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Expansion of models of DP-minimal theories

THESIS

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Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Aigerim Nurlanova

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Dedication

This thesis is dedicated to:

This work is dedicated to my family for their unconditional love and support, and to my supervisor, Dr. Sc. Prof. Bektur Baizhanov, whose guidance inspired and motivated me throughout this journey.

Abstract

This study investigates expansions of models of DP-minimal theories, a main subclass of dependent theories in model theory distinguished by well-controlled combinatorial complexity. Finding the circumstances in which DP-minimality is maintained when structures are extended by more predicates, functions, or relations is the main goal of the project.

Following a thorough explanation of fundamental ideas like DP-rank, definability, and quantifier elimination, the study examines several extensions of the group of integers $(\mathbb{Z}, +, 0)$ and associated ordered algebraic systems. Expansions by linear orders and additional unary or binary predicates are important instances.

The findings show that while some expansions lead to superstable but non-DP-minimal expansions, others, like those corresponding to Presburger arithmetic $(\mathbb{Z}, +, <, 0, 1)$, preserve DP-minimality. By emphasizing the harmony between increased expressive power and minimality condition preservation, these results advance our knowledge of the relationship between model expansions and classification theory.

The final section of the dissertation outlines possible avenues for future study, such as applications to ordered structures and broader classes of expansions.

Аңдатпа

Бұл зерттеу модельдер теориясындағы тәуелді теориялардың маңызды тармағы болып саналатын DP-минималды теориялардың модельдерін кеңейтуді қарастырады. Бұл теориялар жақсы реттелген комбинаторлық күрделілігімен ерекшеленеді. Зерттеудің негізгі мақсаты - құрылымдарға қосымша предикаттар, функциялар немесе қатынастар енгізілген кезде DP-минималдылықтың қандай жағдайларда сақталатынын анықтау.

DP-дәрежесі, анықталынушылық және кванторларды жою сияқты негізгі ұғымдарға толық түсініктеме берілгеннен кейін, бұл зерттеу бүтін сандар тобы $(\mathbb{Z}, +, 0)$ және оған байланысты реттелген алгебралық жүйелердің бірнеше кеңейтулерін қарастырады. Сызықтық реттермен және қосымша унарлы немесе бинарлы предикаттармен жасалған кеңейтулер маңызды мысалдар ретінде қарастырылады.

Зерттеу нәтижелері кейбір кеңейтулер суперстабилді болса да DP-минималдылықты жоғалтатынын, ал кейбіреулері, мысалы, Пресбургер арифметикасына сәйкес келетін $(\mathbb{Z}, +, <, 0, 1)$ сияқты кеңейтулер DP-минималдылықты сақтайтынын көрсетеді. Шартты минималдылықты сақтау мен өрнектеу қуатын арттыру арасындағы үйлесімділікті көрсету арқылы бұл нәтижелер модельдерді кеңейту мен классификациялық теория арасындағы байланысты тереңірек түсінуге мүмкіндік береді.

Диссертацияның соңғы бөлімі болашақ зерттеулерге арналған ықтимал бағыттарды сипаттайды, мысалы, реттелген құрылымдарға және кеңейтулердің ауқымдырақ кластарын қолдану сияқты бағыттар.

Аннотация

В данном исследовании рассматриваются расширения моделей DP-минимальных теорий - основного подкласса зависимых теорий в модельной теории, отличающихся хорошо контролируемой комбинаторной сложностью. Основной целью проекта является выявление условий, при которых DP-минимальность сохраняется при добавлении к структурам новых предикатов, функций или отношений.

После подробного объяснения таких фундаментальных понятий, как DP-ранг, определимость и устранение кванторов, в исследовании рассматриваются несколько расширений группы целых чисел $(\mathbb{Z}, +, 0)$, а также связанных с ней упорядоченных алгебраических систем. Важными примерами служат расширения с добавлением линейного порядка и дополнительных унарных или бинарных предикатов.

Результаты показывают, что в то время как некоторые расширения приводят к суперстабильным, но не DP-минимальным структурам, другие - такие как соответствующие $(\mathbb{Z}, +, <, 0, 1)$ арифметике Пресбургера - сохраняют DP-минимальность. Подчеркивая согласованность между возрастанием выразительной мощности и сохранением условия минимальности, эти результаты углубляют наше понимание взаимосвязи между расширениями моделей и классификационной теорией.

Заключительный раздел диссертации намечает возможные направления для будущих исследований, такие как применение к упорядоченным структурам и более широким классам расширений.

Abbreviations

List of Abbreviations

DP	Dimension Property
NIP	Not Independent Property
IP	Independent Property
SOP	Strict Order Property
NSOP	Not Strict Order Property
QE	Quantifier Elimination
ICT-pattern	Independence of Consistency Type pattern
QF	Quantifier-free
PNF	Prenex Normal Form
Th	Theory

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1. Introduction

In his early work on classification theory, Shelah defined dividing lines based on the presence or absence of certain combinatorial configurations, which allowed him to classify first-order theories. Stability is the most crucial factor. The independence property was discovered by looking at the possible outcomes for the function that connects a subset's size to the total number of types across it. There was not a lot of early study done on the class of theories known as NIP theories, or theories lacking the independent property. Below the DP-minimal class and its extensions are taken into consideration inside the NIP theories.

The notion of NIP (Not Independent Property) is typically formulated in terms of a structure $\phi(x; y)$. A formula is said to exhibit the independent property (IP) if it fails to be NIP. The NIP condition excludes the independence property, which is the basis for the term. In the literature, particularly in the work of Shelah, the terms "dependent" and "independent" are often used interchangeably with NIP and IP. Contrary to common assumptions, most NIP theories are either stable or are closely related to structures such as trees or linear orders.

The study of dp-minimality often intersects with two key frameworks: strong minimality and o-minimality. In model theory, o-minimal structures are widely regarded as a robust and versatile tool. Consider a linearly ordered structure $(M; \dots)$, where the order is given by $<$. Such a structure is called o-minimal if every definable subset of M in one variable can be expressed as a finite union of open or closed intervals. If, in all elementary extensions of M, the definable unary sets are finite unions of convex sets, then the structure is referred to as weakly o-minimal. As noted in [1], many dp-minimal and NIP theories fall under this weakly o-minimal classification.

In addition to a review of the Presburger definable sets and some examples of proper stable expansions, we aim to give a thorough introduction to DP minimality and its extensions.

The project aims to explore the potential applications of abstract model theory ideas, such as DP minimality, in practical settings, such as ordered model theory in valuable or real field research. This comment so functions as some initial investigation proving that an infinite defined subset of a divided ordered Abelian group that is DP-minimal requires an interior.[2]

1.1 History of model theory in Kazakhstan

Over the last six decades, model theory in Kazakhstan has experienced significant development, largely due to the foundational work of academician Asan Dabsovich Taimanov of the Kazakh SSR Academy of Sciences. His academic legacy continued through the efforts of his students and collaborators, such as Bektur Baizhanov, Tatyana Zambarnaya, and Beibut Kulpeshov. Taimanov's contributions helped establish core research directions in Kazakhstan that were consistent with broader international developments in model theory [3].

Kazakh scholars in model theory have addressed core issues including the decidability of logical theories, the axiomatizability of algebraic structures, the classification of non-isomorphic models, various aspects of model expansion and its applications, as well as the integration of model-theoretic methods into other branches of mathematics.

The study of decidability concerns whether an algorithm can determine the truth of statements within a mathematical theory. Work in Kazakhstan on this topic was influenced by Gödel, Tarski, and Mal'tsev. A.D. Taimanov and his students worked on the decidability of algebraic structures, with notable contributions from J.A. Almagambetov, who studied algorithmic problems in model theory in 1965, N.G. Khisamiev, who worked on elementary theories of lattice-ordered algebraic structures in 1968, and T.Sh. Shayakhmetov, who examined undecidability in algebraic structures in 1970. Kazakhstani researchers also made contributions to Tarski's problems, including work on the decidability of free groups. Additionally, A.D. Taimanov independently rediscovered the Ehrenfeucht-Fraïssé method in 1962, providing a topological approach to elementary equivalence.

Axiomatizable classes and model completeness have also been central areas of research. A.D. Taimanov characterized axiomatizable classes and studied theories reducible to special formulas in 1961. A.I. Omarov made significant contributions to the classification of varieties and quasivarieties of lattices in 1988 and 2012. A.T. Nurtazin explored existentially closed models, providing a new approach using forcing techniques. Kazakh researchers also studied filtered products and ultraproducts, finite axiomatizability of categorical theories, and interpretability and similarity of theories.

How many distinct models (up to isomorphism) a complete theory contains is a key question in model theory. In 1970, A.D. Taimanov and T.G. Mustafin conducted research on countable models of uncountably categorical theories. In 1980, B.S. Baizhanov expanded on Lachlan's spectrum function classification for theories that are completely transcendental. In 2017, S.V. Sudoplatov and B.Sh.

Kulpeshov verified the Vaught Conjecture for quite o-minimal theories. In 2018, new classes of trivial types and their effects on spectrum functions were presented by B.S. Baizhanov, John T. Baldwin, and T.S. Zambarnaya.

Kazakhstani logicians have played a major role in constructive model theory, which studies computability properties of mathematical structures. A.I. Mal'tsev introduced constructive models in 1961, and A.V. Ershov developed the notion of strong constructivity in 1980. A.N. Khisamiev solved fundamental problems on constructivizability of Abelian groups in 1990 and 1998. A.T. Nurtazin developed criteria for autostability and constructive prime models in 1974. K.Zh. Kudaibergenov proved results on model companions of structures with automorphisms between 1979 and 2013.

Homogeneous and stable theories have been another area of study. It is examined by Kudaibergenov the number of homogeneous models and solved Keisler and Morley's problem on their cardinality between 1988 and 2002. In 2014 and 2016, he proved results on the small index property of automorphism groups. Baisalov and Meirembekov studied definable minimal rings and Lie algebras between 2006 and 2012.

Kazakhstani researchers have maintained strong international collaborations, participating in major global mathematical conferences, working with leading model theorists such as Baldwin, Poizat, Hrushovski, and Shelah, and publishing in high-impact journals on logic and algebra.

Model theory in Kazakhstan has grown into an internationally recognized field, largely due to the foundational work of Asan Dabsovich Taimanov and his students. The research covers diverse areas, including decidability, axiomatizability, spectrum problems, constructive models, and stable theories. The contributions of A.I. Omarov, B.S. Baizhanov, B.Sh. Kulpeshov, T.S. Zambarnaya, and others have positioned Kazakhstan as a significant center for model theory research.

1.2 Background and Motivation

A key area of study in mathematical logic is model theory, which investigates the connections between syntactic formal languages and the semantic meanings they convey in mathematical structures. The absence of the independence property (NIP) is one of the key dividing lines that have been identified within this framework as a result of the classification of theories according to the behavior of definable sets.

Due to their tame combinatorial behavior, dp-minimal theories occupy a central position within the broader class of NIP theories. Intuitively, a theory is considered dp-minimal if it excludes the existence of two mutually indiscernible sequences that simultaneously demonstrate non-trivial dependence with respect to a given variable. This form of minimality is closely linked to the absence of ICT patterns.

In addition to its inherent definability-theoretic significance, dp-minimality is gaining popularity because it can be used to extend stability theory techniques to more general NIP contexts.

Studying how dp-minimal structures can be expanded has become an important topic in contemporary model theory. Given a dp-minimal structure $\mathcal{M}$, a central question is which additional components can be introduced without violating its dp-minimality. One may, for instance, consider expansions of the form $\mathcal{M}' = (\mathcal{M}, P)$ or $\mathcal{M}' = (\mathcal{M}, R)$, where P is a unary predicate and R is a binary relation. This research particularly emphasizes expansions of the integer group $(\mathbb{Z}, +, 0)$.

This research approach contributes to the larger study of the robustness and bounds of minimality in first-order logic by addressing fundamental classification-theoretic problems, such as figuring out the boundary between minimal and non-minimal behavior and between stable and unstable expansions.

1.3 Research objectives

This dissertation aims to provide a detailed and precise analysis of the behavior of dp-minimal theories under definable expansions, with particular attention to preserving their classification-theoretic properties. The research is structured around the following core objectives:

- **Development of dp-minimality preservation criteria.** Let $\mathcal{M} \models T$ be a dp-minimal structure and let $\mathcal{M}^{'} = (\mathcal{M}, P)$ be an expansion of $\mathcal{M}$ by a predicate P definable. Defining constraints on P that keep $Th(\mathcal{M}^{'})$ dp-minimal is our aim. The formal goal is to demonstrate the implications of the form:

  $$\text{P is externally definable in } \mathcal{M} \Rightarrow Th(\mathcal{M}, P) \text{ is dp-minimal,}$$

  as well as to examine the opposite under stability-theoretic or natural definability constraints.

- **Analysis of integer structures using model theory.** The classic dp-minimal structure $(\mathbb{Z}, +, 0)$ is a crucial test case for studying defined expansions. To investigate the dp-minimality of expansions, we aim to:
  - Unary predicates $P \subseteq \mathbb{Z}$ (set of squares, primes, or congruence classes),
  - Binary relations (modular congruence, inequalities, or partial orders),
  - Predicates that can be defined in more complex languages, like Presburger arithmetic.

For each case, we determine whether the resulting expansion admits ICT-patterns or increases dp-rank beyond 1, indicating a loss of minimality.

- **Interact with concepts from classification theory.** We investigate the relationships between the maintenance or loss of dp-minimality and other characteristics like:

  - Characteristics of **stability and superstability**, such as forking, dividing, and rankings like U-rank or weight,
  - **Quantifier elimination**, which may help or hinder preservation of dp-minimality.

  It aims to describe under which conditions dp-minimality implies or is implied by these more general classification-theoretic properties.

- **Example and counterexample construction.** In order to verify the theoretical findings, created examples that demonstrate:

  - Dp-minimality-failing superstable expansions of $(\mathbb{Z}, +, 0)$
  - Expansions that violate the NIP by introducing the independence property (IP),
  - $(\mathbb{Z}, +, <, 0, 1)$ contains definable enrichments that are not definable in $(\mathbb{Z}, +, 0)$.

  The distinction between expansions that maintain dp-minimality and those that do not is clarified by these examples.

1.4 Significance of the Study

In addition to providing insight into the internal organization of dependent theories, knowing when dp-minimality is maintained under expansion enables one to assess how well the concept of minimality holds up under definitional enrichment. Because of its rich model-theoretic behavior and straightforward algebraic structure, the group $(\mathbb{Z}, +, 0)$ is a crucial testbed in this context.

Conant and Pillay, among others, have recently examined stable expansions of $(\mathbb{Z}, +, 0)$. It has been demonstrated that if $P \subseteq \mathbb{Z}^n$ is definable within a finite dp-rank expansion and the expanded structure is stable, then P is already definable in the structure $(\mathbb{Z}, +, 0)$ itself. This imposes significant limitations on possible stable or superstable expansions and promotes a comprehensive examination of the attributes that either uphold or compromise the maintenance of dp-minimality [4].

This research extends previous work by analyzing how key classification-theoretic properties interact within expanded languages.

2. Theoretical Background and Literature Review

Model theory, a fundamental branch of mathematical logic, focuses on the interpretation of formal languages within mathematical structures [5]. This chapter presents an in-depth overview of foundational techniques in model theory relevant to studying expansions of dp-minimal structures [1].

2.1 Languages and Structures

When a certain statement is true regardless of the meaning of the statements included in it, it forms the subject of the logic of the statements.

"Logical connectives" can be used to link statements together as it shown in figure 2.1. The names and classifications of these connectives are conventional.

connectives	designation	name
A and B	$A\&B$ $A \wedge B$ A and B	conjunction
A or B	$A \vee B$ A or B	disjunction
not A, A false	$\neg A$ $\sim A$ $\bar{A}$ not A	negation
from A follows B, if A then B, A implies B, B - consequence of A	$A \to B$ $A \Rightarrow B$ $A \supset B$ if A then B	implication following

Figure 2.1: Logical connectives

Note also that in the implication $A \Rightarrow B$, the statement A is called the premise, or the antecedent of the implication, and B is called the conclusion, or the consequence.

Denoted elementary statements (from which more complex ones are composed) in small Latin letters and call them *propositional variables*. *Propositional formulas* are constructed from them according to the following rules:

1. There is a formula for each propositional variable.

2. If A is a propositional formula, then so is $\neg A$.

3. Given that A and B are propositional formulas, the expressions $(A \vee B)$, $(A \wedge B)$, and $(A \to B)$ also qualify as propositional formulas.

Some formulas express logical laws, compound statements that are true regardless of the meaning of their parts. Such formulas are called tautologies as it given in [6].

Theorem 2.1.1. *Any Boolean function with n arguments can be written as a propositional formula.*

A literal can be a variable or the opposite of a variable. A finite combination of these literals is called a conjunct. When a formula is a disjunction of conjuncts, it is referred to as being in disjunctive normal form (DNF). Typically, each conjunct contains exactly n literals, where n represents the number of distinct propositional variables. The total number of possible conjuncts, ranging from 0 to 2^n, corresponds to the number of satisfying truth assignments (i.e., input combinations the formula evaluates to true).

For any formulas $A; B; C$, the following expressions are regarded as axioms of propositional logic:

(1) $B \to (A \to B)$;

(2) $(B \to (C \to A)) \to ((B \to C) \to (B \to A))$;

(3) $(B \wedge C) \to B$;

(4) $(B \wedge C) \to C$;

(5) $B \to (C \to (B \wedge C))$;

(6) $B \to (B \vee C)$;

(7) $C \to (B \vee C)$;

(8) $(B \to A) \to ((C \to A) \to (B \vee C \to A))$;

(9) $\neg B \to (B \to C)$;

(10) $(B \to C) \to ((B \to \neg C) \to \neg B;)$

(11) $B \vee \neg B$.

As they say, we have eleven "axiom schemes" here; Various specific axioms can be obtained from each scheme by replacing the letters included in it with propositional formulas.

The only rule of inference of the propositional calculus is the rule with the medieval name "modus ponens"(MP). This rule allows you to get (derive) from formulas A and $A \to B$ formula B.

An inference in propositional logic consists of a finite list of formulas, where each formula is either an axiom or obtained from earlier formulas using the modus ponens inference rule.

Here is an example of the output (in it, the first formula is a special case of scheme (1), the second - scheme (2), and the last is obtained from the previous two according to the modus ponens rule):

$$(A \to (B \to A)),$$
$$(A \to (B \to A)) \to ((A \to B) \to (A \to A)),$$
$$((A \to B) \to (A \to A)).$$

A propositional formula A is called derivable in propositional calculus, or a theorem of propositional calculus, if there is a conclusion in which the latter formula is equal to A. Such an inference is called the inference of formula A. (Principle, it would be possible not to require formula A to be the last one - all further formulas can simply be crossed out.)

A formula A in propositional logic is said to be derivable, or a theorem, if there exists a proof in which A appears as one of the derived formulas - typically as the final one. Such a proof is referred to as a derivation of A. (In principle, it is not strictly necessary for A to be the last formula - any subsequent formulas may simply be ignored.)

Theorem 2.1.2. *Every theorem of propositional calculus is a tautology.*

Proof. It is easy to verify that all axioms are tautologies. For example, let us do this for the longest axiom (more precisely, the axiom scheme) - for the second one. In which the case is the formula

$$(B \to (C \to A)) \to ((B \to C) \to (B \to A))$$

(where$A; B; C$ - some formulas) could it be false? For this, the premise $B \to (C \to A)$ must be true, and the conclusion $(B \to C) \to (B \to A)$ - false. For the result to be false, the structure $B \to C$ must be true, and the structure $B \to A$ - must be false.

In this case, B is assumed to be true while A is false. Given this, the formulas B, $(B \to (C \to A))$ must all hold. As a result, both C and $(C \to A)$ are also true, which implies that A must be true, contradicting our initial assumption. Hence, the original formula cannot be false under any interpretation.

The correctness of the MP rule is also obvious: if formulas $(A \to B)$ and A are always true, then by definition of the implication formula B is also always true.

Thus, all formulas included in the implications (all theorems) are tautologies.

□

Theorem 2.1.3. *Every tautology is a theorem of propositional calculus.*

Proof. Before presenting several alternative proofs of this theorem, it is necessary to gain experience in drawing inferences and applying axioms.

□

A first-order language consists of symbols such as variables, logical connectives, quantifiers, equality, function symbols, predicate symbols, and constants. More formally, a first order-language L includes:

- Variables: $x, y, z, ...$
- Logical connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- Quantifiers: $\forall, \exists$
- Equality: $=$
- Predicate, function, and constant symbols: P, f, c

Terms

1. Symbols of variables: $x_1, v_1, y_5, ...$
2. Symbols of constants: $c_1, c_2, ..., c_n, ...$
3. Given a function symbol f^n of arity n, and terms $v_1, ..., v_n$ that are already constructed, the expression $f^n(v_1, ..., v_n)$ also constitutes a valid term.

Formulas

1. If $v_1, ..., v_n$ are terms and P^n is an n-place predicate symbol, then the expression $P^n(v_1, ..., v_n)$ is classified as an atomic formula.
2. If α and β are formulas, then
   - $\alpha \wedge \beta, \alpha \vee \beta, \alpha \rightarrow \beta, \neg\alpha$ are also formulas.
3. If $\psi(x)$ is a formula with a free variable x, then:
   - $\exists x \psi(x)$ and $\forall x \psi(x)$ are also formulas.

Structures

Definition 2.1.4 (Structure)**.** *A structure $\mathcal{M}$ is defined as a tuple*

$$\mathcal{M} = (M, P^{\mathcal{M}}, f^{\mathcal{M}}, c^{\mathcal{M}}),$$

where:

- *M is the domain (universe) of the structure;*
- *$P^{\mathcal{M}}$ is the interpretation of predicate symbols: for each n-ary predicate symbol P_i^n, the interpretation is a set*

$$P_i^n(\mathcal{M}) \subseteq M^n, \quad \textit{such that} \quad \mathcal{M} \models P_i^n(\bar{a}) \iff \bar{a} \in P_i^n(\mathcal{M});$$

- *$f^{\mathcal{M}}$ is the interpretation of function symbols: for each n-ary function symbol f^n, its graph is given by*

$$P_f^n(\mathcal{M}) := \{(\bar{a}, b) \in M^{n+1} \mid \mathcal{M} \models f(\bar{a}) = b\};$$

- *$c^{\mathcal{M}}$ is the interpretation of constant symbols:*

$$c^{\mathcal{M}} = \{c_a \mid a \in M\}.$$

Truth of sentence on structure

Induction by construction of formulas define the truth of formula:

Let $P^n, \bar{a} \in M^n$,

$$\mathcal{M} \models ((P^n(\bar{a}) \Longrightarrow \bar{a}) \wedge (P^n(\bar{a}) \Longleftarrow \bar{a})) \in P^n(\mathcal{M})$$

$\mathcal{M} \models \psi \wedge \beta \iff \mathcal{M} \models \psi$ and $\mathcal{M} \models \beta$

$\mathcal{M} \models \psi \vee \beta \iff \mathcal{M} \models \psi$ or $\mathcal{M} \models \beta$

$$\mathcal{M} \models \neg\psi \iff \mathcal{M} \not\models \psi$$

Let $\psi(x, \bar{a})$ for any $b \in M$ it is shown $\mathcal{M} \models \psi(b, \bar{a})$. Thus if for same $b_0 \in M, \mathcal{M} \models \psi(b_0, \bar{a})$

Then $\mathcal{M} \models \exists x \psi(x, \bar{b})$

If for any $b \in M$, $\mathcal{M} \models \psi(b, \bar{a})$

Then $\mathcal{M} \models \forall x \psi(x, \bar{a})$.

2.2 Cardinality

Cardinality is used to compare the sizes of sets. It applies to both finite and infinite sets. The idea is based on the existence of functions between sets - injective, subjective, or bijective.

The **Cantor-Bernstein theorem** gives a condition for when two sets have the same cardinality:

Theorem 2.2.1 (Cantor-Bernstein)**.** *A bijection between sets A and B exists if there are injective mappings from A to B and from B to A. In this case, we denote $A \sim B$ and say that A and B have the same cardinality.*

This shows that mutual embeddability implies that the two sets are the same size.

In contrast, **Cantor's theorem** demonstrates a strict inequality between a set and its power set:

Theorem 2.2.2. *For any set A, no function from A to $\mathcal{P}(A)$ can be subjective. To see this, suppose $f : A \to \mathcal{P}(A)$ is any function. Define a subset*

$$B = \{x \in A | x \notin f(x)\}.$$

This set B cannot be in the image of f, since $x \in B$ if and only if $x \notin f(x)$, which is a contradiction. The set B cannot belong to the image of f, because by definition $x \in B$ if and only if $x \notin f(x)$, which leads to a contradiction.

The concept of cardinality, which quantifies both the variety and magnitude of models relative to a given parameter set, plays a crucial role in logic and model theory. For instance, over a model of size k, the cardinality of the space of complete n-types can reach 2^k. This is particularly relevant when discussing the number of definable subsets or formulas in a given language up to equivalency.

It becomes crucial to control cardinality, particularly the number of types or definable families, in the framework of dp-minimality and classification theory. The emergence of exponentially many types, such as the 2^n possible literal combinations, is frequently associated with IP or instability. On the other hand, dp-minimal or NIP behavior is indicated by bounded cardinalities of definable families.

2.3 Compactness and Its Applications

In model theory, the Compactness Theorem is one of the most important results. It is also very important for building models with the right properties. A set of first-order sentences is said to have a model if every one of its finite subsets is satisfiable. The principle let you go from local consistency to global realizability, and it can be used in many different areas of classification theory.

The Compactness Theorem is among the most potent findings in model theory [7]:

Theorem 2.3.1 (Theorem(Compactness))**.** *Let* Γ *be a set of first-order formulas. The set* Γ *is satisfiable if every finite subset of* Γ *is satisfiable.*

This theorem is widely used to demonstrate the existence of models and their basic extensions, and it is crucial for building models with specified features.

Applications to Type Realization. The compactness theorem has significant consequences in type theory. Let T be a complete first-order theory, and let $p(x)$ be a set of formulas with parameters from a model $M \models T$, where x is a free variable. If every finite subset of $p(x)$ is consistent with T, then the entire set $p(x)$ is also consistent with T. Consequently, there exists an element in some elementary extension $\mathcal{M}^* \succ \mathcal{M}$ that realizes the type $p(x)$.

This result underlies the concept of saturation. It is particularly useful in analyzing dp-rank and the independence property (IP). Many constructions - such as trees of formulas or indiscernible sequences - involve infinitary configurations. These configurations are only guaranteed to exist by the Compactness Theorem.

Example 2.3.2. *Building Nonstandard Models Let us consider Peano Arithmetic (PA) and define the set:*

$$\Gamma = Th(\mathbb{N}, +, \cdot) \cup \{c > n | n \in \mathbb{N}\},$$

where the new constant symbol is c. If c is interpreted as any sufficiently large number, then every finite subset of Γ *is satisfiable in* $\mathbb{N}$*. There is an arithmetic model where c is greater than all standard natural numbers since the Compactness Theorem states that the entire set* Γ *is satisfiable. A nonstandard model of arithmetic is produced by this construction, demonstrating how compactness permits the existence of components absent from the original model.*

Compactness and dp-minimality Compactness is crucial to confirming the existence of combinatorial configurations needed to prove or disprove dp-minimality in the context of this study. For example, a binary tree of formulas of the following form can be created to demonstrate that a formula possesses the independence property (IP):

$$\phi(x, a_i), \neg\phi(x, a_i)$$

for $i \in \mathbb{N}$, and show that for every path determined by a binary string $\eta \in \{0, 1\}^n$, the corresponding conjunction of literals is consistent. Then, by compactness, the entire infinite tree defines a consistent type - a hallmark of IP and thus an obstruction to dp-minimality.

Additionally, compactness supports the realization of entire ICT-patterns (indiscernible co-trees), which are used to characterize the absence of dp-minimality. Without compactness, such configurations might not be constructible even if their finite approximations exist.

2.4 Elementary Extensions and Elementary Submodels

Elementary submodels and their extensions are essential resources for comprehending the transfer and preservation of logical properties in the study of first-order structures. These concepts allow for more expressive flexibility in model construction and analysis while offering a precise framework for comparing models with identical theories.

Let us fix a base structure $\mathcal{M} = (\mathbb{Z}, +, 0)$, where $\mathbb{Z}$ is the group of integers. This structure, due to its well-behaved definability and quantifier elimination in appropriate expansions, is a canonical example of a dp-minimal model. To analyze its expansions, we consider the role of elementary extensions. The basic structure for moving model-theoretic properties between structures is made up of elementary extensions and submodels.

Let $\mathcal{M}$ and $\mathcal{N}$ be structures in the same language L. Then, $\mathcal{M}$ is an elementary extension of $\mathcal{N}$ if and only if the following holds:

- $\mathcal{N} \subseteq \mathcal{M}$; and
- For each formula $\psi(x_1, ..., x_n)$ and each tuple $a_1, ..., a_n \in M$:

$$\mathcal{N} \models \psi(a_1, ..., a_n) \Longleftrightarrow \mathcal{M} \models \phi(\dashv_{\infty}, ..., \dashv_{\backslash}).$$

In this case, $\mathcal{N}$ is called an elementary submodel of $\mathcal{M}$.

Since elementary extensions preserve all of the first-order features of the original model, they are significant. This suggests that $\mathcal{N} \models T$ as well if $\mathcal{M} \models T$ and $\mathcal{M} \succ \mathcal{N}$. Elementary extensions allow for the controlled expansion of a model in real-world applications, allowing for the realization of additional types and sequences that may not be present in the standard model [8].

Let $p(x) \in S_1(A)$ be a type that is finitely satisfiable in $\mathcal{N}$ but not realized within it-for instance, when $A \subseteq \mathcal{N}$. By the Compactness Theorem, there exists an elementary extension $\mathcal{M} \succ \mathcal{N}$ in which the type $p(x)$ is realized. Because it gives access to configurations that may observe the presence or absence of an

ICT-pattern or the independence property (IP), this result is crucial for the study of dp-minimality.

Furthermore, $\mathcal{M}$ and $\mathcal{N}$ satisfy the exact same full theory, according to the elementary equivalency concept, which is represented by the notation $\mathcal{M} \equiv \mathcal{N}$. Specifically, any model $\mathcal{N} \equiv \mathcal{M}$ is dp-minimal if $\mathcal{M} = (\mathbb{Z}, +, 0)$ is. This is because dp-minimality is a feature of the entire theory rather than just one specific model, and it is maintained under elementary equivalence [1].

Elementary extensions are especially useful when dealing with saturated or sufficiently saturated models. The model is considered k-saturated if all types over a parameter set of size less than k that are consistent with the theory of $\mathcal{M}$ are realized in $\mathcal{M}$. In the context of dp-minimal theories, saturated elementary extensions provide a suitable setting for various analyses. They enable the effective study of dividing patterns, indiscernible sequences, and dp-rank computations. These extensions also facilitate the analysis of complex type configurations.

To sum up, the fundamental framework for transferring logical properties between structures is made up of elementary submodels and their extensions. When examining the behavior of expansions, they are crucial, especially when figuring out whether definable enrichments maintain dp-minimality. Later chapters will use elementary extensions to study definable sets under enriched languages and to construct examples and counterexamples.

2.5 Model Completeness

A theory is said to have quantifier elimination if all of its formulas are equal to formulas without quantifiers. If there is a quantifier-free formula for each formula $\phi(x)$, then a theory T admits quantifier elimination. $\psi(x)$ so that:

$$T \models \forall x((\phi(x) \to \psi(x)) \wedge (\phi(x) \leftarrow \psi(x))).$$

This property is powerful because it provides direct access to definable sets and allows one to characterize them purely in terms of atomic or boolean combinations of atomic conditions. Quantifier elimination is often closely tied to model completeness.

Definition 2.5.1. *A theory T is said to be model complete if every embedding between models of T is an elementary embedding. Equivalently, T is model complete if for any models $\mathcal{M} \subseteq \mathcal{N} \models T$, it follows that $\mathcal{M} \preceq \mathcal{N}$.*

Although quantifier elimination implies model completeness, this isn't always the case. Quantifier elimination offers a useful method for examining the definable sets and their structural characteristics in a variety of dp-minimal structures.

For instance, Presburger arithmetic $Th(\mathbb{Z}, +, <)$ admits quantifier elimination in the language $\{+, <, 0\}$. This means that any definable set can be described by boolean combinations of linear inequalities and congruences, allowing for precise combinatorial analysis of such sets.

Additionally, quantifier elimination makes it easier to demonstrate that some expansions maintain dp-minimality. One can frequently conclude that the dp-rank stays bounded and that ICT-patterns are excluded if an expansion of a dp-minimal theory maintains quantifier elimination or has quantifier elimination in an appropriate expanded language.

Furthermore, quantifier elimination allows us to verify that a structure is NIP (non-independence property) or even dp-minimal by reducing complex formulas to manageable syntactic components. In particular, model completeness allows the transfer of truth between substructures, which helps establish whether a definable predicate or relation introduces complexity that violates dp-minimality.

In conclusion, two essential components of contemporary model theory are quantifier elimination and model completeness. They provide syntactic and semantic control over logical structure and definability and are essential for determining whether extensions of a given structure preserve low-rank behavior, such as dp-minimality.

3. Methodology

3.1 Independent Property(IP) and Not Independent Property(NIP)

Model-theoretic structures are classified by dividing lines such as NIP, IP, SOP, and NSOP as shown in figure 3.1.

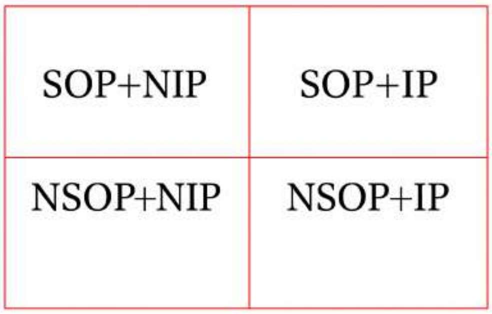


Figure 3.1: Classification theories

During the study, we used only the theoretical method. First, let us consider the Independent Property (IP) [9].

We say that a formula $\phi(\bar{x}, \bar{y})$ has IP, if there exists infinite set

$$\{\bar{b}_1, \bar{b}_2, \bar{b}_3, ..., \bar{b}_n, ...\}_{n<\omega}$$

such that $\forall n \in N$, $\forall \tau \in 2^n, \tau =< \tau_1, \tau_2, \tau_3, ..., \tau_n >$, where $\tau_i = \{0, 1\}$ and ${}^0\phi(x, \bar{a}) = \neg\phi(x, \bar{a})$, ${}^1\phi(x, \bar{a}) = \phi(x, \bar{a})$ the following holds

$$\models \exists x \wedge^{\tau_i} \phi(x, \bar{b}_i)$$
$$1 \leq i \leq n$$

A type is an infinite set of formulas with the property that any finite conjunction from that set is compatible. Compatible means that the element that satisfies that finite conjunction.

Example of the structure with IP

$$< N; =, * >, \phi(x, y) := \exists z(y * z = x)$$

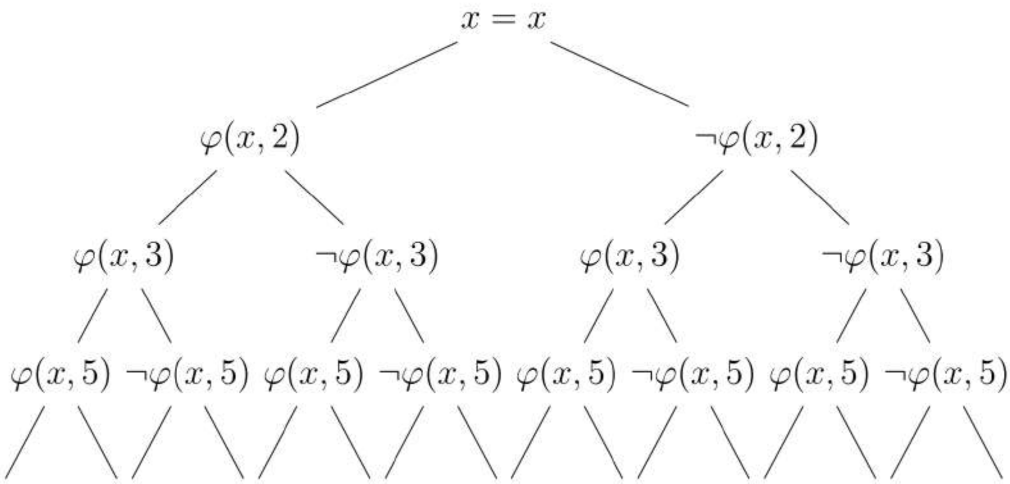


Figure 3.2: Tree diagram for instances of $\varphi(x, i)$

The set of prime numbers 3.2. In the branch 3.3 we take the number where is positive and negative, and multiply them which satisfies them.

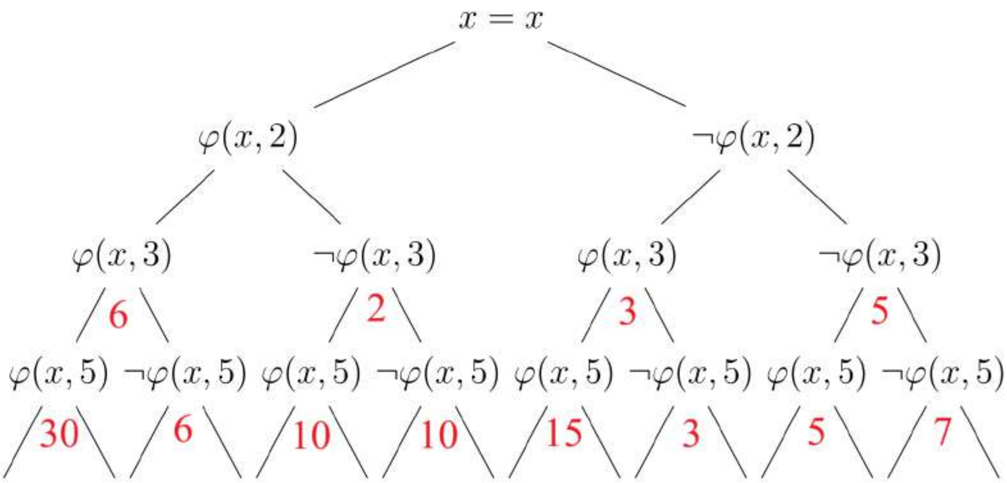


Figure 3.3: Tree diagram for instances of $\varphi(x, i)$

Let us take x = 30, it follows this formula

$$\exists x(\phi(x, 2) \wedge \phi(x, 3) \wedge \phi(x, 5))$$

The formula in the above means that it divided by 2, 3 and 5. Another example will be x = 6:

$$\exists x(\phi(x, 2) \wedge \phi(x, 3) \wedge \neg\phi(x, 5))$$

It means that it divided by 2 and 3, but not divided by 5.

x = 10:

$$\exists x(\phi(x,2) \land \neg\phi(x,3) \land \phi(x,5))$$

It divided by 2 and 5, but not divided by 3.

Second, let us consider DP, which is the negation of IP. DP-rank serves as a measure analogous to the notion of weight in stable theories. Thus a metric for a type in an NIP theory's complexity. If $dp - rank < \aleph_0$ for any type, then the theory is significantly dependent. T is strongly dependent if, for each variable x with $|x| = 1$, $dp - rank(x = x, \emptyset) < \aleph_0$. Dp-minimal theories are an extreme example of strongly dependent theories. A theory T is said to be dp-minimal if the dp-rank of the formula $x = x$ over the empty set is 1, where x is a single variable.

NIP structures as introduced at the beginning of the report, are the main two (tree, linear order). A colored order refers to a structure in which a linear order $<$ is defined on the domain M, and an arbitrary collection of unary predicates is included to distinguish elements by "color" [10].

$$\mathfrak{M}^+ =< M; \sum \cup \{p^n\} >$$

One of the main sources of examples for NIP theories is trees, or constructions built around trees. It is a linearly ordered set.

3.2 SOP - Strictly Ordered Property

$T, m, \phi(\bar{x}, \bar{y})$ has SOP.

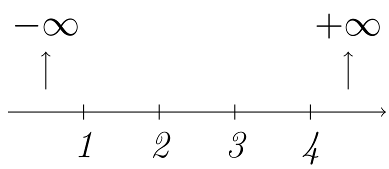
$$\bar{a}_1, \bar{a}_2, ..., \bar{a}_n, ..., n < \omega$$

$$\phi(m, \bar{a}_n) \subseteq \phi(m, \bar{a}_{n+1})$$

Example 3.2.1. *Property of strict order:*

$$< Q; =; <>$$

$$\phi(x, y) := x < y$$



$$\phi(Q, 1) = (-\infty, 1)$$

$$\phi(Q, n) = (-\infty, n)$$

Analogically, we can take $< N, =, <>$. And draw like in the Example 1. These are all properties of strict order. This is an ascending chain.
Let us take a look to the set of elements:

$$\{p/p < 1\}$$

$$\phi(Q, a) = \{b \subseteq Q | Q \models \phi(b, a)\}$$
$$Q \models b < a$$

Definition 3.2.2. *Any linear order with an infinite chain has the properties of a strict order.*

Definition 3.2.3. *There are concepts **ascending** and **descending** chain sets. The ascending set looks like $\phi(\bar{x}, \bar{y})$, consequently, the descending set is $\neg\phi(\bar{x}, \bar{y})$.*

$$\exists \phi(\bar{x}, \bar{y})$$
$$\bar{b}_1, \bar{b}_2, ..., \bar{b}_n$$

s.t. $\phi(M, \bar{b}_1) \subseteq \phi(M, \bar{b}_2) \subseteq \phi(M, \bar{b}_3) \subseteq ...\phi(M, \bar{b}_n) \subseteq \phi(M, \bar{b}_{n+1})$ as shown in 3.4.

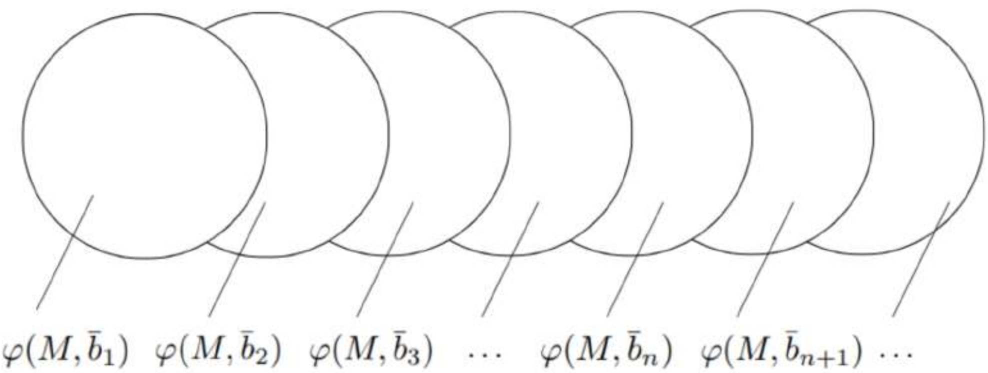


Figure 3.4: Ascending chain of formula sets

Theorem 3.2.4. *If there is an increasing set, then there is also a decreasing set.*

Proof.

$$A \subseteq B$$
$$\bar{B} \subseteq \bar{A} = MA$$
$$b \in \bar{B} => b \notin B => b \notin A => b \in \bar{A}$$

□

Example 3.2.5.

$$\phi(M, \bar{b}) \subseteq M$$
$$\neg\phi(M, \bar{b}) \subseteq M$$

Negation of phi is all values that do not satisfy

$$\phi(M, \bar{b}) = \{a \in M | m \models \phi(a, \bar{b})\}$$

$$\neg\phi(M, \bar{b}) = \{a \in M | m \models \neg\phi(a, \bar{b})\}$$

3.3 Parametrized Structures and Type Examples

Definition 3.3.1. *The relationship of semi-isolation holds for a tuple $\bar{a}$ over a tuple $\bar{b}$ relative to a parameter set A if a formula $\phi(\bar{x}, \bar{y})$ with parameters in A can be found that is both realized by the pair $(\bar{a}, \bar{b})$ and is logically sufficient to entail the complete type of $\bar{b}$ over A. Formally, this condition is met if $\phi(\bar{a}, \bar{y}) \in tp(\bar{b}/A\bar{a})$ and $\phi(\bar{a}, \bar{y}) \vdash tp(\bar{b}/A)$. Such a formula ϕ is referred to as the witnessing formula [11].*

Definition 3.3.2. *Given two types $p(\bar{x})$ and $q(\bar{y})$ over a set A, a formula $\phi(\bar{x}, \bar{y})$ with parameters from A is called a $(p \to q)$ - formula if it establishes a uniform deductive link from realizations of p to the type q. This link is formalized by the requirment that for any tuple $\bar{a}$ such that $\bar{a} \models p(\bar{x})$, the logical entailment is satisfied:*

$$\phi(\bar{a}, \bar{y}) \vdash q(\bar{y}).$$

The formula ϕ thus serves as a uniform witness to this implication.

Specifically, the structure guarantees that whenever $\bar{x}$ realizes the type p, the formula $\phi(\bar{x}, \bar{y})$ entails the type $q(\bar{y})$, as noted in [12].

A formula $\phi(\bar{x}, \bar{y})$ is called a $(p \leftrightarrow q)$ - formula if it works as a logical bridge in both directions at once. It must be both a $(p \to q)$ - formula (implying q from p) and a $(q \to p)$ - formula (implying p from q), thus establishing a symmetric, two-way implication.

When $p = q$, the formula is referred to as p-preserving, or equivalently, a (pßp)-formula, as noted in [12].

Definition 3.3.3. *The quasineighborhood of a set B with respect to a type p, denoted $QV_{p,\mathfrak{M}}(B)$, is defined as the collection of all tuples $\bar{c}$ that are related to the set B in a specific, type-dependent manner. A tuple $\bar{c}$ belongs to this quasineighborhood if and only if there exists an element $\bar{b} \in B$ and a formula $\psi(\bar{x}, \bar{y})$ such that the statement $\mathfrak{M} \models \phi(\bar{b}, \bar{c})$ is true. The essential constraint on this relationship is that the formula ψ must be $(tp(\bar{b}/A), p)$ - preserving. This* *condition means that ψ must function as a $(tp(\bar{b}/A) \to p)$ - formula, thereby ensuring the logical connection between $\bar{b}$ and $\bar{c}$ forces $\bar{c}$ to have the properties of type p.*

In particular, $\bar{c}$ belongs to the quasineighborhood of B in p if it can be connected to some element of B via a formula that transfers the type of $\bar{b}$ into the realization of p through implication. The formula ϕ thus serves as a witness to this type-preserving relation between tuples in B and those in $QV_{p,\mathfrak{M}}(B)$.

The following lemma shows some of the qualities of quasi-neighborhoods as given in [13].

Lemma 3.3.4. *Let $M \models T$ be a model of a complete theory T, and let $\bar{a}, \bar{b}$ be tuples from $\mathfrak{M}$. Suppose $p, q \in S(A)$ for some $A \subseteq M$. Then:*

1. *The tuple $\bar{a}$ always belongs to its own quasineighborhood:*

$$\bar{a} \in QV_{tp(\bar{a}),\mathfrak{M}}(\bar{a}).$$

2. *The tuple $\bar{b}$ lies in the quasineighborhood of $\bar{a}$ with respect to its own type if and only if $\bar{a}$ semi-isolates $\bar{b}$:*

   $\bar{a}$ semi-isolates $\bar{b} \Longleftrightarrow \bar{b} \in QV_{tp(\bar{b}),\mathfrak{M}}(\bar{a})$.

3. *If $\bar{b} \in QV_{p,\mathfrak{M}}(\bar{a})$ and $\bar{c} \in QV_{q,\mathfrak{M}}(\bar{b})$, then $\bar{c}$ is also in the q-quasineighborhood of $\bar{a}$:*

$$\bar{b} \in QV_{p,\mathfrak{M}}(\bar{a}) \wedge \bar{c} \in QV_{q,\mathfrak{M}}(\bar{b}) \Rightarrow \bar{c} \in QV_{q,\mathfrak{M}}(\bar{a}).$$

4. *The quasineighborhood around $\bar{b}$ is includedin the one around $\bar{a}$ if $\bar{b}$ is already in $QV_{p,\mathfrak{M}}(\bar{a})$:*

$$QV_{p,\mathfrak{M}}(\bar{b}) \subseteq QV_{p,\mathfrak{M}}(\bar{a}) \Leftarrow \bar{b} \in QV_{p,\mathfrak{M}}(\bar{a}).$$

Proof. Let $\bar{a} \in M$, and define the quasineighborhood $QV_{tp(\bar{a}),\mathfrak{M}}(\bar{a})$ as the set of all tuples $\bar{b} \in M$ such that there exists a $(tp(\bar{a}), p)$ - preserving formula $\phi(\bar{x}, \bar{y})$ satisfying

$$\mathfrak{M} \models \phi(\bar{a}, \bar{b}).$$

- If $\phi(\bar{x}, \bar{y})$ is simply the identity formula $\bar{x} = \bar{a}$, then clearly $\bar{a} \in QV_{tp(\bar{a}),\mathfrak{M}}(\bar{a})$ by direct substitution.
- Now suppose $\bar{a}$ semi-isolates $\bar{b}$. Then, by definition, there exists a formula $\phi(\bar{x}, \bar{y}) \in tp(\bar{b}/A\bar{a})$ such that

$$\phi(\bar{a}, \bar{y}) \vdash tp(\bar{b}/A).$$

  This means that ϕ is a $(tp(\bar{b}/A), tp(\bar{a}))$ - preserving formula, and hence

$\bar{b} \in QV_{tp(\bar{a}),\mathfrak{M}}(\bar{a})$ by definition.

- Now define

  $QV_{p,\mathfrak{M}}(\bar{a}) := \{\bar{b} \in p(\mathfrak{M}) | \exists p$ - preserving $\phi(\bar{x}, \bar{y})$ s.t. $\bar{a} \in \phi(\bar{x}, \bar{b}) \subseteq p(\mathfrak{M})\}$, and similarly,

  $QV_{q,\mathfrak{M}}(\bar{b}) := \{\bar{c} \in q(\mathfrak{M}) | \exists q$- preserving $\phi(\bar{y}, \bar{z})$ s.t. $\bar{b} \in \phi(\bar{y}, \bar{c}) \subseteq q(\mathfrak{M})\}$.

  Assume $\bar{b} \in QV_{p,\mathfrak{M}}(\bar{a})$, witnessed by some $(tp(\bar{a}), p)$ - preserving formula $\phi_1(\bar{x}, \bar{y})$. Then $\bar{b} \in \phi_1(\bar{a}, M) \subseteq p(\mathfrak{M})$.

  Now let $\bar{c} \in QV_{q,\mathfrak{M}}(\bar{b})$, witnessed by some (p, q) - preserving formula $\phi_2(\bar{y}, \bar{z})$. Then $\bar{c} \in \phi_2(\bar{b}, M) \subseteq q(\mathfrak{M})$.

Hence by composition, $\bar{b}$ semi-isolates $\bar{c}$ via the formula $\phi_2(\bar{b}, \bar{z})$. Let:

$$p(\mathfrak{M}) = \bigcap_{H \in p} H(\mathfrak{M}), q(\mathfrak{M} = \bigcap_{\theta \in p} \theta(\mathfrak{M}).$$

Since $\phi_2(\bar{b}, M) \subseteq q(\mathfrak{M})$, for any $\theta(\bar{y}) \in q(\mathfrak{M})$, we have:

$$\mathfrak{M} \models \forall \bar{y}[\phi_2(\bar{b}, \bar{y}) \to \theta(\bar{y})],$$

and denote this by $K_\theta(\bar{x})$ where:

$$K_\theta(\bar{x}) := \{(\phi_2(\bar{x}, \bar{y}) \to \theta(\bar{y})) | \theta \in q\} \in p(\bar{x}).$$

Now assume $\bar{a}$ is such that for all $H(\bar{x}) \in p$:

$$\mathfrak{M} \models \forall \bar{x}[\phi_1(\bar{a}, \bar{x}) \to H(\bar{x})]$$

Let:

$$R(\bar{a}, \bar{y}) := \exists \bar{x}[\phi_1(\bar{a}, \bar{x}) \wedge \phi_2(\bar{x}, \bar{y})].$$

This clearly:

$$\mathfrak{M} \models \phi_1(\bar{a}, \bar{b}) \wedge \phi_2(\bar{b}, \bar{c}) \Rightarrow \mathfrak{M} \models R(\bar{a}, \bar{c}).$$

Let $\bar{c}_0 \in R(\bar{a}, M)$ be arbitrary. Then there exists $\bar{b}_0 \in M$ such that:

$$\mathfrak{M} \models \phi_1(\bar{a}, \bar{b}_0) \wedge \phi_2(\bar{b}_0, \bar{c}_0).$$

After then, it looks like this:

$$\mathfrak{M} \models \forall \bar{y}[\phi_2(\bar{b}_0, \bar{y}) \to \theta(\bar{y})],$$

for all $\theta(\bar{y}) \in q$.

This implies:

$$\phi_2(\bar{b}_0, M) \subseteq \bigcap_{\theta \in q} \theta(M) = q(M),$$

and hence $\bar{c}_0 \in q(M)$. Since $\bar{c}_0$ was arbitrary, it concluded that $R(\bar{a}, M) \subseteq q(M)$. □

Theorem 3.3.5. *Let $\mathfrak{M} \models T$ be a saturated model of a theory T, and let $A \subseteq M$, with $p \in S(A)$ a type over A. Suppose $\bar{a} \in p(M)$, and the quasi-neighborhood $QV_{p,\mathfrak{M}}(\bar{a})$ is definable. If the relation $\bar{x} \in QV_{p,\mathfrak{M}}(\bar{y})$ fails to be an equivalence relation, then the theory T has the strict order property (SOP).*

Proof. To understand the strict order property(SOP) in the context of quasi-neighborhoods, consider a model $\mathfrak{M} \models T$, a type $p \in S(A)$ over a set $A \subseteq M$, and a sequence of realizations $\bar{a}_1, \bar{a}_2, ..., \bar{a}_n \in p(M)$. The theory T is said to have the strict order property if these realizations give rise to a strictly increasing chain of quasi-neighborhoods:

$$...QV_{p,\mathfrak{M}}(\bar{a}_i) \subsetneq QV_{p,\mathfrak{M}}(\bar{a}_{i+1}) \subsetneq ...QV_{p,\mathfrak{M}}(\bar{a}_n) \subsetneq ...$$

Since $\mathfrak{M}$ is saturated, it is also highly homogeneous: for any finite sequence of elements, there exists an automorphism $g : M \to M$ that maps a_1 to a_2, a_2 to a_3, and so on. This means that if an element b belongs to the quasi-neighborhood of a_1, then its image under g must lie in the quasi-neighborhood of a_2.

This implies that certain pathological configurations are impossible: $QV_{p,\mathfrak{M}}(\alpha) \subsetneq QV_{p,\mathfrak{M}}(\gamma)$ and $QV_{p,\mathfrak{M}}(\beta) \subsetneq QV_{p,\mathfrak{M}}(\gamma)$ but $QV_{p,\mathfrak{M}}(\alpha) \cap QV_{p,\mathfrak{M}}(\beta) = \emptyset$

□

Corollary 3.3.6. *Let $\mathfrak{M} \models T$ be a model of a complete theory T, and let $\bar{a}$ be a tuple from $\mathfrak{M}$. Suppose $p \in S(A)$ for some $A \subseteq M$, and the quasi-neighborhood $QV_{p,\mathfrak{M}}(\bar{a})$ is definable. If the theory T does not exhibit the strict order property, then the quasi-neighborhood of $\bar{a}$ with respect to p coincides with the neighborhood of $\bar{a}$ in the type p; that is:*

$$QV_{p,\mathfrak{M}}(\bar{a}) = V_{p,\mathfrak{M}}(\bar{a}).$$

Proof. By theorem 3.3.5, the relation $\bar{x} \in QV_{p,\mathfrak{M}}(\bar{y})$ defines an equivalence relation under the assumption that T does not have the strict order property. In particular, this implies symmetry. Hence, the quasi-neighborhood and the neighborhood of the tuple $\bar{a}$ in type p must be equal: $QV_{p,\mathfrak{M}}(\bar{a}) = V_{p,\mathfrak{M}}(\bar{a})$

□

3.4 DP-minimal properties and examples

We begin by identifying key subclasses that are encompassed within the framework of **DP-minimality**. Notably, two important examples include the class of **o-minimal theories**, which exhibit well-behaved topological and definability properties, and the class of **strongly minimal theories**, which represent a fundamental notion of minimality in model theory. These classes serve as standard examples of structures satisfying DP-minimality and demonstrate the range of behaviors that such theories can capture.

Definition 3.4.1. *The concept of **o-minimality** applies to a theory T that is based on a language $\mathfrak{L}$ containing a symbol $<$ for a linear order. The theory T is deemed o-minimal if its definable sets are structurally simple. Specifically, for any model M that satisfies the theory, every subset of M that you can define using the language $\mathfrak{L}$ must be nothing more complex than a finite collection of single points and open intervals.*

To explore how model-theoretic properties behave under expansions - especially within dp-minimal frameworks - this research draws on fundamental ideas from the theory of o-minimal structures. O-minimality offers a well-established example of an expansion that remains logically and topologically well-behaved. Its principles serve both as a standard of comparison and as a guide throughout the analysis as it shown in [14].

A linearly ordered structure is called o-minimal if its definable sets are all finite unions of points and open intervals. This simple geometric property is very restrictive, preventing the existence of complex patterns defined by the independent property (IP) and ensuring the theory has the Not Independent Property distinction.

Because of this regularity, o-minimality is used as a reference point for evaluating whether certain expansions retain low combinatorial complexity.In particular, its approach to dimension - defined using coordinate projections with nonempty interior- helps signal whether an expansion increases the dp-rank of types or allows more intricate definable sets to emerge.

One especially useful technique adopted from o-minimality is cell-decomposition. This allows definable sets to be broken into a finite number of manageable parts, or "cells," each of which is either an interval or the graph of a continuous function. This breakdown supports a step-by-step analysis of definability and helps identify when potentially problematic configurations - like ICT-patterns or inp-patterns - may arise.

Another valuable tool is the monotonicity theorem, which states that any definable unary function $f : M \to M$ is piecewise monotonic on finitely many intervals. In this work, such behavior is used as a diagnostic: if functions in an expansion fail to satisfy similar properties, it may indicate a move away from the structural control associated with dp-minimality.

Altogether, o-minimality acts as both as conceptual benchmark and a practical toolkit. Its core ideas help assess whether new expansions preserve the kind of disciplined definability that characterizes dp-minimal theories.

Theorem 3.4.2. *Any theory that is o-minimal is dp-minimal.*

Proof. This proof demonstrates that the theory is dp-minimal by showing a contradiction that arises in o-minimal structures. We start with an o-minimal structure M and an indiscernible sequence of tuples $(b_i : i \in \mathbb{Q})$.

By o-minimality, any formula $\phi(x; b_i)$ defines a finite union of intervals. Since the (b_i) sequence is indiscernible, the structure of these intervals (their number and type) must be uniform for all i. This means their endpoints are determined by a fixed set of definable functions applied to each b_i.

We then proceed by contradiction. Assume a singleton element $a \in M$ exists such that it can distinguish between elements of the sequence (i.e., the type $tp(b_i/a)$ is not constant for all $i \in \mathbb{Q}$). This would mean that a creates a "cut" in the sequence of definable closures, $(dcl(b_i) : i \in \mathbb{Q})$.

However, this leads to an impossibility. A known result states that for mutually indiscernible sequences, their definable closures are also mutually indiscernible, and a single element cannot create a cut in them. Since our assumption leads to this contradiction, it must be false. Therefore, no such singleton a can exist, and the theory is shown to be dp-minimal.

□

Several relaxations of the o-minimality condition have been proposed, motivated in part by the desire to retain desirable model-theoretic behavior while allowing for more flexibility as it given in [1]. Not all variations of these structures guarantee NIP; for instance, those with an o-minimal open core are based on geometric or topological properties and don't necessarily have NIP. However, there are other variations that do maintain NIP, and two of these are discussed briefly.

A structure $(M; <, ...)$ is said to be weakly o-minimal if, in every elementary extension of M, every definable unary set is a finite union of convex subsets. In contrast to o-minimality, which can sometimes be determined by examining just one model, weak o-minimality applies to the entire theory. This means a theory is considered weakly o-minimal only if every one of its models meets the required condition.

A good example of this is the Shelah expansion (M^{Sh}) of an o-minimal structure (M), which results in a weakly o-minimal structure. Importantly, all weakly o-minimal theories are NIP and also dp-minimal. This is demonstrable through a modified version of the proof for Theorem 3.4.2.

Quasi o-minimality is a broader concept where a structure is considered quasi o-minimal if all its definable unary sets are finite Boolean combinations of convex sets and sets definable without parameters. A key example is the structure of integers with addition, $(\mathbb{Z}; <, 0, 1, +)$. Similar to weak o-minimality, quasi o-minimality is a property of an entire theory, meaning all its models must have this characteristic. When a theory is quasi o-minimal, it is also dp-minimal, a fact that can be proven by adapting the same arguments used previously.

It is worth noting that dp-minimal ordered structures form a robust and independently significant class of study. Extensive analysis of these structures has been conducted, including in works such as [15] and [16].

Example 3.4.3. *The first example describes a structure consisting of the rational numbers ($\mathbb{Q}$) with their standard "less than" order ($<$) and a specific set, P, which contains the reciprocals of all natural numbers ($P = \{1, 1/2, 1/3, ...\}$). This specific structure is notable because it is definably complete but fails to be o-minimal, serving as an example that distinguishes between these two properties.*

The second illustration points out that structures with a discrete order behave in a particularly predictable and favorable way. A discrete order is one where every element has an immediate next value(unless in is the final element) and an immediate previous value(unless it's the very first element), much like the integers. This well-behaved nature is a key observation in the context of o-minimal structures.

Definition 3.4.4. *A theory is called strongly minimal if a specific rule applies to all of its models: every subset that can be defined within any given model must be either finite or have a finite complement.*

Example 3.4.5. *The theory of $\mathbb{Q}$-vector spaces can be described by the axioms for torsion-free divisible Abelian groups using only the language of addition and zero. Thanks to a property called quantifier elimination, any definable set in these spaces can be constructed from simple equations of the form $nx = a$ (where n is an integer). Because these groups are torsion-free, the resulting sets are always either finite (having a limited number of elements) or cofinite (missing a limited number of elements).*

3.5 Expansion of models

Expanding models of DP-minimal theories involves creating larger structures or adding new elements, predicates, or functions while ensuring the models still respect the properties of DP-minimality. DP-minimality is a key concept in model theory that lies between stability and NIP (no independence property) theories where it is shown in 3.5. When expanding models of such theories, one must ensure that the model-theoretic tameness, particularly the dependent nature, is preserved.

Expansion formula:

$$\mathfrak{M} =< M, \sum >$$

$$\mathfrak{M}^{+} =< M; \sum \cup \{p^n\} >$$

The difference between these two structures is in the second, where we are adding some new relationships or predicates(+, *, <).

$\mathfrak{M} =< M, \Sigma >$ - structure.

$$F(\mathfrak{M}) = \{\bar{b} | \mathfrak{M} \models F(\bar{b}, \bar{a}), \bar{b} \in M\} \subset F(\mathfrak{M}^{+})$$

For any $\phi(x, \bar{a})$ formula of Σ, $\phi(\bar{M}, \bar{a}) \neq P^n(\mathfrak{M}^{+}$.

$$\Sigma^{+} = \Sigma \cup \{P^n\}$$

When we add new relationships, the properties move into other classes.

First, consider $(\mathbb{Z}, +, 0, 1)$ expansion, which has NIP theories, inside of this DP-minimal properties. The formula inside of the NIP theories depend on each other. Our mission do not go beyond DP-minimal theories. $(\mathbb{Z}, +, <, 0, 1)$ adheres to this principle.

3.6 Quantifier elimination

Definition 3.6.1. *Quantifier elimination - if we can give an equivalent formula E that lacks quantifiers, then any formula in the theory permits quantifier elimination.*

Definition 3.6.2. *Two formulas of First Order Logic are equivalent $\phi \equiv \theta$.*

Proof. a. $\exists x P(x) \equiv \neg \forall x \neg P(x)$, $\forall x P(x) \equiv \neg \exists x \neg P(x)$

b. $(\exists x \phi(x) \vee \exists y \Theta(y)) \equiv \exists x(\phi(x) \vee \Theta(x))$

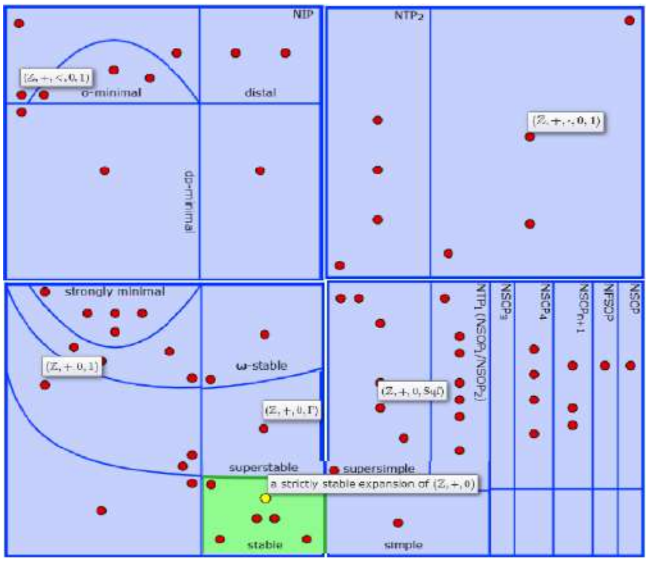


Figure 3.5: Forking and Dividing

c. $(\exists x \phi(x) \wedge \exists y \Theta(y)) \not\equiv \exists x(\phi(x) \wedge \Theta(x))$,

- $(\exists x \phi(x) \wedge \exists y \Theta(y)) \equiv \exists x \exists y(\phi(x) \wedge \Theta(y))$

d. $(\forall x \phi(x) \vee \forall y \Theta(y)) \not\equiv \forall x(\phi(x) \vee \Theta(x))$,

- $(\forall x \phi(x) \vee \forall y \Theta(y)) \equiv \forall x \forall y(\phi(x) \vee \Theta(y))$

e. $(\forall x \phi(x) \wedge \forall y \Theta(y)) \equiv \forall x(\phi(x) \wedge \Theta(x))$

□

The Quantify Elimination plays an important role because it considers all formulas. It is possible to exchange by considering Quantify Free formulas, which very simply understand the nature of structures and theories. The consideration of Quantify Elimination starts from the Definition 3.6.2 of equivalence of two formulas.

Theorem 3.6.3. *Any formula of First Logic is equivalent to PNF(Prenex Normal Form).*

Lemma 3.6.4. *Formula in Prenex Normal Form, when all quantifiers are in the head of formula.*

Theorem 3.6.5. *Test of A. Tarskii Complete theory admits Quantifier Elimination iff for any Quantifier Free formula $\psi(x, y_1, y_2, ..., y_n)$ there exists Quantify Free formula $\psi_0(y_1, ..., y_n)$ such that for any model $\mathcal{M}$ of T*

$$\mathcal{M} \models \forall y_1, \forall y_2, ..., \forall y_n[\exists x \psi(x, \bar{y}) \leftrightarrow \psi_0(\bar{y})]$$

or equivalently,

$$T \vdash \forall y_1, ..., \forall y_n[\exists x \psi(x, \bar{y}) \leftrightarrow \psi_0(\bar{y})]$$

Proof. This condition is a particular case of QE. Because it is just for Elimination Quantify $\exists$ for Quantify Free of formula. Let $\theta(\bar{v})$ is arbitrary formula following Theorem on PNF

$$\mathcal{M} \models \forall \bar{v}[\theta(\bar{v}) \leftrightarrow Q_1 y_1, ..., Q_n y_n H(\bar{y}, \bar{v})],$$

where $H(\bar{y}, \bar{v})$ is Quantify Free Formula and $Q_i \in \{\exists, \forall\}$.

Consider $\forall y H(\bar{y}) \sim \neg\exists y \neg H(\bar{y})$. Notice that the negation of a Quantify Free formula is a Quantify Free formula. Then, replace the step-by-step formula with one Quantify with Quantify Free formula.

Indeed, $Q_n y_n H(y_1, ..., y_{n-1}, y_n, \bar{v})$

$$\exists y_n H(y_1, ..., y_{n-1}, \bar{v}) \equiv H_0(y_1, ..., y_{n-1}, \bar{v})$$

$$\forall y_n H(y_1, ..., y_{n-1}, y_n, \bar{v}) \equiv \neg\exists y \neg H(y_1, ..., y_{n-1}, y_n, \bar{v})$$

by the condition of Test.

$$\neg\exists y \neg H(y_1, ..., y_{n-1}, y_n, \bar{v}) \equiv \neg H_0(y_1, ..., y_{n-1}, \bar{v}) = H_0^{'}(y_1, ..., y_{n-1}, \bar{v}).$$

□

Theorem 3.6.6. *In the system of the rational numbers* ($\mathbb{Q}$) *with equality and their standard ordering* ($<$), *any logical formula can be expressed in an equivalent form without using quantifiers like "for all"* ($\forall$) *or "there exists"* ($\exists$).

Proof. The logical expression being examined is $\exists x \tau(x, x_1, ..., x_n)$. This statement asserts that there is at least one value of x that will make the sub-formula τ true.

Two important assumptions are made about this sub-formula τ:

- It is quantifier-free, meaning it contains no "for all" or "there exists" operators itself.
- Its structure can be treated as a disjunctive normal form (DNF), which is a standardized format of clauses linked by the "OR" operator.

□

Theorem 3.6.7. *Any Boolean function is expressible both as a formula in conjunctive normal form and as one in disjunctive normal form.*

Proof. Essentially, the expression is an OR of multiple AND-clauses, with each clause containing basic logical statements or their negations.

We can make the formula negation-free by substituting any negative statements with their positive equivalents. Specifically, we replace $\neg(x = y)$ with $((x < y) \vee (x > y))$ and $\neg(x < y)$ with $((x = y) \vee (x > y))$. Then, we apply the distributive law to restore the expression to disjunctive normal form, which increases its length but successfully eliminates all negations.

Here, we use the fact that the existential quantifier ($\exists$) distributes over disjunctions ($\vee$). This lets us convert a single formula with multiple 'OR' clauses into a new formula where each clause has its own quantifier. The transformation from $\exists x(\tau_1 \vee \tau_2)$ to $\exists x\tau_1 \vee \exists x\tau_2$ allows us to break the problem down. By applying this rule repeatedly, we can rewrite $\exists x(\tau_1 \vee \tau_2 \vee ... \vee \tau_n)$ as $\exists x\tau_1 \vee \exists x\tau_2 \vee ... \vee \exists x\tau_n$ and then solve each part separately.

The final goal is to create a quantifier-free version of the remaining formula. This formula is a conjunction (an AND-chain) under a single "there exists" quantifier:

$$\exists x(p_1 \wedge p_2 \wedge ... \wedge p_k)$$

The basic parts (p_i) of this conjunction are all simple comparisons using either the equals ($=$) or less-than ($<$) symbol, since we have already dealt with all negations.

This step explains how to handle any terms (p_i) inside the AND-chain that do not involve the variable x. The rule states that if a formula, let's call it α, does not depend on x, it can be moved outside of the "there exists" quantifier. This means an expression of the form: $\exists x(\alpha \wedge \beta)$ can be rewritten as the equivalent, simpler expression: $\alpha \wedge \exists x\beta$. This allows us to separate the parts of the formula that are independent of x from those that are not.

Once the formula only contains comparisons like $x < x_i$, $x = x_i$ and $x > x_i$, we check for equalities. If there's an equality term like $x = y$, the quantifier is easily eliminated. We just remove $\exists x$ from the expression and replace all other occurrences of x with y. For instance, $\exists x((x = y) \wedge A(x))$ simplifies to $A(y)$.

When the variable x only appears in inequalities, the problem is to determine if a value for x can exist between its lower and upper bounds. For a formula like $\exists x((x > a) \wedge (x > b) \wedge (x < c) \wedge (x < d))$, an x can only exist if all lower bounds (a, b) are less than all upper bounds (c, d). This yields the equivalent quantifier-free expression:

$$(a < c) \wedge (a < d) \wedge (b < c) \wedge (b < d)$$

If, however, all the inequalities were of the same type (e.g., all greater-than), the expression would always be true because the set of rational numbers is unbounded.

The basic logic is that to squeeze a number x between a "floor" (lower bounds) and a "ceiling" (upper bounds), the floor must be below the ceiling. Because we don't know the exact height of the floor or ceiling, we make a simple, catch-all rule: every part of the floor must be below every part of the ceiling. This rule is guaranteed to work because the numbers being used (the rationals, $\mathbb{Q}$) are "dense." This simply means that as long as a gap exists at all, you can always find another number to fit inside it, which makes our rule equivalent to finding an existing x.

The reasoning is now finished because the problem was solved by gradually simplifying it step-by-step.

$\square$

During the proof about rational numbers' properties, only the absence of a greatest or least element and the density of the order were used. Therefore, each step applies equally to any dense linear order without endpoints, not just to $\mathbb{Q}$. Applying these transformations to a closed formula (one without parameters) produces an expression that is either identically true or identically false. To write down "identically true" or "identically false" without introducing extra variables, constant symbols for truth and falsehood can be added to the language. As a result, every dense linear order without endpoints satisfies exactly the same sentences in this signature. In other words, such structures are elementarily equivalent in this language. Alternatively, one may apply Löwenheim–Skolem to obtain a countable submodel and then use the classification theorem for countable dense linear orders without endpoints.

In particular, it is proved that the same Signature formulas $(=; <)$ are true for rational and real numbers.

Whether this closed formula is true or false in the interpretation under consideration. To do this, you need to bring it to a quantorless form and see if it turns out to be True or False. In other words, quantifier elimination establishes the solvability of the elementary theory of rational numbers with equality and order relations as it given in [6].

4. Expansion by unary and binary predicates

4.1 Expansion of $\langle \mathbb{Z}; =; + \rangle$ by unary predicate and graph representation

4.1.1 Graph representation

Let $\mathbb{Z}^+ = (Z; =; +; P^1)$ be an expansion of $(Z; =, +)$ by unary relation P^1. Denote $S^2(x, y) := P(x + y)$. Notice that

$$\mathbb{Z}^+ \models \forall x \forall y (S^2(x, y) \equiv S^2(y, x))$$

since

$$(Z; =; +) \models \forall x \forall y, (x + y = y + x)$$

Denote $P(\mathbb{Z}^+) = \{a \in Z | \mathbb{Z}^+ \models P^1(a)\} = A$. We define a graph $G = (Z, S^2)$, where the edge relation is given by S^2. Notice that if $2Z \subset A$, then this graph is reflexive. The properties of expansion depend from the chosen A.

Theorem 4.1.1. *Let $\mathbb{Z}^+ = (Z; =; +; P^1)$ be an expansion of $(Z; =, +)$ by unary relation P^1. Then the foolowing is true:*

(i) $\mathbb{Z}^+$ is superstable iff $G = (Z, S^2)$ is superstable.

(ii) $\mathbb{Z}^+$ is NIP iff $G = (Z, S^2)$ is NIP. Then $\mathbb{Z}^+$ is superstable iff $G = (Z, S^2)$ is superstable..

(iii) For any essential classes (stable, IP+NSOP, NIP+SOP, SOP+IP) of complete theories there are expansions by unary predicate

Proof. Consider the formula:

$$\phi(x, a) := P(x + a),$$

and fix the sequence $(a_i)_{i \in \mathbb{N}} \subseteq A$. For any binary string $\eta : \mathbb{N} \to \{0, 1\}$, define a type:

$$P_\eta(x) = \{\phi(x, a_i)|\eta(i) = 1; \neg\phi(x, a_i)|\eta(i) = 0\}$$

One can demonstrate that for every such η, the corresponding type $p_\eta(x)$ is consistent by using pairwise distinctness and the assumption on $P(a_i + a_j)$. The existence of IP is implied by this construction, which is equivalent to realizing a tree of consistent formulas of infinite depth.

As a result, the expansion is not dp-minimal.

□

4.1.2 Superstable expansion of $(Z; =, +, 0, 1)$

$\langle \mathbb{Z}, +, 0, 1\rangle$ - non ω-stable is countable number of continuum type.

$$\phi_7(x)_i = \exists y(y + y + ... + y = x)$$
$$\phi_7(Z) = 7Z$$

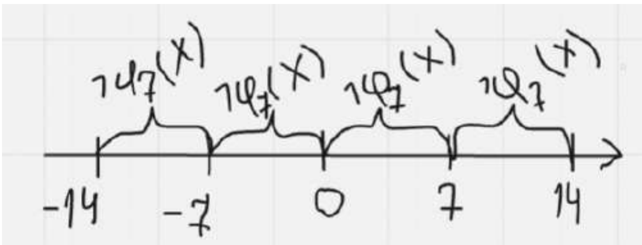


Figure 4.1: Interval Partition of the Real Line for Evaluation of $\phi_7(x)$

Consider prime numbers:

$$\langle 2, 3, 5, 7, 11, ..., \rangle$$

$$\langle p_1, p_2, p_3, p_4, ..., p_n, ...\rangle$$

set of all prime numbers

$$\psi_i(x) = \exists y(p_i, y = x),$$

where $p_i y = y + ... + y$ - p_i times

Sequence:

$$\tau = i_1, i_2, ..., i_n, ...$$

τ is an infinite sequence of zeros and ones.

$$q_\tau = \{\psi_n(x)|i_n = 1\} \cup \{\neg\psi_n(x)|i_n = 0\}, \tau \in 2^N$$

q_τ - consistent set of formulas that called one-type.

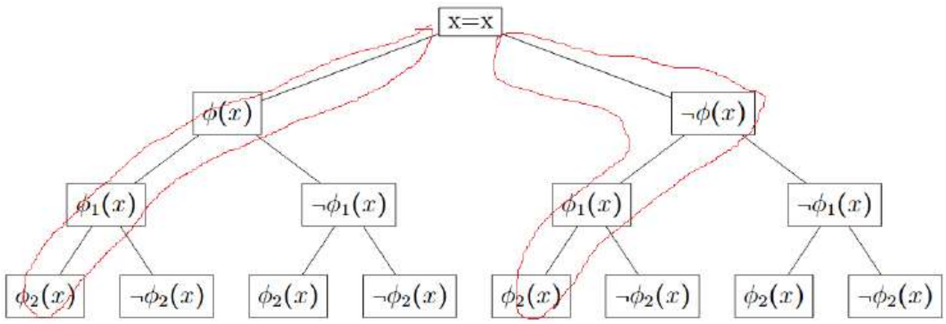


Figure 4.2: Consistent set of formulas

$$\mathfrak{M} \models \exists x(\phi(x) \wedge \phi_1(x) \wedge \phi_2(x)$$
$$\mathfrak{M} \models \exists x(\neg\phi(x) \wedge \phi_1(x) \wedge \phi_2(x))$$

Typical example of the type:

$$\langle \sqrt{2}, \mathbb{Q}, =, < \rangle$$
$$\{a_n < x < b_n\}; n, m \in N$$

T is called to be ω - *stable*, if for any countable A set of all one-types over A is countable. $S_1(A) = \{q$ such that is one-type over A$\}$. Thus elementary theory of $(\mathbb{Z}, +)$ is not ω-stable, because $S_1(\emptyset) = \{q_\tau | \tau \in 2^N\}$ is uncountable.

T is called to be *superstable* if for some λ, $\lambda \in Cardinal$, for any $\beta > \lambda$, for any $|A| = \beta$ we have $|S_1(A)| = \beta$. Let C, such that $|C| > |2^N|$, any element $a \in M - C$ satisfies $\models q_\tau(a)$, consequently $|S_1(C)| = |C| + |2^N| = |C|$, elementary theory of T is superstable.

Theorem 4.1.2. *The theory of the structure* $(\mathbb{Z}, +, 0, \Pi_q)$*, where* Π_q *represents the set of all powers of a given natural number q, is superstable and has Lascar rank* ω*.*

The same techniques can be used to prove that other expansions of the theory of integers are also "superstable." You can add different special sets—such as the set of numbers of the form $\{q^{p^n}_{n<\omega}\}$ or the set of factorials—and the resulting logical system will still be considered well-behaved and non-chaotic.

Proposition 4.1.3. *For any natural number q, the structure* Π_q^{ind}*, viewed relative to* $(\mathbb{Z}, +, 0)$*, is superstable and has Lascar rank 1.*

It is a proper expansion of $(\mathbb{Z}, +, 0)$, it has infinite Lascar rank. Whence, it remains to see that it has Lascar rank ω as it given in [17].

Consider a sufficiently saturated elementary extension of the $(\mathbb{Z}, +, 0, \Pi_q)$ structure , where Π_q is interpreted as $\Pi_q^{'}$. Let $p \in S(\emptyset)$ be the generic type corresponding to the connected component, and let $q = tp(b/B)$ be a type extending p. Let a realize the restriction $p|B$. The structure $\Pi_q^{'}$ is assumed to have Lascar rank equal to one.

Within the framework of model theory for $(\mathbb{Z}, +, 0)$, the type $tp(b/\Pi_q^{'}B)$ is identified as the principal generic type, provided that $b \notin acl(\Pi_q^{'}B)$.

Let d be a finite tuple that is algebraic over $\Pi_q^{'} \cup B$, with this algebraicity witnessed by a tuple $(c_1, ..., c_n)$ drawn from $\Pi_q^{'}$. According to the theory $Th(\mathbb{Z}, +, 0, \Pi_q)$, it then holds that

$$U(d/B) \leq U(\bar{c}/B) > \omega,$$

as the set $(\Pi_q^{'})^n$ is of Lascar rank n. Therefore, the element a does not belong to the algebraic closure $acl(\Pi_q^{'} \cup B)$ within the structure $(\mathbb{Z}, +, 0)$, and the type $tp(a/\Pi_q^{'} \cup B)$ must be the principal generic type.

4.2 Expansion of model of strongly minimal theory by unary predicate

A structure $\mathcal{M} = (M; \Sigma)$ is said to be minimal if every definable subset of M is either finite or cofinite. If all models of elementary theory of minimal structure are minimal then this structure and theory is called to be Strongly minimal.The field of all complex numbers $(C; =, +, \cdot, 0, 1)$ is strongly minimal since its elementary admits Quantify Elimination. Thus any formula $\phi(x, \bar{a})$ with one free variable is equivalent to boolean combination of equality polynomial to zero or negation of such equality. If we consider the equality of polynomial to zero as the elementary proposition and any boolean combination of it is equivalent to the formula in Disjunctive Normal Form (DNF). Thus the formula is equivalent to disjunction of conjunctive of elementary propositions or its negation (system of equalities or non-equalities polynomials to zero). Definable set is equal to union of sets of realizations such systems. Consider the system of non-equalities of polynomials to zero.

$$\begin{cases} f_1(x, \bar{a}) \neq 0 \\ f_2(x, \bar{a}) \neq 0 \\ f_i(x, \bar{a}) \neq 0 \end{cases}$$

The set of realizations of this system is co-finite. If one of polynomial $= 0$, then the set of realization of this system is finite.

Strongly minimal theory is trivial if

$$acl(a_1, a_2, ..., a_n) = acl(a_1) \cup acl(a_2) \cup ... \cup acl(a_n),$$

where $\phi(x, a_1, a_2, ..., a_n) = \phi(x, \bar{a})$ and

$$acl(a_1, ..., a_n) = \{b | \text{exists } \phi(x; \bar{a}) \text{ such that } \mathcal{M} \models \phi(b; \bar{a}) \wedge \exists^{\leq m} x, \phi(x; \bar{a})\}$$

Example 4.2.1. *Example of non-trivial strongly minimal:* $\langle C; =, +, \cdot, 0, 1\rangle$*, if take* $\{e, \pi, e + \pi\}$ *algebraic independent finite number elements*

$$acl(e) \neq acl(\pi), \pi \notin acl(e)$$

$$e + \pi \notin acl(e)$$
$$e + \pi \notin acl(\pi)$$

Example 4.2.2. *Example of trivial strongly minimal structure:* $\langle Z; =, f^1\rangle$*, where* $f(n) = n + 1$*. The models of theory of* $\langle Z; =, f\rangle$ *is exactly the same copies of* $\langle Z; =, f\rangle$*.*

Theorem 4.2.3. *If T is a strongly minimal and trivial theory, then any expansion of T by a unary predicate results in a* ***superstable*** *theory.*

Proof. For any element $A \subset M$:

$$acl(A) = \bigcup_{a \in A} acl(a)$$

since M is trivial. For arbitrary $a \in M$, $acl(A)$ is defined by countable or finite numbers of algebraic formulas, $\{\phi_i(x, a) | i \in \mathbb{N}\}$. Then for any $\phi_i(x, a)$ there are finite number of possibility satisfy to Predicate P^1. Then we have description of all elements from algebraic closure of a. The number of possibility is $2^{\aleph_0}$.

$$|S_1(A)| = |A| + 2^{\aleph_0}$$

□

Theorem 4.2.4. *Let T be a strongly minimal, non-trivial theory and let* $\mathbb{M}$ *be a countable, saturated model of T.*

Then, for each of the fundamental classes of complete theories-namely Stable, theories with the Independent Property but not the Strong Order Property (IP + NSOP), theories with the Strong Order Property but not the Independent Property (SOP + NIP), and theories with both the Independent Property and the Strong

Order Property (IP + SOP)—there exists an expansion of the model $\mathbb{M}$ by a unary relation that results in a theory belonging to that class.

Proof. Since $\mathfrak{M}$ is countable saturated model of strongly mininimal theory there exists countable independent set A. Let $A = \{a_i | i \in \omega\}$. Since T is not trivial there exists finite subset B of A such that there exists $c \in M$, $c \in acl(B)$ and $c \notin alc(B')$, where $c \notin acl(B^1)$, for any $B^1 \subset B, B^1 \neq B$.

Then we suppose that $B = \{a_1, a_2, ..., a_{n-1}, a_n\}$. Consider $b_0 = \{a_1, a_2, ..., a_{n-2}\}$. Since $c \in acl(B)$, there exists the formula $\phi(x; a_1, ..., a_{n-2}, a_{n-1}, a_n)$.

$$\mathcal{M} \models \phi(c; a_1, a_2, ..., a_{n-2}, a_{n-1}, a_n) \wedge \exists \leq^m x\phi(c, \bar{a}_n).$$

Denote

$$S^2(Z_1, Z_2) := \exists x(\phi(x, \bar{a}_{n-2}, Z_1, Z_2) \wedge P(x)).$$
$$\neg S^2(Z_1, Z_2) := \neg\exists x(\phi(x, \bar{a}_{n-1}, Z_1, Z_2) \wedge P(x)) \equiv$$
$$\equiv \forall x[\phi(x, \bar{a}_{n-1}, Z_1, Z_2) \longrightarrow \neg P(x)].$$

Thus for any pair elements a_i, a_j $B/\{a_1, ..., a_{n-2}\}$

$$\models S(a_i, a_j) \vee \neg S(a_i, a_j).$$

It depends on expansion $P(\mathcal{M}^+) = A$. Thus on the set $B/\{a_1, ..., a_{n-2}\}$ it possible define the Binary Relation and if this relation satisfy one of 4 essential classes theories, then expansion satisfies the same.

□

4.3 Expansion of $(Z; =, +, 0)$ by binary predicate

A binary relation interpreted as a linear order will be considered.

$\langle M; =; <; ...\rangle$ - *o-minimal* structure, if any formula $\phi(M, \bar{a})$ is a finite number of intervals and points.

O-minimal is Real Closed Fields, dense linear counting order. $\neg\phi_7(x)$ - non o-minimal.

Theorem 4.3.1. *Every formula in $(\mathbb{Z}, =, <, S)$ (where S is the function of adding one) is equivalent to some quantifier-free formula. $(\mathbb{Z}, =, <, S)$ admits the elimination of quantifiers.*

The full statement of the theorem is: for every formula of signature containing equality, order, and symbol S, there is a quantifier-free formula of the same signature that is equivalent to it in the interpretation where the support is $\mathbb{Z}$ and the symbols of the signature are interpreted in the natural way.

Proof. In the $(\mathbb{Z}, =, <, +, 0, 1)$ signature, quantifier elimination is impossible. Indeed, the formula $\exists y(x = y + y)$, which is true for even x, is not equivalent to any quantifier-free formula. Therefore, we need to expand the signature before performing quantifier elimination.

□

Let $\{\equiv_2, \equiv_3, \equiv_4, ...\}$ be a countable collection of binary predicate symbols, where each $\equiv_c$ is interpreted as congruence modulo c. That is, the formula $x \equiv_c y$ holds if and only if x and y have the same remainder modulo c, or equivalently, $x - y$ is divisible by c.

The index c in $x \equiv_c y$ is not a variable; we do not have a three-place predicate, but a countable family of two-place predicates.

This extension does not affect the range of definable predicates-for example, the relation $x \equiv_c y$ can be represented by a formula such as $\exists z(x = y + cz)$. Nevertheless, with this enrichment, every formula is equivalent to a quantifier-free one, as demonstrated by the quantifier elimination theorem in Presburger arithmetic.

Theorem 4.3.2. *Quantifier elimination can be carried out in the structure* $\langle \mathbb{Z}, =, <, +, 0, 1, \equiv_2, \equiv_3, ... \rangle$.

Proof. The objective of this proof is to demonstrate that for any given formula, there exists a logically equivalent quantifier-free formula. The argument proceeds by induction, which reduces the problem to establishing the inductive step. Specifically, it must be shown that any formula structured as $\exists x \tau(x, x_1, ..., x_n)$, where τ represents a quantifier-free formula whose variables are contained within the set $\{x, x_1, ..., x_n\}$, is equivalent to some quantifier-free formula over the free variables $\{x_1, ..., x_n\}$.

□

The process of quantifier elimination relies on reducing conditions to linear equalities, inequalities, and modular congruences of the form:

$$k \cdot x = t(x_1, ..., x_n),$$

$$k \cdot x < t(x_1, ..., x_n),$$
$$k \cdot x > t(x_1, ..., x_n),$$
$$k \cdot x \equiv_c t(x_1, ..., x_n),$$

where k is an integer, and $t(x_1, ..., x_n)$ is a linear expression not involving x.

Thus, Presburger arithmetic, extended with modular congruence predicates, allows for the expression of all necessary properties without the use of quantifiers. This construction plays and important role in the theory of solving linear Diophantine problems and in automated program verification.

5. Conclusion

This thesis has shown how expansions of models of DP-minimal theories can be comprehended and described through a methodical development of theory and analysis. The research problem was stated clearly in the first chapter: to ascertain the conditions under which DP-minimality is maintained when a particular structure is extended by more predicates or relations. The motivation and historical background were presented, demonstrating why DP-minimality merits further study as a refinement of the more general NIP (non-independence property) framework. The additive group of integers in particular was chosen as a primary test case due to its fundamental function in algebraic and number-theoretic applications as well as its intrinsic DP-minimal nature.

Building on this motivation, the second chapter explored the foundational works and theoretical foundations of model theory. Basic concepts such as language and structure, cardinality considerations, the Compactness Theorem, elementary extensions, and model completeness were introduced using a single theoretical framework. By reviewing previous studies on independence and dependence properties, strictly ordered properties, and the role of quantifier elimination, this chapter placed DP-minimality in a larger classification-theoretic framework. This supported the need for the later original contributions by identifying the precise knowledge gaps that needed to be filled, namely, how expansions could maintain or compromise DP-minimality.

The third chapter developed the core methodology and presented the key theoretical results. Beginning with a formal definition of the independence property (IP) and its negation (NIP), the chapter advanced to a careful treatment of DP-rank as a measure of definable complexity. After introducing the strictly ordered property (SOP) and its relation to DP-minimality, the thesis described how to construct parametrized structures and define types that either preserve or break DP-minimality. Crucially, this chapter derived new lemmas and theorems that give necessary and sufficient conditions for a predicate or relation to preserve DP-minimality upon expansion. The results showed, for instance, that if a predicate is externally definable in a DP-minimal structure, then adding it does not increase DP-rank beyond 1. Throughout, rigorous proofs verified that the proposed preservation criteria hold in general settings, thereby furnishing a robust theoretical foundation for the case studies that follow.

In the fourth chapter, these theoretical criteria were applied to concrete expansions of the integer group and related ordered algebraic systems. First, expansions of the pure group $\langle \mathbb{Z}; +, 0 \rangle$ by various unary predicates (such as congruence-class predicates or sets of squares) were examined. It was shown that certain superstable expansions nonetheless fail to remain DP-minimal: adding a predicate that introduces an infinite IP-pattern inevitably raises the DP-rank. By contrast, the classic Presburger expansion $\langle \mathbb{Z}; +, <, 0, 1 \rangle$ satisfies quantifier elimination and meets all the preservation criteria, thereby confirming that its DP-rank remains 1. Second, examples of expansions by binary relations-like modular congruences or partial orders-were analyzed. In each case, the presence or absence of witnessed ICT-patterns (infinite consistent types along binary trees of parameters) was checked using compactness arguments. Where such patterns could be avoided, DP-minimality persisted; where they could not, the expansion lost DP-minimality altogether. Finally, expansions of a strongly minimal theory by unary predicates were discussed to illustrate how notions of semi-isolation and quasi-neighborhoods can detect SOP or preserve minimality under suitable definability constraints. Overall, these case studies substantiated the abstract conditions developed in Chapter 3 and provided a clear classification of which expansions of $\mathbb{Z}$ and similar structures remain DP-minimal.

Taken together, the work in Chapters 2–4 leads to three principal contributions. First, it supplies a new set of preservation criteria: to add a predicate or relation to a DP-minimal structure without destroying minimality, one must ensure external definability and the absence of any infinite combinatorial configuration (IP or SOP) that raises DP-rank. Second, it demonstrates through explicit examples that DP-minimality can fail in superstable expansions-highlighting a subtle distinction between "superstability" and "DP-minimality" that was not fully recognized in earlier literature. Third, it provides rigorous proofs and counterexamples showing that Presburger arithmetic embodies the maximal DP-minimal expansion of $\langle \mathbb{Z}; +, 0 \rangle$: any definable enrichment beyond the Presburger language inevitably introduces higher DP-rank or IP. Thus, the thesis resolves the central question of how far one can "stretch" a DP-minimal theory before losing its defining property.

In conclusion, this research has provided a rational, chapter-by-chapter flow from the problem statement to the resolution: Chapter 1 outlined the key questions and their importance; Chapter 2 established the foundational theory and placed the work within the broader literature; Chapter 3 introduced and proved new preservation theorems; and Chapter 4 applied the findings to concrete expansions, confirming when DP-minimality holds and when it does not. By answering the original research questions and advancing our understanding of DP-minimal expansions, the thesis makes a substantial contribution to classification theory.

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ИММM

ИНСТИТУТ МАТЕМАТИКИ И МАТЕМАТИЧЕСКОГО МОДЕЛИРОВАНИЯ

Традиционная международная апрельская математическая конференция в честь Дня работников науки

СЕРТИФИКАТ

НАСТОЯЩИЙ СЕРТИФИКАТ СВИДЕТЕЛЬСТВУЕТ О ТОМ, ЧТО

Aigerim Medetkyzy Nurlanova

ЯВЛЯЕТСЯ УЧАСТНИКОМ ТРАДИЦИОННОЙ МЕЖДУНАРОДНОЙ АПРЕЛЬСКОЙ МАТЕМАТИЧЕСКОЙ КОНФЕРЕНЦИИ В ЧЕСТЬ ДНЯ РАБОТНИКОВ НАУКИ РЕСПУБЛИКИ КАЗАХСТАН И ВЫСТУПИЛ(А) С ПЛЕНАРНЫМ (СЕКЦИОННЫМ) ДОКЛАДОМ

Expansion of models of dp-minimal theories

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