_1 l$$

And after integration

$$c_1\gamma_1\frac{d}{dt}\left(\frac{P_1 l^2}{3\lambda_1}+\theta_0 l\right) = c_1\gamma_1\theta_l\frac{dl}{dt} - P_1 + q_1 l \tag{4.29}$$

Integration of (4.9) multiplied by r^2 with respect to r gives

$$\int_{r_0}^{r_m(t)} c_1\gamma_1 \frac{\partial \theta_2}{\partial t} r^2 dr = \int_{r_0}^{r_m(t)} \lambda_1 \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta_2}{\partial r} \right) + \frac{q_2}{r^4} \right] \cdot r^2 dr$$

$$c_1\gamma_1 \frac{d}{dt} \int_{r_0}^{r_m(t)} \theta_2(r,t) r^2 dr - c_1\gamma_1 \frac{dr_m}{dt} \theta_2 \left.\right|_{r=r_m} r_m^2(t) = \lambda_1 \left[r^2 \frac{\partial \theta_2}{\partial r} \right]_{r_0}^{r_m(t)} - \frac{\lambda_1 q_2}{r} \left.\right|_{r_0}^{r_m(t)}$$

$$c_1\gamma_1 \frac{d}{dt}\left[A_2 \frac{r^3}{3} + B_2 \frac{r^2}{2} + C_2 r \right]_{r_0}^{r_m} - c_1\gamma_1 \theta_m r_m^2(t) \frac{dr_m}{dt} = -r_m^2 P_2 + r_0^2 P_1 - \lambda_1 q_2 \left(\frac{1}{r_m} - \frac{1}{r_0} \right)$$

(4.30)

Integration of (4.10) (neglecting $q_3 = 0$) multiplied by $1/r$ gives similarly:

$$c_2\gamma_2 \int_{r_m(t)}^{\infty} \frac{\partial \theta_3}{\partial t} \frac{1}{r} dr = \int_{r_m(t)}^{\infty} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta_3}{\partial r} \right) \frac{1}{r} dr$$

$$c_2\gamma_2 \frac{d}{dt} \int_{r_m(t)}^{\infty} \frac{\theta_3}{r} dr + c_2\gamma_2 \frac{1}{r_m}\frac{dr_m}{dt} \theta_3 \left.\right|_{r=r_m} = \int_{r_m(t)}^{\infty} \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta_3}{\partial r} \right) dr$$

Using (4.25) we get:

$$c_2\gamma_2 \frac{d}{dt}\left[\left(2\theta_m - \frac{P_3 r_m}{\lambda_2} \right) + 2\left(\frac{P_3 r_m}{\lambda_2} - \theta_m \right) \right] + c_2\gamma_2 \frac{\theta_m}{r_m}\frac{dr_m}{dt} = \frac{1}{2r_m^4}\left(\frac{P_3 r_m^2}{\lambda_2} - \theta_m r_m \right)$$

$$\frac{c_2\gamma_2}{\lambda_2}\frac{d}{dt}\left(P_3 r_m \right) = \frac{1}{2r_m^3}\left(\frac{P_3 r_m}{\lambda_2} - \theta_m \right) \qquad (4.31)$$

CONCLUSIONS

The two-phase Stefan problem for a rod with the given heat flux, and the spherical two- and three-phase Stefan problems describing the dynamics of melting and evaporation processes are solved using this method.

In the first chapter for solving problems of heat conduction in spherical geometry by integral method of power balance for the first time evaluated the effectiveness of the temperature type profiles: $T(r,t) = polynomial$ and $T(r,t) = polynomial / r$ with polynomials of degree n. The results show that in dealing with problems of thermal conductivity for different geometry approximate integral method of power balance is not always possible to achieve sufficient accuracy when using temperature profiles of the same type.

Adiabatic model of bridge heating and melting has to be corrected because of important role of heat conduction from filament and bridge to electrodes. Heat transfer from the bridge to electrode is very important for the dynamics of bridge evolution.The time required for melting of filament depends essentially on the thermal properties of contact material. It can be found by solution of corresponding Stefan problem. The shape of a bridge depends on the electrical and thermal parameters of contact material and can be found after solution of the variational problem for minimum of free bridge energy consisting of surface tension, electromagnetic and gravitational components.

Solving the equations (3.4.1) – (3.4.6) we can find unknown values $\theta_0, \theta_1, P_1, P_2, P_3$ and r_m. The time $t = t_b$ when temperature maximum in bridge reaches the boiling point θ_b can be found now from the equation $\theta_l(t_b) = \theta_b$. And the volume of the bridge giving erosion is $V_{br} = \pi r_0^2 l(t_b)$.

REFERENCE

1 Carslow G., Eger D. Thermal conductivity of solids, 1964. – 488 p.

2 LykovA.V. Теория теплопроводности,1967. – 600 p.

3 PehovichA.I., Zhidkih Расчеты теплового режима твердых тел,1976.

4 Goodman T. Application of integral method in nonlinear problems of non-stationary heat exchange,1967. – p. 47–96.

5 VeinikA.I. Приближенный расчет процессов теплопроводности,1959.

6 S.N. Kharin, H. Nouri and T. Davies, "The Mathematical Models of Welding Dynamics in Closed and Switching Electrical Contacts", Proc. 49th IEEE Holm Conference on Electrical Contacts, Washington, USA, 2003, pp. 128-146

7 E.I. Kim, V.T. Omelchenko and N.S. Kharin, "Mathematical Models of Thermal Phenomena in Electrical Contacts", (In Russian), Nauka, Alma-Ata, pp. 46-47, (1977).