Suleyman Demirel University
Engineering Faculty

NARTOVA D.S.

CALCULUS

Brief summary and exercises.

Part 1

Almaty –2005

Suleyman Demirel University
Engineering Faculty

NARTOVA D. S.

CALCULUS

Brief summary and exercises.

Part 1

Almaty – 2005

ББК 22.1я7
N 23

Recommended to print by Academic Council of Suleyman Demirel University.

Nartova D.S.
Calculus. Brief summary and exercises. Part 1. -Almaty, 2005 - 200 p.

ISBN № 9965-453-02-0

ББК 22.1я7

N 1602000000
00(05)-06

ISBN № 9965-453-02-0

CONTENTS

Topic 1. A Function.

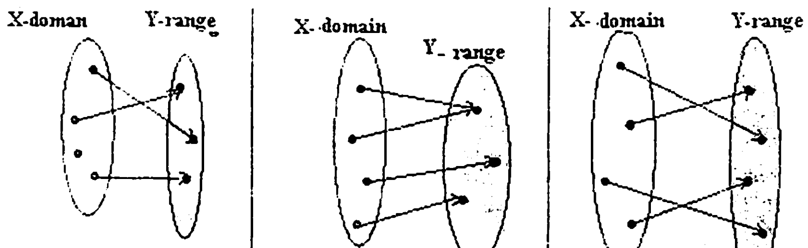
1.1 Definition. 1.2. Ways of representation. 1.3. The classification of the functions. 1.4. Basic properties of the functions.

1.1 Definition.

Let D and R are some sets.

A function is composed of a domain set D, a range set R, and **a rule of correspondence** that assigns to **each** element x of the domain D **exactly one** element y=f(x) of the range R.

The **domain** of function y=f(x) is called the set of independent variables x where the function exists. The **range** is the set of dependent variables y.

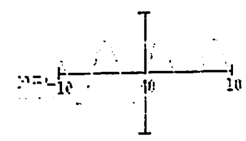


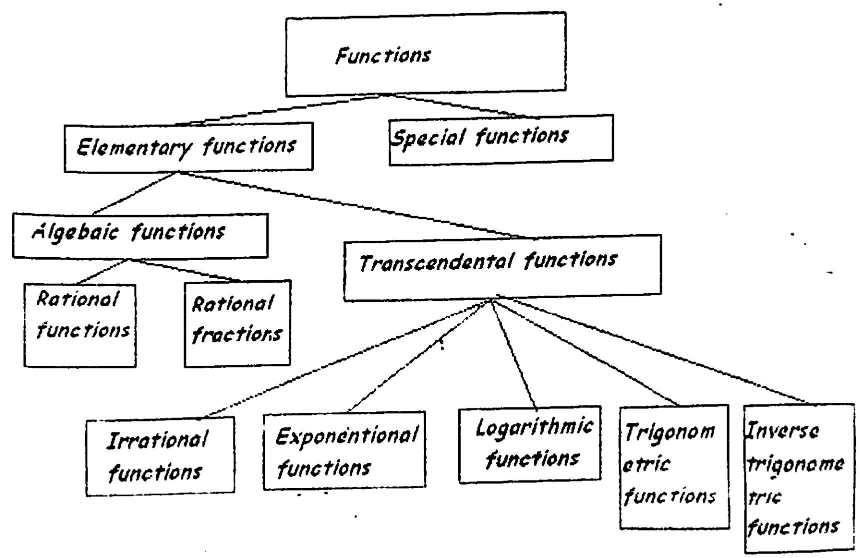
it is not function | it is not function | it is a function

1.2. Ways of representation.

1) **Table** - Representation the set of corresponding pairs.
2) **Graph** - The set of points which coordinates satisfy to the rule of function.
3) **Analytic**-The formulae.

1.3. The classification of the functions.

Me - thods	Advantages	Disadvantages	Example
Table	1)exact value; 2) without calculations	1) not visual; 2) discrete value(disconti nuous)	x: $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$, π; y: -1, 0, 1, 0
Graph	Most visual	1)Approximate values; 2) need calculations.	
Analy tical	1)exact value; 2)total analysis.	1)not visual; 2)need calculations.	$y = \sin(x)$



1.4. Basic properties of the functions.

1.4.1. Even and odd functions.

Function y=f(x) is called **the even function** if for any x from domain **f(-x) = f(x).**

Graphs of even function are symmetrical about Oy axis.

Function y=f(x) is called **the odd function** if for any x from domain **f(-x) = - f(x).**

Graphs of even function are symmetrical about origin O(0,0) point.

If function doesn't satisfy both conditions it will be **neither** function.

Graphs of neither functions is asymmetric.

1.4.2. Periodicity of a function.

Function y = f(x) is called **periodical function** with period $T \neq 0$ if for any x from domain **f(x+T) = f(x).**

1.4.3. Boundedness of a function.

Function y=f(x) is called **bounded above** over the interval X if exist such M>0 that $|f(x)| \leq M$ for any $x \in X$.

Function y=f(x) is called **bounded below** over the interval X if exist such M>0 that $|f(x)| \geq M$ for any $x \in X$.

1.4.4. Monotonicity of a function.

Function y=f(x) is called **monotonic** over the interval X if the function is only **increasing** or only **decreasing** for any x 2$\in$ X.

Function y=f(x) is called **increasing** over the interval X if $f(x_1) > f(x_2)$ for any $x_1 > x_2 \in X$.

Function y=f(x) is called **decreasing** over the interval X if $f(x_1) < f(x_2)$ for any $x_1 > x_2 \in X$.

1.4.5. Implicit and explicit functions

Function is called **implicit function** if for any x from domain dependent variable y couldn't be expressed from independent variable: **F(x,y) = 0.**

Function y=f(x) is called **explicit function** if for any x from domain dependent variable is expressed from independent variable: **y = f(x).**

1.4.6. Inverse functions.

Let given the function $y = f(x)$, where $x \in X$ and $y \in Y$.

If for any $x \in X$ exists a single $y \in Y$ such that $y = f(x)$ and if for any $y \in Y$ exists a single $x \in X$ such that $x = g(y)$ then $y = f(x)$ and $x = g(y)$ are called **mutually inverse.**

An inverse function exists for any strictly monotonic function.

The graphs of mutually inverse functions are symmetric about bisector of the first coordinate quarter.

1.4.7. Composite functions.

Let $y = f(u)$ is a function where $u \in U$ is its domain and $y \in Y$ is its range. In its turn **u** is the function $u = \phi(x)$ where $x \in X$ is its domain and $u \in U$ is its range. Then the function $y = f(\phi(x))$ is called a **composite function** with X as domain and Y as range.

EXERCISES I

1

Find the domain of the function:

1. $y = \sqrt{x} + \lg(2x+5)$;

2. $y = \dfrac{1}{\lg(1-x)} + \sqrt[3]{x+2}$;

3. $y = \dfrac{\sqrt{x+4}}{\lg(2-x)}$;

4. $y = \dfrac{\ln(x-3)}{\sqrt{5-x}}$;

16. $y = \dfrac{\sqrt{x-3}}{\lg(1-x)}$;

17 $y = \arcsin(5x^2 + 3x)$

18. $y = \dfrac{x-2}{x^2-9} + \ln(x+7)$;

19. $y = \arccos(5x^2 - x)$;

5. $y=\frac{4x-3}{x^2-1}+\ln(x+5)$;

6 $y=\arcsin(3x^2-x)$;

7. $y=\ln\frac{1+x}{1-x}$;

8. $y=\ln\frac{x+4}{x-4}$;

9. $y=\ln\frac{4x-3}{x^2-1}$;

10. $y=\frac{x+4}{x-5}+\ln(x+1)$;

11. $y=\frac{\ln(x+4)}{x^2-25}$;

12. $y=\frac{\ln(x-4)}{x^3-9}$;

13. $y=\frac{x+4}{\sqrt{(x-3)}}+\ln(1-x)$;

14. $y=\arccos(4x^2-x)$;

15. $y=\ln\frac{x+4}{x^2-9}$;

20. $y=\ln\frac{3+x}{2-x}$;

21. $y=\ln\frac{5+x}{3-x}$;

22. $y=\frac{\sqrt{x+5}}{\lg(1-x)}$;

23. $y=\arccos(-2x^2+4x)$;

24. $y=\frac{\ln(x-8)}{\sqrt{1-x^2}}$;

25. $y=\frac{\ln(x-6)}{\sqrt{3-x}}$;

26. $y=\frac{\ln(x-4)}{\sqrt{25-x^2}}$;

27. $y=\frac{\ln(x-2)}{\sqrt{81-x^2}}$;

28. $y=\frac{x+3}{\sqrt{(x-5)}}+\ln(8-x)$;

29. $y=\arcsin(2x^2+x)$;

30. $y=\sqrt{x-3}+\lg(4-2x)$.

2

Determine whether the function even, odd or neither:

1. $y=x^3\sin x+2x^2$;
2. $y=x^2\cos x-x$;
3 $y=x^2\sin x+x$;

16. $y=x^2\sin x-x$;
17. $y=x\sin x-x^2$;
18. $y=x^2\sin x-x^3$;

4. $y=x^3\sin x^2$;
5. $y=x\sin x+x$;
6. $y=\frac{x^3}{\sin x}$;
7. $y=\frac{x^2+x^4}{\sin^2 x}$;
8. $y=\frac{x^3}{\sin x}$;
9. $y=\frac{x^3}{x\cos x}$;
10. $y=\frac{x^3}{x\sin x}$;
11. $y=\frac{x^3}{\sin x}+x$;
12. $y=\frac{x^3}{x\cos x}$;
13. $y=\frac{x^2}{\sin^2 x}-x^4$;
14. $y=\frac{\sqrt{x^2-4}}{\cos x}$;
15. $y=\frac{\sqrt{x^2-4}}{\sin x+x}$;

19. $y=x\cos x-x^3$;
20. $y=\frac{x^2-4}{x\sin^2 x}$;
21. $y=\tan x\cdot\frac{x\cos x}{x^3}$;
22. $y=\tan x\cdot x-\cos x$;
23. $y=\frac{x^3}{x\sin x}+x$;
24. $y=\frac{x^3}{\tan x}$;
25. $y=x^2\sin x-\tan x$;
26. $y=\frac{\sqrt{x^2-4}}{\sin x}+\cot x$;
27. $y=\frac{x^3}{x\cos x}+\cot(x^2)$;
28. $y=\frac{\sqrt{x^2-4}}{\tan x}-\sin x$;
29. $y=\frac{\sqrt{x^2-4}}{\sin x}+\cot x$;
30. $y=\frac{x^3}{\cos x}+x$.

3

Sketch the graph of the function:

1. $y=-2(x+3)^2+1$;
2. $y=3(x-2)^2+4$;
3. $y=-(x-2)^2-3$;

16. $y=2\ln(x+1)-4$;
17. $y=3\cdot 2^{(x+4)}-1$;
18. $y=2\cdot\ln(x-2)+3$;

4. $y = -(x-4)^2+2$;
5. $y = 2(x-5)^2-5$;
6. $y = -2(x-4)^2+2$;
7. $y=\dfrac{3}{x-1}-2$;
8. $y=\dfrac{2}{x-1}-1$;
9. $y=\dfrac{-1}{x-1}+3$;
10. $y=\dfrac{1}{x-1}-5$;
11. $y=\dfrac{5}{x-2}+4$;
12. $y=\dfrac{2}{-x+1}-3$;
13. $y=\dfrac{4}{x-2}-3$;
14. $y=\dfrac{-2}{x-3}-3$;
15. $y=\dfrac{2}{x-1}-3$;

19. $y = 4\cdot 3^{(x-1)}+5$;
20. $y = 3\lg(x-1)+2$;
21. $y = -\ln(x-1)+4$;
22. $y = 1/2\cdot\ln(x-2)+2$;
23. $y = -2(x+3)^3+1$
24. $y = 3(x-1)^2+1$;
25. $y = 2\lg(x+4)-2$;
26. $y = 3\cdot\lg(x+2)-4$;
27. $y = -\lg(x+2)+4$;
28. $y = -3^{(x+1)}-5$;
29. $y = 3\cdot 2^{(x-1)}-1$;
30. $y = -4^{(x-1)}-3$;

Topic 2. Limit of the function.

2.1 Definition the limit of function at the point. 2.2. One-sided and two-sided limits. 2.3 Limits involving infinity. 2.4. Limits rules. 2.5. The signs of existence of limits. Sandwich theorem. 2.6. Infinitesimal its properties. Equivalent infinitesimals. 2.7. Infinite quantity, its properties and relation. 2.8. List of indeterminacy. 2.9. First fundamental limit and second fundamental limit.

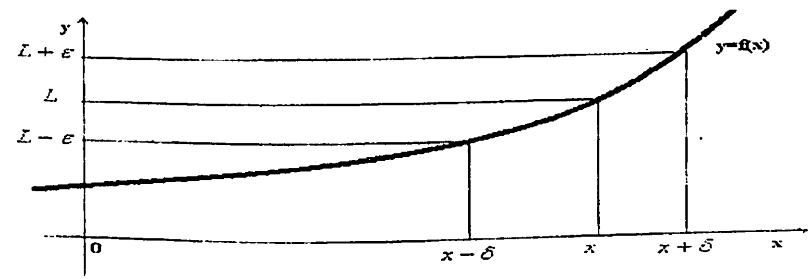
2.1 Definition the limit of function at the point.

The **limit** of f(x) as x approaches x_0 is the number L if the following criterion holds:

Given any radius $\varepsilon>0$ about L there exists a radius $\delta>0$ about x_0 such that for all x,

$$|x - x_0| < \delta \text{ implies } |f(x) - L| < \varepsilon.$$

This fact is denoted by $\lim\limits_{x\to x_0} f(x) = L$.



2.2. Two-sided limits.

2.2.1. The number L is called the **right - hand limit** of f(x) as x approaches x_0 from right side if the following criterion holds:

Given any radius $\varepsilon>0$ about L there exists a radius $\delta>0$ about x_0 such that for all $x\in(x_0, x_0+\delta)$ implies $|f(x) - L| < \varepsilon$.

This fact is denoted by $\lim\limits_{x\to x_0^+} f(x) = L$.

2.2.2. The number L is called the **left- hand limit** of f(x) as x approaches x_0 from left side if the following criterion holds:

Given any radius $\varepsilon>0$ about L there exists a radius $\delta>0$ about x_0 such that for all $x\in(x_0-\delta, x_0)$ implies $|f(x) - L| < \varepsilon$.

This fact is denoted by $\lim_{x \to x_0^-} f(x) = L$.

2.2.3. Theorem:
A function f(x) has a limit as x approach x_0 if and only if the right-hand limit and left-hand limit at x_0 exist and are equal. In symbols,

$$\lim_{x \to x_0} f(x) = L \quad \Leftrightarrow$$

$$\lim_{x \to x_0^+} f(x) = L \quad \text{and} \quad \lim_{x \to x_0^-} f(x) = L$$

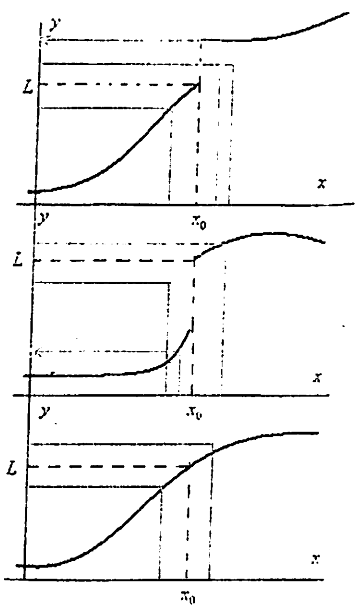
I. Left-hand limit
$\lim_{x \to x_0^-} f(x) = L$

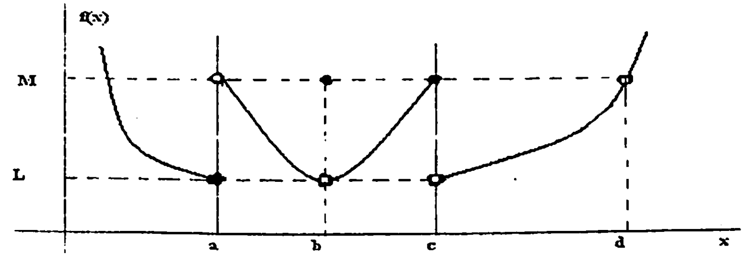
II. Right-hand limit
$\lim_{x \to x_0^+} f(x) = L$

III. $\lim_{x \to x_0} f(x) = L$

exists if and only if

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L$$

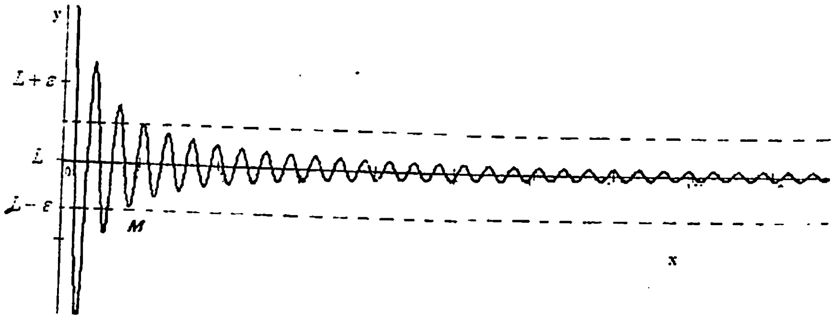




	Left-hand limit	Right-hand limit	Limit at point	Value of function
x=a	$\lim_{x\to a^-} f(x) = \mathbf{L}$	$\lim_{x\to a^+} f(x) = \mathbf{M}$	$\lim_{x\to a} f(x)$ doesn't exist	f(a)=L
x=b	$\lim_{x\to b^-} f(x) = \mathbf{L}$	$\lim_{x\to b+} f(x) = \mathbf{L}$	$\lim_{x\to b} f(x) =$ =L	f(b)=M
x=c	$\lim_{x\to c^-} f(x) = \mathbf{M}$	$\lim_{x\to c+} f(x) = \mathbf{L}$	$\lim_{x\to c} f(x)$ doesn't exist	f(c)=M
x=d	$\lim_{x\to d^-} f(x) = \mathbf{M}$	$\lim_{x\to d+} f(x) = \mathbf{M}$	$\lim_{x\to d} f(x) =$ =M	f(d)-undefined

*2.3. **Limit involving infinity:*** $\lim_{x \to +\infty} f(x) = L$ *and* $\lim_{x \to -\infty} f(x) = L$.

2.3.1. The limit of the function f(x) as x approaches infinity is the number L if following criterion holds:
Given any number $\varepsilon > 0$, there exists a number M such that for all x, x > M implies $|f(x) - L| < \varepsilon$. This fact is denoted by $\lim_{x \to +\infty} f(x) = L$.



2.3.2. The **limit of the function f(x) as x approaches negative infinity** is the number L if following criterion holds:
Given any number $\varepsilon>0$, there exists a number N such that for all x, x <N implies $|f(x)-L|<\varepsilon$. This fact is denoted by $\lim\limits_{x\to-\infty} f(x)=L$.

2.3.2. <u>Theorem.</u>

A function f(x) has a limit as x approach ∞: $\lim\limits_{x\to\infty} f(x)=L$ if and only if the $\lim\limits_{x\to-\infty} f(x)=L$ and $\lim\limits_{x\to+\infty} f(x)=L$ exist and are equal.

2.4. Limits rules.

Let $\lim\limits_{x\to c} f(x)=A$ and $\lim\limits_{x\to c} g(x)=B$ where **c** can be **x_0** (at point) or $\pm\infty$ (involving infinity).(A,B- finite values).

1) $\lim\limits_{x\to c} C=C$;

2) $\lim\limits_{x\to c} C\cdot f(x)=C\cdot\lim\limits_{x\to c} f(x)=C\cdot A$;

3) $\lim\limits_{x\to c}(f(x)\pm g(x))=\lim\limits_{x\to c} f(x)\pm\lim\limits_{x\to c} g(x)=A\pm B$;

4) $\lim\limits_{x\to c}(f(x)\cdot g(x))=\lim\limits_{x\to c} f(x)\cdot\lim\limits_{x\to c} g(x)=A\cdot B$;

5) $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to} g(x)}=\dfrac{A}{B}$ if $B\neq0$;

6) if $\lim\limits_{u\to u_0} f(u)=A$ and $\lim\limits_{x\to c}\phi(x)=u_0$ then the limit of the composite function is: $\lim\limits_{x\to c} f(\phi(x))=A$.

2.5. The signs of existence of limits. Sandwich theorem.

2.5.1. <u>Theorem:</u>
If the function is **monotonic** and **bounded** function then it has a limit.
2.5.2. <u>Sandwich theorem</u>.
Suppose that $g(x)\le f(x)\le h(x)$ for all $x\neq c$ in some interval about c and that $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} h(x)=L$. Then $\lim\limits_{x\to c} f(x)=L$.

2.6. Infinitesimal its properties. Equivalent infinitesimals.

2.6.1. Infinitesimal
Let x approaches c where c is either x_0 or ∞.
A function $y=\alpha(x)$ is called **infinitesimal as x→c** if and only if $\lim\limits_{x\to c}\alpha(x)=0$.

Theorem: $\lim\limits_{x\to c} f(x)=L \Leftrightarrow f(x)=L+\alpha(x)$.

2.6.2. Properties of operations with infinitesimals:

1) If $\alpha_i(x)$ are **infinitesimals** as x→c for any $i=\overline{1,n}$. $\Rightarrow$

$\sum\limits_{i=1}^{n}\alpha_i=\alpha(x)$ is **infinitesimal** as x→c.

2) If $\alpha_i(x)$ are **infinitesimals** as x→c for any $i=\overline{1,n}$ $\Rightarrow$

$\prod\limits_{i=1}^{n}\alpha_i(x)$ is **infinitesimal** as x→c.

3) If $\alpha(x)$ is **infinitesimal** as $x \to c$ and f(x) is **bounded** function over vicinity of c $\Rightarrow$ $\alpha(x) \cdot f(x)$ is **infinitesimal** as $x \to c$.

4) If $\alpha(x)$ is **infinitesimal** as $x \to c$ and $\lim_{x \to c} f(x) \neq 0$ $\Rightarrow$ $\dfrac{\alpha(x)}{f(x)}$ is **infinitesimal** as $x \to c$.

5) $$\lim_{x \to c} \frac{\alpha_1(x)}{\alpha_2(x)} = \begin{cases} \mathbf{0}, & \text{if } \alpha_1(x) = o(\alpha_2(x)),\ \alpha_1(x) \text{ more infinitesimal than } \alpha_2(x); \\ \mathbf{L} \neq 0, & \text{if } \alpha_1(x) \text{ and } \alpha_2(x) \text{ are the same degree infinitesimals;} \\ \infty, & \text{if } \alpha_2(x) = o(\alpha_1(x)),\ \alpha_2(x) \text{ more infinitesimal than } \alpha_1(x). \end{cases}$$

2.6.3. Equivalent infinitesimals and their properties.

If $\lim_{x \to c} \dfrac{\alpha_1(x)}{\alpha_2(x)} = 1$ then $\alpha_1(x)$ and $\alpha_2(x)$ are called **equivalent infinitesimals** and is denoted by $\alpha_1(x) \sim \alpha_2(x)$.

Properties:

1) $\alpha(x) \sim \alpha(x)$;

2) If $\alpha_1(x) \sim \alpha_2(x)$ then $\alpha_2(x) \sim \alpha_1(x)$;

3) If $\alpha_1(x) \sim \alpha_2(x)$ and $\alpha_2(x) \sim \alpha_3(x)$ then $\alpha_1(x) \sim \alpha_3(x)$.

2.6.4. The basic list of equivalent infinitesimals.

1) $\sin\alpha(x) \sim \alpha(x)$;
2) $\tan\alpha(x) \sim \alpha(x)$;
3) $\operatorname{asin}\alpha(x) \sim \alpha(x)$;
4) $\operatorname{atan}\alpha(x) \sim \alpha(x)$;
5) $\log_b(1+\alpha(x)) \sim \alpha(x)/\ln b$ $(b>0, b \neq 1)$, in particular, $\ln(1+\alpha(x)) \sim \alpha(x)$;
6) $b^{\alpha(x)} - 1 \sim \alpha(x)\ln b$ $(b>0)$, in particular $e^{\alpha(x)} - 1 \sim \alpha(x)$;
7) $(1+\alpha(x))^m - 1 \sim m\alpha(x)$ $(m \in R)$.

2.7. Infinite quantity, its properties and relation.

2.7.1. Infinite quantity

Let x approaches c where c is either x_0 or ∞.
A function $y = F(x)$ is called **infinite quantity as $x \to c$** if and only if $\lim_{x \to c} F(x) = \infty$.

2.7.2. Properties of operations with infinite quantities:

1) If $F_i(x)$ are **infinite quantities of the same sign** as $x \to c$ for any $i=\overline{1,n}$ $\Rightarrow$ $\sum_{i=1}^{n} F_i = F(x)$ is **infinite quantity** as $x \to c$.
2) If $F(x)$ is **infinite quantity** as $x \to c$ and g(x) is bounded over some vicinity c $\Rightarrow$ F(x)+g(x) is **infinite quantity** as $x \to c$.
3) If $F(x)$ is **infinite quantity** as $x \to c$ and $g(x) \geq C > 0$ over some vicinity c $\Rightarrow$ F(x)·g(x) is **infinite quantity** as $x \to c$.
4) If $F(x)$ is **infinite quantity** as $x \to c$ and $g(x) \neq 0$ is bounded over some vicinity c $\Rightarrow$ F(x)/g(x) is **infinite quantity** as $x \to c$.

2.7.3. Relation between infinitesimal and infinite quantity.

<u>**Theorem**</u>: If y = F(x) is **infinite quantity** as $x \to c$ then $y = \dfrac{1}{F(x)}$ is **infinitesimal** as $x \to c$.

$(\dfrac{1}{\infty} \sim 0)$

Theorem: If $y = \alpha(x)$ is **infinitesimal** as $x \to c$ then $y = \dfrac{1}{\alpha(x)}$ is **infinite quantity** as $x \to c$.

$\left(\dfrac{1}{0} \sim \infty\right)$.

2.8. List of indeterminacy.

$\dfrac{0}{0}, \dfrac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^{0}, \infty^{0}, 1^{\infty}$

2.9. First fundamental limit and second fundamental limit.

2.9.1.. First fundamental limit. $\left(\dfrac{0}{0}\right)$:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 .$$

2.9.2. Second fundamental limit. (1^{∞}) :

a) $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} = e$ or b) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^{x} = e$,

here e – is constant $e \approx 2{,}718282\ldots$

2.9.3. Short-cut formula for limit of the rational fraction: $\left(\dfrac{\infty}{\infty}\right)$

$$\lim_{x \to \infty} \frac{P_n(x)}{Q_m(x)} = \lim_{x \to \infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0} = \left|\frac{\infty}{\infty}\right| =$$

$$= \begin{cases} \dfrac{a_n}{b_m}, & \textbf{if } \mathbf{n = m}; \\ 0, & \textbf{if } \mathbf{n < m}; \\ \infty, & \textbf{if } \mathbf{n > m}. \end{cases}$$

EXERCISE II.

Find the limits:

1

1. $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 12x + 20}$.
2. $\lim\limits_{x \to 0} \dfrac{x^3 - x^2 + 2x}{x^2 + x}$.
3. $\lim\limits_{x \to 3} \dfrac{6 + x - x^2}{x^3 - 27}$.
4. $\lim\limits_{x \to 1} \dfrac{2x^2 - x - 1}{3x^2 - x - 2}$.
5. $\lim\limits_{x \to 2} \dfrac{2x^2 - 7x + 4}{x^2 - 5x + 6}$.
6. $\lim\limits_{x \to 3} \dfrac{12 + x - x^2}{x^3 - 27}$.
7. $\lim\limits_{x \to 1/3} \dfrac{3x^2 + 2x - 1}{27x^3 - 1}$.
8. $\lim\limits_{x \to -1} \dfrac{x^2 - 4x - 5}{x^2 - 2x - 3}$.
9. $\lim\limits_{x \to -1} \dfrac{3x^2 + 2x - 1}{-x^2 + x + 2}$.
10. $\lim\limits_{x \to 3} \dfrac{3x^2 - 11x + 6}{2x^2 - 5x - 3}$.
11. $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^2 + x - 6}$.
12. $\lim\limits_{x \to -1} \dfrac{x^2 - x - 2}{x^3 + 1}$.
13. $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x^2 + x - 20}$.
14. $\lim\limits_{x \to -3} \dfrac{4x^2 + 11x - 3}{x^2 + 2x - 3}$.
15. $\lim\limits_{x \to 3} \dfrac{3x^2 - 7x - 6}{2x^2 - 7x + 3}$.
16. $\lim\limits_{x \to -2} \dfrac{4x^2 + 7x - 2}{3x^2 + 8x + 4}$.
17. $\lim\limits_{x \to -1} \dfrac{5x^2 + 4x - 1}{3x^2 + x - 2}$.
18. $\lim\limits_{x \to -1} \dfrac{x^2 - 4x - 5}{3x^2 + 2x - 2}$.
19. $\lim\limits_{x \to -1} \dfrac{7x^2 + 4x - 3}{2x^2 + 3x + 1}$.
20. $\lim\limits_{x \to 4} \dfrac{3x^2 - 3x + 2}{x^2 - x - 12}$.
21. $\lim\limits_{x \to 2} \dfrac{2x^2 - 9x + 10}{x^2 + 3x - 10}$.
22. $\lim\limits_{x \to 1} \dfrac{4x^2 + x - 5}{x^2 - 2x + 1}$.
23. $\lim\limits_{x \to 2} \dfrac{-5x^2 + 11x - 2}{3x^2 - x - 10}$.
24. $\lim\limits_{x \to 7} \dfrac{x^2 - 5x - 14}{2x^2 - 9x - 35}$.

25. $\lim\limits_{x\to 5}\dfrac{3x^2-6x-45}{2x^2-3x-35}$.

26. $\lim\limits_{x\to -3}\dfrac{4x^2+3x+15}{x^2-6x-27}$

27. $\lim\limits_{x\to -5}\dfrac{x^2-2x-35}{2x^2+11x+5}$.

28. $\lim\limits_{x\to -8}\dfrac{2x^2+15x-8}{3x^2+25x+8}$

29. $\lim\limits_{x\to 4}\dfrac{3x^2-2x-40}{x^2-3x-4}$.

30. $\lim\limits_{x\to -3}\dfrac{2x^2+5x-3}{3x^2+10x+3}$

2

1. $\lim\limits_{x\to -3}\dfrac{2x^2+11x+15}{3x^2+5x-12}$.

2. $\lim\limits_{x\to 1}\dfrac{2x^2+5x-10}{x^3-1}$

3. $\lim\limits_{x\to 1}\dfrac{x^3-3x+2}{x^2-4x+3}$.

4. $\lim\limits_{x\to 2}\dfrac{3x^2+2x+1}{x^3-8}$.

5. $\lim\limits_{x\to -1}\dfrac{x^4-x^2+x+1}{x^4+1}$.

6. $\lim\limits_{x\to 1}\dfrac{2x^2-3x-1}{x^4-1}$.

7. $\lim\limits_{x\to 2}\dfrac{x^2-x+3}{5x^2+3x-3}$.

8. $\lim\limits_{x\to -2}\dfrac{x^2+2x}{x^2+4x+4}$.

9. $\lim\limits_{x\to -1}\dfrac{x^2-1}{x^2+3x+2}$.

10. $\lim\limits_{x\to -4}\dfrac{2x^2+7x-4}{x^3+64}$.

11. $\lim\limits_{x\to -5}\dfrac{4x^2+19x-5}{2x^2+11x+5}$.

12. $\lim\limits_{x\to 1}\dfrac{x^3-x^2+x-1}{x^3+x-2}$.

13. $\lim\limits_{x\to 1}\dfrac{x^2-2x+1}{2x^2-7x+5}$.

14. $\lim\limits_{x\to 2}\dfrac{x^3-8}{2x^2-9x+10}$.

15. $\lim\limits_{x\to -2}\dfrac{9x^2+17x-2}{x^2+2x}$.

16. $\lim\limits_{x\to 1}\dfrac{x^3+x-2}{x^3-x^2-x+1}$.

17. $\lim\limits_{x\to 0}\dfrac{4x^3-2x^2+5x}{3x^2+7x}$.

18. $\lim\limits_{x\to 1}\dfrac{4x^4-5x^2+1}{x^2-1}$

19. $\lim\limits_{x\to 3}\dfrac{3x^2+5x-1}{x^2-5x+6}$.

20. $\lim\limits_{x\to -5}\dfrac{x^2-x-30}{x^3+125}$

21. $\lim\limits_{x\to 4}\dfrac{x^2+3x-28}{x^3-64}$.

22. $\lim\limits_{x\to 1/2}\dfrac{8x^3-1}{x^2-1/4}$.

23. $\lim\limits_{x\to 4}\dfrac{x^2+3x-28}{x^2-4x}$.

24. $\lim\limits_{x\to -2}\dfrac{3x^2+11x+10}{x^2-5x+14}$

25. $\lim\limits_{x\to -2}\dfrac{x^2-4}{3x^2+x-10}$.

26. $\lim\limits_{x\to 0}\dfrac{3x^2+x}{4x^2-5x+1}$.

27. $\lim\limits_{x\to 6}\dfrac{2x^2-11x-6}{3x^2-20x+12}$.

28. $\lim\limits_{x\to -6}\dfrac{x^2+2x-24}{2x^3+15x+18}$

29. $\lim\limits_{x\to 2}\dfrac{x^3-2x-4}{x^2-11x+18}$.

30. $\lim\limits_{x\to 4}\dfrac{x^3-64}{7x^2-27x-4}$.

3

5. $\lim\limits_{x\to \infty}\dfrac{x^3-4x^2+28x}{5x^3+3x^2+x-1}$.

6. $\lim\limits_{x\to \infty}\dfrac{3x^2+10x+3}{2x^2+5x-3}$.

7. $\lim\limits_{x\to \infty}\dfrac{-3x^4+x^2+x}{x^4+3x-2}$.

8. $\lim\limits_{x\to \infty}\dfrac{2x^2+7x+3}{5x^2-3x+4}$.

9. $\lim\limits_{x\to \infty}\dfrac{-x^2+3x+1}{3x^2+x-5}$.

10. $\lim\limits_{x\to \infty}\dfrac{x^3-3x^2+10}{7x^3+2x+1}$

11. $\lim\limits_{x\to \infty}\dfrac{4x^2+5x-7}{2x^2-x+10}$.

12. $\lim\limits_{x\to \infty}\dfrac{3x^4+2x+1}{x^4-x^3+2x}$.

13. $\lim\limits_{x\to \infty}\dfrac{3x^2+2x+9}{2x^2-x+4}$.

14. $\lim\limits_{x\to \infty}\dfrac{3x^2+5x-7}{3x^2+x+1}$.

15. $\lim\limits_{x\to \infty}\dfrac{2x^3+7x-2}{3x^3-x-4}$.

16. $\lim\limits_{x\to 0}\dfrac{18x^2+5x}{8-3x-9x^2}$.

17. $\lim\limits_{x\to \infty}\dfrac{3x^4-6x^2+2}{x^4+4x-3}$.

18. $\lim\limits_{x\to \infty}\dfrac{8x^2+4x-5}{4x^2-3x+2}$.

19. $\lim\limits_{x\to \infty}\dfrac{8x^4-4x^2+3}{2x^4+1}$.

20. $\lim\limits_{x\to \infty}\dfrac{3x^2-4x+2}{6x^2+5x+1}$.

21. $\lim\limits_{x\to \infty}\dfrac{7x^3+4x}{x^3-3x+2}$.

22. $\lim\limits_{x\to \infty}\dfrac{1+4x-x^4}{x+3x^2+2x^4}$.

23. $\lim\limits_{x\to \infty}\dfrac{2x^3+7x^2-2}{6x^3-4x+3}$.

24. $\lim\limits_{x\to \infty}\dfrac{3x+14x^2}{1+2x+7x^2}$.

25. $\lim\limits_{x\to \infty}\dfrac{x-2x^2+5x^4}{2+3x^2+x^4}$.

26. $\lim\limits_{x\to \infty}\dfrac{3x^4-2x^2-7}{3x^4+3x+5}$.

27. $\lim\limits_{x\to \infty}\dfrac{4-5x^2-3x^5}{x^5+6x+8}$.

28. $\lim\limits_{x\to \infty}\dfrac{5x^3-7x^2+3}{2+2x-x^3}$.

29. $\lim\limits_{x\to \infty}\dfrac{4x^3-2x+1}{2x^3+3x^2+2}$.

30. $\lim\limits_{x\to \infty}\dfrac{5x^2-3x+1}{3x^2+x-5}$.

1. $\lim\limits_{x\to-\infty} \dfrac{x^5-2x+4}{2x^4+3x^2+1}$.

2. $\lim\limits_{x\to\infty} \dfrac{3x^4+2x-5}{2x^2+x+7}$.

3. $\lim\limits_{x\to-\infty} \dfrac{3x^2+7x-4}{x^5+2x-1}$.

4. $\lim\limits_{x\to\infty} \dfrac{3x-x^6}{x^2-2x+5}$.

5. $\lim\limits_{x\to\infty} \dfrac{2x^3+7x-1}{3x^4+2x+5}$.

6. $\lim\limits_{x\to-\infty} \dfrac{2x^3+7x^2+4}{x^4+5x-1}$

7. $\lim\limits_{x\to-\infty} \dfrac{3x^6-5x^2+2}{2x^3+4x-5}$.

8. $\lim\limits_{x\to\infty} \dfrac{x^7+5x^2-4x}{3x^2+11x-7}$.

9. $\lim\limits_{x\to-\infty} \dfrac{7x^2+5x+9}{1+4x-x^3}$.

10. $\lim\limits_{x\to\infty} \dfrac{3x^4+x^2-6}{2x^2+3x+1}$.

11. $\lim\limits_{x\to-\infty} \dfrac{2x^2+5x+7}{3x^4-2x^2+x}$.

12. $\lim\limits_{x\to\infty} \dfrac{3x^3+4x^2-7x}{2x^2+7x-3}$.

13. $\lim\limits_{x\to-1} \dfrac{5x^3-3x^2+7}{2x^4+3x^2+1}$.

14. $\lim\limits_{x\to\infty} \dfrac{5x^2-3x+1}{1+2x-x^4}$

15. $\lim\limits_{x\to-\infty} \dfrac{2x^3+3x^2+5}{3x^2-4x+1}$.

16. $\lim\limits_{x\to\infty} \dfrac{6x^2-5x+2}{4x^3+2x-1}$.

17. $\lim\limits_{x\to-\infty} \dfrac{11x^3+3x}{2x^2-2x+1}$.

18. $\lim\limits_{x\to\infty} \dfrac{8x^2+3x+5}{4x^3-2x^2+1}$

19. $\lim\limits_{x\to-\infty} \dfrac{6x^3+5x^2-3}{2x^2-x+7}$.

20. $\lim\limits_{x\to\infty} \dfrac{3x^2+4x-7}{x^4-2x^3+1}$.

21. $\lim\limits_{x\to-\infty} \dfrac{8x^5-4x^3+3}{2x^3+x-7}$.

22. $\lim\limits_{x\to\infty} \dfrac{2x^2-7x+1}{x^3+4x^2-3}$.

23. $\lim\limits_{x\to-\infty} \dfrac{5x^4-2x^3+3}{2x^2+3x-7}$.

24. $\lim\limits_{x\to\infty} \dfrac{8x^3+x^2-7}{2x^2-5x+3}$.

25. $\lim\limits_{x\to-\infty} \dfrac{3x^4+2x^2-8}{8x^3-4x+5}$.

26. $\lim\limits_{x\to\infty} \dfrac{3x^4+2x-4}{3x^2-4x+1}$

27. $\lim\limits_{x\to-\infty} \dfrac{7x^3-2x+4}{2x^2+x-5}$.

28. $\lim\limits_{x\to\infty} \dfrac{4x^3+5x^2-3x}{3x^2+x-10}$.

29. $\lim\limits_{x\to-\infty} \dfrac{2x^2+10x-11}{3x^4-2x+5}$.

30. $\lim\limits_{x\to\infty} \dfrac{7x^3+3x-4}{2x^2-5x+1}$.

1. $\lim\limits_{x\to\infty} \dfrac{2x^2+3x-5}{7x^3-2x^2+1}$.

2. $\lim\limits_{x\to-\infty} \dfrac{3x^2-7x+2}{x^4+2x-4}$.

3. $\lim\limits_{x\to\infty} \dfrac{7x^4-3x+4}{3x^2-2x+1}$.

4. $\lim\limits_{x\to-\infty} \dfrac{2x^2-x+7}{3x^4-5x^2+10}$.

5. $\lim\limits_{x\to-\infty} \dfrac{4x^3-2x^2+x}{3x^2-x}$.

6. $\lim\limits_{x\to\infty} \dfrac{3x^4-2x+1}{3x^2+2x-5}$.

7. $\lim\limits_{x\to\infty} \dfrac{2x^2-5x+2}{x^4+3x^2-9}$.

8. $\lim\limits_{x\to-\infty} \dfrac{5x^2-4x+2}{4x^3+2x-5}$.

9. $\lim\limits_{x\to\infty} \dfrac{2x^3-3x^2+2x}{x^2+7x+1}$.

10. $\lim\limits_{x\to-\infty} \dfrac{3x^2-7x+5}{4x^5-3x^3+2}$

11. $\lim\limits_{x\to\infty} \dfrac{7x^5+6x^4-x^3}{2x^2+6x+1}$.

12. $\lim\limits_{x\to-\infty} \dfrac{4-3x-2x^2}{3x^4+5x}$

13. $\lim\limits_{x\to-\infty} \dfrac{7-3x^4}{2x^3+3x^2-5}$.

14. $\lim\limits_{x\to\infty} \dfrac{8x^4+7x^3-3}{3x^2-5x+1}$.

15. $\lim\limits_{x\to-\infty} \dfrac{3x+7}{2-3x+4x^2}$.

16. $\lim\limits_{x\to-\infty} \dfrac{2x^3-3x+1}{7x+5}$

17. $\lim\limits_{x\to\infty} \dfrac{10x-7}{3x^4+2x^3+1}$.

18. $\lim\limits_{x\to-\infty} \dfrac{5x^4-3x^2}{1+2x+3x^2}$

19. $\lim\limits_{x\to\infty} \dfrac{5x+3}{x^3-4x^2-x}$.

20. $\lim\limits_{x\to-\infty} \dfrac{3x^4+5x}{2x^2-3x-7}$.

21. $\lim\limits_{x\to\infty} \dfrac{2x^2-5x+3}{3x^4-2x^2+x}$.

22. $\lim\limits_{x\to-\infty} \dfrac{2x^5-x^3}{4x^2+3x-6}$

23. $\lim\limits_{x\to\infty} \dfrac{3x+1}{x^3-5x^2+4x}$.

24. $\lim\limits_{x\to-\infty} \dfrac{2-x-3x^2}{x^3-16}$.

25. $\lim\limits_{x\to\infty} \dfrac{4x^2-10x+7}{2x^3-3x}$.

26. $\lim\limits_{x\to-\infty} \dfrac{2x^3-3x+1}{x^5+4x^3}$

27. $\lim\limits_{x\to\infty} \dfrac{2x-13}{x^7-3x^5-4x}$.

28. $\lim\limits_{x\to-\infty} \dfrac{2x^2-3x+1}{x^3+2x^2+5}$

29. $\lim\limits_{x\to\infty} \dfrac{x^3-81}{3x^2+4x+2}$.

30. $\lim\limits_{x\to-\infty} \dfrac{7x+4}{3x^3-5x+1}$

6

1. $\lim\limits_{x\to 3}\dfrac{x^2+x-12}{\sqrt{x-2}-\sqrt{4-x}}$.

2. $\lim\limits_{x\to -4}\dfrac{\sqrt{x+12}-\sqrt{4-x}}{x^2+2x-8}$

3. $\lim\limits_{x\to -3}\dfrac{\sqrt{x+10}-\sqrt{4-x}}{2x^2-x-21}$.

4. $\lim\limits_{x\to -2}\dfrac{\sqrt{2-x}-\sqrt{x+6}}{x^2-x-6}$.

5. $\lim\limits_{x\to 1}\dfrac{\sqrt{3+2x}-\sqrt{x+4}}{3x^2-4x+1}$.

6. $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{\sqrt{5-x}-\sqrt{x+1}}$.

7. $\lim\limits_{x\to -1}\dfrac{3x^2+4x+1}{\sqrt{x+3}-\sqrt{5+3x}}$.

8. $\lim\limits_{x\to 4}\dfrac{2x^2-9x+4}{\sqrt{5-x}-\sqrt{x-3}}$.

9. $\lim\limits_{x\to 5}\dfrac{\sqrt{2x+1}-\sqrt{x+6}}{2x^2-7x-15}$.

10. $\lim\limits_{x\to -5}\dfrac{\sqrt{3x+17}-\sqrt{2x+12}}{x^2+8x+15}$.

11. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+2}-\sqrt{2}}{\sqrt{x^2+1}-1}$.

12. $\lim\limits_{x\to 0}\dfrac{\sqrt{7-x}-\sqrt{7+x}}{\sqrt{7x}}$.

13. $\lim\limits_{x\to 0}\dfrac{3x}{\sqrt{1+x}-\sqrt{1-x}}$.

14. $\lim\limits_{x\to 4}\dfrac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}}$

15. $\lim\limits_{x\to -1}\dfrac{\sqrt{5+x}-2}{\sqrt{8-x}-3}$.

16. $\lim\limits_{x\to 5}\dfrac{\sqrt{x+4}-3}{\sqrt{x-1}-2}$.

17. $\lim\limits_{x\to 7}\dfrac{\sqrt{x-3}-2}{\sqrt{x+2}-3}$.

18. $\lim\limits_{x\to 3}\dfrac{\sqrt{4x-3}-3}{x^2-9}$.

19. $\lim\limits_{x\to 3}\dfrac{\sqrt{5x+1}-4}{x^2+2x-15}$.

20. $\lim\limits_{x\to 0}\dfrac{2-\sqrt{x^2+4}}{3x^2}$.

21. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{\sqrt{x^2+16}-4}$.

22. $\lim\limits_{x\to 0}\dfrac{3x}{\sqrt{5-x}-\sqrt{5+x}}$

23. $\lim\limits_{x\to 9}\dfrac{\sqrt{2x+7}-5}{3-\sqrt{x}}$.

24. $\lim\limits_{x\to 4}\dfrac{2-\sqrt{x}}{\sqrt{6x+1}-5}$

25. $\lim\limits_{x\to 3}\dfrac{x^3-27}{\sqrt{3x}-x}$.

26. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+3x^2}-1}{x^3+x^2}$.

27. $\lim\limits_{x\to -4}\dfrac{\sqrt{x+20}-4}{x^3+64}$.

28. $\lim\limits_{x\to 1}\dfrac{3x^2-2}{\sqrt{8+x}-3}$

29. $\lim\limits_{x\to 0}\dfrac{\sqrt{9+x}-3}{x^2+x}$.

30. $\lim\limits_{x\to 2}\dfrac{\sqrt{4x+1}-3}{x^3-8}$

7

1. $\lim\limits_{x\to\infty}\left(\dfrac{x+4}{x+8}\right)^{-3x}$.

2. $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+1}\right)^{2x-3}$

3. $\lim\limits_{x\to\infty}\left(\dfrac{2x}{1+2x}\right)^{-4x}$

4. $\lim\limits_{x\to\infty}\left(\dfrac{x-1}{x}\right)^{2-3x}$.

5. $\lim\limits_{x\to\infty}\left(\dfrac{2x+5}{2x+1}\right)^{5x}$.

6. $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x}\right)^{-5x}$.

7. $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x+1}\right)^{1+2x}$.

8. $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x-1}\right)^{x-4}$.

9. $\lim\limits_{x\to\infty}\left(\dfrac{2x}{2x-3}\right)^{3x}$.

10. $\lim\limits_{x\to\infty}\left(\dfrac{x-7}{x}\right)^{2x+1}$.

11. $\lim\limits_{x\to\infty}\left(\dfrac{x-1}{x+4}\right)^{3x+2}$.

12. $\lim\limits_{x\to\infty}\left(\dfrac{2x+1}{2x-1}\right)^{x+2}$.

13. $\lim\limits_{x\to\infty}\left(\dfrac{x-2}{x+1}\right)^{2x-3}$.

14. $\lim\limits_{x\to\infty}\left(\dfrac{x}{x-3}\right)^{x-5}$.

15. $\lim\limits_{x\to\infty}\left(\dfrac{3x-4}{3x+2}\right)^{2x}$.

16. $\lim\limits_{x\to\infty}\left(\dfrac{2x-1}{2x+4}\right)^{3x-1}$

17. $\lim\limits_{x\to\infty}\left(\dfrac{2x-4}{2x}\right)^{-3x}$.

18. $\lim\limits_{x\to\infty}\left(\dfrac{x+5}{x}\right)^{3x+4}$

19. $\lim\limits_{x\to\infty}\left(\dfrac{x-7}{x+1}\right)^{4x-2}$

20. $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x}\right)^{3-2x}$

21. $\lim\limits_{x\to\infty}\left(\frac{2-3x}{5-3x}\right)^{x}$.

22. $\lim\limits_{x\to\infty}\left(\frac{1-x}{2-x}\right)^{3x}$.

23. $\lim\limits_{x\to\infty}\left(\frac{4x-1}{4x+1}\right)^{2x}$.

24. $\lim\limits_{x\to\infty}\left(\frac{3x+4}{3x}\right)^{-2x}$.

25. $\lim\limits_{x\to\infty}\left(\frac{2x-1}{2x+4}\right)^{-x}$.

26. $\lim\limits_{x\to\infty}\left(\frac{3x+4}{3x+5}\right)^{x+1}$.

27. $\lim\limits_{x\to\infty}\left(\frac{1+2x}{3+2x}\right)^{-x}$.

28. $\lim\limits_{x\to\infty}\left(\frac{3x}{3x+2}\right)^{x-2}$.

29. $\lim\limits_{x\to\infty}\left(\frac{x}{x-1}\right)^{3-2x}$.

30. $\lim\limits_{x\to\infty}\left(\frac{4-2x}{1-2x}\right)^{x+1}$.

8

1. $\lim\limits_{x\to\infty}\left(\frac{2x+3}{5x+7}\right)^{x+1}$.

2. $\lim\limits_{x\to\infty}\left(\frac{2x+1}{x-1}\right)^{x}$.

3. $\lim\limits_{x\to\infty}\left(\frac{x+1}{2x-1}\right)^{3x}$.

4. $\lim\limits_{x\to-\infty}\left(\frac{2x-1}{4x+1}\right)^{3x-1}$.

5. $\lim\limits_{x\to\infty}\left(\frac{5x+8}{x-2}\right)^{x+4}$.

6. $\lim\limits_{x\to-\infty}\left(\frac{x+1}{3x-1}\right)^{2x+1}$.

7. $\lim\limits_{x\to-\infty}\left(\frac{2x+1}{x-1}\right)^{4x}$.

8. $\lim\limits_{x\to\infty}\left(\frac{x+1}{2x-1}\right)^{5x}$.

9. $\lim\limits_{x\to-\infty}\left(\frac{x+3}{2x-4}\right)^{x+2}$.

10. $\lim\limits_{x\to-\infty}\left(\frac{2x+1}{3x-1}\right)^{x-1}$.

11. $\lim\limits_{x\to\infty}\left(\frac{5x-3}{x+4}\right)^{x+3}$.

12. $\lim\limits_{x\to-\infty}\left(\frac{2x-3}{7x+4}\right)^{x}$.

13. $\lim\limits_{x\to-\infty}\left(\frac{x-5}{3x+4}\right)^{2x}$.

14. $\lim\limits_{x\to\infty}\left(\frac{x+3}{4x-5}\right)^{2x}$.

15. $\lim\limits_{x\to-\infty}\left(\frac{x-2}{3x+1}\right)^{5x}$.

16. $\lim\limits_{x\to-\infty}\left(\frac{3x-4}{x+6}\right)^{x-1}$.

17. $\lim\limits_{x\to\infty}\left(\frac{x-2}{3x+10}\right)^{3x}$.

18. $\lim\limits_{x\to-\infty}\left(\frac{2x-3}{x+4}\right)^{6x+1}$.

19. $\lim\limits_{x\to-\infty}\left(\frac{x+3}{3x-1}\right)^{2x}$.

20. $\lim\limits_{x\to\infty}\left(\frac{6x+5}{x-10}\right)^{5x}$.

21. $\lim\limits_{x\to-\infty}\left(\frac{3x+7}{x+4}\right)^{4x}$.

22. $\lim\limits_{x\to\infty}\left(\frac{x-1}{4x+5}\right)^{3x}$.

23. $\lim\limits_{x\to-\infty}\left(\frac{5x-7}{x+6}\right)^{2x}$.

24. $\lim\limits_{x\to\infty}\left(\frac{3-4x}{2-x}\right)^{6x}$.

25. $\lim\limits_{x\to\infty}\left(\frac{1-2x}{3-x}\right)^{-x}$.

26. $\lim\limits_{x\to-\infty}\left(\frac{4+3x}{5+x}\right)^{7x}$.

27. $\lim\limits_{x\to-\infty}\left(\frac{3x-1}{2x+5}\right)^{3x}$.

28. $\lim\limits_{x\to\infty}\left(\frac{1-x}{2-10x}\right)^{5x}$.

29. $\lim\limits_{x\to\infty}\left(\frac{3+x}{9x-4}\right)^{2x}$.

30. $\lim\limits_{x\to-\infty}\left(\frac{x+5}{4x-2}\right)^{3x}$.

9

1. $\lim\limits_{x\to0}\frac{1-\cos 8x}{3x^2}$.

2. $\lim\limits_{x\to0}\frac{\sin 3x-\sin x}{5x}$

3. $\lim\limits_{x\to0}\frac{\cos x-\cos 5x}{2x^2}$.

4. $\lim\limits_{x\to0}\frac{\operatorname{tg} 3x}{2\sin x}$.

5. $\lim\limits_{x\to0}\frac{\operatorname{tg} x-\sin x}{3x^2}$.

6. $\lim\limits_{x\to0}\frac{\arcsin 5x}{\sin 3x}$.

7. $\lim\limits_{x\to1}(1-x)\operatorname{tg}\frac{\pi x}{2}$.

8. $\lim\limits_{x\to\pi/2}\frac{1-\sin x}{\pi-2x}$.

9. $\lim\limits_{x\to0}\frac{\operatorname{tg} 2x-\sin 2x}{x^2}$.

10. $\lim\limits_{x\to0}\frac{1-\cos^2 x}{x\operatorname{tg} x}$.

11. $\lim\limits_{x\to0}\left(\frac{1}{\operatorname{tg} x}-\frac{1}{\sin x}\right)$.

12. $\lim\limits_{x\to0}\frac{\sin^2 3x-\sin^2 x}{x^2}$

13. $\lim\limits_{x\to0}\frac{\sin 7x+\sin 3x}{x\sin x}$.

14. $\lim\limits_{x\to0}\frac{1-\cos 5x}{2x^2}$.

15. $\lim\limits_{x\to0}\frac{\cos 2x-\cos 4x}{3x^2}$.

16. $\lim\limits_{x\to0}\frac{\operatorname{arctg} 2x}{\operatorname{tg} 3x}$.

17. $\lim\limits_{x\to0}\frac{\operatorname{tg} 3x-\sin 3x}{2x^2}$.

18. $\lim\limits_{x\to\pi/4}\frac{1-\sin 2x}{\pi-4x}$.

19. $\lim\limits_{x\to 0}\dfrac{\cos 4x-\cos^3 4x}{3x^2}$.

20. $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin 2x}-\dfrac{1}{\operatorname{tg} 2x}\right)$

21. $\lim\limits_{x\to 0}\dfrac{\cos^2 x-\cos^2 2x}{x^2}$.

22. $\lim\limits_{x\to 0}\dfrac{\arcsin 5x}{x^2-x}$.

23. $\lim\limits_{x\to 0}\dfrac{1-\cos^2 2x}{x\arcsin x}$.

24. $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{x\sin x}$.

25. $\lim\limits_{x\to 0}\dfrac{\cos 5x-\cos x}{4x^2}$.

26. $\lim\limits_{x\to 0}\dfrac{\sin 5x+\sin x}{\arcsin x}$.

27. $\lim\limits_{x\to\pi/2}\dfrac{1-\sin x}{(\pi/2-x)^2}$.

28. $\lim\limits_{x\to\pi/2}(\pi/2-x)\operatorname{tg} x$

29. $\lim\limits_{x\to 0}\dfrac{7x}{\sin x+\sin 7x}$.

30. $\lim\limits_{x\to 0}\dfrac{\cos x-\cos^3 x}{5x^2}$.

10

Prove that the functions f(x) and $\phi(x)$ are infinitesimals of the same order as x→ 0.

1. $f(x)=\operatorname{tg} 2x,\ \varphi(x)=\arcsin x$.
2. $f(x)=1-\cos x,\ \varphi(x)=3x^2$.
3. $f(x)=\operatorname{arctg}^2 3x,\ \varphi(x)=4x^2$.
4. $f(x)=\sin 3x-\sin x,\ \varphi(x)=5x$.
5. $f(x)=\cos 3x-\cos x,\ \varphi(x)=7x^2$.
6. $f(x)=x^2-\cos 2x,\ \varphi(x)=6x^2$.
7. $f(x)=\sqrt{1+x}-1,\ \varphi(x)=2x$.
8. $f(x)=\sin x+\sin 5x,\ \varphi(x)=2x$.
9. $f(x)=3x/(1-x),\ \varphi(x)=x/(4+x)$.
10. $f(x)=3x^2/(2+x),\ \varphi(x)=7x^2$.
11. $f(x)=2x^3,\ \varphi(x)=5x^3/(4-x)$.
12. $f(x)=x^2/(5+x),\ \varphi(x)=4x^2/(x-1)$
13. $f(x)=\sin 8x,\ \varphi(x)=\arcsin 5x$.
14. $f(x)=\sin 3x+\sin x,\ \varphi(x)=10x$.
15. $f(x)=\cos 7\lambda-\cos x,\ \varphi(x)=2x^2$.

16. $f(x)=1-\cos 2x,\ \varphi(x)=8x^2$.
17. $f(x)=3\sin^2 4x,\ \varphi(x)=x^2-x^4$.
18. $f(x)=\operatorname{tg}(x^2+2x),\ \varphi(x)=x^2+2x$.
19. $f(x)=\arcsin(x^2-x),\ \varphi(x)=x^3-x$.
20. $f(x)=\sin 7x+\sin x,\ \varphi(x)=4x$.
21. $f(x)=\sqrt{4+x}+2,\ \varphi(x)=3x$.
22. $f(x)=\sin(x^2-2x),\ \varphi(x)=x^4-8x$.
23. $f(x)=2x/(3-x),\ \varphi(x)=2x-x^2$.
24. $f(x)=x^2/(7+x),\ \varphi(x)=3x^3-x^2$.
25. $f(x)=\sin(x^2+5x),\ \varphi(x)=x^3-25x$
26. $f(x)=\cos x-\cos^3 x,\ \varphi(x)=6x^2$.
27. $f(x)=\arcsin 2x,\ \varphi(x)=8x$.
28. $f(x)=1-\cos 4x,\ \varphi(x)=x\sin 2x$.
29. $f(x)=\sqrt{9-x}-3,\ \varphi(x)=2x$.
30. $f(x)=\cos 3x-\cos 5x,\ \varphi(x)=x^2$

11

Find the limits using equivalent infinitesimal functions:

1. $\lim\limits_{x\to 0}\dfrac{\ln(1+3x^2)}{x^3-5x^2}$.

2. $\lim\limits_{x\to 0}\dfrac{\arcsin 5x}{\operatorname{tg} 3x}$

3. $\lim\limits_{x\to 0}\dfrac{\sin 7x}{\operatorname{tg} 2x}$.

4. $\lim\limits_{x\to 0}\dfrac{e^{3x}-1}{x^3+27x}$.

5. $\lim\limits_{x\to 0}\dfrac{\operatorname{arctg} 6x}{2x^2-3x}$.

6. $\lim\limits_{x\to 0}\dfrac{\arcsin 3x}{2x}$

7. $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\operatorname{arctg} 2x}$.

8. $\lim\limits_{x\to 0}\dfrac{\ln(1+3x)}{\sin 2x}$.

9. $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{\operatorname{tg} 3x}$.

10. $\lim\limits_{x\to 3}\dfrac{\sin(x-3)}{x^2-5x+6}$

11. $\lim\limits_{x\to 0}\dfrac{\cos 3x-\cos x}{2x^2}$.

12. $\lim\limits_{x\to 0}\dfrac{1-\cos 6x}{4x^2}$.

13. $\lim\limits_{x\to 0}\dfrac{\operatorname{arctg} 3x}{\ln(1+2x)}$.

14. $\lim\limits_{x\to 0}\dfrac{\arcsin 4x}{\operatorname{tg} 5x}$.

15. $\lim\limits_{x\to 0}\frac{e^{5x}-1}{\sin 2x}$.

16. $\lim\limits_{x\to -2}\frac{\operatorname{tg}(x+2)}{x^2-4}$.

17. $\lim\limits_{x\to -2}\frac{\sin(x+2)}{x^3+8}$.

18. $\lim\limits_{x\to 0}\frac{\arcsin 2x}{\operatorname{tg} 4x}$.

19. $\lim\limits_{x\to 4}\frac{x^3-64}{\operatorname{tg}(x-4)}$.

20. $\lim\limits_{x\to 0}\frac{\cos 2x-\cos 4x}{3x^2}$

21. $\lim\limits_{x\to 0}\frac{\ln(1+4x^3)}{2x^3}$.

22. $\lim\limits_{x\to 0}\frac{\operatorname{arctg} 5x}{\operatorname{tg} 2x}$.

23. $\lim\limits_{x\to 0}\frac{\sin 3x}{\ln(1+2x)}$.

24. $\lim\limits_{x\to 0}\frac{\arcsin 8x}{\operatorname{tg} 4x}$.

25. $\lim\limits_{x\to 0}\frac{e^{5x}-1}{\operatorname{tg} 2x}$.

26. $\lim\limits_{x\to 0}\frac{\ln(1+4x)}{\sin 2x}$

27. $\lim\limits_{x\to 3}\frac{\sin(x-3)}{x^3-27}$.

28. $\lim\limits_{x\to -5}\frac{\operatorname{tg}(x+5)}{x^2-25}$

29. $\lim\limits_{x\to 0}\frac{1-\cos 8x}{2x^2}$.

30. $\lim\limits_{x\to 0}\frac{\ln(1+5x)}{\sin 3x}$.

Topic 3. The continuity of the function.

3.1. Definition of continuous functions. 3.2. Properties of operations with continuous functions. 3.3. Points of discontinuity and their classification. 3.4. Properties of continuous functions at interval. 3.5. Max-min theorem for continuous functions. 3.6. The intermediate value theorem for continuous functions.

3.1. Definition of continuous functions.

A function y=f(x) is **continuous** at point x=c if and only if all three **of the following conditions are true:**

1) f(c) exists (i.e. the point **c** belongs to the domain of f(x));

2) $\lim\limits_{x\to c} f(x)$ exists (i.e. f(x) has a limit as $x\to c$);

3) $\lim\limits_{x\to c} f(x) = f(c)$ (i.e. limit equals the function value).

3.2. Properties of operations with continuous functions.

3.2.1. If the functions f and g are **continuous** at x = c, then the following combinations are **continuous** at x = c: $f\pm g$; $f\cdot g$; $k\cdot f$; $\frac{f}{g}$ if $g\neq 0$.

3.2.2 The continuity of the composite function.

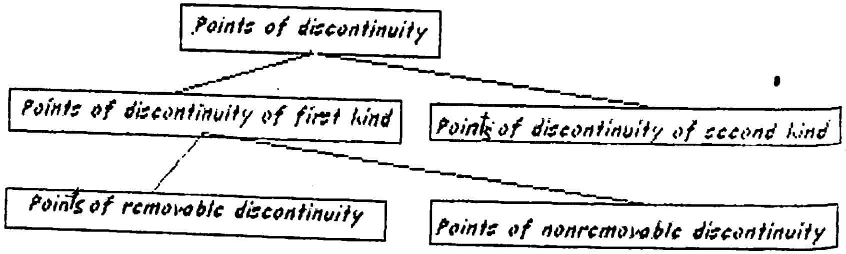
If **f(x)** is continuous at **g(c)** and **g(x)** is continuous at **x=c**, the composite **f(g(x))** is continuous at **x=c**.

3.3. Points of discontinuity and their classification.

3.3.1. Definition of point of discontinuity.

The point $x = x_0$ in which at least one condition of continuity is not satisfied is called **the point of discontinuity.**

3.3.2. The classification of points of discontinuity.



1)Points of discontinuity of first kind:

There are exist the finite limits: 1) $\lim\limits_{x\to x_0-} f(x)$ = $f(x_0 -0)$,

2) $\lim\limits_{x\to x_0+} f(x)$ = $f(x_{0+}0)$,

3) $\lim\limits_{x\to x_0} f(x)$ = $f(x_0)$.

But not all values $f(x_0 -0)$, $f(x_{0+}0)$ and $f(x_0)$ are equal.

a) Points of removable discontinuity:

$f(x_0 -0) = f(x_0+0) \neq f(x_0)$ (third condition is not satisfied).

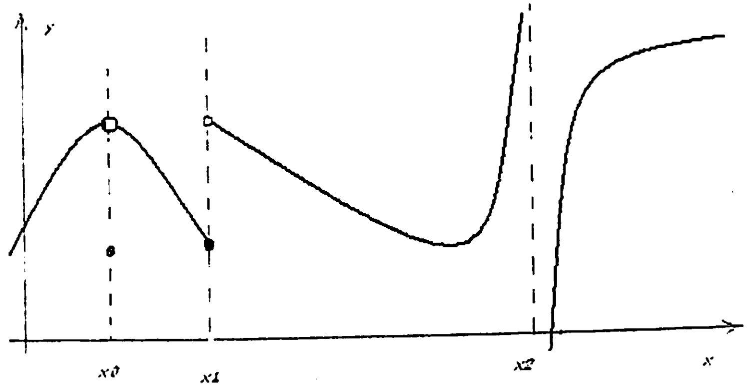
b)Points of nonremovable discontinuity (jump point):

$f(x_0 -0) \neq f(x_0+0)$ (second condition is not satisfied).

The difference $(f(x_0 -0) - f(x_0+0))$ is called the jump of the function at point $x=x_0$.

4) Points of discontinuity of second kind:

If at least one limit $\lim\limits_{x\to x_0+} f(x)$ or $\lim\limits_{x\to x_0-} f(x)$ does not exists or equals to ∞.



There are shown $x=x_0$ is point of the first kind – removable discontinuity;

$x=x_1$ is point of the first kind – nonremovable discontinuity (jump point);

$x=x_2$ is point of the second kind.

3.4. Properties of continuous functions at interval.

3.4.1 Continuous over interval X.

A function $y=f(x)$ is called **continuous over closed interval** [a ,b] if it is continuous at each point of this interval (a ,b) and is continuous at the right of **a** and at the left of **b**.

<u>Theorem</u>: Any elementary function is continuous over its domain.

<u>Theorem:</u> Let f(x) is continuous over [a ,b] and takes on its endpoints the values of different signs

($f(a)\cdot f(b))<0$), then will be discovered at least one point $c\in$ **(a ,b)** such that **f(c)=0.**

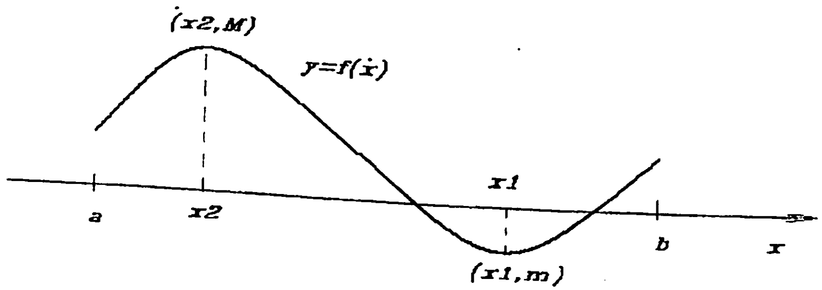
<u>Theorem</u>: If f(x) is **continuous** over [a, b] then it is **bounded** at [a ,b] .

3.5. Max-min theorem for continuous functions.

Definition: $f(x_0)$ is called **maximum** value of the function $y=f(x)$ at interval (a, b) if for any point $x \in (a, b)$ $f(x_0) > f(x)$.

$f(x_0)$ is called **minimum** value of the function $y=f(x)$ at interval (a, b) if for any point $x \in (a, b)$ $f(x_0) < f(x)$.

Max-min theorem: If f is continuous at every point of a closed interval [a ,b], then f takes on both an absolute maximum value M an absolute minimum value m somewhere in the interval. That is, for some numbers x_1 and x_2 in [a ,b] we have $f(x_1)=m$, and $f(x_2)=M$, and $m \le f(x) \le M$ at every other point x of the interval.



3.6. The intermediate value theorem for continuous functions.

A function $y=f(x)$ that is continuous on a closed interval [a ,b] takes on every value between f(a) and f(b).

In other words, continuous function takes on all its intermediate values between f(a) and f(b).

EXERCISE III.

1

Test the function on continuity and graph it.

1. $f(x)=\begin{cases} x+4, & x<-1, \\ x^2+2, & -1\leqslant x<1 \\ 2x, & x\geqslant 1. \end{cases}$

2. $f(x)=\begin{cases} x+1, & x\leqslant 0, \\ (x+1)^2, & 0<x\leqslant 2, \\ -x+4, & x>2. \end{cases}$

3. $f(x)=\begin{cases} x+2, & x\leqslant -1, \\ x^2+1, & -1<x\leqslant 1 \\ -x+3, & x>1. \end{cases}$

4. $f(x)=\begin{cases} -x, & x\leqslant 0, \\ -(x-1)^2, & 0<x<2, \\ x-3, & x\geqslant 2. \end{cases}$

5. $f(x)=\begin{cases} -2(x+1), & x\leqslant -1, \\ (x+1)^3, & -1<x<0. \\ x, & x\geqslant 0. \end{cases}$

6. $f(x)=\begin{cases} -x, & x\leqslant 0, \\ x^2, & 0<x\leqslant 2, \\ x+1, & x>2. \end{cases}$

7. $f(x)=\begin{cases} x^2+1, & x\leqslant 1, \\ 2x, & 1<x\leqslant 3 \\ x+2, & x>3. \end{cases}$

8. $f(x)=\begin{cases} x-3, & x<0, \\ x+1, & 0\leqslant x\leqslant 4, \\ 3+x, & x>4. \end{cases}$

9. $f(x)=\begin{cases} \sqrt{1-x}, & x\leqslant 0, \\ 0, & 0<x\leqslant 2, \\ x-2, & x>2. \end{cases}$

10. $f(x)=\begin{cases} 2x^2, & x\leqslant 0, \\ x, & 0<x\leqslant 1, \\ 2+x, & x>1. \end{cases}$

11. $f(x)=\begin{cases} \sin x, & x<0; \\ x, & 0\leqslant x\leqslant 2, \\ 0, & x>2. \end{cases}$

12. $f(x)=\begin{cases} \cos x, & x\leqslant \pi/2, \\ 0, & \pi/2<x<\pi, \\ 2, & x\geqslant \pi. \end{cases}$

13. $f(x)=\begin{cases} x-1, & x\leqslant 0, \\ x^2, & 0<x<2, \\ 2x, & x\geqslant 2. \end{cases}$

14. $f(x)=\begin{cases} x+1, & x<0, \\ x^2-1, & 0\leqslant x<1 \\ -x, & x\geqslant 1. \end{cases}$

15. $f(x)=\begin{cases} -x, & x<0, \\ x^2+1, & 0\leqslant x<2 \\ x+1, & x\geqslant 2. \end{cases}$

16. $f(x)=\begin{cases} x+3, & x\leqslant 0, \\ 1, & 0<x\leqslant 2, \\ x^2-2, & x>2. \end{cases}$

17. $f(x)=\begin{cases} x-1, & x<0, \\ \sin x, & 0\leqslant x<\pi, \\ 3, & x\geqslant \pi. \end{cases}$

18. $f(x)=\begin{cases} -x+1, & x<-1, \\ x^2+1, & -1\leqslant x\leqslant 2 \\ 2x, & x>2. \end{cases}$

19. $f(x)=\begin{cases} 1, & x\leqslant 0, \\ 2^x, & 0<x\leqslant 2, \\ x+3, & x>2. \end{cases}$

20. $f(x)=\begin{cases} -x+2, & x\leqslant -2, \\ x^3, & -2<x\leqslant 1 \\ 2, & x>1. \end{cases}$

21. $f(x)=\begin{cases} 3x+4, & x\leqslant -1, \\ x^2-2, & -1<x<2, \\ x, & x\geqslant 2. \end{cases}$

22. $f(x)=\begin{cases} x, & x\leqslant 1, \\ (x-2)^2, & 1<x<3, \\ -x+6, & x\geqslant 3. \end{cases}$

23. $f(x)=\begin{cases} x-1, & x<1, \\ x^2+2, & 1\leqslant x\leqslant 2 \\ -2x, & x>2. \end{cases}$

24. $f(x)=\begin{cases} x^3, & x<-1; \\ x-1, & -1\leqslant x\leqslant 3 \\ -x+5, & x>3. \end{cases}$

25. $f(x)=\begin{cases} x, & x<-2, \\ -x+1, & -2\leqslant x\leqslant 1 \\ x^2-1, & x>1. \end{cases}$

26. $f(x)=\begin{cases} x+3, & x\leqslant 0, \\ -x^2+4, & 0<x<2 \\ x-2, & x\geqslant 2. \end{cases}$

27. $f(x)=\begin{cases} 0, & x\leqslant -1, \\ x^2-1, & -1<x\leqslant 2 \\ 2x, & x>2. \end{cases}$

28. $f(x)=\begin{cases} -1, & x<0, \\ \cos x, & 0\leqslant x\leqslant \pi, \\ 1-x, & x>\pi. \end{cases}$

29. $f(x)=\begin{cases} 2, & x<-1, \\ 1-x, & -1\leqslant x\leqslant 1 \\ \ln x, & x>1. \end{cases}$

30. $f(x)=\begin{cases} -x, & x\leqslant 0, \\ x^3, & 0<x\leqslant 2, \\ x+4, & x>2. \end{cases}$

2

Test the function on continuity at indicated points.

1. $f(x)=2^{1/(x-3)}+1;\ x_1=3,\ x_2=4.$
2. $f(x)=5^{1/(x-3)}-1;\ x_1=3,\ x_2=4.$
3. $f(x)=(x+7)/(x-2);\ x_1=2,\ x_2=3.$
4. $f(x)=(x-5)/(x+3);\ x_1=-2,\ x_2=-3$
5. $f(x)=4^{1/(3-x)}+2;\ x_1=2,\ x_2=3.$
6. $f(x)=9^{1/(2-x)};\ x_1=0,\ x_2=2.$
7. $f(x)=2^{1/(x-5)}+1;\ x_1=4,\ x_2=5.$
8. $f(x)=5^{1/(x-4)}-2;\ x_1=3,\ x_2=4.$
9. $f(x)=6^{1/(x-3)}+3;\ x_1=3,\ x_2=4.$
10. $f(x)=7^{1/(5-x)}+1;\ x_1=4,\ x_2=5.$

11. $f(x)=(x-3)(x+4);\ x_1=-5,\ x_2=-4.$
12. $f(x)=(x+5)/(x-2);\ x_1=3,\ x_2=2.$
13. $f(x)=5^{2/(x-3)};\ x_1=3,\ x_2=4.$
14. $f(x)=4^{2/(x-1)}-3;\ x_1=1,\ x_2=2.$
15. $f(x)=2^{5/(1-x)}-1;\ x_1=0,\ x_2=1.$
16. $f(x)=8^{4/(x-2)}-1;\ x_1=2,\ x_2=3.$
17. $f(x)=5^{4/(3-x)}+1;\ x_1=2,\ x_2=3.$
18. $f(x)=3x/(x-4);\ x_1=4,\ x_2=5.$
19. $f(x)=2x/(x^2-1);\ x_1=1,\ x_2=2.$
20. $f(x)=2^{3/(x+2)}+1;\ x_1=-2,\ x_2=-1.$
21. $f(x)=4^{3/(x-2)}+2;\ x_1=2,\ x_2=3.$
22. $f(x)=3^{2/(x+1)}-2;\ x_1=-1,\ x_2=0.$
23. $f(x)=5^{3/(x+4)}+1;\ x_1=-5,\ x_2=-4.$
24. $f(x)=(x-4)/(x+2);\ x_1=-2,\ x_2=-1.$
25. $f(x)=(x-4)/(x+3);\ x_1=-3,\ x_2=-2.$
26. $f(x)=(x+5)/(x-3);\ x_1=3,\ x_2=4.$
27. $f(x)=3^{4/(1-x)}+1;\ x_1=1,\ x_2=2.$
28. $f(x)=4x/(x+5);\ x_1=-5,\ x_2=-4$
29. $f(x)=6^{2/(4-x)};\ x_1=3,\ x_2=4.$
30. $f(x)=(x+1)/(x-2);\ x_1=2,\ x_2=3.$

Topic 4. Derivative of the function with one variable.

4.1. Definition of derivative. 4.2. Geometrical and mechanical meaning of the derivative. Equation of tangent. 4.3. Basic rules of differentiation. 4.4. Table of derivatives.

4.1. Definition of derivative.

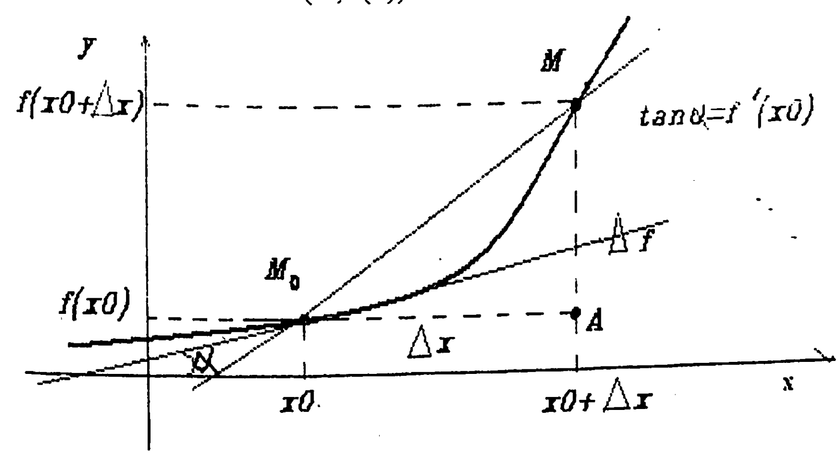
4.1.1. Increment of the independent variable and increment of the function.

Let the function $y=f(x)$ is defined at some vicinity of the point x_0.

The difference $\Delta x=x-x_0$ is called **the increment of independent variable.**

The difference $\Delta f=f(x_0+\Delta x)-f(x_0)$ is called **the increment of the function**.

Geometrical meaning of Δx and Δf is the changing of the abscise and ordinate of the point with coordinate $(x_0,f(x_0))$ when point moves in point with coordinate $(x,f(x))$.



4.1.2. Definition o derivative.

The derivative of a function y=f(x) is the function f' whose value at x is the number

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x\ )}{\Delta x} \quad \text{if the limit exists.}$$

If the limit exists f **has a derivative** at x point. If f has a derivative at every point of its domain f is **differentiable function.**

If the limit doesn't exist at $x=x_0$ that function f is **nondifferentiable at x=a, or f '(a) does not exist.**

Theorem: Relation between differentiable and continuous functions.

If the function y=f(x) is differentiable at the point x_0 then f(x) must be continuous function at this point.

4.2. Geometrical and mechanical meaning and of the derivative. Equation of tangent.

4.2.1. Geometrical meaning of the derivative. Equation of a tangent.

The derivative of the function y=f(x) at the point x_0 **f '(x_0) is the slope of tangent (f '(x_0) =tan α)** to the graph of the function y=f(x) at this point $x=x_0$.

$$y = f(x_0) + f'(x_0)\,(x-x_0)$$

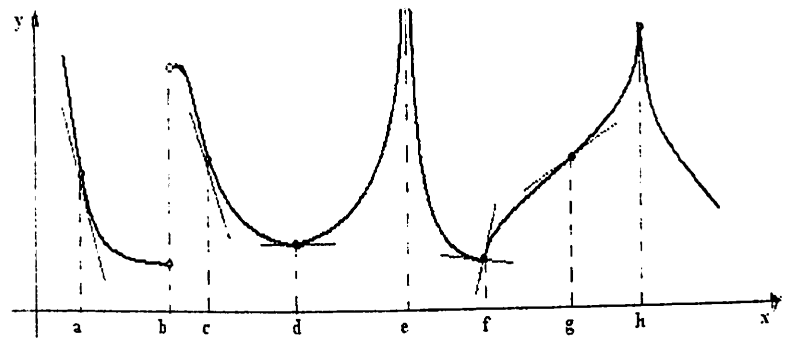
equation of the tangent to the graph of the function y=f(x) at this point $x=x_0$.

If $\alpha = \tan\alpha = 0$ - tangent is horizontal **(f '(x_0) =0);**

If $\alpha = \frac{\pi}{2}$, $\tan\alpha = \infty$ - tangent is vertical **(f '(x_0) – does not exist).**

At the picture f '(x) does not exist at the points: x=b; x=e; x=f ; x=h.

And f '(x) exists at the points: x=a; x=c; x=d; x=g.



4.2.2. Mechanical meaning of the derivative.

Instantaneous velocity is the derivative of position with respect to time. If the position function of a body moving along a line is s = f(t), the body's velocity at time t is

$$V(t) = \frac{ds}{dt} = \lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}.$$

4.3. Basic rules of differentiation.

4.3.1. . Basic rules of differentiation.

If functions f(x) and g(x) are differentiable at the point x. then:

1) $(C)'=0$;
2) $(x)'=1$;
3) $(C\cdot f(x))' = C\cdot f'(x)$;
4) $(f(x)\pm g(x))' = f'(x) \pm g'(x)$;
5) $(f(x)\cdot g(x))' = f'(x)\cdot g(x)+f(x)\cdot g'(x)$

6) $$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)} \quad (g \neq 0).$$

7) $$\left(\frac{C}{g(x)}\right)' = -\frac{C\cdot g'(x)}{g^2(x)} \quad (g \neq 0).$$

4.3.2. The chain rule (Differentiation of composite function).

Suppose that the function $u = u(x)$ is differentiable at the point x_0, $u(x_0)=u_0$; and the function $y = f(u)$ is differentiable at the point u_0; **then the composite function** $y = f(u(x))$ is differentiable at the point x_0 and its derivative is equal:

$$[f(u(x_0))]' = f'(u_0)\cdot u'(x_0).$$

4.4. Table of derivatives.

1) $(u^n)' = n\cdot u^{n-1}\cdot u'$ $(n \in R)$;

2) $(\sqrt{u})' = \dfrac{1}{2\sqrt{u}}\cdot u'$;

3) $\left(\dfrac{1}{u}\right)' = -\dfrac{1}{u^2}\cdot u'$;

4) $(a^u)' = a^u\cdot \ln(a)\cdot u'$;

5) $(e^u)' = e^u\cdot u'$;

6) $(\log_a u)' = \dfrac{1}{u\cdot \ln a}\cdot u'$;

7) $(\ln(u))' = \dfrac{1}{u}\cdot u'$;

8) $(\sin(u))' = \cos(u)\cdot u'$;

9) $(\cos(u))' = -\sin(u)\cdot u'$;

10) $(\tan(u))' = \dfrac{1}{\cos^2 u}\cdot u'$;

11) $(\cot(u))' = -\dfrac{1}{\sin^2 u}\cdot u'$;

12) $(\text{asin}(u))' = \dfrac{1}{\sqrt{1-u^2}}\cdot u'$;

13) $(\text{acos}(u))' = -\dfrac{1}{\sqrt{1-u^2}}\cdot u'$;

14) $(\text{atan}(u))' = \dfrac{1}{1+u^2}\cdot u'$;

15) $(\text{acot}(u))' = -\dfrac{1}{1+u^2}\cdot u'$.

EXERCISE IV.

Find the first order derivative:

1

1. $y = 2x^5 - \frac{4}{x^3} + \frac{1}{x} + 3\sqrt{x}$
2. $y = \frac{3}{x} + \sqrt[5]{x^2} - 4x^3 + \frac{2}{x^4}$.
3. $y = 3x^4 + \sqrt[3]{x^5} - \frac{2}{x} - \frac{4}{x^2}$
4. $y = 7\sqrt{x} - \frac{2}{x^5} - 3x^3 + \frac{4}{x}$.
5. $y = 7x + \frac{5}{x^2} - \sqrt[7]{x^4} + \frac{6}{x}$
6. $y = 5x^2 - \sqrt[3]{x^4} + \frac{4}{x^3} - \frac{5}{x}$.
7. $y = 3x^5 - \frac{3}{x} - \sqrt{x^3} + \frac{10}{x^5}$
8. $y = \sqrt[3]{x^7} + \frac{3}{x} - 4x^6 + \frac{4}{x^5}$
9. $y = 8x^2 + \sqrt[3]{x^4} - \frac{4}{x} - \frac{2}{x^3}$.
10. $y = 4x^6 + \frac{5}{x} - \sqrt[3]{x^7} - \frac{7}{x^4}$.
11. $y = 2\sqrt{x^3} - \frac{7}{x} + 3x^2 - \frac{2}{x^5}$
12. $y = 4x^3 - \frac{3}{x} - \sqrt[5]{x^2} + \frac{6}{x^2}$.
13. $y = 5x^3 - \frac{8}{x^2} + 4\sqrt{x} + \frac{1}{x}$.

6) $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}$ $(g\neq 0)$.

7) $\left(\frac{C}{g(x)}\right)' = -\frac{C\cdot g'(x)}{g^2(x)}$ $(g\neq 0)$.

4.3.2. The chain rule (Differentiation of composite function).

Suppose that the function $u = u(x)$ is differentiable at the point x_0, $u(x_0)=u_0$, and the function $y = f(u)$ is differentiable at the point u_0, **then the composite function** $y = f(u(x))$ is differentiable at the point x_0 and its derivative is equal:

$$[f(u(x_0))]' = f'(u_0)\cdot u'(x_0).$$

4.4. Table of derivatives.

1) $(u^n)' = n\cdot u^{n-1}\cdot u'$ $(n\in R)$;

2) $(\sqrt{u})' = \frac{1}{2\sqrt{u}}\cdot u'$;

3) $(\frac{1}{u})' = -\frac{1}{u^2}\cdot u'$;

4) $(a^u)' = a^u\cdot \ln(a)\cdot u'$;

5) $(e^u)' = e^u\cdot u'$;

6) $(\log_a u)' = \frac{1}{u\cdot \ln a}\cdot u'$;

7) $(\ln(u))' = \frac{1}{u}\cdot u'$;

8) $(\sin(u))' = \cos(u)\cdot u'$;

9) $(\cos(u))' = -\sin(u)\cdot u'$;

10) $(\tan(u))' = \frac{1}{\cos^2 u}\cdot u'$;

11) $(\cot(u))' = -\frac{1}{\sin^2 u}\cdot u'$;

12) $(\text{asin}(u))' = \frac{1}{\sqrt{1-u^2}}\cdot u'$;

13) $(\text{acos}(u))' = -\frac{1}{\sqrt{1-u^2}}\cdot u'$;

14) $(\text{atan}(u))' = \frac{1}{1+u^2}\cdot u'$;

15) $(\text{acot}(u))' = -\frac{1}{1+u^2}\cdot u'$.

EXERCISE IV.

Find the first order derivative:

1

1. $y = 2x^5 - \frac{4}{x^3} + \frac{1}{x} + 3\sqrt{x}$
2. $y = \frac{3}{x} + \sqrt[5]{x^2} - 4x^3 + \frac{2}{x^4}$.
3. $y = 3x^4 + \sqrt[3]{x^5} - \frac{2}{x} - \frac{4}{x^2}$
4. $y = 7\sqrt{x} - \frac{2}{x^5} - 3x^3 + \frac{4}{x}$.
5. $y = 7x + \frac{5}{x^2} - \sqrt[7]{x^4} + \frac{6}{x}$
6. $y = 5x^2 - \sqrt[3]{x^4} + \frac{4}{x^3} - \frac{5}{x}$.
7. $y = 3x^5 - \frac{3}{x} - \sqrt{x^3} + \frac{10}{x^5}$
8. $y = \sqrt[3]{x^7} + \frac{3}{x} - 4x^6 + \frac{4}{x^5}$
9. $y = 8x^2 + \sqrt[3]{x^4} - \frac{4}{x} - \frac{2}{x^3}$.
10. $y = 4x^6 + \frac{5}{x} - \sqrt[3]{x^7} - \frac{7}{x^4}$.
11. $y = 2\sqrt{x^3} - \frac{7}{x} + 3x^2 - \frac{2}{x^5}$
12. $y = 4x^3 - \frac{3}{x} - \sqrt[5]{x^2} + \frac{6}{x^2}$.
13. $y = 5x^3 - \frac{8}{x^2} + 4\sqrt{x} + \frac{1}{x}$.

10. $y=4x^6+\frac{5}{x}-\sqrt[3]{x^7}-\frac{7}{x^4}$.

11. $y=2\sqrt{x^3}-\frac{7}{x}+3x^2-\frac{2}{x^5}$

12. $y=4x^3-\frac{3}{x}-\sqrt[5]{x^2}+\frac{6}{x^2}$.

13. $y=5x^3-\frac{8}{x^2}+4\sqrt{x}+\frac{1}{x}$.

14. $y=\frac{9}{x^3}+\sqrt[3]{x^4}-\frac{2}{x}+5x^4$.

15. $y=\frac{4}{x^5}-\frac{9}{x}+\sqrt[5]{x^2}-7x^3$.

16. $y=\frac{8}{x^3}+\frac{3}{x}-4\sqrt{x^3}+2x^7$.

17. $y=5x^2+\frac{4}{x}-\sqrt[3]{x^7}-2x^6$.

18. $y=10x^2+3\sqrt{x^5}-\frac{4}{x}-\frac{5}{x^4}$.

19. $y=\sqrt{x^5}-\frac{3}{x}+\frac{4}{x^3}-3x^3$.

20. $y=9x^3+\frac{5}{x}-\frac{7}{x^4}+\sqrt[3]{x^7}$.

21. $y=3\sqrt{x}+\frac{4}{x^5}+\sqrt[3]{x^2}-\frac{7}{x}$.

22. $y=\sqrt{x^3}+\frac{2}{x}-\frac{4}{x^5}-5x^3$.

23. $y=7x^2+\frac{3}{x}-\sqrt[5]{x^4}+\frac{8}{x^3}$.

24. $y=8x^3-\frac{4}{x}-\frac{7}{x^4}+\sqrt[7]{x^2}$.

25. $y=8x-\frac{5}{x^4}+\frac{1}{x}-\sqrt[5]{x^4}$.

26. $y=\sqrt[4]{x^3}-\frac{5}{x}+\frac{4}{x^5}+3x$.

27. $y=4x^3+\frac{3}{x}-\sqrt[3]{x^5}-\frac{2}{x^4}$.

28. $y=4x^5-\frac{5}{x}-\sqrt{x^3}+\frac{2}{x^3}$.

29. $y=\frac{7}{x}+\frac{4}{x^3}-\sqrt[5]{x^3}-2x^6$.

30. $y=\frac{6}{x^4}-\frac{3}{x}+3x^3-\sqrt{x^7}$.

2

1. $y=\sqrt[3]{3x^4+2x-5}+\frac{4}{(x-2)^5}$

2. $y=\sqrt[3]{(x-3)^4}-\frac{3}{2x^3-3x+1}$.

3. $y=\sqrt{(x-4)^5}+\frac{5}{(2x^2+4x-1)^2}$.

4. $y=\sqrt[5]{7x^2-3x+5}-\frac{5}{(x-1)^3}$.

5. $y=\sqrt[4]{3x^2-x+5}-\frac{3}{(x-5)^4}$.

6. $y=\sqrt{3x^4-2x^3+x}-\frac{4}{(x+2)^3}$

7. $y=\sqrt[3]{(x-7)^5}+\frac{5}{4x^2+3x-5}$.

8. $y=\sqrt[5]{(x+4)^6}-\frac{2}{2x^2-3x+7}$.

9. $y=\frac{3}{(x-4)^7}-\sqrt{5x^2-4x+3}$.

10. $y=\sqrt[3]{4x^2-3x-4}-\frac{2}{(x-3)^5}$

11. $y=\frac{7}{(x-1)^3}+\sqrt{8x-3+x^2}$.

12. $y=\sqrt[5]{3x^2+4x-5}+\frac{4}{(x-4)^4}$.

13. $y=\sqrt[3]{5x^4-2x-1}+\frac{8}{(x-5)^2}$.

14. $y=\frac{3}{(x+2)^5}-\sqrt[7]{5x-7x^2-3}$

15. $y=\sqrt[4]{(x-1)^5}-\frac{4}{7x^2-3x+2}$.

16. $y=\sqrt[5]{(x-2)^6}-\frac{3}{7x^3-x^2-4}$.

17. $y=\frac{3}{(x+4)^2}-\sqrt[3]{4+3x-x^4}$.

18. $y=\frac{2}{(x-1)^3}-\frac{8}{6x^2+3x-7}$.

19. $y=\sqrt{1+5x-2x^2}+\frac{3}{(x-3)^4}$.

20. $y=\sqrt[3]{5+4x-x^2}-\frac{5}{(x+1)^3}$.

21. $y=\sqrt[4]{5x^2-4x+1}-\frac{7}{(x-5)^2}$

22. $y=\sqrt[5]{3-7x+x^2}-\frac{4}{(x-7)^5}$.

23. $y=\sqrt{(x-3)^7}+\frac{9}{7x^2-5x-8}$.

24. $y=\sqrt[3]{(x-8)^4}-\frac{2}{1+3x-4x^2}$.

25. $y=\frac{3}{4x-3x^2+1}-\sqrt{(x+1)^5}$.

26. $y=\frac{3}{x-4}+\sqrt[6]{(2x^2-3x+1)^5}$.

27. $y=\frac{4}{(x-7)^3}-\sqrt[3]{(3x^2-x+1)^4}$.

28. $y=\sqrt{(x-4)^7}-\frac{10}{(3x^2-5x+1)}$

29. $y=\frac{7}{(x+2)^5}-\sqrt{8-5x+2x^2}$

30. $y=\sqrt[3]{(x-1)^5}+\frac{5}{2x^2-4x+7}$.

3

1. $y=\sin^3 2x\cdot\cos 8x^5$.
2. $y=\cos^5 3x\cdot\operatorname{tg}(4x+1)^3$.
3. $y=\operatorname{tg}^4 x\cdot\arcsin 4x^5$.
4. $y=\arcsin^3 2x\cdot\operatorname{ctg} 7x^4$.
5. $y=\operatorname{ctg} 3x\cdot\arccos 3x^2$.
6. $y=\arccos^2 4x\cdot\ln(x-3)$.
7. $y=\ln^5 x\cdot\operatorname{arctg} 7x^4$.
8. $y=\operatorname{arctg}^3 4x\cdot 3^{\sin x}$.
9. $y=2^{\cos x}\cdot\operatorname{arcctg} 5x^3$.
10. $y=4^{-x}\cdot\ln^5(x+2)$.
11. $y=3^{\operatorname{tg} x}\cdot\arcsin 7x^4$.
12. $y=5^{x^2}\cdot\arccos 2x^5$.

13. $y=\sin^4 3x\cdot\operatorname{arctg} 2x^3$
14. $y=\cos^3 4x\cdot\operatorname{arcctg}\sqrt{x}$.
15. $y=\operatorname{tg}^3 2x\cdot\arcsin x^5$.
16. $y=\operatorname{ctg}^7 x\cdot\arccos 2x^3$.
17. $y=e^{-\sin x}\operatorname{tg} 7x^6$.
18. $y=e^{\cos x}\operatorname{ctg} 8x^3$.
19. $y=\cos^5 x\cdot\arccos 4x$.
20. $y=\sin^3 7x\cdot\operatorname{arcctg} 5x^2$.

21. $y=\sin^2 3x\cdot\operatorname{arcctg} 3x^5$.
22. $y=\cos\sqrt[5]{x}\cdot\operatorname{arctg} x^4$.
23. $y=\operatorname{tg}^6 2x\cdot\cos 7x^2$.
24. $y=\operatorname{ctg}^3 4x\cdot\arcsin\sqrt{x}$
25. $y=\operatorname{ctg}\frac{1}{x}\cdot\arccos x^4$.
26. $y=\operatorname{tg}\sqrt{x}\cdot\operatorname{arctg} 3x^5$.
27. $y=\operatorname{tg}^3 2x\cdot\arccos 2x^3$.
28. $y=2^{\operatorname{tg} x}\operatorname{arctg}^5 3x$.
29. $y=\sin^5 3x\cdot\operatorname{arctg}\sqrt{x}$.
30 $y=\cos^4 3x\cdot\arcsin 3x^2$.

4

1. $y=\operatorname{arcctg}^2 5x\cdot\ln(x-4)$.
2. $y=\operatorname{arctg}^3 2x\cdot\ln(x+5)$.
3. $y=\arccos^4 x\cdot\ln(x^2+x-1)$
4. $y=\sqrt{\arccos 2x}\cdot 3^{-x}$.
5. $y=\operatorname{tg}^4 3x\cdot\operatorname{arctg} 7x^2$.
6. $y=5^{-x^2}\arcsin 3x^3$.
7. $y=\operatorname{arctg}^5 x\cdot\log_2(x-3)$.
8. $y=\log_3(x+5)\cdot\arccos 3x$.
9. $y=e^{-x}\cdot\arcsin^2 5x$.
10. $y=\log_4(x-1)\cdot\arcsin^4 x$.
11. $y=(x-4)^5\cdot\operatorname{arcctg} 3x^2$.
12. $y=\operatorname{ctg}^3 4x\cdot\operatorname{arctg} 2x^3$.
13. $y=e^{-\cos x}\operatorname{arctg} 7x^5$.
14. $y=(x+1)\arccos 3x^4$.
15. $y=2^{\sin x}\operatorname{arcctg} x^4$.

16. $y=3^{-x^3}\operatorname{arctg} 2x^5$.
17. $y=3^{\cos x}\arcsin^2 3x$.
18. $y=\ln(x-10)\cdot\arccos^2 4x$.
19. $y=\lg(x-2)\cdot\arcsin^5 x$.
20. $y=\log_3(x+1)\cdot\operatorname{arctg}^5 7x$.
21. $y=\ln(x+9)\cdot\operatorname{arcctg}^3 2x$.
22. $y=\lg(x+2)\cdot\arcsin^2 3x$.
23. $y=4^{-\sin x}\operatorname{arctg} 3x$.
24. $y=2^{\cos x}\operatorname{arcctg}^3 x$.
25. $y=\lg(x-3)\cdot\arcsin^2 5x$.
26. $y=\log_2(x+3)\cdot\arccos^2 x$
27. $y=2^{-x}\operatorname{arctg}^3 4x$.
28. $y=\ln(x-4)\cdot\operatorname{arcctg}^4 3x$.
29. $y=\lg(x+3)\cdot\operatorname{arcctg}^2 5x$.
30. $y=\log_5(x+1)\cdot\operatorname{arctg}^2 x^3$.

5

1. $y=\operatorname{tg}^4 3x\cdot\arcsin 2x^3$.
2. $y=(x-2)^4\arcsin 5x^4$.
3. $y=2^{-x^3}\operatorname{arctg} 7x^4$.
4. $y=(x+6)^5\operatorname{arcctg} 3x^5$.
5. $y=3^{\cos x}\ln(x^2-3x+7)$.
6. $y=\log_2(x-7)\cdot\operatorname{arctg}\sqrt{x}$.
7. $y=\arccos^3 5x\cdot\operatorname{tg} x^4$.
8. $y=(x-5)^7\operatorname{arcctg} 7x^3$
9. $y=\arccos x^2\cdot\operatorname{ctg} 7x^3$.
10. $y=5^{-x^3}\arccos 5x^4$.
11. $y=\operatorname{arctg}^4 x\cdot\cos 7x^4$.
12. $y=4(x-7)^6\arcsin 3x^5$.
13. $y=(x+5)^2\arccos^3 5x$.
14. $y=2^{-\sin x}\arcsin^3 2x$.
15. $y=(x+2)^7\arccos\sqrt{x}$.
16. $y=(x-7)^5\arcsin 7x^4$.
17. $y=\ln(x-3)\cdot\arccos 3x^4$.
18. $y=\log_2(x-4)\cdot\operatorname{arctg}^3 4x$.
19. $y=(x-7)^4\operatorname{arcctg}^2 7x$.

20. $y=\sqrt[3]{x-3}\arccos^4 2x$.
21. $y=\sqrt[3]{x-4}\arcsin^4 5x$.
22. $y=(x-3)^5\arccos 3x^6$
23. $y=\sqrt{(x+3)^5}\arcsin 2x^3$
24. $y=\sqrt[3]{(x+1)^2}\arccos 3x$
25. $y=\operatorname{tg}^3 x\cdot\operatorname{arcctg} 3x$.
26. $y=\sqrt{(x-2)^3}\operatorname{arctg}(7x-1)$
27. $y=\sqrt[5]{(x+4)^2}\arcsin 7x^2$
28. $y=\arcsin^3 4x\cdot\operatorname{ctg} 3x$.
29. $y=e^{-\cos x}\arcsin 2x$.
30. $y=\sqrt{(x+5)^3}\arccos^4 x$.

6

1. $y=(x-3)^4\arccos 5x^3$.
2. $y=(3x-4)^3\arccos 3x^2$
3. $y=\operatorname{sh}^3 4x\cdot\arccos\sqrt{x}$.
4. $y=\operatorname{th}^2\sqrt{x}\cdot\operatorname{arcctg} 3x^2$.
5. $y=\operatorname{cth}^3 5x\cdot\arcsin 3x^2$.
6. $y=\operatorname{ch}\frac{1}{x}\cdot\operatorname{arctg}(7x+2)$.
7. $y=\operatorname{ch}^3 4x\cdot\arccos 4x^2$.
8. $y=\operatorname{sh}^3 3x\cdot\operatorname{arcctg} 5x^2$.
9. $y=\operatorname{th}^5 3x\cdot\arcsin\sqrt{x}$.
10. $y=\operatorname{cth}^2(x+1)\cdot\arccos\frac{1}{x}$
11. $y=\operatorname{sh}^4 2x\cdot\arccos x^2$.
12. $y=\operatorname{ch}^3(3x+2)\cdot\operatorname{arctg} 3x$.
13. $y=\operatorname{th}^3 4x\cdot\operatorname{arcctg} 3x^4$.
14. $y=\operatorname{cth}^4 7x\cdot\arcsin\sqrt{x}$.
15. $y=\operatorname{sh}^3 2x\cdot\arcsin 7x^2$.
16. $y=\operatorname{th}^5 4x\cdot\arccos 3x^4$.
17. $y=\operatorname{ch}^2 5x\cdot\operatorname{arctg}\sqrt{x}$.
18. $y=\operatorname{cth}^4 2x\cdot\operatorname{arctg} x^3$.
19. $y=\operatorname{sh}^4 5x\cdot\arccos 3x^2$.
20. $y=\operatorname{ch}^3 9x\cdot\operatorname{arctg}(5x-1)$.

21. $y=\operatorname{th}^4 x\cdot\operatorname{arcctg}\frac{1}{x}$.
22. $y=\operatorname{cth}^3 4x\cdot\arcsin(3x+1)$
23. $y=\operatorname{ch}^2 5x\cdot\operatorname{arctg} x^4$.
24. $y=\operatorname{th}^4 7x\cdot\arccos x^3$.
25. $y=\operatorname{cth} 4x^5\cdot\arccos 2x$.
26. $y=\operatorname{cth} 3x\cdot\arcsin^4 2x$.
27. $y=\operatorname{th}^5 3x\cdot\operatorname{arcctg}\sqrt{x}$.
28. $y=\operatorname{sh}^4 3x\cdot\arccos 5x^4$.
29. $y=\operatorname{cth}^2 4x\cdot\arcsin x^3$.
30. $y=\operatorname{th}^3 5x\cdot\operatorname{arctg}(2x-5)$.

7

1. $y=\frac{e^{\arccos^3 x}}{\sqrt{x+5}}$.
2. $y=\frac{(x-4)^2}{e^{\operatorname{arcctg} x}}$.
3. $y=\frac{e^{-x^3}}{\sqrt{x^2+5x-1}}$.
4. $y=\frac{e^{-\operatorname{ctg} 5x}}{(3x^2-4x+2)}$.
5. $y=\frac{\sqrt{7x^3-5x+2}}{e^{\cos x}}$.
6. $y=\frac{e^{\operatorname{tg} 3x}}{\sqrt{3x^2-x+4}}$.
7. $y=\frac{e^{\sin x}}{(x-5)^7}$.
8. $y=\frac{\sqrt[3]{2x^2-3x+1}}{e^{-x}}$.
9. $y=\frac{\sqrt{x^3+4x-5}}{e^{x^4}}$.
10. $y=\frac{e^{\operatorname{ctg} 5x}}{(x+4)^3}$.
11. $y=\frac{\sqrt{3+2x-x^2}}{e^x}$.
12. $y=\frac{e^{3x}}{\sqrt{3x^2-4x-7}}$.
13. $y=\frac{e^{-\sin 2x}}{(x+5)^4}$.
14. $y=\frac{e^{\cos 5x}}{\sqrt{x^2-5x-2}}$.
15. $y=\frac{(2x+5)^3}{e^{\operatorname{tg} x}}$.
16. $y=\frac{e^{-\operatorname{tg} 3x}}{4x^2-3x+5}$.
17. $y=\frac{e^{-\sin 4x}}{(2x-5)^6}$.
18. $y=\frac{3x^2-5x+10}{e^{-x^4}}$.

19. $y=\frac{e^{-x}}{(2x^2-x+4)^2}$.
20. $y=\frac{e^{4x}}{(3x+5)^3}$.
21. $y=\frac{e^{\operatorname{ctg} 5x}}{(3x-5)^4}$.
22. $y=\frac{(2x-3)^7}{e^{-2x}}$.
23. $y=\frac{(3x+1)^4}{e^{4x}}$.
24. $y=\frac{5x^2+4x-2}{e^{-x}}$.
25. $y=\frac{\sqrt{5x^2-x+1}}{e^{3x}}$.
26. $y=\frac{e^{-x^2}}{(2x-5)^7}$.
27. $y=\frac{e^{\cos 3x}}{(2x+4)^5}$.
28. $y=\frac{e^{\sin 5x}}{(3x-2)^2}$.
29. $y=\frac{\sqrt{x^2-3x-7}}{e^{-x^3}}$.
30. $y=\frac{e^{-\operatorname{tg} x}}{4x^2+7x-5}$

8

1. $y=\frac{\log_5(3x-7)}{\operatorname{ctg} 7x^3}$.
2. $y=\frac{\ln(5x-3)}{4\operatorname{tg} 3x^4}$
3. $y=\frac{\ln(7x+2)}{5\cos 42x}$.
4. $y=\frac{\sin^3 5x}{\ln(2x-3)}$
5. $y=\frac{\cos^2 3x}{\lg(3x-4)}$.
6. $y=\frac{\operatorname{tg}^3 2x}{\lg(5x+1)}$
7. $y=\frac{\log_3(4x+5)}{2\operatorname{ctg}\sqrt{x}}$.
8. $y=\frac{\ln(7x-3)}{3\operatorname{tg}^2 4x}$
9. $y=\frac{\lg(11x+3)}{\cos^2 5x}$.
10. $y=\frac{\operatorname{ctg}^2 5x}{\ln(7x-2)}$.
11. $y=\frac{\operatorname{tg}^2(x-2)}{\lg(x+3)}$.
12. $y=\frac{\sin^3(5x+1)}{\lg(3x-2)}$
13. $y=\frac{\cos^4(7x-1)}{\lg(x+5)}$.
14. $y=\frac{\sin^3(4x+3)}{\ln(7x+1)}$
15. $y=\frac{\operatorname{ctg}^3(2x-3)}{\log_3(x+2)}$.
16. $y=\frac{\lg^3 x}{\sin 5x^2}$.

17. $y=\dfrac{\ln^2(x+1)}{\cos 3x^4}$.

18. $y=\dfrac{\log_2(7x-5)}{\operatorname{tg}\sqrt{x}}$

19. $y=\dfrac{\log_3(4x-2)}{\operatorname{ctg} 2x}$.

20. $y=\dfrac{\ln^3(x-5)}{\operatorname{tg}(1/x)}$.

21. $y=\dfrac{\lg(x+2)}{\sin 2x^5}$.

22. $y=\dfrac{\operatorname{tg}^3 7x}{\ln(3x+2)}$.

23. $y=\dfrac{\operatorname{ctg}\sqrt{x-2}}{\lg(3x+5)}$.

24. $y=\dfrac{\operatorname{tg}(3x-5)}{\ln^2(x+3)}$.

25. $y=\dfrac{\cos^2 x}{\lg(x^2-2x+1)}$.

26. $y=\dfrac{\log_2(3x+7)}{\operatorname{tg} 3x}$.

27. $y=\dfrac{\ln^3 x}{\operatorname{ctg}(x-3)}$.

28. $y=\dfrac{\operatorname{tg}^4 5x}{\ln(x+7)}$.

29. $y=\dfrac{\log_3(x+4)}{\cos^5 x}$.

30. $y=\dfrac{\operatorname{tg}^4 3x}{\lg(x^2-x+4)}$

9

1. $y=\dfrac{\operatorname{arcctg}^4 5x}{\operatorname{sh}\sqrt{x}}$.

2. $y=\dfrac{\operatorname{arctg}^3 2x}{\operatorname{ch}(1/x)}$.

3. $y=\dfrac{\arccos 3x^4}{\operatorname{th}^2 x}$.

4. $y=\dfrac{\arcsin 5x^3}{\operatorname{ch}\sqrt{x}}$

5. $y=\dfrac{\operatorname{cth}^3(x+1)}{\arccos 2x}$.

6. $y=\dfrac{\operatorname{th} 3x^5}{\operatorname{arctg}^2 3x}$.

7. $y=\dfrac{\arccos^7 2x}{\operatorname{th} x^5}$.

8. $y=\dfrac{\arcsin^3 4x}{\operatorname{sh}(3x+1)}$

9. $y=\dfrac{\operatorname{th}^4(2x+5)}{\arccos 3x}$.

10. $y=\dfrac{\sqrt[3]{\operatorname{arctg} 2x}}{\operatorname{sh}^2 x}$.

11. $y=\dfrac{\arcsin^2 4x}{\operatorname{th}(5x-3)}$.

12. $y=\dfrac{\operatorname{ch}^2(4x+2)}{\operatorname{arctg} x^3}$.

13. $y=\dfrac{\arcsin 4x^5}{\operatorname{th}^3 x}$.

14. $y=\dfrac{\operatorname{arctg}^3(2x+1)}{\operatorname{ch}\sqrt{x}}$

15. $y=\dfrac{\arccos 4x^3}{\operatorname{sh}^4 x}$.

16. $y=\dfrac{\operatorname{cth}^2(x-2)}{\arccos 3x}$.

17. $y=\dfrac{\operatorname{th}^3(2x+2)}{\arcsin 5x}$

18. $y=\dfrac{\operatorname{cth}^2(3x-1)}{\arccos x^2}$

19. $y=\dfrac{\operatorname{sh}^5 x}{\arccos 4x}$.

20. $y=\dfrac{\sqrt{\operatorname{ch}^3 x}}{\operatorname{arctg} 5x}$.

21. $y=\dfrac{\operatorname{th}^2(x+3)}{\operatorname{arcctg}\sqrt{x}}$.

22. $y=\dfrac{\arcsin^2 3x}{\operatorname{ch}(x-5)}$

23. $y=\dfrac{\operatorname{arcctg}^3 x}{\operatorname{sh}(2x-5)}$.

24. $y=\dfrac{\arccos^3 5x}{\operatorname{th}(x-2)}$

25. $y=\dfrac{\sqrt{\arccos 3x}}{\operatorname{sh}^2 x}$.

26. $y=\dfrac{\arcsin^2 3x}{\sqrt{\operatorname{th} x}}$

27. $y=\dfrac{\operatorname{arctg}^2 5x}{\sqrt[3]{\operatorname{cth} x}}$.

28. $y=\dfrac{\operatorname{arctg}^2 5x}{\operatorname{th}(x+3)}$.

29. $y=\dfrac{\sqrt{\operatorname{sh}^3 x}}{\operatorname{arcctg} 5x}$.

30. $y=\dfrac{\sqrt[5]{\operatorname{ch} 3x}}{\operatorname{arctg}(x+2)}$

10

1. $y=\dfrac{9\operatorname{arctg}(x+7)}{(x-1)^2}$.

2. $y=\dfrac{8\operatorname{arctg}(2x+3)}{(x+1)^3}$

3. $y=\dfrac{7\arccos(4x-1)}{(x+2)^4}$.

4. $y=\dfrac{6\arcsin(x+5)}{(x-2)^5}$

5. $y=\dfrac{3\operatorname{arcctg}(2x-5)}{(x+1)^4}$.

6. $y=\dfrac{2\operatorname{arctg}(3x+2)}{(x-3)^2}$

7. $y=\dfrac{4\arccos 3x}{(x+2)^5}$.

8. $y=\dfrac{\arcsin(3x+8)}{(x-7)^3}$.

9. $y=\dfrac{7\operatorname{arctg}(4x+1)}{(x-4)^2}$.

10. $y=\dfrac{3\arcsin(2x-7)}{(x+2)^4}$

11. $y=\dfrac{2\lg(4x+5)}{(x+6)^4}$.

12. $y=\dfrac{5\ln(5x+7)}{(x-7)^2}$.

13. $y=\dfrac{4\log_3(3x+1)}{(x+1)^2}$.

14. $y=\dfrac{7\log_4(2x-5)}{(x-1)^5}$.

15. $y=\dfrac{\ln(7x+2)}{(x-6)^4}$.

16. $y=\dfrac{4\lg(3x+7)}{(x+1)^7}$.

17. $y = \frac{5\log_2(x^2+1)}{(x-3)^4}$.

18. $y = \frac{6\log_3(2x+9)}{(x+4)^2}$.

19. $y = \frac{3\log_2(5x-4)}{(x-3)^5}$.

20. $y = \frac{7\log_5(x^2+x)}{(x+3)^3}$.

21. $y = \frac{\log_7(2x^2+5)}{(x-4)^2}$.

22. $y = \frac{2\ln(3x-10)}{(x+5)^7}$.

23. $y = \frac{8\lg(4x+5)}{(x-1)^5}$.

24. $y = \frac{2\log_3(4x-7)}{(x+3)^4}$.

25. $y = \frac{3\log_4(2x+9)}{(x-7)^2}$.

26. $y = \frac{\lg(x^2+2x)}{(x+8)^4}$.

27. $y = \frac{3\ln(x^2+5)}{(x-7)^3}$.

28. $y = \frac{4\log_2(3x-5)}{(x-2)^2}$.

29. $y = \frac{2\ln(2x^2+3)}{(x-7)^4}$.

30. $y = \frac{4\lg(3x+7)}{(x-5)^3}$.

Topic 5. Differentiation of different functions. Differential.

5.1. Differentiation of inverse, parameter and implicit functions. 5.2. Logarithmic differentiation. 5.3. Differential of the function. 5.4. Properties of differential.

5.1. Differentiation of inverse, parameter and implicit functions.

5.1.1. Differentiation of the inverse function.

If for some function y=f(x) the inverse function exists x=g(y) and $g'(y) \neq 0$, then

$$f'(x) = \frac{1}{g'(y)}.$$

5.1.2. Differentiation of parameter functions.

Let the function is given as parameter function: $\begin{cases} x = \varphi(t) \\ y = \psi(t), \end{cases}$

If the functions $\varphi'(t)$ $\psi'(t)$ exist at the point t and $\varphi'(t) \neq 0$, then y=f(x) is differentiable at the point x and $\mathbf{f'(x)} = \frac{\psi'(t)}{\varphi'(t)}$.

5.1.3. Differentiation of implicit functions.

To find the derivative of implicit function F(on the other handx,y)=0 need differentiate both sides of given equation, taking into account that y is function of x, then express y'.

On the other hand $y'_x = -\frac{F'_x}{F'_y}$.

5.2. Logarithmic differentiation.

The consistent application 1) taking the logarithm and 2) differentiation to function y=f(x) is called **logarithmic differentiation**.

Logarithmic derivative of the function y=f(x) is called the derivative of logarithm of this function, i.e.

$$(\ln f(x))' = \frac{f'(x)}{f(x)}.$$

Sometimes preliminary taking the logarithm of the function simplifies the finding of its derivative.

For example: **I. for function** $y = u(x)^{v(x)}$

1) taking the natural logarithm : $\ln y = \ln(u(x)^{v(x)})$

$$\ln y = v(x)\cdot \ln(u(x))$$

2) differentiation of both sides of the equation, taking into account that at the left side y is function of x and at the right side the derivative of multiplication of two functions.

$$\frac{y'}{y} = v'\cdot\ln u + v\cdot\frac{u'}{u} \quad\Rightarrow\quad y' = y(v'\cdot\ln u + v\cdot\frac{u'}{u})$$

Substitute $y=u(x)^{v(x)}$ $\Rightarrow$ $y' = u^{v}(v'\cdot\ln u + v\cdot\frac{u'}{u})$.

II. for function $y = \frac{u^k(x)\cdot v^l(x)}{s^m(x)\cdot p^n(x)}$

1) taking the natural logarithm :

$\ln y = k \ln(u(x))+l\cdot \ln(v(x))- m \ln(s(x))+ n \ln(p(x))$;

2) differentiation of both sides of the equation

$$\frac{y'}{y} = k\frac{u'}{u} + l\frac{v'}{v} - m\frac{s'}{s} - n\frac{p'}{p} \Rightarrow y' = y\,(k\frac{u'}{u} + l\frac{v'}{v} - m\frac{s'}{s} - n\frac{p'}{p})$$

Substitute $y=\frac{u^k(x)\cdot v^l(x)}{s^m(x)\cdot p^n(x)}$ $\Rightarrow$

$$y' = \frac{u^k(x)\cdot v^l(x)}{s^m(x)\cdot p^n(x)}\cdot(k\frac{u'}{u} + l\frac{v'}{v} - m\frac{s'}{s} - n\frac{p'}{p})$$

5.3. Differential of the function.

5.3.1. Definition and geometrical meaning of the differential.

Let y=f(x) is defined and differentiable over X, then the derivative of f(x) exists $f'(x)=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ this means that the ratio of increments can be expressed as sum of derivative and infinitesimal with respect Δx, $\alpha(\Delta x)$: $\frac{\Delta y}{\Delta x} = f'(x) + \alpha(\Delta x)$ $\Rightarrow$

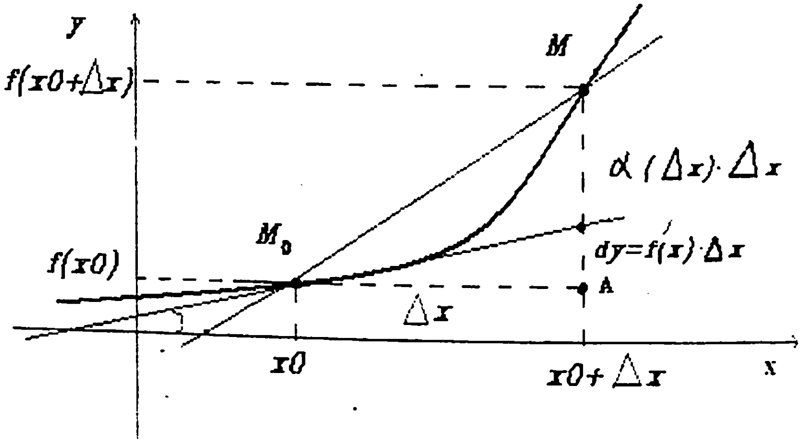
$\Delta y = f'(x)\cdot\Delta x + \alpha(\Delta x)\cdot\Delta x$ - The increment of the function consists from linear part $f'(x)\cdot\Delta x$ and nonlinear part $\alpha(\Delta x)\cdot\Delta x$ regarding Δx.

<u>**Definition:**</u> The main linear regarding Δx part of increment of the function is called **the differential** of the function is denoted by **dy** or **df(x)**. The differential is equal:

$$dy = f'(x)\cdot\Delta x.$$

Geometrical meaning of the differential.

The differential of the function y=f(x) **dy** at the point x is the increment of the ordinate of the tangent to the graph of the function y=f(x) at this point x.



Differential of independent variables is equal to the increment of this variable. Really $\quad dx = x'\cdot\Delta x \;\Rightarrow\; dx = \Delta x$.

Thus the differential of the function can be represented as:

$$\mathbf{dy = f'(x)\cdot dx.}$$

Theorem: Differential of the function at the point $x=x_0$ exists if and only if the function is differentiable at the point $x=x_0$ and

$$\mathbf{df(x_0) = f'(x_0)\cdot dx.}$$

5.3.2. Properties of differential of the function.

1) $d(f\pm g) = df \pm dg$;

2) $d(f\cdot g) = d(f)\cdot g + f\cdot d(g)$;

3) $d\left(\dfrac{f}{g}\right) = \dfrac{df\cdot g - f\cdot dg}{g^2}$ $(g\neq 0)$;

4) Differential of **the composite function** (Invariance of differential).

If the function $u = u(x)$ is differentiable at the point x_0, $u(x_0)=u_0$, and the function $y = f(u)$ is differentiable at the point u_0, then the composite function $y = f(u(x))$ is differentiable at the point x_0 and its differential is equal:

$$\mathbf{df(u) = f'(u)\cdot u'(x)\,dx = f'(u)\cdot du\,.}$$

EXERCISE V.

Differentiate the functions:

1

1. $y=(\operatorname{cth} 3x)^{\arcsin x}$.
2. $y=(\cos(x+2))^{\ln x}$.
3. $y=(\sin 3x)^{\arccos x}$.
4. $y=(\operatorname{th} 5x)^{\arcsin(x+1)}$.
5. $y=(\operatorname{sh}(x+2))^{\arcsin 2x}$.
6. $y=(\cos 5x)^{\operatorname{arctg}\sqrt{x}}$.
7. $y=(\sqrt{3x+2})^{\operatorname{arcctg} 3x}$.
8. $y=(\ln(x+3))^{\sin\sqrt{x}}$.
9. $y=(\log_2(x+4))^{\operatorname{ctg} 7x}$.
10. $y=(\operatorname{sh} 3x)^{\operatorname{arctg}(x+2)}$.
11. $y=(\operatorname{ch} 3x)^{\operatorname{ctg} 1/x}$.
12. $y=(\arcsin 5x)^{\operatorname{tg}\sqrt{x}}$.
13. $y=(\arccos 5x)^{\ln x}$.
14. $y=(\operatorname{arctg} 2x)^{\sin x}$.
15. $y=(\ln(x+7))^{\operatorname{ctg} 2x}$.
16. $y=(\operatorname{ctg}(7x+4))^{\sqrt{x+3}}$.
17. $y=(\operatorname{th}\sqrt{x+1})^{\operatorname{arctg} 2x}$.
18. $y=\left(\operatorname{cth}\dfrac{1}{x}\right)^{\arcsin 7x}$
19. $y=(\cos(x+5))^{\arcsin 3x}$.
20. $y=(\sqrt{x+5})^{\arccos 3x}$
21. $y=(\sin 4x)^{\operatorname{arctg} 1/x}$.
22. $y=(\operatorname{tg} 3x^4)^{\sqrt{x+3}}$.
23. $y=(\operatorname{ctg} 2x^3)^{\sin\sqrt{x}}$.
24. $y=(\operatorname{tg} 7x^5)^{\sqrt{x+2}}$.
25. $y=(\arccos x)^{\sqrt{\cos x}}$.
26. $y=(\operatorname{ctg} 7x)^{\operatorname{sh}(x+3)}$.
27. $y=(\operatorname{sh} 5x)^{\operatorname{arctg}(x+2)}$.
28. $y=(\operatorname{arctg} x)^{\operatorname{th}(3x+1)}$
29. $y=(\operatorname{cth}\sqrt{x})^{\sin(x+3)}$.
30. $y=(\operatorname{sh} 3x)^{\operatorname{arcctg} 2x}$

2

1. $y=(\arccos(x+2))^{\operatorname{tg} 3x}$.
2. $y=(\arcsin 2x)^{\operatorname{ctg}(x+1)}$.
3. $y=(\operatorname{arctg}(x+7))^{\cos 2x}$.
4. $y=(\operatorname{arcctg}(x-3))^{\sin 4x}$.
5. $y=(\operatorname{ctg}(3x-2))^{\arcsin 3x}$.
6. $y=(\operatorname{tg}(4x-3))^{\arccos 2x}$.
7. $y=(\cos(2x-5))^{\operatorname{arctg} 5x}$.
8. $y=(\sin(7x+4))^{\operatorname{arcctg} x}$.
9. $y=(\arcsin 2x)^{\ln(x+3)}$.
10. $y=(\arccos 3x)^{\lg(5x-1)}$.
11. $y=(\operatorname{arctg} 5x)^{\log_2(x+4)}$.
12. $y=(\operatorname{arctg} 7x)^{\lg(x+1)}$.
13. $y=(\log_4(2x+3))^{\arcsin x}$.
14. $y=(\log_5(3x+2))^{\arccos x}$
15. $y=(\lg(7x-5))^{\operatorname{arctg} 2x}$.
16. $y=(\ln(5x-4))^{\operatorname{arcctg} x}$.
17. $y=(\log_2(6x+5))^{\arcsin 2x}$.
18. $y=(\lg(4x-3))^{\arccos 4x}$.
19. $y=(\ln(7x-3))^{\operatorname{arctg} 5x}$.
20. $y=(\log_5(2x+5))^{\operatorname{arctg} x}$.

21. $y=(\sin(8x-7))^{\operatorname{cth}(x+3)}$.

22. $y=(\cos(3x+8))^{\operatorname{th}(x-7)}$.

23. $y=(\operatorname{tg}(9x+2))^{\operatorname{ch}(2x-1)}$.

24. $y=(\operatorname{ctg}(7x+5))^{\operatorname{sh}3x}$.

25. $y=(\operatorname{sh}(3x-7))^{\cos(x+4)}$.

26. $y=(\operatorname{ch}(2x-3))^{\operatorname{tg}(x+5)}$

27. $y=(\operatorname{th}(7x-5))^{\sin(x+2)}$.

28. $y=(\operatorname{ch}(3x+2))^{\cos(x+4)}$.

29. $y=(\ln(7x+4))^{\operatorname{tg}x}$.

30. $y=(\lg(8x+3))^{\operatorname{tg}5x}$.

3

1. $y=\dfrac{\sqrt{x+7}(x-3)^4}{(x+2)^5}$.

2. $y=\dfrac{(x-3)^5(x+2)^3}{\sqrt{(x-1)^3}}$

3. $y=\dfrac{(x-2)^3\sqrt{(x+1)^5}}{(x-4)^2}$.

4. $y=\dfrac{(x+3)\sqrt[5]{(x-2)^2}}{(x+1)^7}$

5. $y=\dfrac{(x+2)^7(x-3)^3}{\sqrt{(x+1)^5}}$.

6. $y=\dfrac{(x-1)^4(x+2)^5}{\sqrt[3]{(x-4)^2}}$.

7. $y=\dfrac{(x-3)^2\sqrt{x+4}}{(x+2)^7}$.

8. $y=\dfrac{(x-7)^{10}\sqrt{3x-1}}{(x+3)^5}$

9. $y=\dfrac{(x+1)^8(x-3)^2}{\sqrt{(x+2)^5}}$.

10. $y=\dfrac{(x+2)(x-7)^4}{\sqrt[3]{(x-1)^4}}$.

11. $y=\dfrac{\sqrt[5]{(x+4)^3}}{(x-1)^2(x+3)^5}$.

12. $y=\dfrac{\sqrt[3]{(x-1)^7}}{(x+1)^5(x-5)^3}$

13. $y=\dfrac{\sqrt{(x+2)^3}(x-1)^4}{(x+2)^7}$.

14. $y=\dfrac{\sqrt[3]{(x-2)^5}(x+3)^2}{(x-7)^3}$

15. $y=\dfrac{\sqrt[4]{x-8}(x+2)^6}{(x-1)^5}$.

16. $y=\dfrac{\sqrt[5]{x+1}(x-3)^7}{(x+8)^3}$.

17. $y=\dfrac{\sqrt[7]{(x-2)^4}}{(x+1)^2(x-6)^5}$.

18. $y=\dfrac{\sqrt[5]{(x+1)^2}}{(x-3)^4(x-4)^3}$

19. $y=\dfrac{\sqrt{x^2+2x-3}}{(x+3)^7(x-4)^2}$.

20. $y=\dfrac{\sqrt[3]{(x-2)^4}}{(x-5)(x+1)^7}$.

21. $y=\dfrac{(x+4)^3(x-2)^4}{\sqrt[3]{(x-2)^5}}$.

22. $y=\dfrac{(x-1)^6(x+2)^3}{\sqrt[5]{(x+3)^2}}$

23. $y=\dfrac{(x-1)^4(x-7)^2}{\sqrt[3]{(x+2)^5}}$.

24. $y=\dfrac{(x+7)^2(x-3)^5}{\sqrt{x^2+3x-1}}$

25. $y=\dfrac{\sqrt[3]{x-3}(x+7)^5}{(x-4)^2}$.

26. $y=\dfrac{\sqrt{x+10}(x-8)^3}{(x-1)^5}$.

27. $y=\dfrac{\sqrt[5]{(x-2)^3}(x-1)}{(x+3)^4}$.

28. $y=\dfrac{\sqrt[4]{(x+1)^3}(x-2)^5}{(x-3)^2}$

29. $y=\dfrac{\sqrt[6]{(x-1)^5}}{(x+2)^4(x-5)^7}$.

30. $y=\dfrac{\sqrt[5]{(x+2)^3}}{(x-1)^4(x-3)^5}$.

Topic 6. Higher order derivatives and differentials. Applications of differential in approximate calculations.

6.1. Higher order derivatives. Leibniz's rule. 6.2. Higher order differentials. 6.3. Applications of differential in approximate calculations.

6.1. Higher order derivatives.

6.1.1.<u>Definition:</u> The derivative $y'=\frac{dy}{dx}$ is the first derivative of y with respect to x . The first derivative may also be differentiable function of x. If so its derivative $y''=\frac{dy'}{dx}=\frac{d}{dx}(\frac{dy}{dx})=\frac{d^2y}{dx^2}$ is **the second order derivative** of y with respect to x. If y'' is differentiable its derivative

$y'''=\frac{dy''}{dx}=\frac{d}{dx}(\frac{d^2y}{dx^2})=\frac{d^3y}{dx^3}$ is **the third order derivative** of y with respect to x. If continue the process we obtain general formulae

$y^{(k)}=\frac{dy^{(k-1)}}{dx}=\frac{d}{dx}(\frac{d^{k-1}y}{dx^{k-1}})=\frac{d^ky}{dx^k}$ -is **the k – order derivative** of y with respect to x.

6.1.2. k-order derivative of some elementary functions:

1) $(x^n)^{(k)}=n(n-1)(n-2)\ldots(n-(k-1))\,x^{n-k}$, in particular, $(x^n)^{(n)}=n!$;

2) $(a^x)^{(k)}=a^x\cdot(\ln a)^k$; in particular $(e^x)^{(k)}=e^x$;

3) $(\ln x)^{(k)}=\frac{(-1)^{k-1}(k-1)!}{x^k}$;

4) $(\sin x)^{(k)}=\sin(x+n\cdot\frac{\pi}{2})$;

5) $(\cos x)^{(k)}=\cos(x+n\cdot\frac{\pi}{2})$.

Properties of derivative of n- order of the function. If f(x) and g(x) k times differentiable functions, then:

1) $(f(x)\pm g(x))^{(k)}=f^{(k)}(x)\pm g^{(k)}(x)$;

2) $(C\cdot f(x))^{(k)}=C\cdot f^{(k)}(x)$.

6.1.3. Leibniz's rule.

If f(x) and g(x) k- times differentiable functions, then **k –order derivative of multiplication** of two functions is expressed by **Leibniz's rule:**

$$(f\cdot g)^{(k)}=f^{(k)}g+kf^{(k-1)}g'+\frac{k(k-1)}{2}f^{(k-2)}g''+\ldots+\frac{k!}{m!(k-m)!}f^{(k-m)}g^{(m)}+\ldots+kf'g^{(k-1)}+fg^{(k)}=$$

$$=\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}f^{(k-m)}g^{(m)} - Leibniz's.......rule$$

6.2. Higher order differentials.

$dy=f'dx$ - is **the first differential** of f with respect to **x** .

$d^2y=d(dy)=d(f'dx)$ | assumed that $dx=\Delta x$ - const | $=d(f')\cdot dx=f''\cdot dx\cdot dx=f''\cdot(dx)^2$, thus,

$d^2y=f''\cdot(dx)^2$ - is **the second order differential** of f with respect to x.

If continue the process we obtain general formulae:

$d^ky=d(d^{k-1}f)=d(f^{(k-1)}(dx)^{k-1})=$ | assumed that $dx=\Delta x$ - const| $=f^{(k)}dx\,(dx)^{k-1}=f^{(k)}(dx)^k$, thus

$d^ky=f^{(k)}(dx)^k$ is **the k – order differential** of f with respect to x.

For differentials of k-order also analogical operations are true:

1) $d^k(f(x) \pm g(x)) = d^k f(x) \pm d^k g(x)$;

2) $d^k(C \cdot f(x)) = C \cdot d^k f(x)$;

3) $d^k(f \cdot g) = \sum_{m=1}^{k} \frac{k!}{m!(k-m)!} d^{k-m}(f) \cdot d^m(g)$ - **Leibniz's rule.**

6.3. Applications of differential in approximate calculations.

It is known that the increment of the function can be represented as: $\Delta y = f'(x) \cdot \Delta x + \alpha(\Delta x) \cdot \Delta x$.

The second term can be neglected because it more infinitesimal than first term as $\Delta x \to 0$ =>

$\Delta y \approx f'(x) \cdot \Delta x$. Taking into account that $\Delta x = dx$ and $\Delta y = f(x + \Delta x) - f(x)$ we obtain next:

$f(x + \Delta x) - f(x) \approx f'(x) \cdot dx$ or

$\mathbf{f(x + \Delta x) \approx f(x) +} f'(x) \cdot dx$ **- formula for approximate calculations the value of the functions at point $(x + \Delta x)$.**

EXERCISE 6.

1

Find y' and y''.

1. $y^2 = 8x$.
2. $x^2/5 + y^2/7 = 1$.
3. $y = x + \text{arctg}\, y$.
4. $x^2/5 + y^2/3 = 1$.
5. $y^2 = 25x - 4$.
6. $\text{arcctg}\, y = 4x + 5y$.
7. $y^2 - x = \cos y$.
8. $3x + \sin y = 5y$.
9. $\text{tg}\, y = 3x + 5y$.
10. $xy = \text{ctg}\, y$.
11. $y = e^y + 4x$.
12. $\ln y - y/x = 7$.
13. $y^2 + x^2 = \sin y$.
14. $e^y = 4x - 7y$.
15. $4 \sin^2(x + y) = x$.
16. $\sin y = 7x + 3y$.
17. $\text{tg}\, y = 4y - 5x$.
18. $y = 7x - \text{ctg}\, y$.
19. $xy - 6 = \cos y$.
20. $3y = 7 + xy^3$.
21. $y^2 = x + \ln(y/x)$.
22. $xy^2 - y^3 = 4x - 5$.
23. $x^2y^2 + x = 5y$.
24. $x^4 + x^2y^2 + y = 4$.
25. $\sin y = xy^2 + 5$.
26. $x^3 + y^3 = 5x$.
27. $\sqrt{x} + \sqrt{y} = \sqrt{7}$.
28. $y^2 = (x - y)/(x + y)$.
29. $\sin^2(3x + y^2) = 5$.
30. $\text{ctg}^2(x + y) = 5x$.

2

Find y' and y''.

1. $\begin{cases} x = (2t + 3)\cos t, \\ y = 3t^3. \end{cases}$
2. $\begin{cases} x = 2\cos^2 t, \\ y = 3\sin^2 t. \end{cases}$
3. $\begin{cases} x = 6\cos^3 t, \\ y = 2\sin^3 t. \end{cases}$
4. $\begin{cases} x = 1/(t + 2), \\ y = (t/(t + 2))^2. \end{cases}$
5. $\begin{cases} x = e^{-2t}, \\ y = e^{4t}. \end{cases}$
6. $\begin{cases} x = \sqrt{t}, \\ y = \sqrt[5]{t}. \end{cases}$
7. $\begin{cases} x = 2t/(1 + t^3), \\ y = t^2/(1 + t^2). \end{cases}$
8. $\begin{cases} x = \sqrt{t^2 - 1}, \\ y = (t + 1)/\sqrt{t^2 - 1}. \end{cases}$

9. $\begin{cases} x = 4t + 2t^2, \\ y = 5t^3 - 3t^2. \end{cases}$

10. $\begin{cases} x = (\ln t)/t, \\ y = t \ln t. \end{cases}$

11. $\begin{cases} x = e^t \cos t, \\ y = e^t \sin t. \end{cases}$

12. $\begin{cases} x = t^4, \\ y = \ln t. \end{cases}$

13. $\begin{cases} x = 5 \cos t, \\ y = 4 \sin t. \end{cases}$

14. $\begin{cases} x = 5 \cos^2 t, \\ y = 3 \sin^2 t. \end{cases}$

15. $\begin{cases} y = \operatorname{arctg} t, \\ y = \ln (1 + t^2). \end{cases}$

16. $\begin{cases} x = \arcsin t, \\ y = \sqrt{1 - t^2}. \end{cases}$

17. $\begin{cases} x = 3(t - \sin t), \\ y = 3(1 - \cos t). \end{cases}$

18. $\begin{cases} x = 3(\sin t - t \cos t), \\ y = 3(\cos t + t \sin t). \end{cases}$

19. $\begin{cases} x = \sin 2t, \\ y = \cos^2 t. \end{cases}$

20. $\begin{cases} x = e^{3t}, \\ y = e^{-3t}. \end{cases}$

21. $\begin{cases} x = (\ln t)/t, \\ y = t^2 \ln t. \end{cases}$

22. $\begin{cases} x = \arccos t, \\ y = \sqrt{1 - t^2}. \end{cases}$

23. $\begin{cases} x = 1/(t + 1), \\ y = (t/(t + 1))^2. \end{cases}$

24. $\begin{cases} x = 5 \sin^3 t, \\ y = 3 \cos^3 t. \end{cases}$

25. $\begin{cases} x = e^{-3t}, \\ y = e^{8t}. \end{cases}$

26. $\begin{cases} x = \sqrt[3]{(t - 1)^2}, \\ y = \sqrt{t - 1}. \end{cases}$

27. $\begin{cases} x = \ln^2 t, \\ y = t + \ln t. \end{cases}$

28. $\begin{cases} x = te^t, \\ y = t/e^t. \end{cases}$

29. $\begin{cases} x = 6t^2 - 4, \\ y = 3t^5. \end{cases}$

30. $\begin{cases} x = \arcsin t, \\ y = \ln t. \end{cases}$

3

For given function y and its argument x_0 calculate the $y'''(x_0)$:

1. $y = \sin^2 x, \; x_0 = \pi/2.$
2. $y = \operatorname{arctg} x, \; x_0 = 1.$
3. $y = \ln (2 + x^2), \; x_0 = 0.$
4. $y = e^x \cos x, \; x_0 = 0$
5. $y = e^x \sin 2x, \; x_0 = 0.$
6. $y = e^{-x} \cos x, \; x_0 = 0.$
7. $y = \sin 2x, \; x_0 = \pi.$
8. $y = (2x + 1)^5, \; x_0 = 1.$
9. $y = \ln (1 + x), \; x_0 = 2.$
10. $y = \frac{1}{2} x^2 e^x, \; x_0 = 0$
11. $y = \arcsin x, \; x_0 = 0.$
12. $y = (5x - 4)^5, \; x_0 = 2.$
13. $y = x \sin x, \; x_0 = \pi/2.$
14. $y = x^2 \ln x, \; x_0 = 1/3.$
15. $y = x \sin 2x, \; x_0 = -\pi/4.$
16. $y = x \cos 2x, \; x_0 = \pi/12.$

17. $y = x^4 \ln x, \; x_0 = 1.$
18. $y = x + \operatorname{arctg} x, \; x_0 = 1.$
19. $y = \cos^2 x, \; x_0 = \pi/4.$
20. $y = \ln (x^2 - 4), \; x_0 = 3.$
21. $y = x^2 \cos x, \; x_0 = \pi/2$
22. $y = x \arccos x, \; x_0 = \sqrt{3}/2.$
23. $y = (x + 1) \ln (x + 1), \; x_0 = -1/2.$
24. $y = \ln^3 x, \; x_0 = 1.$
25. $y = 2^{x^2}, \; x_0 = 1.$
26. $y = (4x - 3)^5, \; x_0 = 1.$
27. $y = x \operatorname{arcctg} x, \; x_0 = 2.$
28. $y = (7x - 4)^6, \; x_0 = 1.$
29. $y = x \sin 2x, \; x_0 = \pi/4.$
30. $y = \sin (x^3 + \pi), \; x_0 = \sqrt[3]{\pi}.$

4

Find the n – order derivative of the functions:

1. $y = \ln x.$
2. $y = 1/x.$
3. $y = 2^x.$
4. $y = \cos x.$
5. $y = \sin x.$
6. $y = 1/(x + 5).$
7. $y = e^{-2x}.$
8. $y = \ln (3 + x).$
9. $y = \sqrt{x}.$
10. $y = xe^{3x}.$
11. $y = 1(x - 3).$
12. $y = \ln (5 + x^2).$
13. $y = e^{4x}.$
14. $y = 1/(x - 7).$
15. $y = 5^x.$
16. $y = e^{-5x}.$
17. $y = \ln (4 + x).$
18. $y = 1/(x - 6).$
19. $y = 10^x.$
20. $y = 7^x.$
21. $y = \cos 3x.$
22. $y = \ln (3x - 5).$
23. $y = \frac{x}{x + 5}.$
24. $y = \ln \frac{1}{4 - x}.$
25. $y = \sqrt{x + 7}.$
26. $y = xe^{6x}.$
27. $y = \frac{4}{x + 3}.$
28. $y = \frac{1 + x}{\sqrt{x}}.$
29. $y = \frac{1}{1 + x}.$
30. $y = \ln (5x - 1)$

5

Calculate approximately applying the differential and estimate the accuracy to within 0,01

1. $\sqrt[5]{34}$.
2. $\sqrt[3]{26,19}$.
3. $\sqrt[4]{16,64}$.
4. $\sqrt{8,76}$.
5. $\sqrt[5]{31}$.
6. $\sqrt[3]{70}$.
7. $(2,01)^3+(2,01)^2$.
8. $\sqrt[3]{65}$.
9. $2,9/\sqrt{(2,9)^2+16}$.
10. $\sqrt{\frac{4-3,02}{1+3,02}}$
11. $\sqrt[4]{15,8}$.
12. $\sqrt[3]{10}$.
13. $\sqrt[5]{200}$.
14. $(3,03)^5$.
15. $\sqrt{\frac{(2,037)^2-3}{(2,037)^2+5}}$.
16. $\sqrt[7]{130}$.
17. $\sqrt[3]{27,5}$.
18. $\sqrt{17}$.
19. $\sqrt{640}$.
20. $\sqrt{1,2}$.
21. $\sqrt[10]{1025}$.
22. $(3,02)^4+(3,02)^3$.
23. $(5,07)^3$.
24. $(4,01)^{1,5}$.
25. $\sqrt[3]{1,02}$.
26. $\cos 151°$.
27. arctg 1,05.
28. $\cos 61°$.
29. tg 44°.
30. arctg 0,98

6

Calculate approximately applying the differential and estimate the accuracy to within 0,01

1. arcsin 0,6.
2. arctg 0,95.
3. $e^{0,2}$.
4. lg 11.
5. arcsin 0,54.
6. $\cos 59°$.
7. $e^{2,01}$.
8. ln tg 46°.
9. arctg $\sqrt{1,02}$.
10. arctg $\sqrt{0,97}$.
11. arctg 1,01.
12. $\ln(e^2+0,2)$.
13. arctg 1,03.
14. ln tg 47°15′.
15. lg 9,5.

16. arctg $\sqrt{3,1}$.
17. $2^{2,1}$.
18. $4^{1,2}$.
19. tg 59°.
20. $\log_2 1,9$.
21. arctg $\sqrt{3,2}$.
22. ctg 29°.
23. sin 93°.
24. lg 1,5.
25. sin 29°.
26. lg 101.
27. sin 31°.
28. lg 0,9.
29. $e^{0,25}$.
30. $\sqrt{15}$.

Topic 7. Theorems about differentiable functions.

7.1. Maximum and Minimum. 7.2. The first derivative theorem. Rolle's theorem. The mean value theorem. 7.3.

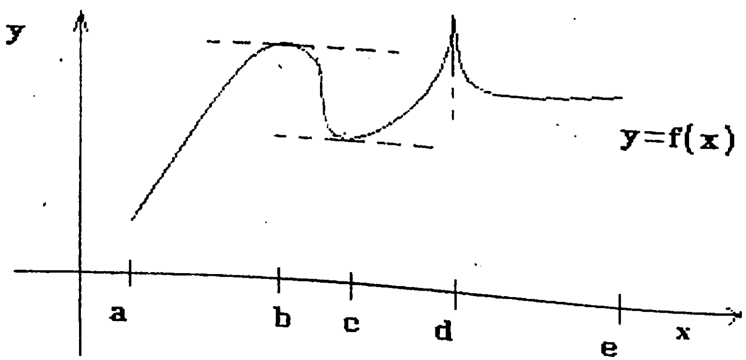
L'Hopital's rule.

7.1. Maximum and Minimum.

Definition: A function f has a **local maximum** value at an interior point **c** of its domain if **f(c)**$\geq$**f(x)** for all x in some open interval U about **c**. The function has an **absolute maximum** value at **c** if **f(c)** $\geq$**f(x)** for all x in the domain.

Similarly, f has a **local minimum** value at an interior point **c** of its domain if **f(c)**$\leq$**f(x)** for all x in some open interval U about **c**. The function has an **absolute minimum** value at **c** if **f(c)**$\leq$**f(x)** for all x in the domain.

Maximum and minimum values are called **extreme values** of the function.

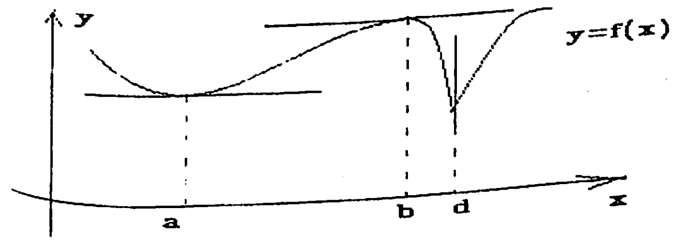


f(a) - absolute minimum; f(b) – local maximum; f(c) – local minimum; f(d) –absolute maximum; f(e) –local minimum.

7.2. The first derivative theorem. Rolle's theorem. The mean value theorem.

7.2.1. The first derivative theorem for local extreme values.

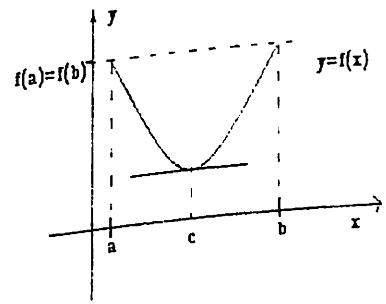
If a function f has a local maximum or a local minimum value at an interior point **c** of an interval where it is defined, and if f ' is defined at c, then **f ' (c) =0 .**



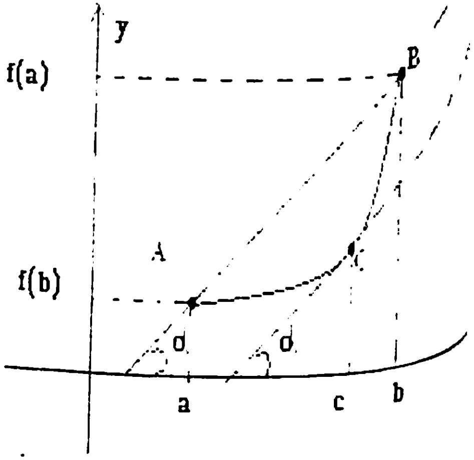
Here f(a) and f(d) – minimums and f(b) – maximum, and f '(a) =0 and f '(b) =0 both are defined and tangents to graph of the function at x=a and x=b are parallel. But f(d) is maximum, but f '(d) is not defined (tangent is vertical) that is why f '(d)$\neq$0 .

7.2.2. Rolle's theorem.

Suppose that y=f(x) is continuous at every point of the closed interval [a,b] and differentiable at every point of its interior (a,b).If f(a)=f(b), then there is at least one number **c** between **a** and **b** at which f '(c)=0 .



7.2.3. The mean value theorem.



If y=f(x) is continuous at every point of the closed interval [a,b] and differentiable at every point of its interior (a,b). then there is at least one number c between a and b at which

$$\frac{f(b)-f(a)}{b-a}=f'(c).$$

7.3. *L 'Hopital's rule.*

Let y=f(x) and y=g(x) two infinitesimals or infinite quantities as x→ a which are differentiable at the vicinity U(a)\ (a), and g(x)≠0 and g'(x)≠0 at U(a)\ (a). Then, if $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ exists then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ exists and they are equal: $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\left|\frac{0}{0}...or...\frac{\infty}{\infty}\right| = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$.

That rule can be applied scores of times until indeterminacy will be eliminated.

Analogical statements are true for x→ a+; x→ a-; x→ +∞ ; x→ -∞; x→ ∞.

To another kind indeterminacy: $0\cdot\infty;\infty-\infty;1^{\infty};0^{0};\infty^{0}$ the L 'Hopital's rule can be applied , but before the function must be reduced to the form: $\left|\frac{0}{0}...or...\frac{\infty}{\infty}\right|$.

EXERSICE VII.

Find the limits applying the L 'Hopital's rule.

1

1. $\lim\limits_{x\to\infty}\frac{\ln(x+5)}{\sqrt[4]{x+3}}$.
2. $\lim\limits_{x\to 0}\frac{a^{\ln x}-x}{x-1}$.
3. $\lim\limits_{x\to 0}\frac{\operatorname{tg} x-x}{x-\sin x}$.
4. $\lim\limits_{x\to 1}\frac{1-4\sin^2(\pi x/6)}{1-x^2}$.
5. $\lim\limits_{x\to a}\arcsin\frac{x-a}{a}\cdot\operatorname{ctg}(x-a)$.
6. $\lim\limits_{x\to\infty}(\pi-2\operatorname{arctg} x)\ln x$.
7. $\lim\limits_{x\to\infty}(a^{1/x}-1)x$.
8. $\lim\limits_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right)$
9. $\lim\limits_{x\to 0}\frac{1-\cos x^2}{x^2-\sin x^2}$.
10. $\lim\limits_{x\to 0}\frac{\operatorname{tg} x-x}{2\sin x+x}$.
11. $\lim\limits_{x\to\infty}\frac{e^{1/x^2}-1}{2\operatorname{arctg} x^2-\pi}$.
12. $\lim\limits_{x\to 1}\frac{x^3-2x^2-x+2}{x^3-7x+6}$
13. $\lim\limits_{x\to 0}\frac{x\cos x-\sin x}{x^3}$.
14. $\lim\limits_{x\to\infty}\frac{e^x}{x^5}$.
15. $\lim\limits_{x\to 1}\frac{1-x}{1-\sin(\pi x/2)}$.
16. $\lim\limits_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}$.
17. $\lim\limits_{x\to 0}\frac{\operatorname{ch} x-1}{1-\cos x}$.
18. $\lim\limits_{x\to 0}\frac{\pi/x}{\operatorname{ctg}(\pi x/2)}$
19. $\lim\limits_{x\to\pi/4}\frac{1/\cos^2 x-2\operatorname{tg} x}{1+\cos 4x}$.
20. $\lim\limits_{x\to 0}\frac{\ln(\sin mx)}{\ln(\sin x)}$.
21. $\lim\limits_{x\to\pi/2}\frac{\operatorname{tg} x}{\operatorname{tg} 5x}$.
22. $\lim\limits_{x\to 0}(1-\cos x)\operatorname{ctg} x$.
23. $\lim\limits_{x\to 1}(1-x)\operatorname{tg}(\pi x/2)$.
24. $\lim\limits_{x\to\infty}x\ \sin(3/x)$
25. $\lim\limits_{x\to -1}\frac{\sqrt[3]{1+2x}+1}{\sqrt{2+x}+x}$.
26. $\lim\limits_{x\to 0}\frac{x\cos x-\sin x}{x^3}$

27. $\lim_{x\to1}\frac{1-x}{1-\sin(\pi x/2)}$.

28. $\lim_{x\to0}\frac{\operatorname{tg}x-\sin x}{4x-\sin x}$.

29. $\lim_{x\to\pi/2}\frac{\operatorname{tg}3x}{\operatorname{tg}5x}$.

30. $\lim_{x\to\pi/4}\frac{\sec^2 x-2\operatorname{tg}x}{1+\cos 4x}$

2

1. $\lim_{x\to0}\frac{1-\cos 8x}{\operatorname{tg}^2 2x}$.

2. $\lim_{x\to\infty} x^4\sin(a/x)$.

3. $\lim_{x\to1}\ln x\cdot\ln(x-1)$.

4. $\lim_{x\to3}\left(\frac{1}{x-3}-\frac{5}{x^2-x-6}\right)$

5. $\lim_{x\to1}\left(\frac{1}{2(1-\sqrt{x})}-\frac{1}{3(1-\sqrt[3]{x})}\right)$.

6. $\lim_{x\to0}\frac{e^{ax}-e^{bx}}{\sin x}$.

7. $\lim_{x\to\pi/2}\left(\frac{x}{\operatorname{ctg}x}-\frac{\pi}{2\cos x}\right)$

8. $\lim_{x\to\pi}(\pi-x)\ \operatorname{tg}(x/2)$.

9. $\lim_{x\to0}\frac{x-\operatorname{arctg}x}{x^3}$.

10. $\lim_{x\to\pi/(2a)}\frac{1-\sin ax}{(2ax-\pi)^2}$.

11. $\lim_{x\to\pi/6}\frac{1-2\sin x}{\cos 3x}$

12. $\lim_{x\to0}\frac{e^{2x}-1}{\ln(1+2x)}$.

13. $\lim_{x\to0}\frac{a^x-1}{c^x-1}$.

14. $\lim_{x\to1}\frac{\ln x}{1-x^3}$.

15. $\lim_{x\to1}\frac{\ln x}{\operatorname{ctg}x}$.

16. $\lim_{x\to0}\frac{1-\cos ax}{1-\cos bx}$.

17. $\lim_{x\to a}\frac{x-a}{x^n-a^n}$

18. $\lim_{x\to0}\frac{e^x-1}{\sin 2x}$.

19. $\lim_{x\to0}(x\ln x)$.

20. $\lim_{x\to0}\left(\frac{1}{x\sin x}-\frac{1}{x^2}\right)$.

21. $\lim_{x\to0}(1-e^{2x})\ \operatorname{ctg}x$.

22. $\lim_{x\to0}\frac{a^x-b^x}{x\sqrt{1-x^2}}$.

23. $\lim_{x\to0}\frac{e^{x^3}-1-x^3}{\sin^2 2x}$

24. $\lim_{x\to0}\frac{e^{a\sqrt{x}}-1}{\sqrt{\sin bx}}$.

25. $\lim_{x\to0}\frac{\ln(1+x^2)}{\cos 3x-e^{-x}}$

26. $\lim_{x\to\infty}\frac{e^x}{x^5}$.

27. $\lim_{x\to+\infty}\frac{\ln(x+7)}{\sqrt[7]{x-3}}$.

28. $\lim_{x\to0}\frac{\pi/x}{\operatorname{ctg}(5x/2)}$.

29. $\lim_{x\to0}(1-\cos 2x)\ \operatorname{ctg}4x$.

30. $\lim_{x\to\infty}(x^2\sin b/x)$.

3

1. $\lim_{x\to0}\frac{\arcsin 4x}{5-5e^{-3x}}$.

2. $\lim_{x\to0}\frac{\ln\cos x}{x}$.

3. $\lim_{x\to0}\frac{e^{x^2}-1}{\cos x-1}$.

4. $\lim_{x\to0}\frac{e^x-x^2/2-x-1}{\cos x-x^2/2-1}$.

5. $\lim_{x\to0}\frac{e^{\operatorname{tg}x}-1}{\operatorname{tg}x-x}$.

6. $\lim_{x\to1}\frac{\ln(1-x)+\operatorname{tg}(\pi x/2)}{\operatorname{ctg}\pi x}$

7. $\lim_{x\to a}\frac{\cos x\cdot\ln(x-a)}{\ln(e^x-e^a)}$.

8. $\lim_{x\to1}\frac{1}{\cos(\pi x/2)\cdot\ln(1-x)}$

9. $\lim_{x\to0}\frac{\cos(e^{x^2}-1)}{\cos x-1}$.

10. $\lim_{x\to0}\frac{e^{\alpha x}-\cos\alpha x}{e^{\beta x}-\cos\beta x}$.

11. $\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}$.

12. $\lim_{x\to\infty}x\ \sin\frac{a}{6x}$.

13. $\lim_{x\to0}\frac{3\operatorname{tg}4x-12\operatorname{tg}x}{3\sin 4x-12\sin x}$.

14. $\lim_{x\to\pi/4}\frac{\sqrt{\operatorname{tg}x}-1}{2\sin^2 x-1}$

15. $\lim_{x\to0}\frac{x(e^x+1)-2(e^x-1)}{x^3}$.

16. $\lim_{x\to0}\frac{\arcsin 2x-2\arcsin x}{x^3}$

17. $\lim_{x\to0}\frac{a^x-a^{\sin x}}{x^3}$.

18. $\lim_{x\to\pi/4}(\operatorname{tg}x)^{\operatorname{tg}2x}$.

19. $\lim_{x\to0}\frac{\ln(\cos ax)}{\ln(\cos bx)}$.

20. $\lim_{x\to\pi/4}\frac{\sqrt[3]{\operatorname{tg}x}-1}{2\sin^2 x-1}$

21. $\lim_{x\to0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$.

22. $\lim_{x\to\infty}x^2\ e^{-0,01x}$.

23. $\lim_{x\to0}\frac{\ln(1+xe^x)}{\ln(x+\sqrt{1+x^2})}$.

24. $\lim_{x\to1}(1-x)^{\log_2 x}$.

25. $\lim_{x\to\infty}\frac{e^{1/x^2}-1}{2\operatorname{arctg}x^2-\pi}$.

26. $\lim_{x\to1/2}\ln 2x\cdot\ln(2x-1)$

27. $\lim_{x\to1/2}\left(\frac{x}{3x-1}-\frac{1}{\ln 3x}\right)$.

28. $\lim_{x\to0}\arcsin x\cdot\operatorname{tg}x$.

29. $\lim_{x\to\infty}(x^3e^{-x})$.

30. $\lim_{x\to1}(x-1)^{x-1}$.

1. $\lim\limits_{x\to 0}(1-\sin 2x)^{\operatorname{ctg} x}$.
2. $\lim\limits_{x\to 0}(\ln(1/x))^{x}$.
3. $\lim\limits_{x\to 0}(\cos x)^{\operatorname{ctg}^2 x}$.
4. $\lim\limits_{x\to 0} x^{x}$.
5. $\lim\limits_{x\to\infty}(\ln 2x)^{1/\ln x}$.
6. $\lim\limits_{x\to 0}(1+\sin^2 x)^{1/\operatorname{tg}^2 x}$
7. $\lim\limits_{x\to 1}(1-x)^{\ln x}$.
8. $\lim\limits_{x\to 0}(\ln(x+e))^{1/x}$.
9. $\lim\limits_{x\to 0}(\sin x)^{\operatorname{tg} x}$.
10. $\lim\limits_{x\to\infty}\sqrt[x]{x}$.
11. $\lim\limits_{x\to 0} x^{\sin x}$.
12. $\lim\limits_{x\to 1}(1-x)^{\cos(\pi x/2)}$.
13. $\lim\limits_{x\to 0}(1+x^2)^{1/x}$.
14. $\lim\limits_{x\to 1} x^{1/(x-1)}$.
15. $\lim\limits_{x\to 1}\left(\operatorname{tg}\dfrac{\pi x}{4}\right)^{\operatorname{tg}(\pi x/2)}$.
16. $\lim\limits_{x\to 1}\left(\operatorname{ctg}\dfrac{\pi x}{4}\right)^{\operatorname{tg}(\pi x/2)}$
17. $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{\operatorname{tg} x}$.
18. $\lim\limits_{x\to\infty}\left(\dfrac{x-4}{x+3}\right)^{3x}$.
19. $\lim\limits_{x\to 0}(\operatorname{ctg} x)^{\sin x}$.
20. $\lim\limits_{x\to\infty}(\ln x)^{1/x}$.
21. $\lim\limits_{x\to\infty} x^{6/(1+2\ln x)}$.
22. $\lim\limits_{x\to\infty}(1-e^{x})^{1/x}$
23. $\lim\limits_{x\to\infty}(x-1)^{1/\ln(2(x-1))}$.
24. $\lim\limits_{x\to\infty}\left(\cos\dfrac{m}{x}\right)^{x}$.
25. $\lim\limits_{x\to 0}(\operatorname{ctg} 2x)^{1/\ln x}$.
26. $\lim\limits_{x\to 5}\left(\dfrac{1}{x-5}-\dfrac{5}{x^2-x-20}\right)$
27. $\lim\limits_{x\to\infty} x^2\ \sin\dfrac{a}{x}$.
28. $\lim\limits_{x\to 1}\left(\dfrac{1}{2(1-\sqrt{x})}-\dfrac{1}{3(1-\sqrt[3]{x})}\right)$.
29. $\lim\limits_{x\to 1}(1-x)^{\cos(\pi x/2)}$.
30. $\lim\limits_{x\to 0}(\operatorname{ctg} x)^{\sin x}$

1. $\lim\limits_{x\to\infty} x(\ln(2+x)-\ln(x+1))$
2. $\lim\limits_{x\to\infty}\left(\cos\dfrac{m}{x}+\lambda\sin\dfrac{m}{x}\right)^{x}$
3. $\lim\limits_{x\to\infty}(x+2^{x})^{1/x}$.
4. $\lim\limits_{x\to 0}(1+3\operatorname{tg}^2 x)^{\operatorname{ctg}^2 x}$.
5. $\lim\limits_{x\to\infty}(\cos(m/\sqrt{x}))^{x}$.
6. $\lim\limits_{x\to 0}(\cos 2x)^{3/x^2}$
7. $\lim\limits_{x\to 0}(\ln\operatorname{ctg} x)^{\operatorname{tg} x}$.
8. $\lim\limits_{x\to a}(2-x/a)^{\operatorname{tg}(\pi x/(2a))}$.
9. $\lim\limits_{x\to 0} x^{1/\ln(e^x-1)}$.
10. $\lim\limits_{x\to 0}\left(\dfrac{5}{2+\sqrt{9+x}}\right)^{1/\sin x}$
11. $\lim\limits_{x\to\infty}(1+3/x)^{x}$.
12. $\lim\limits_{x\to 0}(e^{x}+x)^{1/x}$.
13. $\lim\limits_{x\to\pi/2}(\operatorname{tg} x)^{2x-\pi}$.
14. $\lim\limits_{x\to\infty}\left(\dfrac{2}{\pi}\operatorname{arctg} x\right)^{x}$
15. $\lim\limits_{x\to 0}\left(\dfrac{\cos x}{\cos 2x}\right)^{1/x^2}$.
16. $\lim\limits_{x\to\infty}\left(\dfrac{1+\operatorname{tg} x}{1+\sin x}\right)$
17. $\lim\limits_{x\to\infty}(\cos(1/x)+\sin(1/x))^{x}$.
18. $\lim\limits_{x\to 1}(x-1)^{e^{1/(x-1)}}$.
19. $\lim\limits_{x\to 0}\left(\dfrac{\operatorname{tg} x}{x}\right)^{1/x^2}$
20. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x^2-2}\right)^{x^2}$.
21. $\lim\limits_{x\to 0}\sqrt[x]{1-2x}$.
22. $\lim\limits_{x\to\infty}\left(\sin\dfrac{2}{x}+\cos\dfrac{2}{x}\right)^{x}$.
23. $\lim\limits_{x\to 0}\sqrt[x]{\cos\sqrt{x}}$.
24. $\lim\limits_{x\to 1}(1+\sin\pi x)^{\operatorname{ctg}\pi x}$
25. $\lim\limits_{x\to\infty}\left(\dfrac{x+a}{x-a}\right)^{x}$.
26. $\lim\limits_{x\to\infty} x^{1/x}$.
27. $\lim\limits_{x\to 0} x^{3/(4+\ln x)}$.
28. $\lim\limits_{x\to 0} x^{\sin x}$.
29. $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{\operatorname{tg} x}$.
30. $\lim\limits_{x\to 1}\dfrac{x^3-2x^2-x+2}{x^3-7x+6}$

Topic 8. Optimization.

Monotonicity and extreme points.

8.1. Monotonicity. 8.2. Extreme points. Necessary and sufficient conditions of extreme points. 8.3. The smallest and the largest value at interval.

8.1. Monotonicity.

8.1.1. Definition:

Test the function $y=f(x)$ on **monotonicity** means to define the intervals where the function **is increasing** and where **is decreasing**.

<u>**Definition:**</u> A function $y=f(x)$ defined throughout an interval X is said to **increase** on X if, for any two points x_1 and x_2 in X, $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$.

Similarly, f is said **to decrease** on X if, for any two points x_1 and x_2 in X,

$$x_2 > x_1 \Rightarrow f(x_2) < f(x_1).$$

8.1.2. Test for increasing and decreasing.

Suppose that f is continuous at each point of the closed interval [a, b] and differentiable at each point of its interior (a, b).

If $f' > 0$ at each point of (a, b), then f **increases** throughout [a, b].

If $f' < 0$ at each point of (a, b), then f **decreases** throughout [a, b].

In either case, f is one-to-one on [a, b].

8.2. Extreme points. Necessary and sufficient conditions for extreme points.

8.2.1. Necessary condition for extreme points.

If $y=f(x)$ is continuous on the interval (a,b) and f(c) is a local extreme point, then either

1) **$f'(c)=0$** or
2) **$f'(c)$ does not exist (is not defined).**

The values of x **in the domain** of f where $f'(c)=0$ or $f'(c)$ does not exist are called **the critical values of f.** (Critical value has the added requirement that it must belongs to the domain of the function.)

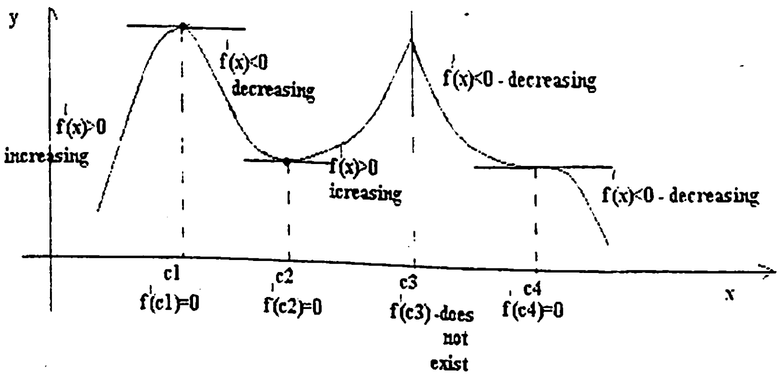
This condition states that a local extremum can occur only at a critical value, but it does not imply that every critical value produces a local extremum.

A critical point can be an extreme point if it is satisfied to the sufficient conditions.

8.2.2. Sufficient conditions for extreme points.

Let the function $y=f(x)$ is continuous at the vicinity U(c) of the critical point $x=c$ and is differentiable at U(c) \(c). Then

1) **if at U(c) $f'(x)>0$ when $x < c$ and $f'(x)<0$ when $x > c$, then f(c) is local maximum.**
2) **if at U(c) $f'(x)<0$ when $x < c$ and $f'(x)>0$ when $x > c$, then f(c) is local minimum.**
3) if $f'(x)$ does not change its sign at $x=c$, then f(c) is neither a local maximum or local minimum.



All points: x=c1, x=c2, x=c3, x=c4 are critical points because they satisfy to necessary condition:

f '(c1)=0 , f '(c2)=0 , f '(c4)=0 and f '(c3) – does not exist.

There are exist only three extreme among four : f (c1), f (c3) – local maximums and f (c2) – local minimum, because these points are satisfied to sufficient condition, but the critical value f(c4) is not extreme because does not satisfy to sufficient condition.

8.2.3. The second derivative test for extreme points.

Let the function y=f(x) is twice-differentiable at the vicinity of critical point x=c. Then,

If f ''(c) > 0, then f(c) - is local minimum;

If f ''(c) < 0, then f(c) - is local maximum.

8.3. The smallest and the largest value at interval.

Testing the extreme the smallest and the largest value at segment [a,b] can be defined:

$\max\limits_{[a,b]} f(x)$ =max{ f(a),f(b),$f(x_i)$:$x_i \in$ (a,b) - points of maximum}- **the largest value at segment [a,b];**

$\min\limits_{[a,b]} f(x)$ =min{f(a),f(b),$f(x_i)$: $x_i \in$ (a,b) - points of minimum}- **the smallest value at segment [a,b].**

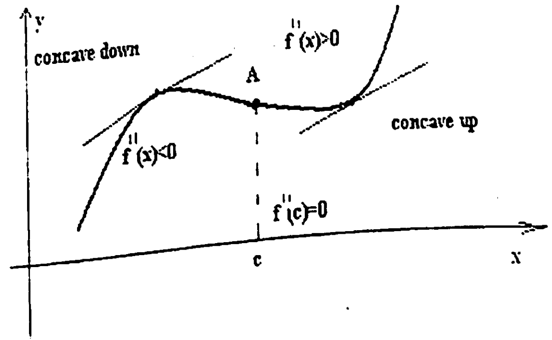
Topic 9. Concavity and points of inflections. Asymptotes of graph of functions.

9.1. Concavity and points of inflections. 9.2. Necessary and sufficient condition of points of inflections. 9.3. Asymptotes of graph of functions.

9.1. Concavity and points of inflections.

9.1.1. Definition:

The graph of the function is **concave down** on an interval where the slope y' decreases. It is **concave up** on an interval where the slope y' increases.



Geometrically, the graph is **concave down** on an interval if it lies below its tangent lines in the interval and is **concave up** if it lies above its tangent lines in the interval.

Definition: A point on the graph of a differentiable function where the concavity changes is called a point **of inflection.**

A – point of inflection.

9.1.2. The second derivative test on concavity.

Suppose that **f** is continuous at each point of the closed interval [a, b] and twice- differentiable at each point of its interior (a, b) .
If f '' > 0 at each point of (a, b), then f **concave up** throughout [a, b].
If f ''< 0 at each point of (a, b), then f **concave down** throughout [a,b].

9.2. Necessary and sufficient condition of the points of inflections.

9.2.1. Necessary condition of the points of inflection.

If y=f(x) is continuous on the interval (a,b) and f(c) is the point of inflection, then either

1) f ''(c) =0 or

2) f ''(c) does not exist (is not defined).

The values of x in the domain of f where f ''(c) =0 or f ''(c) does not exist are called **the critical values of the second kind of f.**

9.2.2. Sufficient conditions for point of inflection.

Let the function y=f(x) be continuous at the vicinity U(c) of the critical point of the second kind **x=c** and be twice- differentiable at U(c) \(c). Then

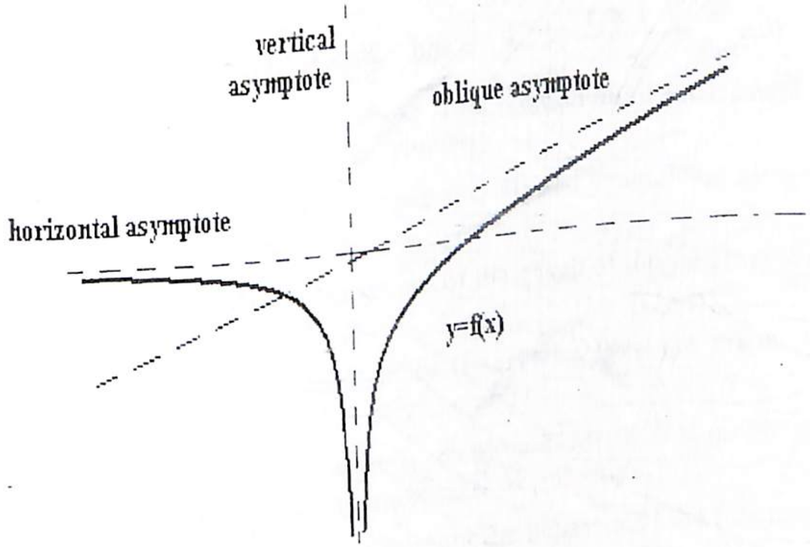
if at U(c) f ''(x)>0 when x < c and f ''(x)<0 when x > c, or

if at U(c) f ''(x)<0 when x < c and f ''(x)>0 when x > c, then f(c) is the point of inflection.

In other words, if f ''(x) changes its sign at the critical point of the second kind x=c, then f(c) is the point of inflection.

9.3. Asymptotes of graph of functions.

9.3.1. __**Definition:**__ If the distance between the graph of a function and some fixed line approaches zero as the graph moves father and father from the origin, we say that the graph approaches the line asymptotically and that the line **an asymptote** of the graph.



Kinds of asymptotes:

1) **horizontal;**

2) **vertical;**

3) **oblique.**

9.3.2. Horizontal asymptote.

A line **y = b** is a **horizontal asymptote** for the graph of y=f(x) if

$\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.

9.3.3. Vertical asymptote.

A line **x=a** is a **vertical asymptote** for the graph of y=f(x) if

$\lim_{x\to a+} f(x) = \pm\infty$ or $\lim_{x\to a-} f(x) = \pm\infty$.

9.3.4. Oblique asymptote.

A line y = **k·x+b** is **an oblique asymptote** for the graph of y=f(x) if

$\lim_{x\to \pm\infty} \frac{f(x)}{x} = k$ and $\lim_{x\to\infty} (f(x) - kx) = b$, where **k** and **b** are finite number.

Topic 10. Total investigation of a function. Graphing the functions.

A total investigation and graphing strategy

for function y = f(x) :

1) Find the domain of the function.

2) Test is the function even, odd or neither. Periodicity. Establish has the graph of the function the symmetry.

3) Find the x- intercept point (where f(x)=0) and y- intercept points (where x=0) of the graph.

4) Determine all kinds of asymptotes of the graph of the function: 1) horizontal; 2) vertical; 3) oblique.

5) Using f '(x) test on monotonicity (intervals where the graph is increasing and decreasing) and extreme points.

6) Using f "(x) test on concavity (intervals where the graph is concave up and concave down) and points of inflection.

7)Graph the function taking into account the behavior of the function at each interval.

EXERCISES X.

otal investigate the function and graph it.

1

1. $y=\dfrac{x^2-2x+2}{x-1}.$
2. $y=\dfrac{x+1}{(x-1)^2}.$
3. $y=e^{1/(5+x)}.$
4. $y=x/(9-x)$
5. $y=\dfrac{4x-x^2-4}{x}.$
6. $y=\dfrac{x^2}{4x^2-1}.$
7. $y=\dfrac{\ln x}{\sqrt{x}}.$
8. $y=x+\dfrac{\ln x}{x}.$
9. $y=x-\ln(1+x^2).$
10. $y=\dfrac{x^3}{x^2-x+1}$
11. $y=x^2-2\ln x.$
12. $y=x^3e^{-x^2/2}.$
13. $y=\dfrac{x^2-x-1}{x^2-2x}.$
14. $y=\dfrac{(x-2)^2}{x+1}.$
15. $y=-\ln\dfrac{1+x}{1-x}.$
16. $y=\ln(x^2+1)$
17. $y=\dfrac{x^2+6}{x^2+1}.$
18. $y=x\ln x.$
19. $y=(x-1)e^{3x+1}.$
20. $y=\dfrac{x^2-3x+2}{x+1}$
21. $y=\dfrac{2x-1}{(x-1)^2}.$
22. $y=\dfrac{x^5}{x^4-1}.$
23. $y=(x^3+4)/x^2.$
24. $y=\frac{1}{3}\sqrt[3]{x^2}(x-5).$
25. $y=x^3/(x^4-1).$
26. $y=(e^{2x}+1)/e^x.$
27. $y=x^2+1/x^2.$
28. $y=(5x^4+3)/x.$
29. $y=\dfrac{4-2x}{1-x^2}.$
30. $y=\dfrac{5x}{4-x^2}.$

2

1. $y=e^{2x-x^2}.$
2. $y=x+\ln(x^2-4)$
3. $y=\dfrac{2(x+1)^2}{x-2}.$
4. $y=x\ln^2 x.$
5. $y=(4e^{x^2}-1)/e^{x^2}.$
6. $y=x^2e^{-x^2/2}.$
7. $y=xe^{1/x}.$
8. $y=\dfrac{2+x}{(x+1)^2}.$
9. $y=\dfrac{(1-x)^3}{(x-2)^2}.$
10. $y=xe^x.$
11. $y=x^2e^{1/x}.$
12. $y=x^2/(x+2)^2$
13. $y=(x+2)e^{1-x}.$
14. $y=\dfrac{\ln x}{x}.$
15. $y=\left(\dfrac{x-2}{x+1}\right)^2.$
16. $y=\dfrac{x^3}{9-x^3}.$
17. $y=(x+1)e^{2x}.$
18. $y=4x/(4+x^2).$
19. $y=x^4/(x^3-1).$
20. $y=\ln(x^2-2x+6)$
21. $y=\ln(1-1/x^2).$
22. $y=x^3e^{x+1}.$
23. $y=x-\ln(1+x^2).$
24. $y=1-\ln^3 x.$
25. $y=(x-1)e^{4x+2}.$
26. $y=\dfrac{2x^2+2+4x}{2-x}.$
27. $y=-x\ln^2 x.$
28. $y=x^2-2\ln x.$
29. $y=e^{1/(2-x)}.$
30. $y=\ln(4-x^2).$

3

Find the largest and smallest values of the function at interval [a,b]:

1. $y=\ln(x^2-2x+2)$, $[0;\ 3]$.
2. $y=3x/(x^2+1)$, $[0;\ 5]$.
3. $y=(2x-1)/(x-1)^2$, $[-1/2;\ 0]$.
4. $y=(x+2)e^{1-x}$, $[-2;\ 2]$.
5. $y=\ln(x^2-2x+4)$, $[-1;\ 3/2]$
6. $y=x^3/(x^2-x+1)$, $[-1;\ 1]$.
7. $y=((x+1)/x)^3$, $[1;\ 2]$.

8. $y=\sqrt{x-x^3}$, $[-2;\ 2]$.
9. $y=4-e^{-x^2}$, $[0;\ 1]$.
10. $y=(x^3+4)/x^2$, $[1;\ 2]$.
11. $y=xe^x$, $[-2;\ 0]$.
12. $y=(x-2)e^x$, $[-2;\ 1]$.
13. $y=(x-1)e^{-x}$, $[0;\ 3]$.
14. $y=x/(9-x^2)$, $[-2;\ 2]$.
15. $y=(1+\ln x)/x$, $[1/e;\ e]$.
16. $y=e^{4x-x^2}$, $[1;\ 3]$.
17. $y=(x^5-8)/x^4$, $[-3;\ -1]$.
18. $y=\dfrac{e^{2x}+1}{e^x}$, $[-1;\ 2]$.
19. $y=x\ln x$, $[1/e^2;\ 1]$.
20. $y=x^3e^{x+1}$, $[-4;\ 0]$.
21. $y=x^2-2x+2/(x-1)$, $[-1;\ 3]$
22. $y=(x+1)\sqrt[3]{x^2}$, $[-4/5;\ 3]$.
23. $y=e^{6x-x^2}$, $[-3;\ 3]$.
24. $y=(\ln x)/x$, $[1;\ 4]$.
25. $y=3x^4-16x^3+2$, $[-3;\ 1]$
26. $y=x^5-5x^4+5x^3+1$, $[-1;\ 2]$.
27. $y=(3-x)e^{-x}$, $[0;\ 5]$.
28. $y=\sqrt{3}/2+\cos x$, $[0;\ \pi/2]$.
29. $y=108x-x^4$, $[-1;\ 4]$.
30. $y=x^4/4-6x^3+7$, $[16;\ 20]$

Topic 11. Indefinite integrals.

11.1. Antiderivative. 11.2. Indefinite integrals. 11.3. Properties of indefinite integrals. 11.4. Table of indefinite integrals.

11.1. Antiderivative.

Let the function y=f(x) is defined at the interval (a,b). A function F(x) is called the **antiderivative** of the function y=f(x) at the interval (a,b) if

$$F'(x)\equiv f(x) \quad \text{for any } x\in(a,b).$$

Theorem: If $y=F_1(x)$ and $y=F_2(x)$ two antiderivates of the same function y=f(x) over (a,b), then there is exists such constant number C that $F_2(x)\equiv F_1(x)+C$.

This theorem states that at least one antiderivative is known $y=F_1(x)$, the any another antiderivatives will be defined as $y = F_1(x)+C$ for some constant C.

11.2. Indefinite integrals.

Definition: The set of all antiderivateves of the function y = f(x) over the interval (a,b) is called its **indefinite integral** and is denoted by:

$$\int f(x)dx = F(x)+C, \qquad C\in R.$$

Here $\int$ - integral sign. f(x) – integrand and dx – differential of the variable, x – dummy variable, F(x) – antiderivative, C constant of integration.

11.3. Properties of indefinite integrals.

1) $(\int f(x)dx)' = f(x)$

2) $d(\int f(x)dx) = f(x)dx$

3) $\int f'(x)dx = f(x)+C$

4) $\int kf(x)dx = k\int f(x)dx$ where k-const

5) $\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$

6) If $\int f(x)dx = F(x) + C$, and a ≠ 0, b – const

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C.$$

11.4. Table of basic indefinite integrals.

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C, (n \neq -1)$

2. $\int \dfrac{dx}{x} = \ln|x| + C$

3. $\int a^x dx = \dfrac{a^x}{\ln a} + C$ (a>0,a ≠ 1)

4. $\int e^x dx = e^x + C$

5. $\int \sin(x)dx = -\cos(x) + C$

6. $\int \cos(x)dx = \sin(x) + C$.

7. $\int \dfrac{dx}{\cos^2(x)} = \tan(x) + C$

8. $\int \dfrac{dx}{\sin^2(x)} = -\cot(x) + C$

9. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\arctan\dfrac{x}{a} + C$

10. $\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a}\ln\left|\dfrac{x+a}{x-a}\right| + C$,

11. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \arcsin\dfrac{x}{a} + C$,

EXERCISE XI.

1

Evaluate the indefinite integrals:

1. $\int \dfrac{3 + \sqrt[3]{x^2} - 2x}{\sqrt{x}} dx.$

2. $\int \dfrac{2x^2 + 3\sqrt{x} - 1}{2x} dx.$

3. $\int \dfrac{3\sqrt{x} + 4x^2 - 5}{2x^2} dx.$

4. $\int \dfrac{2\sqrt{x} - x^2 + 3}{\sqrt[3]{x}} dx.$

5. $\int \dfrac{\sqrt[4]{x} - 2x + 5}{x^2} dx.$

6. $\int \dfrac{2x^3 - \sqrt{x} + 4}{\sqrt{x}} dx.$

7. $\int \left(\sqrt[3]{x} - \dfrac{2\sqrt[4]{x}}{x} + 3\right) dx.$

8. $\int \dfrac{2x^3 - \sqrt{x^5} + 1}{\sqrt{x}} dx.$

9. $\int \dfrac{3x^2 - \sqrt[5]{x} + 2}{x} dx.$

10. $\int \dfrac{2x^4 - \sqrt{x} + 4}{x^2} dx.$

11. $\int \dfrac{\sqrt[6]{x^5} - 5x^2 + 3}{x} dx.$

12. $\int \left(x\sqrt{x} - \dfrac{1}{\sqrt{x^3}} + 1\right) dx.$

13. $\int \left(x^2 - \dfrac{\sqrt[6]{x}}{x} - 3\right) dx.$

14. $\int \dfrac{\sqrt[3]{x^2} - 2x^5 + 3}{x} dx.$

15. $\int \left(\dfrac{\sqrt[3]{x}}{x} + 2x^3 - 4\right) dx.$

16. $\int \dfrac{\sqrt{x^3} - 3x^4 + 2}{x} dx.$

17. $\int \left(2x^3 - 3\sqrt{x^5} + \dfrac{4}{x}\right) dx.$

18. $\int \dfrac{2x^3 - \sqrt{x^5} + 5}{x^2} dx.$

19. $\int \dfrac{3x^2 - \sqrt{x^3} + 7}{x^3} dx.$

20. $\int \dfrac{3x^4 - \sqrt[3]{x^2} + 1}{x^2} dx.$

21. $\int \left(\sqrt[5]{x^2} - \dfrac{2}{x^3} + 4\right) dx.$

22. $\int \dfrac{\sqrt{x} - 2x^3 + 6}{x} dx.$

23. $\int \dfrac{\sqrt[5]{x} - 2x^4 + 4}{x^2} dx.$

24. $\int \left(\sqrt{x} - \dfrac{3x^2}{\sqrt{x^3}} + 2\right) dx.$

25. $\int \left(\sqrt[5]{x} - \dfrac{4}{x^5} + 2\right) dx.$

26. $\int \dfrac{\sqrt[7]{x^6} - 2x^2 + 3}{x} dx.$

27. $\int \left(\dfrac{\sqrt[3]{x}}{x} - \dfrac{2}{x^3} + 1\right) dx.$

28. $\int \left(\dfrac{2x^2}{\sqrt{x}} - \dfrac{5}{x} + 6\right) dx.$

29. $\int \left(\dfrac{\sqrt[3]{x^2}}{x} - \dfrac{7}{x^3} + 5\right) dx.$

30. $\int \left(\dfrac{5x^2}{\sqrt{x}} - \sqrt[3]{x^2} + 2\right) dx.$

2

1. $\sqrt{3+x}\,dx.$
2. $\int\sqrt[3]{1+x}\,dx.$
3. $\int\sqrt[3]{(1+x)^2}\,dx.$
4. $\int\frac{dx}{\sqrt{1+x}}.$
5. $\int\frac{dx}{\sqrt{(1-x)^3}}.$
6. $\int\frac{dx}{\sqrt[3]{2+x}}.$
7. $\int(1-4x)^7\,dx.$
8. $\int(1+4x)^5\,dx.$
9. $\int(1-3x)^4\,dx.$
10. $\int\sqrt{1+3x}\,dx.$
11. $\int\sqrt{5-4x}\,dx.$
12. $\int\frac{dx}{\sqrt[3]{5+3x}}.$
13. $\int\frac{dx}{\sqrt[3]{(1-4x)^5}}.$
14. $\int\frac{dx}{\sqrt[3]{(3-4x)^2}}.$
15. $\int\frac{dx}{\sqrt[3]{2-5x}}.$
16. $\int\sqrt[5]{3-2x}\,dx.$
17. $\int\sqrt[4]{1+3x}\,dx.$
18. $\int\sqrt[3]{1+3x}\,dx.$
19. $\int\frac{dx}{\sqrt{(3-x)^5}}.$
20. $\int\frac{dx}{\sqrt[3]{3+x}}.$
21. $\int\frac{dx}{(2+x)^3}.$
22. $\int\sqrt[3]{5-2x}\,dx.$
23. $\int\sqrt{5-4x}\,dx.$
24. $\int\sqrt[5]{(6-5x)^2}\,dx.$
25. $\int\sqrt[4]{2-5x}\,dx.$
26. $\int\sqrt[3]{4-2x}\,dx.$
27. $\int\sqrt{3-4x}\,dx.$
28. $\int\sqrt[5]{3+2x}\,dx.$
29. $\int\sqrt[4]{(3+5x)^3}\,dx.$
30. $\int\sqrt[3]{(2-x)^2}\,dx.$

3

1. $\int\frac{dx}{3-x}.$
2. $\int\frac{dx}{3x+9}.$
3. $\int\frac{dx}{2-3x}.$
4. $\int\frac{dx}{1-4x}.$
5. $\int\frac{dx}{2+3x}.$
6. $\int\frac{dx}{2-5x}.$
7. $\int\frac{dx}{3x-2}.$
8. $\int\frac{dx}{2x+3}.$
9. $\int\frac{dx}{3x-4}.$

10. $\int\frac{dx}{4-3x}.$
11. $\int\frac{dx}{3x+4}.$
12. $\int\frac{dx}{4x-2}.$
13. $\int\frac{dx}{5-3x}.$
14. $\int\frac{dx}{4-7x}.$
15. $\int\frac{dx}{5x-3}.$
16. $\int\frac{dx}{3-2x}.$
17. $\int\frac{dx}{5+3x}.$
18. $\int\frac{dx}{3-5x}.$
19. $\int\frac{dx}{5+4x}.$
20. $\int\frac{dx}{6-3x}.$
21. $\int\frac{dx}{6+5x}.$
22. $\int\frac{dx}{1-7x}.$
23. $\int\frac{dx}{1+6x}.$
24. $\int\frac{dx}{2+7x}.$
25. $\int\frac{dx}{7-3x}.$
26. $\int\frac{dx}{5-2x}.$
27. $\int\frac{dx}{2x+7}.$
28. $\int\frac{dx}{2x+9}.$
29. $\int\frac{dx}{7x-3}.$
30. $\int\frac{dx}{6x+1}.$

4

1. $\int\sin(2-3x)\,dx.$
2. $\int\sin(3-2x)\,dx.$
3. $\int\sin(5-3x)\,dx.$
4. $\int\cos(2+3x)\,dx.$
5. $\int\cos(3+2x)\,dx.$
6. $\int\sin(4-2x)\,dx.$
7. $\int\cos(5-2x)\,dx.$
8. $\int\cos(7x+3)\,dx.$
9. $\int\sin(8x-3)\,dx.$
10. $\int\sin(3+4x)\,dx.$
11. $\int\sin(3-4x)\,dx.$
12. $\int\cos(4x+3)\,dx.$
13. $\int\cos(3-4x)\,dx.$
14. $\int\cos(2+5x)\,dx$
15. $\int\cos(3x+5)\,dx.$
16. $\int\sin(5x-3)\,dx.$
17. $\int\sin(5-3x)\,dx.$
18. $\int\sin(3x+6)\,dx.$
19. $\int\cos(5x-8)\,dx.$
20. $\int\cos(3x-7)\,dx.$
21. $\int\cos(5x-6)\,dx.$
22. $\int\sin(7x+1)\,dx.$
23. $\int\cos(7x+3)\,dx.$
24. $\int\sin(7-4x)\,dx.$
25. $\int\cos(3x-7)\,dx.$
26. $\int\sin(8x-5)\,dx.$
27. $\int\cos(8x-4)\,dx.$
28. $\int\sin(9x-1)\,dx.$
29. $\int\cos(10x-3)\,dx.$
30. $\int\sin(9x+7)\,dx$

5

1. $\int \frac{\sqrt{3}dx}{9x^2-3}$. 2. $\int \frac{dx}{\sqrt{9x^2+3}}$. 3. $\int \frac{dx}{9x^2+3}$.

4. $\int \frac{9dx}{\sqrt{9x^2-3}}$. 5. $\int \frac{dx}{\sqrt{3-9x^2}}$. 6. $\int \frac{dx}{7x^2-4}$.

7. $\int \frac{3dx}{\sqrt{7x^2-4}}$. 8. $\int \frac{dx}{5x^2+3}$. 9. $\int \frac{dx}{5x^2-3}$.

10. $\int \frac{dx}{\sqrt{3-5x^2}}$. 11. $\int \frac{dx}{\sqrt{5x^2+3}}$. 12. $\int \frac{dx}{\sqrt{4-7x^2}}$.

13. $\int \frac{\sqrt{5}dx}{\sqrt{3-4x^2}}$. 14. $\int \frac{dx}{\sqrt{2x^2-9}}$. 15. $\int \frac{dx}{2x^2+7}$.

16. $\int \frac{dx}{\sqrt{3x^2+1}}$. 17. $\int \frac{dx}{3x^2+2}$. 18. $\int \frac{\sqrt{2}dx}{\sqrt{7-2x^2}}$.

19. $\int \frac{\sqrt{14}dx}{2x^2-7}$. 20. $\int \frac{dx}{8x^2+9}$. 21. $\int \frac{dx}{3x^2-2}$.

22. $\int \frac{dx}{4x^2+3}$. 23. $\int \frac{dx}{\sqrt{4x^2+3}}$. 24. $\int \frac{dx}{\sqrt{3-4x^2}}$.

25. $\int \frac{dx}{\sqrt{9-8x^2}}$. 26. $\int \frac{dx}{4x^2-3}$. 27. $\int \frac{dx}{8x^2-9}$.

28. $\int \frac{dx}{4x^2+7}$. 29. $\int \frac{2dx}{4+3x^2}$. 30. $\int \frac{2dx}{\sqrt{4x^2-3}}$.

6

1. $\int \frac{2xdx}{\sqrt{5-4x^2}}$. 2. $\int \frac{xdx}{\sqrt{5-3x^2}}$. 3. $\int \frac{3xdx}{4x^2+1}$.

4. $\int \frac{4xdx}{\sqrt{3-4x^2}}$. 5. $\int \frac{2xdx}{\sqrt{8x^2-9}}$. 6. $\int \frac{4xdx}{\sqrt{4x^2+3}}$.

7. $\int \frac{xdx}{\sqrt{9-8x^2}}$. 8. $\int \frac{\sqrt{3}xdx}{\sqrt{3x^2-2}}$. 9. $\int \frac{2xdx}{\sqrt{3x^2-2}}$.

10. $\int \frac{2xdx}{\sqrt{7-2x^2}}$. 11. $\int \frac{xdx}{2x^2-7}$. 12. $\int \frac{xdx}{3x^2+8}$.

13. $\int \frac{2xdx}{3x^2-7}$. 14. $\int \frac{2xdx}{\sqrt{2x^2+5}}$. 15. $\int \frac{xdx}{\sqrt{7-3x^2}}$.

16. $\int \frac{xdx}{2x^2+9}$. 17. $\int \frac{5xdx}{\sqrt{3-5x^2}}$. 18. $\int \frac{xdx}{\sqrt{3x^2+8}}$.

19. $\int \frac{5xdx}{\sqrt{5x^2+3}}$. 20. $\int \frac{xdx}{3x^2-6}$. 21. $\int \frac{xdx}{5x^2+1}$.

22. $\int \frac{5xdx}{5x^2-3}$. 23. $\int \frac{xdx}{2x^2-7}$. 24. $\int \frac{9xdx}{\sqrt{1-9x^2}}$.

25. $\int \frac{3xdx}{9x^2+2}$. 26. $\int \frac{5xdx}{\sqrt{7x^2-1}}$. 27. $\int \frac{3xdx}{\sqrt{9x^2+5}}$.

28. $\int \frac{2xdx}{5x^2-3}$. 29. $\int \frac{xdx}{3x^2-2}$. 30. $\int \frac{7xdx}{7x^2+1}$.

7

1. $\int \frac{dx}{\sqrt{2-5x^2}}$. 2. $\int \frac{dx}{2x^2-5}$. 3. $\int \frac{dx}{\sqrt{7x^2-3}}$.

4. $\int \frac{dx}{5x^2+2}$. 5. $\int \frac{dx}{2x^2+3}$. 6. $\int \frac{dx}{\sqrt{5x^2+1}}$.

7. $\int \frac{dx}{2x^2+9}$. 8. $\int \frac{dx}{\sqrt{9-2x^2}}$. 9. $\int \frac{dx}{\sqrt{9x^2+2}}$.

10. $\int \frac{dx}{5x^2-4}$. 11. $\int \frac{dx}{3x^2-7}$. 12. $\int \frac{dx}{3x^2+7}$.

13. $\int \frac{dx}{6x^2-7}$. 14. $\int \frac{dx}{7x^2+6}$. 15. $\int \frac{dx}{\sqrt{7-3x^2}}$.

16. $\int \frac{dx}{6x^2+1}$. 17. $\int \frac{dx}{\sqrt{5x^2-1}}$. 18. $\int \frac{dx}{3x^2-5}$.

19. $\int \frac{dx}{\sqrt{2-3x^2}}$. 20. $\int \frac{dx}{\sqrt{8-3x^2}}$. 21. $\int \frac{dx}{\sqrt{3x^2+8}}$.

22. $\int \frac{dx}{\sqrt{3x^2+2}}$. 23. $\int \frac{dx}{2x^2+7}$. 24. $\int \frac{dx}{4x^2-3}$.

25. $\int \frac{dx}{3x^2+4}$. 26. $\int \frac{dx}{\sqrt{8x^2-9}}$. 27. $\int \frac{dx}{\sqrt{5-4x^2}}$.

28. $\int \frac{dx}{\sqrt{1-3x^2}}$. 29. $\int \frac{dx}{\sqrt{4x^2+5}}$. 30. $\int \frac{dx}{3x^2-2}$.

8

1. $\int e^{2x-7}dx.$	2. $\int e^{3+5x}dx.$	3. $\int e^{2-3x}dx.$
4. $\int e^{2x+1}dx.$	5. $\int e^{7x-2}dx.$	6. $\int e^{5x-7}dx.$
7. $\int e^{5x+7}dx.$	8. $\int e^{7-2x}dx.$	9. $\int e^{3-4x}dx.$
10. $\int e^{10x+2}dx.$	11. $\int e^{2x-10}dx.$	12. $\int e^{4x+3}dx.$
13. $\int e^{4x+5}dx.$	14. $\int e^{6x-1}dx.$	15. $\int e^{5-2x}dx.$
16. $\int e^{4-3x}dx.$	17. $\int e^{3-5x}dx.$	18. $\int e^{1-4x}dx.$
19. $\int e^{2-5x}dx.$	20. $\int e^{6x-4}dx.$	21. $\int e^{8x+1}dx.$
22. $\int e^{2-6x}dx.$	23. $\int e^{2-4x}dx.$	24. $\int e^{3-6x}dx.$
25. $\int e^{4-5x}dx.$	26. $\int e^{5-x}dx.$	27. $\int e^{7+3x}dx.$
28. $\int e^{2x+3}dx.$	29. $\int e^{8x+1}dx.$	30. $\int e^{4-7x}dx.$

Topic 12. Basic methods of integration.

12.1. Substitution method. 12.2. Integration by parts.

12.3. Integration of rational functions.

12.1. Substitution method.

If the function y=f(x) is continuous and x=x(t) is continuously differentiable, then

$$\int f(x)dx = \int f(x(t)) \cdot x'(t) \cdot dt .$$

Corollary: Let functions f(x), x (t) , x' (t) are continuous, then

$$\int f(x(t)) \cdot x'(t) \cdot dt = \int f(x) \cdot dx .$$

12.2. Integration by parts.

Let functions y=u(x) and y=v(x) is continuously differentiable, then

$$\int u(x)\, dv(x) = u(x) \cdot v(x) - \int v(x) \cdot du(x) .$$

Short-cut form: $\int u\, dv = u \cdot v - \int v \cdot du$.

Integration by parts method is applied when integrand function can be represented as

$f(x) = P_n(x) \cdot g(x)$ here $P_n(x)$ is polynomial and :

I. g(x) one of the trigonometric or exponential function:

$$g(x) = \begin{cases} \sin(x) \\ cox(x) \\ e^x \\ a^x \end{cases}$$

Recommendation: $u = P_n(x)$, $dv = g(x)dx$

II. g(x) one of the inverse trigonometric or logarithmic function:

$$g(x) = \begin{cases} asin(x) \\ acox(x) \\ acotx \\ \ln x \\ \log_a x \end{cases}$$

Recommendation: $u = g(x)$, $dv = P_n(x)dx$

12.3. *Integration of rational functions.*

12.3.1. Improper and proper fractions.

Function is called rational fraction if it is represented as a ratio of two polynomials

$f(x) = \frac{P_n(x)}{Q_m(x)}$. Any **improper rational fraction** (which n≥m) can be represented as sum of the polynomial and **proper fraction** (which n<m) by division numerator on denominator of the fraction.

12.3.2. Integration of partial fractions.

I. $\int \frac{A}{x-a} dx = A \cdot \ln|x-a| + C;$

II. $\int \frac{A}{(x-a)^n} dx = \frac{-A}{n-1} \cdot \frac{1}{(x-a)^{n-1}} + C;$ (n>1)

III.

$$\int \frac{dx}{x^2+px+q} = \int \frac{dx}{(x+\frac{p}{2})^2 + q - \frac{p^2}{4}} = \frac{2}{\sqrt{4q-p^2}} a\tan\left(\frac{2x+p}{\sqrt{4q-p^2}}\right) + C;$$

if $D = p^2 - 4q < 0$

IV. $\int \frac{Ax+B}{x^2+px+q} dx =$

$$= \frac{A}{2} \int \frac{d(x^2+px+q)}{x^2+px+q} + \left(B - \frac{Ap}{2}\right) \int \frac{dx}{x^2+px+q} =$$

$$= \int \frac{A}{2} \ln|x^2+px+q| + \left(B - \frac{Ap}{2}\right) \cdot \frac{2}{\sqrt{4q-p^2}} \cdot a\tan\left(\frac{2x+p}{\sqrt{4q-p^2}}\right) + C.$$

if $D = p^2 - 4q < 0$. substitution: $x + \frac{p}{2} = t$; $dx = dt$.

V. $\int \frac{Ax+B}{(x^2+px+q)^k} dx = \int \frac{A(x+\frac{p}{2}) + B - A\frac{p}{2}}{((x+\frac{p}{2})^2 + q - \frac{p^2}{4})^k} dx =$

$B_1 = B - A\frac{p}{2}$; $m^2 = q - \frac{p^2}{4}$ $= \int \frac{At + B_1}{(t^2+m^2)^k} dt =$

$$\frac{A}{2} \int \frac{d(t^2+m^2)}{(t^2+m^2)^k} + B_1 \int \frac{dt}{(t^2+m^2)^k} = \frac{A}{2} \frac{(t^2+m^2)^{-k+1}}{-k+1} + B_1 I_k,$$

where is I_k can be evaluated by recurrence relations:

$I_k =$

$$\int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2} \int \frac{(t^2+m^2) - t^2}{(t^2+m^2)^k} dt = \frac{1}{m^2} \int \frac{dt}{(t^2+m^2)^{k-1}} -$$

$$\frac{1}{m^2} \int \frac{t^2}{(t^2+m^2)^k} dt = \frac{1}{m^2} I_{k-1} -$$

$$- \frac{1}{m^2} \int \frac{t \cdot tdt}{(t^2+m^2)^k} = \frac{1}{m^2} I_{k-1} -$$

$$\frac{1}{2m^2} \int t (t^2+m^2)^{-k} d(t^2+m^2) = \frac{1}{m^2} I_{k-1} -$$

$$- \frac{1}{2m^2} \int t \cdot d\frac{(t^2+m^2)^{-k+1}}{-k+1} = \left| u = t,\ v = \frac{(t^2+m^2)^{-k+1}}{-k+1} \right| =$$

$$= \frac{1}{m^2} I_{k-1} - \frac{1}{2m^2(1-k)} \left(\frac{t}{(t^2+m^2)^{-k+1}} - I_{k-1} \right).$$, i.e. the integral

I_k is expressed from the integral which degree one unit less. Applying (k-1) times will reduce that integral to integral in first degree

$$I_1 = \int \frac{dt}{t^2+m^2} = \frac{1}{m} a\tan\left(\frac{t}{m}\right) + C.$$

12.3.3. Integration of the rational fractions. $\int \frac{P_n(x)}{Q_m(x)}dx$

1) If n>m : Represent the rational fraction as a sum of the polynomial and proper fraction (which n<m) by division numerator on denominator of the fraction.

$$\int \frac{P_n(x)}{Q_m(x)}dx = I_{n-m}(x) + \int \frac{P_{m-1}(x)}{Q_m(x)}dx.$$

2) Factorize the denominator: $Q_m(x) = (x-a)^s \cdot (x^2+px+q)^t$. Where $D = p^2-4q<0$.

3) Using **the method of undetermined coefficients** represent obtained proper fraction like the sum of the partial fractions:

$$\frac{P_{m-1}(x)}{Q_m(x)} =$$

$$\frac{P_{m-1}(x)}{(x-a)^s \cdot (x^2+px+q)^t} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_s}{(x-a)^s} + \dots$$

$$\dots + \frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \dots + \frac{B_tx+C_t}{(x^2+px+q)^t}.$$

Where A_i $i=\overline{1,s}$; B_j, C_j $j=\overline{1,t}$ undefined constants.

4) **To determine the constants A_i, B_j, C_j** reduce the fractions from right side to a common denominator and compare the numerators of both sides of the equation, grouping the similar terms set the system of liner equations. Solving obtained system determine the values of the coefficients A_i $i=\overline{1,s}$; B_j, C_j $j=\overline{1,t}$.

5) Substituting instead A_i $i=\overline{1,s}$; B_j, C_j $j=\overline{1,t}$ obtained numerical values, **evaluate the integral of partial fractions.**

EXERCISE XII.

1

Evaluate the indefinite integrals:

1. $\int \frac{dx}{(2x+1)\sqrt[3]{\ln^2(2x+1)}}$.

2. $\int \frac{\sqrt[3]{\ln^2(1-x)}}{x-1}dx$.

3. $\int \frac{dx}{(1-x)\sqrt[3]{\ln^2(1-x)}}$.

4. $\int \frac{dx}{(1-x)\sqrt{\ln^3(1-x)}}$.

5. $\int \frac{\ln^3(1-x)}{x-1}dx$.

6. $\int \frac{\sqrt{\ln(2x-1)}}{2x-1}dx$.

7. $\int \frac{\sqrt[3]{\ln(3x+1)}}{3x+1}dx$.

8. $\int \frac{dx}{(x+1)\ln^2(x+1)}$.

9. $\int \frac{dx}{(x+1)\sqrt[3]{\ln(x+1)}}$.

10. $\int \frac{\sqrt[5]{\ln^2(x+1)}}{x+1}dx$.

11. $\int \frac{\sqrt{\ln^5(x+1)}}{x+1}dx$.

12. $\int \frac{\sqrt[7]{\ln^3(x+1)}}{x+1}dx$.

13. $\int \frac{\sqrt{\ln^3(x+1)}}{x+1}dx$.

14. $\int \frac{dx}{(x+1)\sqrt[5]{\ln(x+1)}}$.

15. $\int \frac{\sqrt{\ln^7(x+1)}}{x+1}dx$.

16. $\int \frac{dx}{(x+2)\sqrt{\ln(x+2)}}$.

17. $\int \frac{\ln^4(3x+1)}{3x+1}dx$.

18. $\int \frac{dx}{(x-3)\ln^4(x-3)}$.

19. $\int \frac{dx}{(x+5)\ln^3(x+5)}$.

20. $\int \frac{\ln^3(x-5)}{x-5}dx$.

21. $\int \frac{\sqrt[3]{\ln(x+4)}}{x+4}dx$.

22. $\int \frac{\ln^5(x-7)}{x-7}dx$.

23. $\int \frac{\sqrt{\ln^3(x+3)}}{x+3}dx$.

24. $\int \frac{\sqrt[3]{\ln^4(x-5)}}{x-5}dx$.

25. $\int \frac{dx}{(x+3)\ln^4(x+3)}$.

26. $\int \frac{\ln^5(x-8)}{x-8}\,dx$.

27. $\int \frac{\sqrt{\ln^3(x+6)}}{x+6}\,dx$.

28. $\int \frac{dx}{(x-4)\ln^5(x-4)}$.

29. $\int \frac{\ln^6(x+9)}{x+9}\,dx$.

30. $\int \frac{\ln(3x+5)}{3x+5}\,dx$.

2

1. $\int \sin^4 2x \cos 2x\,dx$.

2. $\int \frac{\cos 2x}{\sin^3 2x}\,dx$.

3. $\int \frac{\sin 3x}{\cos^4 3x}\,dx$.

4. $\int \frac{\sin x}{\sqrt[3]{\cos x}}\,dx$.

5. $\int \frac{\sin x}{\cos^5 x}\,dx$.

6. $\int \cos^7 2x \sin 2x\,dx$.

7. $\int \frac{\cos x\,dx}{\sin x+2}$.

8. $\int \frac{\cos x\,dx}{3-\sin x}$.

9. $\int \frac{\sin x\,dx}{\sqrt{\cos x+3}}$.

10. $\int \frac{\sin x\,dx}{\sqrt[3]{\cos x+1}}$.

11. $\int \frac{\cos x\,dx}{\sqrt{(\sin x-4)^3}}$.

12. $\int \frac{\sin 3x}{\cos^2 3x}\,dx$.

13. $\int \frac{\sin 5x}{\sqrt{\cos 5x}}\,dx$.

14. $\int \frac{\cos 4x}{\sin^3 4x}\,dx$.

15. $\int \sin^3 4x \cos 4x\,dx$.

16. $\int \sqrt[3]{\cos 2x}\,\sin 2x\,dx$.

17. $\int \sqrt{\cos^3 2x}\,\sin 2x\,dx$.

18. $\int \frac{\sin 4x}{\sqrt[3]{\cos^2 4x}}\,dx$.

19. $\int \sin^3 5x \cos 5x\,dx$.

20. $\int \frac{\cos 5x}{\sqrt{\sin^3 5x}}\,dx$.

21. $\int \frac{\sin 5x}{\cos^4 5x}\,dx$.

22. $\int \sqrt{\cos 7x}\,\sin 7x\,dx$.

23. $\int \sin^6 3x \cos 3x\,dx$.

24. $\int \frac{\cos 6x}{\sin^7 6x}\,dx$.

25. $\int \sqrt{\cos^3 2x}\,\sin 2x\,dx$.

26. $\int \sin^4 8x \cos 8x\,dx$.

27. $\int \sin^5 4x \cos 4x\,dx$.

28. $\int \frac{\sin 4x}{\sqrt[3]{\cos 4x}}\,dx$.

29. $\int \frac{\sin 2x}{\sqrt[3]{\cos^4 2x}}\,dx$.

30. $\int \frac{\cos 6x}{\sin^4 6x}\,dx$.

3

1. $\int \frac{\sqrt{\mathrm{tg}^3 x}}{\cos^2 x}\,dx$.

2. $\int \frac{dx}{\cos^2 x\sqrt{\mathrm{tg}^3 x}}$.

3. $\int \frac{dx}{\sin^2 x\,\mathrm{ctg}^4 x}$.

4. $\int \frac{\mathrm{ctg}^5 2x}{\sin^2 2x}\,dx$.

5. $\int \frac{\mathrm{tg}^3 4x}{\cos^2 4x}\,dx$.

6. $\int \frac{\sqrt[3]{\mathrm{tg}\,5x}}{\cos^2 5x}\,dx$.

7. $\int \frac{\sqrt[3]{\mathrm{ctg}^2 x}}{\sin^2 x}\,dx$.

8. $\int \frac{dx}{\sin^2 x\,\mathrm{ctg}^3 x}$.

9. $\int \frac{dx}{\cos^2 3x\,\mathrm{tg}^4 3x}$.

10. $\int \frac{\sqrt{\mathrm{ctg}\,7x}}{\sin^2 7x}\,dx$.

11. $\int \frac{\sqrt[5]{\mathrm{ctg}\,3x}}{\sin^2 3x}\,dx$.

12. $\int \frac{\mathrm{tg}^4 7x}{\cos^2 7x}\,dx$.

13. $\int \frac{\mathrm{ctg}^5 6x}{\sin^2 6x}\,dx$.

14. $\int \frac{\sqrt[3]{\mathrm{tg}^5 4x}}{\cos^2 4x}\,dx$.

15. $\int \frac{\mathrm{ctg}^4 3x}{\sin^2 3x}\,dx$.

16. $\int \frac{dx}{\cos^2 4x\sqrt{\mathrm{tg}\,4x}}$.

17. $\int \frac{dx}{\sin^2 3x\,\mathrm{ctg}^3 3x}$.

18. $\int \frac{\mathrm{tg}\,6x}{\cos^2 6x}\,dx$.

19. $\int \frac{dx}{\sin^2 x\,\mathrm{ctg}^3 x}$.

20. $\int \frac{\sqrt{\mathrm{ctg}\,4x}}{\sin^2 4x}\,dx$.

21. $\int \frac{\mathrm{ctg}^5 4x}{\sin^2 4x}\,dx$.

22. $\int \frac{\sqrt[3]{\mathrm{tg}\,7x}}{\cos^2 7x}\,dx$.

23. $\int \frac{\sqrt[5]{\operatorname{tg}^2 3x}}{\cos^2 3x} dx.$

24. $\int \frac{\sqrt{\operatorname{ctg}^3 5x}}{\sin^2 5x} dx.$

25. $\int \frac{dx}{\sin^2 x \sqrt[5]{\operatorname{ctg}^4 x}}.$

26. $\int \frac{dx}{\cos^2 x \sqrt[5]{\operatorname{tg}^2 x}}.$

27. $\int \frac{\operatorname{tg}^6 2x}{\cos^2 2x} dx.$

28. $\int \frac{\sqrt{\operatorname{ctg}^5 x}}{\sin^2 x} dx.$

29. $\int \frac{\sqrt[5]{\operatorname{ctg}^2 x}}{\sin^2 x} dx.$

30. $\int \frac{\operatorname{tg}^7 3x}{\cos^2 3x} dx.$

4

1. $\int \frac{\sqrt{\operatorname{arctg}^6 3x}}{1 + 9x^2} dx.$

2. $\int \frac{\sqrt[3]{\arcsin x}}{\sqrt{1 - x^2}} dx.$

3. $\int \frac{\arccos^2 3x}{\sqrt{1 - 9x^2}} dx.$

4. $\int \frac{\arccos \operatorname{tg}^3 2x}{1 + 4x^2} dx.$

5. $\int \frac{\sqrt[3]{\arccos^2 x}}{\sqrt{1 + x^2}} dx.$

6. $\int \frac{dx}{(1 + x^2) \operatorname{arctg}^3 x}.$

7. $\int \frac{\arccos^3 x}{\sqrt{1 - 9x^2}} dx.$

8. $\int \frac{\sqrt[3]{\operatorname{arctg}^2 x}}{1 + x^2} dx.$

9. $\int \frac{\arcsin^5 2x}{\sqrt{1 - 4x^2}} dx.$

10. $\int \frac{dx}{\sqrt{1 - x^2} \arcsin^4 x}.$

11. $\int \frac{\arccos^3 2x}{\sqrt{1 - 4x^2}} dx.$

12. $\int \frac{\operatorname{arcctg}^7 3x}{1 + 9x^2} dx.$

13. $\int \frac{\arccos 4x}{\sqrt{1 - 16x^2}} dx.$

14. $\int \frac{\arcsin^4 x}{\sqrt{1 - x^2}} dx.$

15. $\int \frac{\arcsin^3 2x}{\sqrt{1 - 4x^2}} dx.$

16. $\int \frac{dx}{(1 + x^2) \operatorname{arctg}^7 x}.$

17. $\int \frac{\sqrt[3]{\operatorname{arctg} 2x}}{1 + 4x^2} dx.$

18. $\int \frac{\arccos^6 3x}{1 + 9x^2} dx.$

19. $\int \frac{\sqrt{\operatorname{arctg}^3 x}}{1 + x^2} dx.$

20. $\int \frac{dx}{(1 + x^2) \sqrt{\operatorname{arctg} x}}.$

21. $\int \frac{dx}{(1 + x^2) \operatorname{arctg}^5 x}.$

22. $\int \frac{\arccos^7 x dx}{\sqrt{1 - x^2}}.$

23. $\int \frac{\sqrt[3]{\arccos 2x}}{\sqrt{1 - 4x^2}} dx.$

24. $\int \frac{\operatorname{arcctg}^4 5x}{1 + 25x^2} dx.$

25. $\int \frac{\arcsin^2 5x}{\sqrt{1 - 25x^2}} dx.$

26. $\int \frac{dx}{\sqrt{1 - 25x^2} \arcsin 5x}.$

27. $\int \frac{\operatorname{arctg}^6 3x}{1 + 9x^2} dx.$

28. $\int \frac{\arccos^2 7x}{\sqrt{1 - 49x^2}} dx.$

29. $\int \frac{\sqrt[5]{\operatorname{arctg}^3 x}}{1 + x^2} dx.$

30. $\int \frac{\operatorname{arcctg}^4 8x}{1 + 64x^2} dx.$

5

1. $\int \frac{xdx}{e^{3x^2+4}}.$

2. $\int \frac{xdx}{e^{x^2+3}}.$

3. $\int \frac{x^2dx}{e^{x^3+1}}.$

4. $\int e^{\cos x} \sin xdx.$

5. $\int e^{2x^3-1} x^2dx.$

6. $\int \frac{\sin x}{e^{\cos x}} dx.$

7. $\int e^{7x^2+2} xdx.$

8. $\int e^{3-x^2} xdx.$

9. $\int e^{4x^2+5} xdx.$

10. $\int \frac{dx}{\sqrt{1 - x^2} e^{\arcsin x}}.$

11. $\int e^{5x^2-3} xdx.$

12. $\int e^{1-4x^2} xdx.$

13. $\int e^{3x^2+4} xdx.$

14. $\int e^{\sin x+1} \cos xdx.$

15. $e^{4-x^2} xdx.$

16. $\int e^{\operatorname{tg} x} \frac{1}{\cos^2 x} dx.$

17. $\int e^{3 \cos x+2} \sin xdx.$

18. $\int e^{4\sin x-1}\cos x dx$.

19. $\int e^{5x^2-3} x dx$.

20. $\int e^{5-2x^2} x dx$.

21. $\int e^{4-3x^2} x dx$.

22. $\int e^{\cos 2x}\sin 2x dx$.

23. $\int e^{1-6x^2} x dx$.

24. $\int e^{x^3+1} x^2 dx$.

25. $\int \frac{e^{\operatorname{arctg} x}}{1+x^2} dx$.

26. $\int e^{3x^3} x^2 dx$.

27. $\int \frac{x^4 dx}{e^{x^5+1}}$.

28. $\int \frac{x dx}{e^{x^2-3}}$.

29. $\int \frac{x dx}{e^{2x^2+1}}$.

30. $e^{4-5x^2} x dx$.

6

1. $\int \frac{x-1}{7x^2+4} dx$.

2. $\int \frac{1-2x}{5x^2-1} dx$.

3. $\int \frac{2x+1}{5x^2+1} dx$.

4. $\int \frac{x+3}{\sqrt{x^2+4}} dx$.

5. $\int \frac{3x-2}{2x^2+7} dx$.

6. $\int \frac{5-x}{3x^2+1} dx$.

7. $\int \frac{5+x}{3x^2+1} dx$.

8. $\int \frac{2x-5}{\sqrt{7x^2+3}} dx$.

9. $\int \frac{2x-3}{\sqrt{x^2+9}} dx$.

10. $\int \frac{3x-2}{3x^2+1} dx$.

11. $\int \frac{x-1}{5-2x^2} dx$.

12. $\int \frac{2x+3}{1-3x^2} dx$.

13. $\int \frac{2x+3}{5x^2+2} dx$.

14. $\int \frac{x-3}{4x^2+1} dx$.

15. $\int \frac{x-3}{1-4x^2} dx$.

16. $\int \frac{3x-1}{4-x^2} dx$.

17. $\int \frac{5x-2}{x^2+9} dx$.

18. $\int \frac{2x+5}{\sqrt{5x^2+1}} dx$.

19. $\int \frac{1-2x}{\sqrt{3x^2+2}} dx$.

20. $\int \frac{2x-4}{x^2+16} dx$.

21. $\int \frac{2x-3}{\sqrt{4-x^2}} dx$.

22. $\int \frac{2x-1}{\sqrt{5-3x^2}} dx$.

23. $\int \frac{3x+4}{5-2x^2} dx$.

24. $\int \frac{3x-3}{\sqrt{1-x^2}} dx$.

25. $\int \frac{5x+2}{\sqrt{x^2+9}} dx$.

26. $\int \frac{3-2x}{x^2-8} dx$.

27. $\int \frac{x-5}{8-4x^2} dx$.

28. $\int \frac{x+4}{7x^2+3} dx$.

29. $\int \frac{3x+2}{\sqrt{2x^2-1}} dx$.

30. $\int \frac{x-5}{\sqrt{4-9x^2}} dx$.

7

1. $\int \frac{2-3x}{x^2+2} dx$.

(Answer: $\sqrt{2}\operatorname{arctg}\frac{x}{\sqrt{2}} - \frac{3}{2}\ln|x^2+2| + C.$) $| + C.)$

2. $\int \frac{3-5x}{\sqrt{1-x^2}} dx$.

(Answer: $3\arcsin x + 5\sqrt{1-x^2} + C.$)

3. $\int \frac{8-13x}{\sqrt{x^2-1}} dx$.

(Answer: $8\ln|x+\sqrt{x^2-1}| - 13\sqrt{x^2-1} + C.$) $+ C.)$

4. $\int \frac{6x+1}{2x^2-1} dx$.

(Answer: $\frac{3}{2}\ln|2x^2-1| + \frac{\sqrt{2}}{4}\ln\left|\frac{\sqrt{2}x-1}{\sqrt{2}x+1}\right| + C.$)

5. $\int \frac{x-2}{\sqrt{2-x^2}}dx.$

$\left(\text{Answer: } -\sqrt{2-x^2} - 2\arcsin\frac{x}{\sqrt{2}} + C.\right)$ 1 + C.) + C.)

6. $\int \frac{3-7x}{\sqrt{1-4x^2}}dx.$

$\left(\text{Answer: } \frac{3}{2}\arcsin 2x + \frac{7}{4}\sqrt{1-4x^2} + C.\right)$ + C.)

7. $\int \frac{5-3x}{\sqrt{2x^2+1}}dx.$

$\left(\text{Answer: } \frac{5}{\sqrt{2}}\ln\left|\sqrt{2}x + \sqrt{2x^2+1}\right| - \frac{3}{2}\sqrt{2x^2+1} + C.\right)$

8. $\int \frac{1+x}{\sqrt{2-x^2}}dx.$

$\left(\text{Answer: } \arcsin\frac{x}{\sqrt{2}} - \sqrt{2-x^2} + C.\right)$

9. $\int \frac{3x+2}{2x^2+1}dx.$

$\left(\text{Answer } \frac{3}{4}\ln|2x^2+1| + \sqrt{2}\,\mathrm{arctg}\,\sqrt{2}x + C.\right)$

10. $\int \frac{1-5x}{1+25x^2}dx.$

$\left(\text{Answer}: \frac{1}{5}\,\mathrm{arctg}\,5x - \frac{1}{10}\ln|1+25x^2| + C.\right)$

11. $\int \frac{4x-3}{3x^2-4}dx.$

$\left(\text{Answer: } \frac{2}{3}\ln|3x^2-4| - \frac{\sqrt{3}}{4}\ln\left|\frac{\sqrt{3}x-2}{\sqrt{3}x+2}\right| + C.\right)$

12. $\int \frac{5x+1}{\sqrt{x^2-6}}dx.$

$\left(\text{Answer: } 5\sqrt{x^2-6} + \ln\left|x + \sqrt{x^2-6}\right| + C.\right)$

13. $\int \frac{x-3}{9x^2+7}dx.$

$\left(\text{Answer: } \frac{1}{18}\ln|9x^2+7| - \frac{1}{\sqrt{7}}\,\mathrm{arctg}\frac{3x}{\sqrt{7}} + C.\right)$

14. $\int \frac{5-3x}{\sqrt{4-3x^2}}dx.$

$\left(\text{Answer: } \frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}x}{2} + \sqrt{4-3x^2} + C.\right)$

15. $\int \frac{4-2x}{\sqrt{1-4x^2}}dx.$

(Answer: $2\arcsin 2x + \frac{1}{2}\sqrt{1-4x^2} + C.$)

16. $\int \frac{5-x}{2+x^2}dx.$

(Answer: $\frac{5}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}} - \frac{1}{2}\ln|2+x^2| + C.$)

17. $\int \frac{1+3x}{\sqrt{1+4x^2}}dx.$

(Answer: $\frac{1}{2}\ln|2x+\sqrt{1+4x^2}| + \frac{3}{4}\sqrt{1+4x^2} + C.$)

18. $\int \frac{5-4x}{\sqrt{1-x^2}}dx.$

(Answer: $5\arcsin x + 4\sqrt{1-x^2} + C.$)

19. $\int \frac{5x-1}{\sqrt{x^2-3}}dx.$

(Answer: $5\sqrt{x^2-3} - \ln|x+\sqrt{x^2-3}| + C.$)

20. $\int \frac{1-3x}{4x^2-1}dx.$

(Answer: $\frac{1}{4}\ln\left|\frac{2x-1}{2x+1}\right| - \frac{3}{8}\ln|4x^2-1| + C.$)

21. $\int \frac{x-5}{3-2x^2}dx.$ (Ответ: $-\frac{1}{4}\ln|3-2x^2| +$

$+ \frac{5}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}x-\sqrt{3}}{\sqrt{2}x+\sqrt{3}}\right| + C.$)

22. $\int \frac{x+4}{\sqrt{9-x^2}}dx.$

(Answer: $-\sqrt{9-x^2} + 4\arcsin\frac{x}{3} + C.$)

23. $\int \frac{2x-7}{x^2-5}dx.$

(Answer: $\ln|x^2-5| - \frac{7}{2\sqrt{5}}\ln\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C.$)

24. $\int \frac{7x-2}{\sqrt{x^2-1}}dx.$

(Answer: $7\sqrt{x^2-1} - 2\ln|x+\sqrt{x^2-1}| + C.$)

25. $\int \frac{1+3x}{\sqrt{x^2+1}}dx.$

(Answer: $\ln|x+\sqrt{x^2+1}| + 3\sqrt{x^2+1} + C.$)

26. $\int \frac{x-5}{x^2+7}dx.$

(Answer: $\frac{1}{2}\ln|x^2+7| - \frac{5}{\sqrt{7}}\operatorname{arctg}\frac{x}{\sqrt{7}} + C.$)

27. $\int \frac{3-7x}{1+x^2}dx.$

(Answer: $3\operatorname{arctg} x - \frac{7}{2}\ln|1+x^2| + C.$)

28. $\int \frac{8-2x}{1+3x^2}dx.$

(Answer: $\frac{8}{\sqrt{3}}\operatorname{arctg}\sqrt{3}x - \frac{1}{3}\ln|1+3x^2| + C.$)

29. $\int \frac{3x+7}{\sqrt{x^2+4}}dx.$

(Answer: $3\sqrt{x^2+4} + 7\ln|x+\sqrt{x^2+4}| + C.$)

30. $\int \frac{2x-1}{\sqrt{3x^2-4}}dx.$

(Answer : $\frac{2}{3}\sqrt{3x^2-4}-\frac{1}{\sqrt{3}}\ln|\sqrt{3}x+\sqrt{3x^2-4}|+C.$)

8

1. $\int \frac{\sin 2x}{1+3\cos 2x}dx.$

(Answer : $-\frac{1}{6}\ln|1+3\cos 2x|+C.$)

2. $\int \frac{3x^3}{1-x^4}dx.$

(Answer : $-\frac{3}{4}\ln|1-x^4|+C.$)

3. $\int \frac{\sin 3x}{3-\cos 3x}dx.$

(Answer: $\frac{1}{3}\ln|3-\cos 3x|+C.$)

4. $\int \frac{e^x dx}{2e^x+3}.$

(Answer: $\frac{1}{2}\ln|2e^x+3|+C.$)

5. $\int \frac{\sin 2x}{\cos^2 x-4}dx.$

(Answer: $-\ln|\cos^2 x-4|+C.$)

$\int \frac{e^x dx}{4-3e^x}.$

(Answer: $-\frac{1}{3}\ln|4-3e^x|+C.$)

7. $\int \frac{x^2 dx}{7-5x^3}.$

(Answer: $-\frac{1}{15}\ln|7-5x^3|+C.$)

8. $\int \frac{\sin 2x}{3\sin^2 x+4}dx.$

(Answer: $\frac{1}{3}\ln|3\sin^2 x+4|+C.$)

9. $\int \frac{e^{2x}}{5+e^{2x}}dx.$

(Answer: $\frac{1}{2}\ln|5+e^{2x}|+C.$)

10. $\int \frac{4x^3}{7+2x^4}dx.$

(Answer: $\frac{1}{2}\ln|7+2x^4|+C.$)

11. $\int \frac{4x-5}{2x^2-5x+17}dx.$

(Answer: $\ln|2x^2-5x+17|+C.$)

12. $\int \frac{7x^3}{2x^4-5}dx.$

(Answer: $\frac{7}{8}\ln|2x^4-5|+C.$)

13. $\int \frac{\cos 3x}{\sqrt{\sin 3x-2}}dx.$

(Answer : $\frac{2}{3}\sqrt{\sin 3x-2}+C.$)

14. $\int \frac{\sin 2x}{\sqrt{1+\cos^2 x}}dx.$ (Answer: $-2\sqrt{1+\cos^2 x}+C.$)

15. $\int \frac{\sin x}{1+3\cos x}dx.$

(Answer : $-\frac{1}{3}\ln|1+3\cos x|+C.$)

16. $\int \frac{\sin 2x}{4-\sin^2 x}dx.$

(Answer : $-\ln|4-\sin^2 x|+C.$)

17. $\int \frac{e^{3x}}{e^{3x}-5}dx.$

(Answer: $\frac{1}{3}\ln|e^{3x}-5|+C.$)

18. $\int \frac{x^2}{7+3x^3}dx.$

(Answer: $\frac{1}{9}\ln|7+3x^3|+C.$)

19. $\int \frac{3x+3}{x^2+2x}dx.$

(Answer: $\frac{3}{2}\ln|x^2+2x|+C.$)

20. $\int \frac{e^{2x}}{\sqrt{e^{2x}+3}}dx.$

(Answer: $\sqrt{e^{2x}+3}+C.$)

21. $\int \frac{3x^2+1}{x^3+x-10}dx.$ (Answer: $\ln|x^3+x-10|+C.$)

22. $\int \frac{x^5}{3x^6-7}dx.$ (Answer: $\frac{1}{18}\ln|3x^6-7|+C.$)

23. $\int \frac{x^4dx}{\sqrt{x^5+3}}.$ (Answer: $\frac{2}{5}\sqrt{x^5+3}+C.$)

24. $\int \frac{3x^2-2}{\sqrt{2x^3-4x}}dx.$ (Answer: $\sqrt{2x^3-4x}+C.$)

25. $\int \frac{\cos 7x}{\sqrt{5-\sin 7x}}dx.$

(Answer: $-\frac{2}{7}\sqrt{5-\sin 7x}+C.$)

26. $\int \frac{\sin 4x}{\sqrt{\cos 4x+3}}dx.$ (Answer: $-\frac{1}{2}\sqrt{\cos 4x+3}+C.$)

27. $\int \frac{12x^2+5x^4}{4x^3+x^5}dx.$ (Answer: $\ln|4x^3+x^5|+C.$)

28. $\int \frac{4e^{2x}}{\sqrt{1-e^{2x}}}dx.$ (Answer: $-4\sqrt{1-e^{2x}}+C.$)

29. $\int \frac{\sin 2x}{\sqrt{6-\cos^2 x}}dx.$ (Answer: $2\sqrt{6-\cos^2 x}+C.$)

30. $\int \frac{7x}{\sqrt{5x^2-4}}dx.$ (Answer: $\frac{7}{5}\sqrt{5x^2-4}+C.$)

9 .

1. $\int \frac{1-2x-x^3}{1+x^2}dx.$

(Answer: $-\frac{x^2}{2}-\frac{1}{2}\ln|x^2+1|+\operatorname{arctg}x+C.$)

2. $\int \frac{7-x^2}{1-x}dx.$ (Answer: $\frac{x^2}{2}+x-6\ln|1-x|+C.$)

3. $\int \frac{x^3+2}{x^2-1}dx.$

(Answer: $\frac{x^2}{2}+\frac{1}{2}\ln|x^2-1|+\ln\left|\frac{x-1}{x+1}\right|+C.$)

4. $\int \frac{8x^3-1}{2x+1}dx.$

(Answer: $\frac{4}{3}x^3-x^2+x-\ln|2x+1|+C.$)

5. $\int \frac{x^5-2}{x^2-4}dx.$

(Answer: $\frac{1}{4}x^4+2x^2+8\ln|x^2-4|-\frac{1}{2}\ln\left|\frac{x-2}{x+2}\right|+C.$)

6. $\int \frac{2x^4-3}{x^2+1}dx.$

(Answer: $\frac{2}{3}x^3-2x-\operatorname{arctg}x+C.$)

7. $\int \frac{x^3-1}{2x+1}dx.$

(Answer: $\frac{1}{6}x^3-\frac{1}{8}x^2+\frac{1}{8}x-\frac{9}{16}\ln|2x+1|+C.$)

8. $\int \frac{x^5}{1-x^3}dx.$ (Answer: $-\frac{1}{3}x^3-\frac{1}{3}\ln|1-x^3|+C.$)

9. $\int \frac{x^2}{x^2+3}dx.$ (Answer: $x-\sqrt{3}\,\mathrm{arctg}\,\frac{x}{\sqrt{3}}+C.$)

10. $\int \frac{6x^3+x^2-2x+1}{2x-1}dx.$

(Answer: $x^3+x^2+\frac{1}{2}\ln|2x-1|+C.$)

11. $\int \frac{x^4}{x^2-3}dx.$

(Answer: $\frac{x^3}{3}+3x+\frac{9}{2\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|+C.$)

12. $\int \frac{x^3+5x}{x^2+1}dx.$ (Answer: $\frac{x^2}{2}+2\ln|x^2+1|+C.$)

13. $\int \frac{x^2-5x+6}{x^2+4}dx.$

(Answer: $x-\frac{5}{2}\ln|x^2-4|+\mathrm{arctg}\,\frac{x}{2}+C.$)

14. $\int \frac{x^3-1}{x+3}dx.$

(Answer: $\frac{x^3}{3}-\frac{3}{2}x^2+9x-28\ln|x+3|+C.$)

15. $\int \frac{x^3}{x^2-1}dx.$ (Answer: $\frac{1}{2}x^2+\frac{1}{2}\ln|x^2-1|+C.$)

16. $\int \frac{x^4+1}{x^2+1}dx.$ (Answer: $\frac{1}{3}x^3-x+2\,\mathrm{arctg}\,x+C.$)

17. $\int \frac{x^4-2x^2-1}{x^2+1}dx.$

(Answer: $\frac{x^3}{3}-3x+2\,\mathrm{arctg}\,x+C.$)

18. $\int \frac{x^4+2}{x^2-4}dx.$ (Answer: $\frac{x^3}{3}+4x+\frac{9}{2}\ln\left|\frac{x-2}{x+2}\right|+C.$)

19. $\int \frac{x^3-3}{x+5}dx.$

(Answer: $\frac{x^3}{3}-\frac{5}{2}x^2+25x-128\ln|x+5|+C.$)

20. $\int \frac{x^3+1}{x^2+1}dx.$

(Answer: $\frac{1}{2}x^2-\frac{1}{2}\ln|x^2+1|+\mathrm{arctg}\,x+C.$)

21. $\int \frac{1-2x^4}{x^2+1}dx.$ (Answer: $-\frac{2}{3}x^3+2x-\mathrm{arctg}\,x+C.$)

22. $\int \frac{2x^3-3}{x-2}dx.$

(Answer: $\frac{2}{3}x^3+2x^2+8x+13\ln|x-2|+C.$)

23. $\int \frac{2x^2+5}{x+1}dx.$ (Answer: $2x+3\,\mathrm{arctg}\,x+C.$)

24. $\int \frac{x^3+3x+1}{x^2+2}dx.$

(Answer: $\frac{x^2}{2}+\frac{1}{2}\ln|x^2+2|+\frac{1}{\sqrt{2}}\mathrm{arctg}\,\frac{x}{\sqrt{2}}+C.$)

25. $\int \frac{x^2+x}{2-x}dx.$ (Answer: $-\frac{x^2}{2}-3x-6\ln|x-2|+C.$)

26. $\int \frac{2x^2+5}{x-7}dx.$

(Answer: $x^2+14x+103\ln|x-7|+C.$)

27. $\int \frac{2x^3+3}{x-1}dx.$

(Answer: $\frac{2}{3}x^3+x^2+2x+5\ln|x-1|+C.$)

28. $\int \frac{1-x^4}{x^2+4}dx$

(Answer: $-\frac{x^3}{3}+4x-\frac{15}{2}\mathrm{arctg}\,\frac{x}{2}+C.$)

29. $\int \frac{x^2+4}{x-3}dx$. (Answer: $\frac{x^2}{2}+3x+13\ln|x-3|+C$.)

30. $\int \frac{2x^2+3}{2x^2-1}dx$.

(Answer: $x+\sqrt{2}\ln\left|\frac{\sqrt{2}x-1}{\sqrt{2}x+1}\right|+C$.)

10

1. $\int \frac{\ln(\cos x)}{\cos^2 x}dx$. (Answer: $\operatorname{tg} x\ln(\cos x)+\operatorname{tg} x-x+C$.)

2. $\int \cos(\ln x)dx$. (Answer: $\frac{x}{2}(\sin(\ln x)+\cos(\ln x))+C$.)

3. $\int \frac{\ln x}{x^2}dx$. (Answer: $C-\frac{\ln x+1}{x}$.)

4. $\int \ln(x+2)dx$. (Answer: $x\ln(x+2)-x+2\ln(x+2)+C$.)

5. $\int \frac{\ln(\cos x)}{\sin^2 x}dx$. (Answer: $C-\operatorname{ctg} x\ln(\cos x)-x$.)

9. $\int x\ln\frac{1-x}{1+x}dx$. (Answer: $\frac{x^2}{2}\ln\frac{1-x}{1+x}-x-\frac{1}{2}\ln\frac{1-x}{1+x}+C$.)

10. $\int \ln(x+\sqrt{1+x^2})dx$. (Answer: $x\ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}+C$.)

11. $\int \ln(x+4)dx$. (Answer: $x\ln(x+4)-x+4\ln(x+4)+C$.)

12. $\int \frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx$. (Answer: $\sqrt{1+x^2}\ln(x+\sqrt{1+x^2})-x+C$.)

13. $\int \frac{\ln(\sin x)}{\sin^2 x}dx$. (Answer: $C-x-\operatorname{ctg} x-\operatorname{ctg} x\ln(\sin x)$.)

14. $\int x^2\ln(x+1)dx$. (Answer: $\frac{x^3}{3}\ln(x+1)-\frac{x^3}{9}+\frac{x^2}{6}-\frac{x}{3}+\frac{1}{3}\ln(x+1)+C$.)

15. $\int \frac{\ln x\ln(\ln x)}{x}dx$. (Answer: $\frac{1}{2}\ln^2 x\ln(\ln x)-\frac{1}{4}\ln^2 x+C$.)

16. $\int \ln(x^2+1)dx$. (Answer: $x\ln(x^2+1)-2x+2\operatorname{arctg} x+C$.)

17. $\int \frac{\ln x}{x^3}dx$. (Answer: $C-\frac{\ln x}{2x^2}-\frac{1}{4x^2}$.)

18. $\int \sqrt{x}\ln^2 x\,dx$. (Answer: $\frac{2}{3}\sqrt{x^3}\ln^2 x-\frac{8}{9}\sqrt{x^3}\ln x+\frac{16}{27}\sqrt{x^3}+C$.)

19. $\int \ln\frac{1-x}{1+x}dx$. (Answer: $x\ln\frac{1-x}{1+x}-\ln(x^2-1)+C$.)

20. $\int (x^2-x+1)\ln x\,dx$. (Answer: $\left(\frac{x^3}{3}-\frac{x^2}{2}+x\right)\ln x-\frac{x^3}{9}+\frac{x^2}{4}-x+C$.)

21. $\int \sqrt{x}\ln x\,dx$. (Answer: $\frac{2}{3}\sqrt{x^3}\ln x-\frac{4}{9}\sqrt{x^3}+C$.)

22. $\int \frac{\ln(\sin x)}{\cos^2 x}dx$. (Answer: $\operatorname{tg} x\ln(\sin x)-x+C$.)

23. $\int x \ln (x^2+1)\,dx$. (Answer: $\frac{x^2}{2}\ln(x^2+1)-\frac{x^2}{2}+$
$+\frac{1}{2}\ln(x^2+1)+C.$)

24. $\int x\ln^2 x\,dx$. (Answer: $\frac{x^2}{2}\ln^2 x-\frac{x^2}{2}\ln x+$
$+\frac{x^2}{4}+C.$)

25. $\int x^2\ln x\,dx$. (Answer: $\frac{x^3}{3}\ln x-\frac{x^3}{9}+C.$)

26. $\int x\ln(x+1)\,dx$. (Answer: $\frac{x^2}{2}\ln(x+1)-\frac{x^2}{4}+$
$+\frac{x}{2}-\frac{1}{2}\ln(x+1)+C.$)

27. $\int \sin(\ln x)\,dx$. (Answer: $\frac{x}{2}(\sin(\ln x)-$
$-\cos(\ln x))+C.$)

28. $\int (x^2-4)\sin 5x\,dx$. (Answer: $\frac{2}{25}x\sin 5x-$
$-\frac{x^2-21}{5}\cos 5x+C.$)

29. $\int \ln(x+5)\,dx$. (Answer: $x\ln(x+5)-x+$
$+5\ln(x+5)+C.$)

30. $\int \ln\frac{2-x}{2+x}\,dx$. (Answer: $x\ln\frac{2-x}{2+x}-2\ln|4-$
$-x^2|+C.$)
$-\frac{x^2-21}{5}\cos 5x+C.$)

1. $\int\sqrt{1-x}\arccos\sqrt{x}\,dx$. (Answer: $\frac{2}{9}\sqrt{x^3}-\frac{2}{3}\sqrt{x}-$
$-\frac{2}{3}\sqrt{(1-x)^3}\arccos\sqrt{x}+C.$)

2. $\int\sqrt{1-x}\arcsin\sqrt{x}\,dx$. (Answer: $\frac{2}{3}\sqrt{x}-$
$-\frac{2}{9}\sqrt{x^3}-\frac{2}{3}\sqrt{(1-x)^3}\arcsin\sqrt{x}+C.$)

3. $\int x\,\mathrm{arctg}\,2x\,dx$. (Answer: $\frac{x^2}{2}\mathrm{arctg}\,2x-\frac{x}{4}+$
$+\frac{1}{8}\mathrm{arctg}\,2x+C.$)

4. $\int\frac{\arcsin x}{\sqrt{x+1}}\,dx$. (Answer: $2\sqrt{x+1}\arcsin x+$
$+4\sqrt{1-x}+C.$)

5. $\int\frac{\arcsin x}{\sqrt{1-x}}\,dx$. (Answer: $4\sqrt{1-x}-$
$+\frac{1}{8}\mathrm{arctg}\,2x+C.$)
$+C.$)

6. $\int\frac{\arcsin\sqrt{x}}{\sqrt{1-x}}\,dx$. (Answer: $2\sqrt{x}-$
$-2\sqrt{1-x}\arcsin\sqrt{x}+C.$)

7. $\int\frac{x\,\mathrm{arctg}\,x}{\sqrt{1+x^2}}\,dx$. (Answer: $\sqrt{1+x^2}\,\mathrm{arctg}\,x-$
$-\ln|x+\sqrt{1+x^2}|+C.$)

8. $\int\frac{x\arcsin x}{\sqrt{1-x^2}}\,dx$. (Answer: $x-\sqrt{1-x^2}\arcsin x+C.$)

9. $\int x\,\mathrm{arctg}\,x\,dx$. (Answer: $\frac{x^2}{2}\mathrm{arctg}\,x-\frac{x}{2}+$
$+\frac{1}{2}\mathrm{arctg}\,x+C.$)

10. $\int x \operatorname{arcctg} x dx$. (Answer: $\frac{x^2}{2} \operatorname{arcctg} x + \frac{x}{2} + \frac{1}{2} \operatorname{arcctg} x + C$.)

11. $\int \frac{x \arccos 2x}{\sqrt{1-4x^2}} dx$. (Answer: $C - \frac{x}{2} - \frac{1}{4}\sqrt{1-4x^2} \arccos 2x$.)

$+ \frac{1}{2} \operatorname{arctg} x + C$.)

12. $\int \arccos 2x dx$. (Answer: $\arccos 2x - \frac{1}{2}\sqrt{1-4x^2} + C$.)

13. $\int \operatorname{arctg} x dx$. (Answer: $x \operatorname{arctg} x - \frac{1}{2}\ln(1+x^2) + C$.)

14. $\int \frac{\arccos \sqrt{x}}{\sqrt{1-x}} dx$. (Answer: $C - 2\sqrt{x} - 2\sqrt{1-x} \arccos \sqrt{x}$.)

15. $\int \frac{x \arccos x}{\sqrt{1-x^2}} dx$. (Answer: $C - x - \sqrt{1-x^2} \arccos x$.)

16. $\int \frac{\arccos x}{\sqrt{1-x}} dx$. (Answer: $C - 4\sqrt{1+x} - 2\sqrt{1-x} \arccos x$.)

17. $\int \operatorname{arcctg} 2x dx$. (Answer: $x \operatorname{arcctg} 2x + \frac{1}{4}\ln(1+4x^2) + C$.)

18. $\int \frac{x \operatorname{arctg} x}{\sqrt{1+x^2}} dx$. (Answer: $\sqrt{1+x^2} \operatorname{arctg} x + \ln|x+\sqrt{1+x^2}| + C$.)

19. $\int \arcsin 2x dx$. (Answer: $x \arcsin 2x + \frac{1}{2}\sqrt{1-4x^2} + C$.)

20. $\int \frac{x \arcsin 2x}{\sqrt{1-4x^2}} dx$. (Answer: $\frac{1}{2}x - \frac{1}{4}\sqrt{1-4x^2} \arcsin 2x + C$.)

21. $\int \frac{\arccos x}{\sqrt{1+x}} dx$. (Answer: $2\sqrt{1+x} \arccos x - 4\sqrt{1-x} + C$.)

22. $\int x^2 \operatorname{arctg} x dx$. (Answer: $\frac{x^3}{3} \operatorname{arctg} x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1) + C$.)

23. $\int x \operatorname{arctg} 2x dx$. (Answer: $\frac{x^2}{2} \operatorname{arctg} 2x + \frac{x}{4} + \frac{1}{8} \operatorname{arctg} 2x + C$.)

24. $\int \operatorname{arctg}(x+5) dx$. (Answer: $x \operatorname{arctg}(x+5) - \frac{1}{2}\ln|x^2+10x+26| + 5 \operatorname{arctg}(x+5) + C$.)

25. $\int x^2 \operatorname{arcctg} x dx$. (Answer: $\frac{x^3}{3} \operatorname{arcctg} x + \frac{x^2}{6} - \frac{1}{6}\ln(x^2+1) + C$.)

26. $\int x \operatorname{arctg}^2 x dx$. (Answer: $\frac{x^2}{2} \operatorname{arctg}^2 x + \frac{1}{2} \operatorname{arctg}^2 x - x \operatorname{arctg} x + \frac{1}{2}\ln(x^2+1) + C$.)

27. $\int x^2 \cos\frac{x}{3} dx$. (Answer: $3x^2 \sin\frac{x}{3} + 18x \cos\frac{x}{3} - 54 \sin\frac{x}{3} + C$.)

28. $\int x \operatorname{arcctg}^2 x dx$. (Answer: $\frac{x^2}{2} \operatorname{arcctg}^2 x + \frac{1}{2} \operatorname{arcctg}^2 x + x \operatorname{arcctg} x + \frac{1}{2} \ln(x^2+1) + C.$)

29. $\int x^2 \sin 2x dx$. (Answer: $\frac{x}{2} \sin 2x - \frac{x^2}{2} \cos 2x + \frac{1}{4} \cos 2x + C.$)

30. $\int (x^2+4) e^{2x} dx$. (Answer: $\frac{1}{2}(x^2+4) e^{2x} + \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C.$)

12

1. $\int x^2 \cos 2x dx$. (Answer: $\frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{1}{4} \sin 2x + C.$)

2. $\int x \sin^2 x dx$. (Answer: $\frac{x^2}{4} - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x + C.$)

3. $\int x \sin x \cos x dx$. (Answer: $\frac{1}{8} \sin 2x - \frac{x}{4} \cos 2x + C.$)

4. $\int x^2 (\sin 2x - 3) dx$. (Answer: $\frac{x}{2} \sin 2x - \frac{x^2}{2} \cos 2x + \frac{1}{4} \cos 2x - x^3 + C.$)

5. $\int x^2 (\sin x + 1) dx$. (Answer: $2x \sin x - x^2 \cos x + 2 \cos x + \frac{x^3}{3} + C.$)

6. $\int (x^2+x) e^{-x} dx$. (Answer: $C - (x^2+3x+3) e^{-x}.$)

7. $\int (x^2+x) e^{x} dx$. (Answer $(x^2-x+1) e^x + C.$)

8. $\int (x^2-x+1) e^{-x} dx$. (Answer: $C - (x^2+x+2) e^{-x}$)

9. $\int (x^2-x+1) e^{x} dx$. (Answer: $(x^2-3x+4) e^x + C.$)

10. $\int x \operatorname{ctg}^2 x dx$. (Answer: $\ln|\sin x| - x \operatorname{ctg} x - \frac{x^2}{2} + C.$)

11. $\int x^2 e^{-x} dx$. (Answer: $C - (x^2+2x+2) e^{-x}.$)

12. $\int \frac{x dx}{\sin^2 x}$. (Answer: $\ln|\sin x| - x \operatorname{ctg} x + C.$)

13. $\int \frac{x dx}{\cos^2 x}$. (Answer: $x \operatorname{tg} x + \ln|\cos x| + C.$)

14. $\int x \operatorname{tg}^2 x dx$. (Answer: $x \operatorname{tg} x + \ln|\cos x| - \frac{x^2}{2} + C.$)

15. $\int (x^2+2) e^{-x} dx$. (Answer: $C - (x^2+2x+4) e^{-x}.$)

16. $\int x^2 \sin^2 x dx$. (Answer: $\frac{x^3}{6} - \frac{x^2}{4} \sin 2x + \frac{x}{4} \cos 2x + \frac{1}{8} \sin 2x + C.$)

17. $\int x^2 (\cos 2x + 3) dx$. (Answer: $x^3 + \frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{1}{4} \sin 2x + C.$)

18. $\int (x^2+2) e^{-x} dx$. (Answer: $(x^2-2x+4) e^x + C.$)

19. $\int (x^3+3) \sin x dx$. (Answer: $2x \sin x - (x^2+1) \cos x + C.$)

20. $\int (x^2-3) \cos x dx$. (Answer: $(x^2-4) \sin x + 2x \cos x + C.$)

21. $\int (x^2+1) e^{-x} dx$. (Answer: $C - (x^2+2x+3) e^{-x}.$)

22. $\int (x^2-1) e^{x} dx$. (Answer $(x-1)^2 e^x + C.$)

23. $\int x^2 \cos^2 x dx$. (Answer: $\frac{x^3}{6} + \frac{x^2}{4} \sin 2x + \frac{x}{4} \cos 2x - \frac{1}{8} \sin 2x + C.$)

24. $\int (x^2+x)\sin x dx$. (Answer: $(2x+1)\sin x - (x^2+x-2)\cos x + C$.)

25. $\int (x^2+x)\cos x dx$. (Answer: $(x^2+x-1)\sin x + (2x+1)\cos x + C$.)

26. $\int (x^2+1)e^x dx$. (Answer: $(x^2-2x+3)e^x + C$.)

27. $\int (x^2-1)e^{-x} dx$. (Answer: $C-(x+1)^2 e^{-x}$.)

28. $\int x\sin^2 x dx$. $\left(\text{Answer: } \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C.\right)$

29. $\int \arcsin 9x dx$. $\left(\text{Answer: } x\arcsin 9x + \frac{1}{9}\sqrt{1-81x^2} + C.\right)$

30. $\int x \,\mathrm{arctg}\, 2x dx$. $\left(\text{Answer: } \frac{x^2}{2}\mathrm{arctg}\, 2x - \frac{x}{4} + \frac{1}{2}\mathrm{arctg}\, 2x + C.\right)$

13

1. $\int (x+1)e^{2x} dx$.
2. $\int (x-2)e^x dx$.
3. $\int (x-7)\cos 2x dx$.
4. $\int (x-1)\cos 5x dx$.
5. $\int (x+2)\cos 3x dx$.
6. $\int (x-2)\cos 4x dx$.
7. $\int (x-4)\sin 2x dx$.
8. $\int (x-3)\cos x dx$.

$-\frac{x}{4} + \frac{1}{2}\mathrm{arctg}\, 2x + C.)$

9. $\int (x+4)\sin 2x dx$.
10. $\int x\sin 3x dx$.
11. $\int (x+5)\sin x dx$.
12. $\int (x-5)\cos x dx$.
13. $\int (x+9)\sin x dx$.
14. $\int (x+7)\sin 2x dx$.
15. $\int (x+4)\sin 3x dx$.
16. $\int (x+3)\sin 5x dx$.

17. $\int (x-4)\cos 2x dx$.
18. $\int (x-8)\sin x dx$.
19. $\int (x+4)\cos 3x dx$.
20. $\int (x+8)\sin 3x dx$.
21. $\int (x+6)\cos 4x dx$.
22. $\int (x-6)\sin\frac{x}{2} dx$.
23. $\int (x+1)\cos 7x dx$.
24. $\int (x+2)\sin\frac{x}{2} dx$.
25. $\int x\sin\frac{x}{5} dx$.
26. $\int (x+4)\cos\frac{x}{2} dx$.
27. $\int (x+1)\sin\frac{x}{3} dx$.
28. $\int (x+2)\cos\frac{x}{4} dx$.
29. $\int (x+3)\sin\frac{x}{4} dx$.
30. $\int (x-9)\sin\frac{x}{2} dx$.

14

1. $\int \ln(x-5) dx$.
2. $\int \mathrm{arctg}\, 2x dx$.
3. $\int x^2 e^{-x} dx$.
4. $\int (x+1)e^{-4x} dx$.
5. $\int x^2 e^{-2x} dx$.
6. $\int \mathrm{arctg}\, 3x dx$.
7. $\int x\cos 8x dx$.
8. $\int \mathrm{arctg}\, 4x dx$.
9. $\int \arcsin 5x dx$.
10. $\int (x+1)e^{-x} dx$.
1. $\int x\,\mathrm{arctg}\, x dx$.
12. $\int x^2 e^{3x} dx$.
3. $\int x\cos(x+4) dx$.
14. $\int x\cos(x-2) dx$.
5. $\int x\cos(x+3) dx$.
16. $\int x e^{x+2} dx$.
7. $\int x e^{-7x} dx$.
18. $\int \arcsin 2x dx$.
19. $\int x\sin(x+7) dx$.
20. $\int x\cos(x-4) dx$.
21. $\int x\sin(x+4) dx$.
22. $\int x\cos(x+9) dx$.
23. $\int (x+3)e^{-x} dx$.
24. $\int \arccos x dx$.
25. $\int (x^2-3)e^x dx$.
26. $\int x e^{-4x} dx$.
27. $\int x\cos(x+7) dx$.
28. $\int x e^{-5x} dx$.
29. $\int x e^{x+3} dx$.
30. $\int x\cos(2-x) dx$.

15

1. $\int \operatorname{arctg} 2x dx.$

2. $\int x \cos 6x dx.$

3. $\int \arcsin 3x dx.$

4. $\int \arccos 2x dx.$

5. $\int \operatorname{arctg} 8x dx.$

6. $\int x \sin (x-2)\, dx.$

7. $\int \arcsin 8x dx.$

8. $\int x \sin (x+3)\, dx.$

9. $\int x \cos (x+4)\, dx.$

10. $\int \arccos 7x dx.$

11. $\int x \cos (x-7)\, dx.$

12. $\int x \sin (x-5)\, dx.$

13. $\int (x-4) e^x dx.$

14. $\int x e^{-6x} dx.$

15. $\int \operatorname{arctg} 7x dx.$

16. $\int \arcsin 5x dx.$

17. $\int \ln (x-7)\, dx.$

18. $\int x \cos (x+6)\, dx.$

19. $\int \operatorname{arctg} \frac{x}{2}\, dx.$

20. $\int \ln (x+8)\, dx.$

21. $\int \operatorname{arctg} \frac{x}{5}\, dx.$

22. $\int \ln (x+12)\, dx.$

23. $\int \arcsin \frac{x}{5}\, dx.$

24. $\int \ln (2x-1)\, dx.$

25. $\int \ln(2x+3) dx.$

26. $\int \arccos \frac{x}{5}\, dx.$

27. $\int \operatorname{arctg} \frac{x}{4}\, dx.$

28. $\int \arcsin \frac{x}{7}\, dx.$

29. $\int \operatorname{arctg} 6x dx.$

30. $\int \arccos \frac{x}{3}\, dx.$

16

1. $\int \frac{dx}{4x^2-5x+4}.$

(Answer: $\frac{2}{\sqrt{39}} \operatorname{arctg} \frac{8x-5}{\sqrt{39}} + C.$)

2. $\int \frac{dx}{x^2-4x+10}.$

(Answer: $\frac{1}{\sqrt{6}} \operatorname{arctg} \frac{x+2}{\sqrt{6}} + C.$)

3. $\int \frac{dx}{2x^2-7x+1}.$

(Answer: $\frac{1}{\sqrt{41}} \ln\left|\frac{4x-7-\sqrt{41}}{4x-7+\sqrt{41}}\right| + C.$)

4. $\int \frac{dx}{2x^2+x-6}.$ (Answer: $\frac{1}{7} \ln\left|\frac{2x-3}{2x+4}\right| + C.$)

5. $\int \frac{dx}{5x^2+2x+7}.$ (Answer: $\frac{1}{\sqrt{34}} \operatorname{arctg} \frac{5x+1}{\sqrt{34}} + C.$)

6. $\int \frac{dx}{2x^2-2x+1}.$ (Answer: $\operatorname{arctg} (2x-1) + C.$)

7. $\int \frac{dx}{2x^2-11x+2}.$

(Answer: $\frac{1}{\sqrt{105}} \ln\left|\frac{4x-11-\sqrt{105}}{4x-11+\sqrt{105}}\right| + C.$)

8. $\int \frac{dx}{2x^2+x+2}.$ (Answer: $\frac{2}{\sqrt{15}} \operatorname{arctg} \frac{4x+1}{\sqrt{15}} + C.$)

9. $\int \frac{dx}{3x^2-12x+3}.$

(Answer: $\frac{1}{6\sqrt{3}} \ln\left|\frac{x-2-\sqrt{3}}{x-2+\sqrt{3}}\right| + C.$)

10. $\int \frac{dx}{2x^2+3x}.$ (Answer: $\frac{1}{3} \ln\left|\frac{x}{x+3/2}\right| + C.$)

11. $\int \frac{dx}{x^2-5x+6}.$ (Answer: $\ln\left|\frac{x-3}{x-2}\right| + C.$)

12. $\int \frac{dx}{2x-3-4x^2}.$

(Answer: $-\frac{1}{\sqrt{11}} \operatorname{arctg} \frac{4x-1}{\sqrt{11}} + C.$)

13. $\int \frac{dx}{3x^2-8x-3}$. (Answer: $\frac{1}{10}\ln\left|\frac{3x-9}{3x+1}\right|+C.$)

14. $\int \frac{dx}{8-2x-x^2}$. (Answer: $-\frac{1}{6}\ln\left|\frac{x-2}{x+4}\right|+C.$)

15. $\int \frac{dx}{5x-x^2-6}$. (Answer: $-\ln\left|\frac{x-3}{x-2}\right|+C.$)

16. $\int \frac{dx}{x^2+4x+25}$. (Answer: $\frac{1}{\sqrt{21}}\operatorname{arctg}\frac{x+2}{\sqrt{21}}+C.$)

17. $\int \frac{dx}{2x^2-8x+30}$. (Answer: $\frac{1}{2\sqrt{11}}\operatorname{arctg}\frac{x-2}{\sqrt{11}}+C.$)

18. $\int \frac{dx}{3x^2-9x+6}$. (Answer: $\frac{1}{3}\ln\left|\frac{x-2}{x-1}\right|+C.$)

19. $\int \frac{dx}{2x^2-2x+5}$. (Answer: $\frac{1}{3}\operatorname{arctg}\frac{2x-1}{3}+C.$)

20. $\int \frac{dx}{2x^2-3x-2}$. (Answer: $\frac{1}{5}\ln\left|\frac{2x-4}{2x+1}\right|+C.$)

21. $\int \frac{dx}{2x^2-6x+1}$.

(Answer: $\frac{1}{2\sqrt{7}}\ln\left|\frac{2x-3-\sqrt{7}}{2x-3+\sqrt{7}}\right|+C.$)

22. $\int \frac{dx}{2x^2-3x+2}$. (Answer: $\frac{2}{\sqrt{7}}\operatorname{arctg}\frac{4x-3}{\sqrt{7}}+C.$)

23. $\int \frac{dx}{x^2+7x+11}$.

Answer: $\frac{1}{\sqrt{5}}\ln\left|\frac{2x+7-\sqrt{5}}{2x+7+\sqrt{5}}\right|+C.$)

24. $\int \frac{dx}{2x^2-3x+1}$. (Answer: $\ln\left|\frac{2x-2}{2x-1}\right|+C.$)

25. $\int \frac{dx}{5x^2-10x+25}$. (Answer: $\frac{1}{10}\operatorname{arctg}\frac{x-1}{2}+C.$)

26. $\int \frac{dx}{2x^2+6x+3}$.

(Answer: $\frac{dx}{2\sqrt{3}}\ln\left|\frac{2x+3-\sqrt{3}}{2x+3+\sqrt{3}}\right|+C$)

27. $\int \frac{dx}{x^2-6x+8}$. (Answer: $\frac{1}{2}\ln\left|\frac{x-4}{x-2}\right|+C.$)

28. $\int \frac{dx}{1-2x-3x^2}$. (Answer: $-\frac{1}{4}\ln\left|\frac{3x-1}{3x+3}\right|+C.$)

29. $\int \frac{dx}{2x^2+3x+6}$. (Answer: $\frac{2}{\sqrt{39}}\operatorname{arctg}\frac{4x+3}{\sqrt{39}}+C.$)

30. $\int \frac{dx}{3x^2+5x+1}$.

(Answer: $\frac{1}{\sqrt{13}}\ln\left|\frac{6x+5-\sqrt{13}}{6x+5+\sqrt{13}}\right|+C.$)

17

1. $\int \frac{x+1}{2x^2+3x-4}dx$. (Answer: $\frac{1}{4}\ln|2x^2+3x-4|+$
$+\frac{1}{4\sqrt{41}}\ln\left|\frac{4x+3-\sqrt{41}}{4x+3+\sqrt{41}}\right|+C.$)

2. $\int \frac{x+6}{3x^2+x+1}dx$. (Answer: $\frac{1}{6}\ln|3x^2+x+1|+$
$+\frac{35}{3\sqrt{11}}\operatorname{arctg}\frac{6x+1}{\sqrt{11}}+C.$)

3. $\int \frac{2x-1}{3x^2-2x+6}dx$. (Answer: $\frac{1}{3}\ln|3x^2-2x+6|-$
$-\frac{1}{3\sqrt{17}}\operatorname{arctg}\frac{3x-1}{\sqrt{17}}+C.$)

4. $\int \frac{xdx}{2x^2+x+5}$. (Answer : $\frac{1}{4}\ln|2x^2+x+5| - \frac{1}{2\sqrt{39}}\operatorname{arctg}\frac{4x+1}{\sqrt{39}} + C.$)

5. $\int \frac{x+5}{x^2+x-2}dx$. (Answer : $\frac{1}{2}\ln|x^2+x-2| + \frac{3}{2}\ln\left|\frac{x-1}{x+2}\right| + C.$)

6. $\int \frac{3x-2}{5x^2-3x+2}dx$. (Answer : $\frac{3}{10}\ln|5x^2-3x+2| - \frac{11}{5\sqrt{31}}\operatorname{arctg}\frac{10x-3}{\sqrt{31}} + C.$)

7. $\int \frac{x+4}{2x^2-6x-8}dx$. (Answer : $\frac{1}{4}\ln|2x^2-6x-8| + \frac{11}{20}\ln\left|\frac{x-4}{x+1}\right| + C.$)

8. $\int \frac{x+4}{2x^2-7x+1}dx$. (Answer : $\frac{1}{4}\ln|2x^2-7x+1| + \frac{23}{4\sqrt{41}}\ln\left|\frac{4x-7-\sqrt{41}}{4x-7+\sqrt{41}}\right| + C.$)

9. $\int \frac{5x-2}{2x^2-5x+2}dx$. (Answer : $\frac{5}{4}\ln|2x^2-5x+2| + \frac{17}{12}\ln\left|\frac{2x-4}{2x-1}\right| + C.$)

10. $\int \frac{4x-1}{4x^2-4x+5}dx$. (Answer : $\frac{1}{2}\ln|4x^2-4x+5| + \frac{1}{4}\operatorname{arctg}\frac{2x-1}{2} + C.$)

11. $\int \frac{x+1}{2x^2+x+1}dx$. (Answer : $\frac{1}{4}\ln|2x^2+x+1| + \frac{3}{2\sqrt{7}}\operatorname{arctg}\frac{4x+1}{\sqrt{7}} + C.$)

12. $\int \frac{x+1}{3x^2-2x-3}dx$. (Answer : $\frac{1}{6}\ln|3x^2-2x-3| + \frac{2}{3\sqrt{10}}\ln\left|\frac{3x-1-\sqrt{10}}{3x-1+\sqrt{10}}\right| + C.$)

13. $\int \frac{4x+8}{4x^2+6x-13}dx$. (Answer : $\frac{1}{2}\ln|4x^2+6x-13| + \frac{5}{2\sqrt{61}}\ln\left|\frac{4x+3-\sqrt{61}}{4x+3+\sqrt{61}}\right| + C.$)

14. $\int \frac{5x+1}{x^2-4x+1}dx$. (Answer : $\frac{5}{2}\ln|x^2-4x+1| + \frac{11}{2\sqrt{3}}\ln\left|\frac{x-2-\sqrt{3}}{x-2+\sqrt{3}}\right| + C.$)

15. $\int \frac{xdx}{2x^2+2x+5}$. (Answer : $\frac{1}{4}\ln|2x^2+2x+5| - \frac{1}{6}\operatorname{arctg}\frac{2x+1}{3} + C.$)

16. $\int \frac{x-3}{x^2-5x+4}dx$. (Answer : $\frac{1}{2}\ln|x^2-5x+4| - \frac{1}{6}\ln\left|\frac{x-4}{x-1}\right| + C.$)

17. $\int \frac{2x-1}{2x^2+8x-6}dx$. (Answer : $\frac{1}{2}\ln|2x^2+8x-6| - \frac{5}{4\sqrt{7}}\ln\left|\frac{x+2-\sqrt{7}}{x+2+\sqrt{7}}\right| + C.$)

18. $\int \frac{2-x}{4x^2+16x-12}dx$. (Answer: $-\frac{1}{8}\ln|4x^2+16x-12|+\frac{1}{2\sqrt{7}}\ln\left|\frac{x+2-\sqrt{7}}{x+2+\sqrt{7}}\right|+C.$)

19. $\int \frac{2x-1}{3x^2-6x-9}dx$. (Answer: $\frac{1}{3}\ln|3x^2-6x-9|+\frac{1}{12}\ln\left|\frac{x-3}{x+1}\right|+C.$)

20. $\int \frac{2x-1}{3+x-2x^2}dx$. (Answer: $-\frac{1}{2}\ln|2x^2-x-3|+\frac{1}{10}\ln\left|\frac{2x-3}{2x+2}\right|+C.$)

21. $\int \frac{x-4}{3x^2+x-1}dx$. (Answer: $\frac{1}{6}\ln|3x^2+x-1|-\frac{25}{6\sqrt{13}}\ln\left|\frac{6x+1-\sqrt{13}}{6x+1+\sqrt{13}}\right|+C.$)

22. $\int \frac{3x+1}{x^2-4x-2}dx$. (Answer: $\frac{3}{2}\ln|x^2-4x-2|+\frac{7}{2\sqrt{6}}\ln\left|\frac{x-2-\sqrt{6}}{x-2+\sqrt{6}}\right|+C.$)

23. $\int \frac{x-5}{2x^2+x-4}dx$. (Answer: $\frac{1}{4}\ln|2x^2+x-4|+\frac{21}{4\sqrt{33}}\ln\left|\frac{4x+1-\sqrt{33}}{4x+1+\sqrt{33}}\right|+C.$)

24. $\int \frac{2x+3}{3x^2+2x-7}dx$. (Answer: $\frac{1}{3}\ln|3x^2+2x-7|+\frac{7}{6\sqrt{22}}\ln\left|\frac{3x+1-\sqrt{22}}{3x+1+\sqrt{22}}\right|+C.$)

25. $\int \frac{x-3}{4x^2+2x-3}dx$. (Answer: $\frac{1}{8}\ln|4x^2+2x-3|-\frac{\sqrt{13}}{8}\ln\left|\frac{4x+1-\sqrt{13}}{4x+1+\sqrt{13}}\right|+C.$)

26. $\int \frac{x+2}{3x^2-x+5}dx$. (Answer: $\frac{1}{6}\ln|3x^2-x+5|+\frac{13}{3\sqrt{59}}\operatorname{arctg}\frac{6x-1}{\sqrt{59}}+C.$)

27. $\int \frac{3x-2}{x^2+5x-1}dx$. (Answer: $\frac{3}{2}\ln|x^2+5x-1|-\frac{19}{2\sqrt{29}}\ln\left|\frac{2x+5-\sqrt{29}}{2x+5+\sqrt{29}}\right|+C.$)

28. $\int \frac{x-7}{4x^2+3x-1}dx$. (Answer: $\frac{1}{8}\ln|4x^2+3x-1|-\frac{59}{40}\ln\left|\frac{8x-2}{8x+8}\right|+C.$)

29. $\int \frac{2x+1}{5x^2+2x+10}dx$. (Answer: $\frac{1}{5}\ln|5x^2+2x-10|+\frac{3}{5\sqrt{49}}\operatorname{arctg}\frac{5x+1}{\sqrt{49}}+C.$)

30. $\int \frac{x-4}{5x^2-x+7}dx$. (Answer: $\frac{1}{10}\ln|5x^2-x+7|-\frac{39}{5\sqrt{139}}\operatorname{arctg}\frac{10x-1}{\sqrt{139}}+C.$)

Topic 13. Integration of some irrational functions. Integration of some trigonometric functions.

13.1. Integration of some irrational functions. 13.2. Integration of some trigonometric functions.

13.1. Integration of some irrational functions.

The evaluation of the integrals of irrational function implements by reducing to the integral of the rational function using the substitution method.

13.1.1. Integration of the irrational functions of kind $\int R\left(x, x^{\frac{p_1}{q_1}}, x^{\frac{p_2}{q_2}}, \ldots, x^{\frac{p_n}{q_n}}\right)dx$.

Here R(...) is the rational function of the variable x and irrational functions: $x^{\frac{p_i}{q_i}} = \sqrt[q_i]{x^{p_i}}$.

The available substitution is : $\mathbf{x = t^k}$ **and** $\mathbf{dx = k\cdot t^{k-1}dt}$, where k is the least common denominator of the fractions: $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}$.

$$\int R\left(x, x^{\frac{p_1}{q_1}}, x^{\frac{p_2}{q_2}}, \ldots, x^{\frac{p_n}{q_n}}\right)dx = \left| \begin{array}{c} x = t^k \\ dx = k\cdot t^{k-1}dt \end{array} \right| =$$

$$\int R\left(t^k, t^{\frac{kp_1}{q_1}}, t^{\frac{kp_2}{q_2}}, \ldots, t^{\frac{kp_n}{q_n}}\right)\cdot k\cdot t^{k-1}dt$$ - integral of rational function, because all $\frac{kp_i}{q_i}$ are integer.

13.1.2. Integration of the irrational functions of kind

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_n}{q_n}}\right)dx$$

There is the available substitution: $\frac{ax+b}{cx+d} = t^k$, where k is the lowest common denominator of the fractions: $\frac{p_1}{q_1}, \frac{p_2}{q_q}, \ldots, \frac{p_n}{q_n}$

13.1.3.Integration of the differential binomial:

$$\int x^m (a + bx^n)^p dx .$$

Here a,b nonzero constants and m,n,p –rational numbers. Next substitutions are true:

1	p- integer	$\mathbf{x = t^k}$ **and** $\mathbf{dx = k\cdot t^{k-1}dt}$, where **k** is the lowest common denominator of the fractions **m** and **n**.
2	$p = \frac{r}{s}$ – fraction and (m+1)/n - integer	$\mathbf{a + b x^n = t^s}$ (s>0) , where **s** is the denominator of **p**
3	p ; (m+1)/n - fraction and (m+1)/n + p - integer	$\mathbf{a + b x^n = t^s \cdot x^n}$ (s>0) , where **s** is the denominator of **p**

13.1.4. Integration of the irrational functions of kind: $\int R\left(x, \sqrt{ax^2 + bx + c}\right)dx$.

The radicand complete in square by substitution $x + \frac{b}{2a} = t$, then reduce the integral to one of three integrals which can be reduced to integral of the trigonometric function applying the next substitutions:

1. $\int R_1(t,\sqrt{a^2-t^2})dt = \left| \ t = a\cdot \sin(z)\ ,\ dt = a\cdot\cos(z)\cdot dz \ \right| =$

$= \int R_1(a\sin z, a\cos z)a\cos z dz$;

2. $\int R_1(t,\sqrt{t^2-a^2})dt = \left| \ t = \dfrac{a}{\sin(z)}\ ;\quad dt = -\dfrac{a\cdot\cos z}{\sin^2 z}dz \ \right| = -$

$= \int R_1(\dfrac{a}{\sin z}, a\cdot\cot z)\cdot\dfrac{a\cdot\cos z}{\sin^2 z}dz$;

3. $\int R_1(t,\sqrt{t^2+a^2})dt = \left| t = a\cdot \tan(z)\ ,\ dt = \dfrac{a}{\cos^2 z}\ dz \right| =$

$= \int R_1(a\cdot\tan z, \dfrac{a}{\cos z})\cdot\dfrac{a}{\cos^2 z}dz$;

4. $\int \dfrac{dx}{x\sqrt{ax^2+bx+c}} = \left| x=\dfrac{1}{t} \quad dx = -\dfrac{dt}{t^2} \right|$.

13.2. Integration of some trigonometric functions.

The evaluation of the integrals of trigonometric function implements by reducing to the integral of the rational function using the substitution method.

13.2.1. Integration of the trigonometric functions of kind: $\int R(\sin x,\cos x)dx$

Here R(...) is the rational function.

There is suitable universal substitution: $\tan(\dfrac{x}{2}) = t$ =>

x= 2atan(t) => $dx = \dfrac{2dt}{1+t^2}$.

Well known from trigonometry formulas must be used:

$\sin(x) = \dfrac{2t}{1+t^2}$; $\cos(x) = \dfrac{1-t^2}{1+t^2}$.

13.2.2. Integration of the trigonometric functions of kind: $\int R(\sin x)\cdot\cos x\cdot dx$ **and** $\int R(\cos x)\cdot\sin x\cdot dx$.

Here R(...) is the rational function.

Next substitution is recommended:

for $\int R(\sin x)\cdot\cos x\cdot dx = \int R(\sin x)\cdot d(\sin x) = \left| \text{sinx=t} \right| =$ $\int R(t)\cdot dt$.

for $\int R(\cos x)\cdot\sin x\cdot dx = -\int R(\cos x)\cdot d(\cos x) = \left| \text{cosx=t} \right| = -$ $\int R(t)\cdot dt$.

13.2.3. Integration of the trigonometric functions of kind: $\int R(-\sin x,-\cos x)dx = \int R(\sin x\cdot\cos x)dx$.

If the integrand satisfies that condition it can be expressed from the trigonometric functions:

$\int R(\sin x\cdot\cos x)dx = \int R_1(\tan x)dx = \left| \text{tanx=t ; x=atanx;dx=}\dfrac{dt}{1+t^2} \right| =$

$= \int R_1(t)\cdot\dfrac{dt}{1+t^2}$

Integral is reduced by substitution method to integral of rational function.

EXERCISE XIII.

Evaluate the integrals:

1. $\int \dfrac{\sqrt{1-x^2}}{x}dx$. $\left(\text{Answer}: \dfrac{1}{2}\ln\left|\dfrac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right| + \sqrt{1-x^2}+C.\right)$

2. $\int \dfrac{\sqrt{x^2-1}}{x}dx$. $\left(\text{Answer}: \sqrt{x^2-1}-\arccos\dfrac{1}{x}+C.\right)$

3. $\int \frac{\sqrt{x^2+4}}{x}dx$. (Answer: $\left(\sqrt{4+x^2} + \ln\left|\frac{2-\sqrt{4+x^2}}{2+\sqrt{4+x^2}}\right| + C.\right)$

4. $\int \frac{\sqrt{1-x^2}}{x^4}dx$. (Answer: $C - \frac{1}{3}\frac{\sqrt{(1-x^2)^3}}{x^3}$.)

5. $\int \sqrt{4-x^2}\,dx$. (Answer: $2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C$.)

6. $\int \frac{\sqrt{x^2+9}}{x}dx$. (Answer: $\sqrt{x^2+9} + \frac{3}{2}\ln\left|\frac{3-\sqrt{x^2+9}}{3+\sqrt{x^2+9}}\right| + C$.)

7. $\int \frac{\sqrt{x^2+4}}{x^2}dx$. (Answer: $\ln\left|\frac{x+\sqrt{4+x^2}}{x-\sqrt{4+x^2}}\right| - \frac{\sqrt{4-x^2}}{x}$

8. $\int \frac{\sqrt{4-x^2}}{x^4}dx$. (Answer: $C - \frac{1}{12}\frac{\sqrt{(4-x^2)^3}}{x^3}$.)

9. $\int \frac{dx}{\sqrt{(1+x^2)^3}}$. (Answer: $\frac{x}{\sqrt{1+x^2}} + C$.)

10. $\int \frac{\sqrt{x^2+4}}{x^4}dx$. (Answer: $C - \frac{1}{12}\frac{\sqrt{(4+x^2)^3}}{x^3}$.)

11. $\int \frac{\sqrt{(4-x^2)^3}}{x^6}dx$. (Answer: $C - \frac{1}{20}\frac{\sqrt{(4-x^2)^5}}{x^5}$.)

12. $\int \frac{dx}{\sqrt{(1+x^2)^5}}$. (Answer: $\frac{x}{\sqrt{1+x^2}} - \frac{1}{3}\frac{x^3}{\sqrt{(1+x^2)^3}} + C$.)

13. $\int \frac{\sqrt{x^2-9}}{x}dx$. (Answer: $\sqrt{x^2-9} - 3\arccos\frac{3}{x} + C$.)

14. $\int \frac{dx}{\sqrt{(x^2-1)^3}}$. (Answer: $C - \frac{x}{\sqrt{x^2-1}}$.)

15. $\int x^3\sqrt{9-x^2}\,dx$. (Answer: $\frac{1}{5}\sqrt{(9-x^2)^5} - 3\sqrt{(9-x^2)^3} + C$.)

16. $\int \frac{dx}{x^2\sqrt{(x^2-1)^3}}$. (Answer: $C - \frac{x}{\sqrt{x^2-1}} - \frac{\sqrt{x^2-1}}{x}$.)

17. $\int \frac{dx}{x^2\sqrt{x^2-1}}$. (Answer: $\frac{\sqrt{x^2-1}}{x} + C$.)

18. $\int \frac{\sqrt{x^2-9}}{x^2}dx$. (Answer: $\frac{1}{2}\ln\left|\frac{\sqrt{x^2-9}+x}{\sqrt{x^2-9}-x}\right| - \frac{\sqrt{x^2-9}}{x} + C$.)

19. $\int \frac{dx}{x^3\sqrt{x^2-1}}$. (Answer: $\frac{1}{2}\arccos\frac{1}{x} + \frac{\sqrt{x^2-1}}{2x^2} + C$.)

20. $\int \frac{\sqrt{9-x^2}}{x^4}dx$. (Answer: $C - \frac{1}{27}\frac{\sqrt{(9-x^2)^3}}{x^3}$.)

21. $\int \frac{dx}{x^2\sqrt{x^2+9}}$. (Answer: $C - \frac{\sqrt{9+x^2}}{9x}$.)

22. $\int x^2\sqrt{1-x^2}\,dx$. (Answer: $\frac{1}{8}\arcsin x - \frac{1}{8}x\sqrt{1-x^2}(1-2x^2) + C$.)

23. $\int x^3\sqrt{1-x^2}\,dx$. (Answer: $\frac{1}{5}\sqrt{(1-x^2)^5} - \frac{1}{3}\sqrt{(1-x^2)^3} + C$.)

24. $\int \frac{\sqrt{(4-x^2)^3}\,dx}{x^4}$. (Answer: $\arcsin\frac{x}{2} + \frac{\sqrt{4-x^2}}{x} - \frac{1}{3}\frac{\sqrt{(4-x^2)^3}}{x^3} + C.$)

25. $\int \frac{dx}{\sqrt{(4+x^2)^3}}$. (Answer: $\frac{x}{4\sqrt{4+x^2}} + C.$)

26. $\int \frac{\sqrt{x^2+9}}{x^4}\,dx$. (Answer: $C - \frac{1}{27}\frac{\sqrt{(9+x^2)^3}}{x^3}.$)

27. $\int \frac{dx}{\sqrt{(9+x^2)^3}}$. (Answer: $\frac{1}{9}\frac{x}{\sqrt{9+x^2}} + C.$)

28. $\int \frac{x^2dx}{\sqrt{9-x^2}}$. (Answer: $\frac{9}{2}\arcsin\frac{x}{3} - \frac{1}{2}x\sqrt{9-x^2} + C.$)

29. $\int \frac{\sqrt{16-x^2}}{x^4}\,dx$. (Answer: $C - \frac{1}{48}\frac{x^3}{\sqrt{16-x^2}}.$)

30. $\int \frac{\sqrt{16-x^2}}{x^2}\,dx$. (Answer: $C - \arcsin\frac{x}{4} - \frac{x}{\sqrt{16-x^2}} + C.$)

2

1. $\int \frac{dx}{(x+1)\sqrt{1+x^2}}$. (Answer: $C - \frac{1}{\sqrt{2}}\ln\left|\frac{1}{x+1} - \frac{1}{2} - \frac{\sqrt{1+x^2}}{\sqrt{2}(x+1)}\right|.$)

2. $\int \frac{dx}{(x+1)\sqrt{x^2-1}}$. (Answer: $\sqrt{\frac{x-1}{x+1}} + C.$)

3. $\int \frac{dx}{(x-1)\sqrt{x^2-1}}$. (Answer: $C - \sqrt{\frac{x+1}{x-1}}.$)

4. $\int \frac{dx}{x\sqrt{1-x^2}}$. (Answer: $C - \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|.$)

5. $\int \frac{dx}{x\sqrt{1+x^2}}$. (Answer: $C - \ln\left|\frac{1+\sqrt{1+x^2}}{x}\right|.$)

6. $\int \frac{dx}{x\sqrt{x^2-1}}$. (Answer: $C - \arcsin\frac{1}{x}.$)

7. $\int \frac{dx}{x\sqrt{x^2+x+1}}$. (Answer: $C - \ln\left|\frac{1}{x} + \frac{1}{2} + \frac{\sqrt{x^2+x+1}}{x}\right|.$)

8. $\int \frac{dx}{x\sqrt{x^2-x+1}}$. (Answer: $C - \ln\left|\frac{1+\sqrt{x^2-x+1}}{x} - \frac{1}{2}\right|.$)

9. $\int \frac{dx}{x\sqrt{x^2+x-1}}$. (Answer: $C - \arcsin\frac{2-x}{\sqrt{5}\,x}.$)

10. $\int \frac{dx}{x\sqrt{x^2-x-1}}$. (Answer: $C - \arcsin\frac{x+2}{\sqrt{5}\,x} + \frac{\sqrt{x^2+x+1}}{x}\Big|.$)

11. $\int \frac{dx}{x\sqrt{1+x-x^2}}$. (Answer: $C-\ln\left|\frac{1}{x}+\frac{1}{2}+\frac{\sqrt{1+x-x^2}}{x}\right|$.)

12. $\int \frac{dx}{x\sqrt{x^2+x-2}}$. (Answer: $C-\frac{1}{\sqrt{2}}\arcsin\frac{4-x}{3x}$.)

$+\frac{\sqrt{x^2+x+1}}{x}\Big|$.)

13. $\int \frac{dx}{(x+1)\sqrt{x^2-x+1}}$. (Answer: $C-\frac{1}{\sqrt{3}}\ln\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{x^2-x+1}}{\sqrt{3}(x+1)}\right|$.)

14. $\int \frac{dx}{(x+1)\sqrt{x^2-x-1}}$. (Answer: $C-\ln\left|\frac{1}{x+1}-\frac{3}{2}+\frac{\sqrt{x^2-x-1}}{x+1}\right|$.)

15. $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$. (Answer: $C-\ln\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{x+1}\right|$.)

16. $\int \frac{dx}{(x+1)\sqrt{x^2+x-1}}$. (Answer: $C-\arcsin\frac{x+3}{\sqrt{5}(x+1)}$.)

17. $\int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$. (Answer: $\arcsin\frac{3x+1}{\sqrt{5}(x+1)}+C$.)

18. $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$. (Answer: $C-\frac{1}{\sqrt{3}}\ln\left|\frac{1}{x-1}+\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{\sqrt{3}(x-1)}\right|$.)

19. $\int \frac{dx}{(x-1)\sqrt{x^2-x+1}}$. (Answer: $C-\ln\left|\frac{1}{x-1}+\frac{1}{2}+\frac{\sqrt{x^2-x+1}}{x-1}\right|$.)

20. $\int \frac{dx}{(x-1)\sqrt{x^2+x-1}}$. (Answer: $C-\ln\left|\frac{1}{x-1}+\frac{3}{2}+\frac{\sqrt{x^2+x-1}}{x-1}\right|$.)

21. $\int \frac{dx}{(x-1)\sqrt{x^2-x-1}}$. (Answer: $C-\arcsin\frac{3-x}{\sqrt{5}(x-1)}$.)

22. $\int \frac{dx}{(x-1)\sqrt{1+x-x^2}}$. (Answer: $C-\ln\left|\frac{1}{x-1}-\frac{1}{2}+\frac{\sqrt{1+x-x^2}}{x-1}\right|$.)

23. $\int \frac{dx}{(x+1)\sqrt{1-x-x^2}}$. (Ответ: $C-\ln\left|\frac{1}{x+1}+\frac{1}{2}+\frac{\sqrt{1-x-x^2}}{x+1}\right|$.)

24. $\int \frac{dx}{(x-1)\sqrt{1-x-x^2}}$. (Answer: $C-\arcsin\frac{3x-1}{\sqrt{5}(x-1)}$.)

25. $\int \frac{dx}{x\sqrt{1-x-x^2}}$. (Answer: $C-\ln\left|\frac{1}{x}-\frac{1}{2}+\frac{\sqrt{1-x-x^2}}{x}\right|$.)

26. $\int \frac{dx}{x\sqrt{x^2+x-3}}$. (Answer: $C-\frac{1}{\sqrt{3}}\arcsin\frac{6-x}{x\sqrt{3}}$.)

27. $\int \frac{dx}{(x+1)\sqrt{x^2+x-2}}$. (Answer: $C-$

$-\frac{1}{\sqrt{2}}\arcsin\frac{x+5}{3(x+1)}$.)

28. $\int \frac{dx}{x\sqrt{x^2-3x+2}}$. (Answer: $C-\frac{1}{\sqrt{2}}\ln\left|\frac{1}{x}-\frac{3}{4}+\right.$

$\left.+\frac{\sqrt{x^2-3x+2}}{2x}\right|$.)

29. $\int \frac{dx}{(x+1)\sqrt{2-x-x^2}}$. (Answer: $C-$

$-\frac{1}{\sqrt{2}}\ln\left|\frac{1}{x+1}+\frac{1}{4}+\frac{\sqrt{2-x-x^2}}{x+1}\right|$.)

30. $\int \frac{dx}{x\sqrt{1-3x-2x^2}}$. (Answer: $C-\ln\left|\frac{1}{x}-\frac{3}{2}+\right.$

$\left.+\frac{\sqrt{1-3x-2x^2}}{x}\right|$.)

3

1. $\int \sin^2(1-x)dx$. (Answer: $\frac{1}{2}x+\frac{1}{4}\sin 2(1-x)+C$.)

2. $\int \sin^3(1-x)dx$.

(Answer: $\cos(1-x)-\frac{1}{3}\cos^3(1-x)+C$.)

3. $\int\left(1-2\sin\frac{x}{5}\right)^2dx$.

(Answer: $3x+20\cos\frac{x}{5}-5\sin\frac{2x}{5}+C$.)

4. $\int \cos^3 5x\sin 5x dx$. (Answer: $-\frac{1}{20}\cos^4 5x+C$.)

5. $\int \cos^3(1-x)dx$.

(Answer: $-\sin(1-x)+\frac{1}{3}\sin^3(1-x)+C$.)

6. $\int(3-\sin 2x)^2dx$.

(Answer: $\frac{19}{2}x+3\cos 2x-\frac{1}{8}\sin 4x+C$.)

7. $\int \sin^2\frac{3x}{2}dx$. (Answer: $\frac{1}{2}x-\frac{1}{6}\sin 3x+C$.)

8. $\int(\cos x+3)^2dx$. (Answer: $\frac{19}{2}x+6\sin x+\frac{1}{4}\sin 2x+C$.)

9. $\int \cos^3(x+3)dx$. (Answer: $\sin(x+3)-\frac{1}{3}\sin^3(x+$

$+3)+C$.)

10. $\int \sin^3\frac{4x}{5}dx$. (Answer: $-\frac{5}{4}\cos\frac{4x}{5}+\frac{5}{12}\cos^3\frac{4x}{5}+$

$+C$.)

11. $\int(1-\cos x)^2dx$. (Answer: $\frac{3}{2}x-2\sin x+$

$+\frac{1}{4}\sin 2x+C$.)

12. $\int \sin^2(2x-1)dx$. (Answer: $\frac{x}{2}-\frac{1}{8}\sin(4x-2)+C$.)

13. $\int \sin^3 6x dx$. (Answer: $-\frac{1}{6}\cos 6x+\frac{1}{18}\cos^3 6x+C$.)

14. $\int \sin^2 0{,}5x dx$. (Answer: $\frac{1}{2}x-\frac{1}{2}\sin x+C$.)

15. $\int \sin^2\left(\frac{x}{2}+1\right)dx$. (Answer: $\frac{1}{2}x-\frac{1}{2}\sin(x+$

$+2)+C$.)

16. $\int \cos^2 2x dx$. (Answer: $\frac{1}{2}x+\frac{1}{8}\sin 4x+C$.)

17. $\int\left(1+2\cos\frac{x}{2}\right)^2 dx$. (Answer: $3x+8\sin\frac{x}{2}+2\sin x+C.$)

18. $\int\cos^2 3x dx$. (Answer: $\frac{1}{2}x+\frac{1}{12}\sin 6x+C.$)

19. $\int\sin^4 2x dx$. (Answer: $\frac{3}{8}x-\frac{1}{8}\sin 4x+\frac{1}{64}\sin 8x+C.$)

20. $\int\sin^2 3x dx$. (Answer: $\frac{1}{2}x-\frac{1}{12}\sin 6x+C.$)

21. $\int(1-\cos 3x)^2 dx$. (Answer: $\frac{3}{2}x-\frac{2}{3}\sin 3x+\frac{1}{12}\sin 6x+C.$)

22. $\int\cos^2\frac{2x}{5}dx$. (Answer: $\frac{1}{2}x+\frac{5}{8}\sin\frac{4x}{5}+C.$)

23. $\int\sin^3 5x dx$. (Answer: $-\frac{1}{5}\cos 5x+\frac{1}{15}\cos^3 5x+C.$)

24. $\int\sin^4 x dx$.

(Answer: $\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C.$)

25. $\int\cos^4 x dx$.

(Answer: $\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C.$)

26. $\int\cos^3 4x dx$.

(Answer: $\frac{1}{4}\sin 4x-\frac{1}{12}\sin^3 4x+C.$)

27. $\int\cos^2 7x dx$. (Answer: $\frac{1}{2}x+\frac{1}{28}\sin 14x+C.$)

28. $\int(\sin x-5)^2 dx$.

(Answer: $\frac{51}{2}x-\frac{1}{4}\sin 2x+10\cos x+C.$)

29. $\int\sin^3 4x dx$.

(Answer: $-\frac{1}{4}\cos 4x+\frac{1}{12}\cos^3 4x+C.$)

30. $\int\sin^2\frac{3x}{4}dx$. (Answer: $\frac{1}{2}x-\frac{1}{3}\sin\frac{3x}{2}+C.$)

4

1. $\int tg^2 x dx$. (Answer: $tg\, x-x+C.$)

2. $\int ctg^3(x-6)dx$.

(Answer: $-\frac{1}{2}ctg^2(x-6)-\ln|\sin(x-6)|+C.$)

3. $\int tg^4 3x dx$.

(Answer: $\frac{1}{9}tg^3 3x-\frac{1}{3}tg\, 3x+x+C.$)

4. $\int tg^2 7x dx$. (Answer: $\frac{1}{7}tg\, 7x-x+C.$)

5. $\int tg^5 x dx$.

(Answer: $\frac{1}{4}tg^4 x-\frac{1}{2}tg^2 x-\ln|\cos x|+C.$)

6. $\int x\, tg^2 x^2 dx$. (Answer: $\frac{1}{2}tg\, x^2-\frac{1}{2}x^2+C.$)

7. $\int ctg^3 x dx$.

(Answer: $-\frac{1}{2}ctg^2 x-\ln|\sin x|+C.$)

8. $\int tg^2\frac{x}{2}dx$. (Answer: $2\, tg\frac{x}{2}-x+C.$)

9. $\int tg^3\frac{x}{2}dx$.

(Answer: $tg^2\frac{x}{2}+2\ln\left|\cos\frac{x}{2}\right|+C.$)

10. $\int tg^2 4x dx$. (Answer: $\frac{1}{4}tg\, 4x-x+C.$)

11. $\int ctg^3 x dx$.

(Answer: $-\frac{1}{2}ctg^2 x-\ln|\sin x|+C.$)

12. $\int \mathrm{ctg}^2 5x dx$. (Answer: $-\frac{1}{5}\mathrm{ctg}\, 5x - x + C.$)

13. $\int \mathrm{tg}^3 \frac{x}{3} dx$.

(Answer: $\frac{3}{2}\mathrm{tg}^2 \frac{x}{3} + 3 \ln\left|\cos\frac{x}{3}\right| + C.$)

14. $\int (1 - \mathrm{tg}\, 2x)^2 dx$.

(Answer $\ln|\cos 2x| + \frac{1}{2}\mathrm{tg}\, 2x + C.$)

15. $\int \mathrm{tg}^5 2x dx$.

(Answer: $\frac{1}{8}\mathrm{tg}^4 2x - \frac{1}{4}\mathrm{tg}^2 2x - \frac{1}{2}\ln|\cos 2x| + C.$)

16. $\int (2x + \mathrm{tg}^2 7x) dx$.

(Answer: $x^2 + \frac{1}{7}\mathrm{tg}\, 7x - x + C.$)

17. $\int \mathrm{tg}^4 \frac{2x}{3} dx$.

(Answer: $\frac{1}{2}\mathrm{tg}^3 \frac{2x}{3} - \frac{3}{2}\mathrm{tg}\frac{2x}{3} + x + C.$)

18. $\int (\mathrm{tg}\, 2x + \mathrm{ctg}\, 2x)^2 dx$.

(Answer $\frac{1}{2}\mathrm{tg}\, 2x - \frac{1}{2}\mathrm{ctg}\, 2x + C.$)

19. $\int (1 - \mathrm{ctg}\, x)^2 dx$.

(Answer: $-2\ln|\sin x| - \mathrm{ctg}\, x + C.$)

20. $\int \mathrm{ctg}^3 3x dx$.

(Answer: $-\frac{1}{6}\mathrm{ctg}^2 3x - \frac{1}{3}\ln|\sin 3x| + C.$)

21. $\int \mathrm{ctg}^4 x dx$.

(Answer: $-\frac{1}{3}\mathrm{ctg}^3 x + \mathrm{ctg}\, x + x + C.$)

22. $\int \mathrm{tg}^2 \frac{x}{2} dx$. (Answer: $6\,\mathrm{tg}\frac{x}{6} - x + C.$)

23. $\int \mathrm{tg}^4 (x-6) dx$.

(Answer: $\frac{1}{3}\mathrm{tg}^3(x-6) - \mathrm{tg}(x-6) + x + C.$)

24. $\int \mathrm{tg}^3 4x dx$.

(Answer: $\frac{1}{8}\mathrm{tg}^2 4x + \frac{1}{4}\ln|\cos 4x| + C.$)

25. $\int \mathrm{tg}^4 \frac{x}{4} dx$.

(Answer: $\frac{4}{3}\mathrm{tg}^3 \frac{x}{4} - 4\,\mathrm{tg}\frac{x}{4} + x + C.$)

26. $\int \mathrm{tg}^4 (x+5) dx$.

(Answer: $\frac{\mathrm{tg}^3 (x+5)}{3} - \mathrm{tg}\,(x+5) + x + C.$)

27. $\int \mathrm{tg}^3 (x-3) dx$.

(Answer: $\frac{1}{2}\mathrm{tg}^2 (x-3) + \ln|\cos(x-3)| + C.$)

28. $\int \mathrm{tg}^2 (5x+1) dx$.

(Answer: $\frac{1}{5}\mathrm{tg}\,(5x+1) - x + C.$)

29. $\int \mathrm{tg}^2 \frac{7x}{4} dx$. (Answer: $\frac{4}{7}\mathrm{tg}\frac{7x}{4} - x + C.$)

30. $\int \mathrm{tg}^5 4x dx$.

(Answer: $\frac{1}{46}\mathrm{tg}^4 4x - \frac{1}{8}\mathrm{tg}^2 4x + bx + C.$)

5

1. $\int \sin 3x \cos x dx$.

(Answer: $-\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C.$)

2. $\int \sin^5 2x \cos 2x dx$. (Answer: $\frac{1}{12}\sin^6 2x + C.$)

3. $\int \sin^2 3x \cos 3x dx$. (Answer: $\frac{1}{9}\sin^3 3x + C.$)

4. $\int \cos^3 5x \sin 5x dx$. (Answer: $-\frac{1}{20}\cos^4 5x + C.$)

5. $\int \sin\frac{x}{2}\cos\frac{x}{4} dx$.

(Answer: $-\frac{2}{3}\cos\frac{3x}{4} - 2\cos\frac{x}{4} + C.$)

6. $\int \cos x \sin 9x\,dx.$

$\left(\text{Answer: } -\frac{1}{20}\cos 10x - \frac{1}{16}\cos 8x + C.\right)$

7. $\int \sin^4 2x \cos 2x\,dx.$

$\left(\text{Answer: } \frac{1}{10}\sin^5 2x + C.\right)$

8. $\int \sin\frac{x}{2}\cos\frac{3x}{2}\,dx.$

$\left(\text{Answer: } -\frac{1}{4}\cos 2x + \frac{1}{2}\cos x + C.\right)$

9. $\int \cos^5 x \sin x\,dx.$ $\left(\text{Answer: } -\frac{1}{6}\cos^6 x + C.\right)$

10. $\int \cos 2x \cos 3x\,dx.$

$\left(\text{Answer: } \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C.\right)$

11. $\int \sin 5x \sin 7x\,dx.$

$\left(\text{Answer: } \frac{1}{4}\sin 2x - \frac{1}{24}\sin 12x + C.\right)$

12. $\int \sin 4x \cos 2x\,dx.$

$\left(\text{Answer: } -\frac{1}{12}\cos 6x - \frac{1}{4}\cos 2x + C.\right)$

13. $\int \cos^3 4x \sin 4x\,dx.$ $\left(\text{Answer: } -\frac{1}{16}\cos^4 4x + C.\right)$

14. $\int \cos^{-3} 2x \sin 2x\,dx.$ $\left(\text{Answer: } \frac{1}{4}\cos^{-2} 2x + C.\right)$

15. $\int \cos x \sin 9x\,dx.$

$\left(\text{Answer: } -\frac{1}{20}\cos 10x - \frac{1}{16}\cos 8x + C.\right)$

16. $\int \sin 4x \cos 2x\,dx.$

$\left(\text{Answer: } -\frac{1}{2}\cos 6x - \frac{1}{4}\cos 2x + C.\right)$

17. $\int \sin 3x \cos 2x\,dx.$

$\left(\text{Answer: } -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C.\right)$

18. $\int \sin^3 7x \cos 7x\,dx.$ $\left(\text{Answer: } \frac{1}{28}\sin^4 7x + C.\right)$

19. $\int \frac{\sin x}{\cos^3 x}\,dx.$ $\left(\text{Answer: } \frac{1}{2}\cos^{-2} x + C.\right)$

20. $\int \frac{\cos 2x\,dx}{\sin^4 2x}.$ $\left(\text{Answer: } \frac{1}{6\sin^3 2x} + C.\right)$

21. $\int \cos 2x \cos 5x\,dx.$

$\left(\text{Answer: } \frac{1}{6}\sin 3x + \frac{1}{14}\sin 7x + C.\right)$

. $\int \sin^2 2x \cos x\,dx.$

$\left(\text{Answer: } \frac{4}{3}\sin^3 x - \frac{4}{5}\sin^5 x + C.\right)$

23. $\int \frac{\cos x}{\sin^4 x}\,dx.$ $\left(\text{Answer: } -\frac{1}{3\sin^3 x} + C.\right)$

24. $\int \sin 2x \sin 3x\,dx.$

$\left(\text{Answer: } \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C.\right)$

25. $\int \sin x \cos^3 x\,dx.$ $\left(\text{Answer: } -\frac{\cos^4 x}{4} + C.\right)$

26. $\int \sin 5x \cos x\,dx.$

$\left(\text{Answer: } \frac{1}{12}\cos 6x - \frac{1}{8}\cos 4x + C.\right)$

27. $\int \sin x \cos 4x\,dx.$

$\left(\text{Answer: } -\frac{1}{10}\cos 5x + \frac{1}{6}\cos 3x + C.\right)$

28. $\int \cos 3x \cos x\,dx.$

$\left(\text{Answer: } \frac{1}{4}\sin 2x + \frac{1}{8}\sin 4x + C.\right)$

29. $\int \cos^4 2x \sin 2x\,dx.$ $\left(\text{Answer: } -\frac{1}{10}\cos^5 2x + C.\right)$

30. $\int \cos 7x \cos 5x\,dx.$

$\left(\text{Answer: } \frac{1}{4}\sin 2x - \frac{1}{24}\sin 12x + C.\right)$

6

1. $\int \frac{dx}{\sqrt{4+8x-x^2}}$. (Answer: $\arcsin\frac{x-4}{\sqrt{20}}+C.$)

2. $\int \frac{dx}{\sqrt{3x^2-4x+1}}$.

(Answer: $\frac{1}{\sqrt{3}}\ln\left|x-\frac{2}{3}+\sqrt{x^2-\frac{4}{3}x+\frac{1}{3}}\right|+C.$)

3. $\int \frac{dx}{\sqrt{2-3x-2x^2}}$. (Answer: $\frac{1}{\sqrt{2}}\arcsin\frac{4x+3}{5}+C.$)

4. $\int \frac{dx}{\sqrt{x^2+6x+8}}$.

5. $\int \frac{dx}{\sqrt{2+8x-2x^2}}$. (Answer: $\frac{1}{\sqrt{2}}\arcsin\frac{x-2}{\sqrt{5}}+C.$)

6. $\int \frac{dx}{\sqrt{3+2x-2x^2}}$. (Answer: $\frac{1}{\sqrt{2}}\arcsin\frac{2x-1}{\sqrt{7}}+C.$)

7. $\int \frac{dx}{\sqrt{2-2x-3x^2}}$. (Answer: $\frac{1}{\sqrt{3}}\arcsin\frac{3x+1}{\sqrt{7}}+C.$)

8. $\int \frac{dx}{\sqrt{1+x-x^2}}$. (Answer: $\arcsin\frac{2x-1}{\sqrt{5}}+C.$)

9. $\int \frac{dx}{\sqrt{5x^2-10x+4}}$.

(Answer: $\frac{1}{\sqrt{5}}\ln\left|x-1+\sqrt{x^2-2x+\frac{4}{5}}\right|+C.$)

10. $\int \frac{dx}{\sqrt{2x+3-x^2}}$. (Answer: $\arcsin\frac{x-1}{2}+C.$)

11. $\int \frac{dx}{\sqrt{4x^2-8x+3}}$.

(Answer: $\frac{1}{2}\ln\left|x-1+\sqrt{x^2-2x+\frac{3}{4}}\right|+C.$)

12. $\int \frac{dx}{\sqrt{1+2x-x^2}}$. (Answer: $\arcsin\frac{x-1}{\sqrt{2}}+C.$)

13. $\int \frac{dx}{\sqrt{4x^2-x+4}}$.

(Answer: $\frac{1}{2}\ln\left|x-\frac{1}{8}+\sqrt{x^2-\frac{1}{4}x+1}\right|+C.$)

14. $\int \frac{dx}{\sqrt{2+4x-3x^2}}$. (Answer: $\frac{1}{\sqrt{3}}\arcsin\frac{3x-2}{\sqrt{10}}+C.$)

15. $\int \frac{dx}{\sqrt{4x^2+2x+4}}$.

(Answer: $\frac{1}{2}\ln\left|x+\frac{1}{4}+\sqrt{x^2+\frac{1}{2}x+1}\right|+C.$)

16. $\int \frac{dx}{\sqrt{3x+2-2x^2}}$. (Answer: $\frac{1}{\sqrt{2}}\arcsin\frac{4x-3}{5}+C.$)

17. $\int \frac{dx}{\sqrt{2x^2-8x+1}}$.

(Answer: $\frac{1}{\sqrt{2}}\ln\left|x-2+\sqrt{x^2-4x+\frac{1}{2}}\right|+C.$)

18. $\int \frac{dx}{\sqrt{x^2-5x+6}}$.

(Ответ: $\ln\left|x-\frac{5}{2}+\sqrt{x^2-5x+6}\right|+C.$)

19. $\int \frac{dx}{\sqrt{3x-2x^2}}$. (Answer: $\frac{1}{\sqrt{2}}\arcsin\frac{4x-3}{3}+C.$)

20. $\int \frac{dx}{\sqrt{2x^2-x+3}}$.

(Answer: $\frac{1}{\sqrt{2}}\ln\left|x-\frac{1}{4}+\sqrt{x^2-\frac{1}{2}x+\frac{3}{2}}\right|+C.$)

21. $\int \frac{dx}{\sqrt{2-x-2x^2}}$. (Answer: $\frac{1}{\sqrt{2}}\arcsin\frac{4x+1}{\sqrt{17}}+C.$)

22. $\int \frac{dx}{\sqrt{x^2+3x-1}}$.

(Ответ: $\ln\left|x+\frac{3}{2}+\sqrt{x^2+3x-1}\right|+C.$)

23. $\int\frac{dx}{\sqrt{5-7x-3x^2}}$. (Answer: $\frac{1}{\sqrt{3}}\arcsin\frac{6x+7}{\sqrt{109}}+C.$)

24. $\int\frac{dx}{\sqrt{3x^2-x+5}}$.

(Answer: $\frac{1}{\sqrt{3}}\ln\left|x-\frac{1}{4}+\sqrt{x^2-\frac{1}{3}x+\frac{5}{3}}\right|+C.$)

25. $\int\frac{dx}{\sqrt{1-x-x^2}}$. (Answer: $\arcsin\frac{2x+1}{\sqrt{5}}+C.$)

26. $\int\frac{dx}{\sqrt{1-2x-x^2}}$. (Answer: $\arcsin\frac{x+1}{\sqrt{2}}+C.$)

27. $\int\frac{dx}{\sqrt{4-3x-x^2}}$. (Answer: $\arcsin\frac{2x+3}{5}+C.$)

28. $\int\frac{dx}{\sqrt{x^2+5x+1}}$.

(Answer: $\ln\left|x+\frac{5}{2}+\sqrt{x^2+5x+1}\right|+C.$)

29. $\int\frac{dx}{\sqrt{3-x-x^2}}$. (Answer: $\arcsin\frac{2x+1}{\sqrt{13}}+C.$)

30. $\int\frac{dx}{\sqrt{x^2+4x+1}}$.

(Answer: $\ln\left|x+2+\sqrt{x^2+4x+1}\right|+C.$)

7

1. $\int\frac{2x-13}{\sqrt{3x^2-3x-16}}dx$. (Answer: $\frac{2}{3}\sqrt{3x^2-3x-16}-$

$-4\sqrt{3}\ln\left|x-\frac{1}{2}+\sqrt{x^2-x-\frac{16}{3}}\right|+C.$)

2. $\int\frac{x-3}{\sqrt{2x^2-4x-1}}dx$. (Answer: $\frac{1}{2}\sqrt{2x^2-4x-1}-$

$-\sqrt{2}\ln\left|x-1+\sqrt{x^2-2x-\frac{1}{2}}\right|+C.$)

3. $\int\frac{x-1}{\sqrt{3x^2-x+5}}dx$. (Answer: $\frac{1}{3}\sqrt{3x^2-x+5}-$

$-\frac{5}{6\sqrt{3}}\ln\left|x-\frac{1}{6}+\sqrt{x^2-\frac{x}{3}+\frac{5}{3}}\right|+C.$)

4. $\int\frac{2x+1}{\sqrt{1+x-3x^2}}dx$. (Answer: $\frac{2}{3}\sqrt{1+x-3x^2}+$

$+\frac{4}{3\sqrt{3}}\arcsin\frac{6x-1}{\sqrt{3}}+C.$)

5. $\int\frac{2x+5}{\sqrt{4x^2+8x+9}}dx$. (Answer: $\frac{1}{2}\sqrt{4x^2+8x+9}+$

$+\frac{3}{2}\ln\left|x+1+\sqrt{x^2+2x+\frac{9}{4}}\right|+C.$)

6. $\int\frac{2x-10}{\sqrt{1+x-x^2}}dx$. (Answer: $-2\sqrt{1+x-x^2}-$

$-9\arcsin\frac{2x-1}{\sqrt{5}}+C.$)

7. $\int\frac{2x-8}{\sqrt{1-x+x^2}}dx$. (Answer: $2\sqrt{1-x+x^2}-$

$-7\ln\left|x-\frac{1}{2}+\sqrt{x^2-x+1}\right|+C.$)

8. $\int\frac{3x+4}{\sqrt{x^2+6x+13}}dx$. (Ответ: $3\sqrt{x^2+6x+13}-$

$-5\ln\left|x+3+\sqrt{x^2+6x+13}\right|+C.$)

9. $\int \frac{3x-1}{\sqrt{2x^2-5x+1}}\,dx.$ $\left(\text{Answer: } \frac{3}{2}\sqrt{2x^2-5x+1}+ \right.$

$\left. +\frac{11}{4\sqrt2}\ln\left|x-\frac54+\sqrt{x^2-\frac52x+\frac12}\right|+C.\right)$

10. $\int \frac{5x+2}{\sqrt{x^2+3x-4}}\,dx.$ $\left(\text{Answer: } 5\sqrt{x^2+3x-4}- \right.$

$\left. -\frac{11}{2}\ln\left|x+\frac32+\sqrt{x^2+3x-4}\right|+C.\right)$

11. $\int \frac{x-4}{\sqrt{2x^2-x+7}}\,dx.$ $\left(\text{Answer: } \frac12\sqrt{2x^2-x+7}- \right.$

$\left. -\frac{15}{4\sqrt2}\ln\left|x-\frac14+\sqrt{x^2-\frac{x}{2}-\frac72}\right|+C.\right)$

12. $\int \frac{2x-1}{\sqrt{x^2-3x+4}}\,dx.$ $\left(\text{Answer: } 2\sqrt{x^2-3x+4}+ \right.$

$\left. +2\ln\left|x-\frac32+\sqrt{x^2-3x+4}\right|+C.\right)$

13. $\int \frac{4x+1}{\sqrt{2+x-x^2}}\,dx.$ $\left(\text{Answer: } -4\sqrt{2+x-x^2}+ \right.$

$\left. +3\arcsin\frac{2x-1}{3}+C.\right)$

14. $\int \frac{5x-3}{\sqrt{2x^2+4x-5}}\,dx.$ $\left(\text{Answer: } \frac52\sqrt{2x^2+4x-5}- \right.$

$\left. -4\sqrt2\ln\left|x+1+\sqrt{x^2+2x-\frac52}\right|+C.\right)$

15. $\int \frac{3x+2}{\sqrt{4+2x-x^2}}\,dx.$ $\left(\text{Answer: } -3\sqrt{4+2x-x^2}+ \right.$

$\left. +5\arcsin\frac{x-1}{\sqrt5}+C.\right)$

16. $\int \frac{x-7}{\sqrt{3x^2-2x+1}}\,dx.$ $\left(\text{Answer: } \frac13\sqrt{3x^2-2x+1}- \right.$

$\left. -\frac{20}{3\sqrt3}\ln\left|x-\frac13+\sqrt{x^2-\frac23x+\frac13}\right|+C.\right)$

17. $\int \frac{x+5}{\sqrt{3-6x-x^2}}\,dx.$ $\left(\text{Answer: } -\sqrt{3-6x-x^2}+ \right.$

$\left. +2\arcsin\frac{x+3}{\sqrt{12}}+C.\right)$

18. $\int \frac{2x+4}{\sqrt{3x^2+x-5}}\,dx.$ $\left(\text{Answer: } \frac23\sqrt{3x^2+x-5}+ \right.$

$\left. +\frac{11}{3\sqrt3}\ln\left|x+\frac16+\sqrt{x^2+\frac{x}{3}-\frac53}\right|+C.\right)$

19. $\int \frac{7x-2}{\sqrt{x^2-5x+1}}\,dx.$ $\left(\text{Answer: } 7\sqrt{x^2-5x+1}+ \right.$

$\left. +\frac{31}{2}\ln\left|x-\frac52+\sqrt{x^2-5x+1}\right|+C.\right)$

20. $\int \frac{x-8}{\sqrt{4x^2+x-5}}\,dx.$ $\left(\text{Answer: } \frac14\sqrt{4x^2+x-5}- \right.$

$\left. -\frac{65}{16}\ln\left|x+\frac18+\sqrt{x^2+\frac14x-\frac54}+C.\right)\right.$

21. $\int \frac{3x+4}{\sqrt{2+3x-x^2}}\,dx.$ $\left(\text{Answer: } -3\sqrt{2+3x-x^2}+ \right.$

$\left. +\frac{17}{2}\arcsin\frac{2x-3}{\sqrt{17}}+C.\right)$

22. $\int \frac{x-6}{\sqrt{3-2x-x^2}}dx$. (Answer : $-\sqrt{3-2x-x^2}-$

$-7\arcsin\frac{x+1}{2}+C.$)

23. $\int \frac{2x+3}{\sqrt{2x^2-x+6}}dx$. (Answer : $\sqrt{2x^2-x+6}+$

$+\frac{7}{2\sqrt{2}}\ln\left|x-\frac{1}{4}+\sqrt{x^2-\frac{x}{2}+3}\right|+C.$)

24. $\int \frac{x-9}{\sqrt{4+2x-x^2}}dx$. (Answer : $-\sqrt{4+2x-x^2}-$

$-8\arcsin\frac{x-1}{\sqrt{5}}+C.$)

25. $\int \frac{2x+7}{\sqrt{x^2+5x-4}}dx$. (Answer : $2\sqrt{x^2+5x-4}+$

$+2\ln\left|x+\frac{5}{2}+\sqrt{x^2+5x-4}\right|+C.$)

26. $\int \frac{3x-4}{\sqrt{2x^2-6x+1}}dx$. (Answer : $\frac{3}{2}\sqrt{2x^2-6x+1}+$

$+\frac{1}{2\sqrt{2}}\ln\left|x-\frac{3}{2}+\sqrt{x^2-3x+\frac{1}{2}}\right|+C.$)

27. $\int \frac{2x+5}{\sqrt{3x^2+9x-4}}dx$. (Answer: $\frac{2}{3}\sqrt{3x^2+9x-4}+$

$+\frac{2}{\sqrt{3}}\ln\left|x+\frac{3}{2}+\sqrt{x^2+3x-\frac{4}{3}}\right|+C.$)

28. $\int \frac{4x+3}{\sqrt{2x^2-x+5}}dx$. (Answer: $2\sqrt{2x^2-x+5}+$

$+2\sqrt{2}\ln\left|x-\frac{1}{4}+\sqrt{x^2-\frac{x}{2}+\frac{5}{2}}\right|+C.$)

29. $\int \frac{3x-7}{\sqrt{x^2-5x+1}}dx$. (Answer : $3\sqrt{x^2-5x+1}+$

$+\frac{1}{2}\ln\left|x-\frac{5}{2}+\sqrt{x^2-5x+1}\right|+C.$)

30. $\int \frac{7x-1}{\sqrt{2-3x-x^2}}dx$. (Answer : $-7\sqrt{2-3x-x^2}-$

$-\frac{23}{2}\arcsin\frac{2x+3}{\sqrt{17}}+C.$)

Topic 14. Definite integrals.

14.1. Definite integrals. Area under a curve. Riemann sum. 14.2. Properties of definite integrals. 14.3. Fundamental theorem of integral calculus. 14.4. Substitution method and Integration by parts in definite integrals.

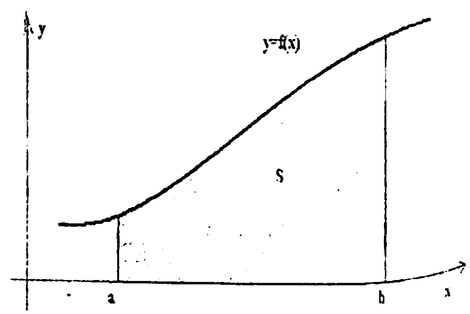
14.1. Area under a curve. Riemann sum. Definite integrals.

14.1.1. Area under a curve.

Definition: Suppose that y=f(x) is continuous and nonnegative throughout [a, b]. The area of the region enclosed:

1) by the graph of f(x),
2) the x-axis and
3) the vertical lines x=a and x=b

is called **the area under curve y=f(x) from a to b.**



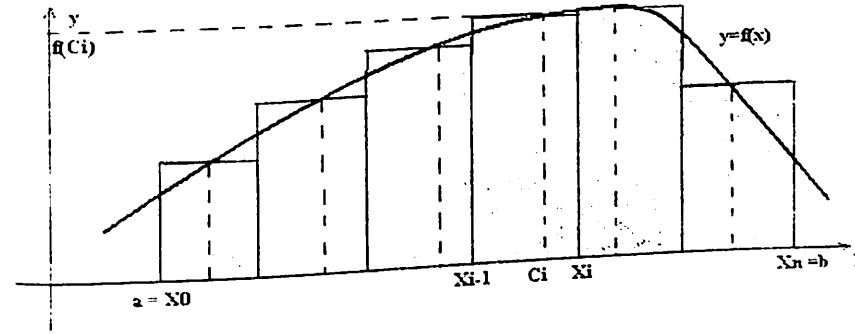
14.1.2.. Riemann sum.

Partition T of the interval [a, b] on n subintervals is called the set of numbers

$x_0 < x_1 < x_2 < \dots < x_{i-1} < x_i < \dots < x_n$, where $x_0=a$ and $x_n=b$. There is chosen some arbitrary point c_i at each subinterval $c_i \in [x_{i-1}, x_i]$. The length of subinterval $\Delta x_i = x_i - x_{i-1}$.

Definition: A sum of products of the values of the function at the chosen points $f(c_i)$ on the length of subintervals Δx_i is called **a Riemann sum of the function y=f(x)** which is constructed on partition T of the interval [a, b]:

$$\sum_{i=1}^{n} f(c_i)\cdot\Delta x_i .$$



14.1.3. Definite integrals.

Definition: The limit of Riemann sum of the function y=f(x) on partition T of the interval [a, b] as max $\Delta x_i \to 0$ is called **the definite integral** of the function y=f(x) at the interval [a, b], is denoted by

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\cdot\Delta x_i .$$

Geometrical meaning of definite integral: If f(x)>0 at [a,b] then a definite integral is equal the area under curve, enclosed by the graph of the function y=f(x) over [a,b] : $\int_a^b f(x)\,dx = S$.

Theorem: (existence) If the function y=f(x) is **continuous** at [a,b] or there are exist finite number of discontinuous points of the first kind, then the function is **integrable** over [a,b] , i.e. $\int_a^b f(x)\,dx$ exists.

14.2. Properties of definite integrals.

Further suppose that all considered functions are integrable.

1) $\int_a^b C\cdot dx = C\cdot(b-a)$, where C – const;

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$;

3) $\int_a^a f(x)dx = 0$;

4) If f(x) ≤ g(x) at [a,b] , then $\int_a^b f(x)dx \le \int_a^b g(x)dx$;

5) **Lower and upper estimation of the definite integral**:
If $m \le f(x) \le M$ at [a,b] , then

$$m\cdot(b-a) \le \int_a^b f(x)dx \le M\cdot(b-a);$$

6) **Mean value Theorem**: If the function y = f(x) is continuous at [a,b]. There is exist the point **c** at [a,b],that

$$\int_a^b f(x)dx = c\cdot(b-a);$$

7) **Module estimation of the definite integral**:

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx;$$

8) **Linear property**:

$$\int_a^b (A\cdot f(x) + B\cdot g(x))dx = A\cdot\int_a^b f(x)dx + B\cdot\int_a^b g(x)dx;$$

9) **Additive property**: For arbitrary point **c** :

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

14.3. Fundamental theorem of integral calculus.

14.3.1. An integral with variable upper limit.
Let a function is integrable at [a,b] and $a \le x \le b$. New function y = F(x) is introduced as

$$F(x) = \int_a^x f(t)\,dt \quad \text{for } x \in [a,b].$$

Theorem: If a function y = f(x) is continuous at [a,b], then the function $F(x) = \int_a^x f(t)\,dt$ is **antiderivative** for function y = f(x) at every point of [a,b] , i.e. $F'(x) = f(x)$.

14.3.2. Fundamental theorem.
If y = f(x) is continuous at every point of [a,b] and y = F(x) is any antiderivative of f on [a,b], then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

14.4. Substitution method and Integration by parts in definite integrals.

14.4.1. Substitution method in definite integral.
Theorem: Suppose that y = f(x) is continuous at [a,b], and a function $x = \phi(t)$ is monotonic and continuously differentiable at $[\alpha,\beta]$, where $\varphi(\alpha) = a$ *and* $\phi(\beta) = b$, then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))\cdot\phi'(t)\ dt.$$

14.4.2. Integration by parts in definite integrals.
Theorem: Let two functions y = u(x) and y = v(x) are continuously differentiable at [a,b], then the following equality is true:

$$\int_a^b u(x)\cdot dv(x) = u(x)\cdot v(x)\Big|_a^b - \int_a^b v(x)\cdot du(x).$$

EXERCISE XIV

Evaluate the definite integral (with accuracy within two significance figure after point):

1

1. $\int_0^{\sqrt{3}} x\sqrt[3]{1+x^2}\,dx$

2. $\int_0^{\sqrt[12]{3}} \frac{12x^5dx}{\sqrt{x^6+1}}$

3. $\int_0^1 \frac{x^2dx}{x^2+1}$

4. $\int_0^{\pi/2} \sin x \cos^2 x dx.$

5. $\int_0^{\pi/2} \frac{\cos x}{1+\cos x}\,dx$

6. $\int_{3/4}^{4/3} \frac{dx}{x^2+1}$

7. $\int_0^{-3} \frac{dx}{\sqrt{25+3x}}.$

8. $\int_0^2 \frac{x^3dx}{\sqrt{x^4+4}}.$

9. $\int_1^e \frac{1+\ln x}{x}\,dx$

10. $\int_0^1 \frac{z^3}{z^8+1}\,dz$

11. $\int_{\pi/4}^{\pi/2} \frac{dx}{1-\cos^2 x}.$

12. $\int_2^5 \frac{dx}{\sqrt{5+4x-x^2}}.$

13. $\int_0^1 x^3\sqrt{4+5x^4}\,dx$

14. $\int_{-\pi}^{\pi} \sin^2\frac{x}{2}\,dx$

15. $\int_1^2 \frac{e^{1/x}}{x^2}\,dx.$

16. $\int_0^{1/2} \frac{xdx}{\sqrt{1-x^2}}$

17. $\int_0^1 3(x^2+x^2e^{x^3})dx.$

18. $\int_{\pi^2/9}^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

19. $\int_1^{\sqrt{3}} \frac{x^2dx}{1+x^6}.$

20. $\int_1^e \frac{\sin \ln x}{x}\,dx.$

21. $\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1-\ln^2 x}}$

22. $\int_3^8 \sqrt{x+1}\cdot dx$

23. $\int_{\pi/6}^{\pi/2} \sin\alpha\cos^3\alpha\, d\alpha$

24. $\int_{\pi/18}^{\pi/6} 12\,\mathrm{ctg}\,3x dx$

25. $\int_0^1 \frac{dx}{\sqrt{4-3x}}.$

26. $\int_1^{\sqrt{2}} \frac{xdx}{\sqrt{4-x^2}}$

27. $\int_1^e \frac{\ln^2 x}{x}\,dx$

28. $\int_{-1}^0 \frac{dx}{4x^2-9}$

29. $\int_{\pi/6}^{\pi/2} \cos\alpha\sin^3\alpha\, d\alpha$

30. $\int_0^{\sqrt{\pi/4}} \frac{xdx}{\cos^2(x^2)}$

3

1. $\int_2^3 y\ln(y-1)dy.$

2. $\int_{-2}^0 x^2e^{-x/2}dx.$

3. $\int_0^{\pi/2} x\cos x dx$

4. $\int_0^{\pi} x^2\sin x dx.$

5. $\int_{-1/2}^{1/2} \arccos 2x dx$

6. $\int_1^2 (y-1)\ln y dy.$

7. $\int_{-1/2}^0 xe^{-2x}dx$

8. $\int_{-\pi}^{\pi} x\sin x\cos x dx.$

9. $\int_{-1/3}^{-2/3} \frac{x}{e^{3x}}\,dx.$

10. $\int_1^e \frac{\ln^2 x}{x^2}\,dx$

11. $\int\limits_{1}^{e^2} \sqrt{x} \ln x\,dx$

12. $\int\limits_{0}^{1} \operatorname{arctg} \sqrt{x}\,dx$

13. $\int\limits_{0}^{\pi} (x+2) \cos \frac{x}{2}\,dx$

14. $\int\limits_{0}^{\pi/8} x^2 \sin 4x\,dx$

15. $\int\limits_{1}^{2} y^2 \ln y\,dy$

16. $\int\limits_{1}^{2} \frac{\ln(x+1)}{(x+1)^2}\,dx$

17. $\int\limits_{3/2}^{2} \operatorname{arctg}(2x-3)\,dx$

18. $\int\limits_{0}^{\pi/2} (x+3) \sin x\,dx$

19. $\int\limits_{1}^{e} x \ln^2 x\,dx$

20. $\int\limits_{-3}^{0} (x-2) e^{-x/3}\,dx$

21. $\int\limits_{0}^{\pi/9} \frac{x\,dx}{\cos^2 3x}$

22. $\int\limits_{1/2}^{1} \arcsin(1-x)\,dx$

23. $\int\limits_{1}^{\sqrt{3}} \operatorname{arctg} \frac{1}{x}\,dx$

24. $\int\limits_{-1}^{0} x \ln(1-x)\,dx$

25. $\int\limits_{0}^{1} \frac{\arcsin(x/2)}{\sqrt{2-x}}\,dx$

26. $\int\limits_{1}^{2} \ln(3x+2)\,dx$

27. $\int\limits_{0}^{4} x^3 \sqrt{x^2+9}\,dx$

28. $\int\limits_{-1}^{0} (x+1) e^{-2x}\,dx$

29. $\int\limits_{0}^{\pi/4} x \operatorname{tg}^2 x\,dx$

30. $\int\limits_{0}^{1} x \operatorname{arctg} x\,dx$

4

1. $\int\limits_{0}^{1} \frac{3x^4+3x^2+1}{x^2+1}\,dx$

2. $\int\limits_{2}^{3} \frac{2x^4-5x^2+3}{x^2-1}\,dx$

3. $\int\limits_{2}^{3} \frac{x+2}{x^2(x-1)}\,dx$

4. $\int\limits_{2}^{3} \frac{dx}{x^2(x-1)}$

5 $\int\limits_{-1}^{1} \frac{y^5\,dy}{y+2}$

6. $\int\limits_{2}^{3} \frac{3x^2+2x-3}{x^3-x}\,dx$

7. $\int\limits_{1/3}^{1/2} \frac{x\,dx}{(x-1)^3}$

8. $\int\limits_{4}^{5} \frac{dx}{(x-1)(x+2)}$

9. $\int\limits_{3}^{4} \frac{dx}{(x+1)(x-2)}$

10. $\int\limits_{0}^{1} \frac{(2x+3)\,dx}{(x-2)^3}$

11. $\int\limits_{2}^{3} \frac{dx}{(x-1)^2(x+1)}$

12. $\int\limits_{3}^{5} \frac{(x^2+2)\,dx}{(x+1)^2(x-1)}$

13. $\int\limits_{0}^{1} \frac{x^4+3x^3-1}{(x+1)^2}\,dx$

14. $\int\limits_{-1}^{0} \frac{x^5-2x^2+3}{(x-2)^2}\,dx$

15. $\int\limits_{0}^{1} \frac{x\,dx}{x^2+3x+2}$

16. $\int\limits_{8}^{10} \frac{(x^2+3)\,dx}{x^3-x^2-6x}$

17. $\int\limits_{1}^{\sqrt{3}} \frac{dx}{x^4+x^2}$

18. $\int\limits_{2}^{3} \frac{x^7\,dx}{1-x^4}$

19. $\int\limits_{2}^{3} \frac{dx}{x^4-1}$

20. $\int\limits_{-1}^{0} \frac{x\,dx}{x^3-1}$

21. $\int\limits_0^{\sqrt{3}/3} \frac{2x^2+4}{x^3-x^2+x+1}$ 22. $\int\limits_4^5 \frac{dx}{x^2(x-1)}$

23. $\int\limits_0^2 \frac{dx}{(x+1)(x^2+4)}$ 24. $\int\limits_7^9 \frac{x^2-x+2}{x^4-5x^2+4}\,dx.$

25. $\int\limits_4^6 \frac{x\,dx}{x^3-6x^2+16-6}$ 26. $\int\limits_1^2 \frac{dx}{x^3+1}$

27. $\int\limits_1^{\sqrt{3}} \frac{x^5+1}{x^6+x^4}\,dx;$ 28. $\int\limits_2^3 \frac{x^3+x^2+2}{x(x^2-1)^2}\,dx.$

29. $\int\limits_3^5 \frac{x^3-2x^2+4}{x^3(x-2)^2}\,dx$ 30. $\int\limits_0^{1/\sqrt{3}} \frac{x^2dx}{x^4-1}$

5

1. $\int\limits_0^2 x^2\sqrt{x-x^2}\,dx$ 2. $\int\limits_{\sqrt{2}}^1 \frac{\sqrt{4-x^2}}{x^2}\,dx$

3. $\int\limits_3^6 \frac{\sqrt{x^2-9}}{x^4}\,dx$ 4. $\int\limits_0^1 \sqrt{4-x^2}\,dx$

5. $\int\limits_1^{\sqrt{3}} \frac{x^3+1}{x^2\sqrt{4-x^2}}\,dx$ 6. $\int\limits_0^{\sqrt{3}} \sqrt{3-x^2}\,dx$

7. $\int\limits_{-3}^3 x^2\sqrt{9-x^2}\,dx.$ 8. $\int\limits_{\sqrt{2}}^1 \frac{\sqrt{1-x^2}}{x^6}\,dx$

9. $\int\limits_0^1 \sqrt{(1-x^2)^3}\,dx.$ 10. $\int\limits_{\sqrt{3}/3}^1 \frac{dx}{x^2\sqrt{(1+x^2)^3}}.$

11. $\int\limits_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$ 12. $\int\limits_0^1 \frac{dx}{(x^2+3)^{3/2}}$

13. $\int\limits_1^{\sqrt{2}} \sqrt{2-x^2}\,dx.$ 14. $\int\limits_0^1 \frac{x^2dx}{(x^2+1)^2}$

15. $\int\limits_{2\sqrt{3}}^6 \frac{dx}{x^2\sqrt{x^2-9}}$ 16. $\int\limits_{1/\sqrt{3}}^1 \frac{dx}{x^2\sqrt{1+x^2}}$

17. $\int\limits_{1/2}^{\sqrt{3}/2} \sqrt{1-x^2}\,dx.$ 18. $\int\limits_0^3 \frac{dx}{(9+x^2)\sqrt{9+x^2}}$

19. $\int\limits_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$ 20. $\int\limits_{-1/2}^{1/2} \frac{dx}{(1-x^2)\sqrt{1-x^2}}$

21. $\int\limits_0^{\sqrt{2.5}} \frac{dx}{(5-x^2)^3}$ 22. $\int\limits_0^{1/2} \frac{x^4dx}{\sqrt{(1-x^2)^3}}$

23. $\int\limits_{\sqrt{3}}^2 \frac{dx}{x^4\sqrt{x^2-3}}$ 24. $\int\limits_2^4 \frac{\sqrt{16-x^2}}{x^4}\,dx.$

25. $\int\limits_0^{\sqrt{7/3}} x^3\sqrt{7+x^2}\,dx.$ 26. $\int\limits_{4\sqrt{2/3}}^{\sqrt{8}} \frac{\sqrt{x^2-8}}{x^4}\,dx.$

27. $\int\limits_1^{\sqrt{2}} \frac{dx}{x^5\sqrt{x^2-1}}$ 28. $\int\limits_0^3 x^4\sqrt{9-x^2}\,dx.$

29. $\int\limits_0^3 \frac{x^3dx}{\sqrt{9+x^2}}$ 30. $\int\limits_0^{\sqrt{6}} \sqrt{6-x^2}\,dx.$

6

1. $\int_{-\pi/2}^{-\pi/4} \frac{\cos^3 x}{\sqrt{\sin x}} dx$. 2. $\int_{0}^{\pi/2} \frac{dx}{2+\cos x}$.

3. $\int_{0}^{\pi/4} \sin^3 2x dx$. 4. $\int_{0}^{\pi} \sin^4 \frac{x}{2} dx$

5. $\int_{0}^{\pi/3} \cos^3 x \sin 2x dx$. 6. $\int_{0}^{\pi/3} \mathrm{tg}^2 x dx$.

7. $\int_{\pi/2}^{\pi} \frac{\sin x}{(1-\cos x)^3} dx$ 8. $\int_{0}^{\pi/4} 2\cos x \sin 3x dx$

9. $\int_{0}^{\pi} \cos \frac{x}{2} \cos \frac{x}{3} dx$ 10. $\int_{0}^{\pi/32} (32\cos^2 4x - 16) dx$

11. $\int_{0}^{\pi/2} \frac{\cos x dx}{\sin^2 x + 1}$ 12. $\int_{\pi/4}^{\pi/3} \mathrm{tg}^4 \varphi d\varphi$.

13. $\int_{0}^{\pi} \cos \frac{x}{2} \cos \frac{3x}{2} dx$ 14. $\int_{0}^{\pi/4} \sin 3x \cos 5x dx$

15. $\int_{0}^{\pi/3} \frac{\sin^3 x}{\cos^4 x} dx$ 16. $\int_{0}^{\pi/6} \frac{dx}{\cos x}$

17. $\int_{\pi/6}^{\pi/2} \mathrm{ctg}^3 x dx$ 18. $\int_{0}^{\pi/2} \cos x \cos 3x \cos 5x dx$.

19. $\int_{0}^{\pi} \cos^4 x \sin^2 x dx$. 20. $\int_{0}^{\pi/2} \sin^6 x dx$

21. $\int_{\pi/2}^{\pi} \sqrt{1+\sin x} dx$ 22. $\int_{\pi/6}^{\pi/4} \frac{1+\mathrm{tg}\, x}{\sin 2x}$

23. $\int_{\pi/6}^{\pi/3} \frac{\sin 2x}{\cos^3 x} dx$ 24. $\int_{0}^{\pi/8} \sin x \sin 3x dx$

25. $\int_{\pi/4}^{\pi} \sin x \sin 2x \sin 3x dx$ 26. $\int_{\pi/3}^{\pi/2} \frac{dx}{\sin x}$

27. $\int_{0}^{\pi/2} \cos^5 x dx$ 28. $\int_{\pi/2}^{\pi} \cos^2 x \sin^4 x dx$

29. $\int_{\pi/3}^{\pi/2} \frac{dx}{\sin^3 x}$ 30. $\int_{0}^{\pi} \sin^4 \frac{x}{2} dx$

7

1. $\int_{2}^{3} \frac{dx}{2x^2+3x-2}$ 2. $\int_{-2}^{0} \frac{dx}{\sqrt{x^2+2x+4}}$

3. $\int_{-5}^{-2} \frac{dx}{x^2+4x-21}$ 4. $\int_{1}^{\sqrt{5}} \frac{x^2 dx}{13-6x^3+x^6}$

5. $\int_{1}^{2} \frac{dx}{x^2+x}$ 6. $\int_{-1/2}^{1/2} \frac{dx}{4x^2+4x+5}$

7. $\int_{-1/2}^{1} \frac{dx}{\sqrt{8+2x-x^2}}$ 8. $\int_{1}^{2} \frac{dt}{t^2+5t+4}$

9. $\int_{0}^{2} \frac{x dx}{x^2+3x+2}$ 10. $\int_{1}^{2} \frac{x-5}{x^2-2x+2}$

11. $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$ 12. $\int_{6}^{8} \frac{dx}{x^2+2x}$

13. $\int_{1/2}^{1} \frac{dx}{\sqrt{x-x^2}}$ 14. $\int_{-1/2}^{0} \frac{2x-8}{\sqrt{1-x-x^2}}$

15. $\int_{3/4}^{2} \frac{dx}{\sqrt{2+3x-2x^2}}$ 16. $\int_{1/6}^{2} \frac{dx}{3x^2-x+1}$

17. $\int_{3}^{4}\frac{x^2dx}{x^2-6x+10}$ 18. $\int_{3.5}^{5}\frac{xdx}{x^2-7x+13}$

19. $\int_{2}^{3}\frac{3x-2}{x^2-4x+5}$ 20. $\int_{-3/2}^{2}\frac{(x-1)^2}{x^2+3x+4}$

21. $\int_{4}^{5}\frac{xdx}{x^4-4x^2+3}$ 22. $\int_{-1/2}^{1}\frac{x^3}{x^2+x+1}dx.$

23. $\int_{7}^{10}\frac{x^3dx}{x^2-3x+2}$ 24. $\int_{3}^{5}\frac{x^2dx}{\sqrt{8x-x^2-15}}$

25. $\int_{0}^{1}\frac{dx}{x^2+4x+5}$ 26. $\int_{-1/3}^{0}\frac{dx}{\sqrt{2-6x-9x^2}}$

27. $\int_{4}^{7}\frac{dx}{x^2+3x-10}$ 28. $\int_{1/3}^{4/3}\frac{dx}{\sqrt{8+6x-9x^2}}$

29. $\int_{2}^{3}\frac{dx}{\sqrt{4x-3-x^2}}$ 30. $\int_{-1}^{1}\frac{dx}{x^2+2x+3}$

8

1. $\int_{3}^{29}\frac{\sqrt[3]{(x-2)^2}}{3+\sqrt[3]{(x-2)^2}}dx$ 2. $\int_{0}^{\ln 2}\frac{dx}{e^x(3+e^{-x})}$

3. $\int_{0}^{5}\frac{dx}{2x+\sqrt{3x+1}}$ 4. $\int_{3}^{8}\frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}dx.$

5. $\int_{3}^{8}\frac{xdx}{\sqrt{x+1}}$ 6. $\int_{0}^{\ln 5}\frac{e^x\sqrt{e^x-1}}{e^x+3}dx.$

7. $\int_{\ln 2}^{2\ln 2}\frac{dx}{e^x-1}$ 8. $\int_{0}^{\ln 2}\sqrt{e^x-1}\,dx.$

9. $\int_{0}^{5}\frac{xdx}{\sqrt{x+4}}$ 10. $\int_{0}^{4}\frac{dx}{1+\sqrt{2x+1}}$

11. $\int_{2/3}^{7/3}\frac{xdx}{\sqrt{2+3x}}$ 12. $\int_{\ln 2}^{\ln 3}\frac{dx}{e^x-e^{-x}}$

13. $\int_{0}^{1}\frac{x^2dx}{(1+x)^4}$ 14. $\int_{-1}^{0}\frac{dx}{1+\sqrt[3]{x+1}}$

15. $\int_{0}^{\frac{1}{2}\ln 2}\frac{e^xdx}{e^x+e^{-x}}$ 16. $\int_{0}^{\sqrt[4]{7}}\frac{z^2dz}{\sqrt{9+z^3}}$

17. $\int_{0}^{5}\frac{xdx}{\sqrt{1+3x}}$ 18. $\int_{0}^{2}\frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}}$

19. $\int_{\ln 3}^{0}\frac{1-e^x}{1+e^x}dx$ 20. $\int_{0}^{\pi/2}\frac{\cos y\,dy}{4+\sqrt{\sin y}}$

21. $\int_{2}^{5}\frac{x^2dx}{(x-1)\sqrt{x-1}}$ 22. $\int_{0}^{\ln 2}\frac{dx}{e^x\sqrt{1-e^{-2x}}}$

23. $\int_{1}^{e^x}\frac{dx}{x\sqrt{1+\ln x}}$ 24. $\int_{\ln 2}^{\ln x}\frac{dx}{\sqrt{1+e^x}}$

25. $\int_{e^2}^{e^3}\frac{\ln x\,dx}{x(1-\ln^2 x)}$ 26. $\int_{4}^{9}\frac{\sqrt{x}}{\sqrt{x}-1}dx.$

27. $\int_{\sqrt{7}}^{\sqrt{26}}\frac{x^3dx}{(x^2+1)^{2/3}}$ 28. $\int_{0}^{13}\frac{x+1}{\sqrt[3]{2x+1}}dx$

29. $\int_{\ln 5}^{\ln 12}\frac{dx}{\sqrt{e^x+4}}$ 30. $\int_{-1}^{1}\frac{xdx}{\sqrt{5-4x}}$

Topic 15. Improper integrals.

15.1. Improper integrals with indefinite limits. Convergence and divergence of improper integral. 15.2. Improper integrals of discontinuous function. 15.3. The signs of convergence of improper integrals.

15.1. Improper integrals with indefinite limits.

Definition: $\int_a^b f(x)dx$ denotes **an improper integral** if:

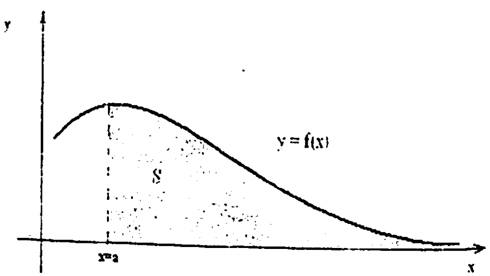
1) one or both of **the limits** of integration is **infinite** and / or
2) **f(x)** becomes **infinite** at one or more points of the interval of integration.

1) Let a function y = f(x) is continuous at $[a, +\infty)$. **An improper integral from a to** $+\infty$ is called the limit :

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx .$$

If the limit exists and equal to finite number, then the improper integral will be called **convergent**, if the limit is equal to infinite or doesn't exist, then the improper integral will be called **divergent**.

Fundamental theorem:
If f(x) is continuous at $[a, +\infty)$, then



$$\int_a^{+\infty} f(x)dx = F(x)\Big|_a^{+\infty} = \lim_{x\to\infty} F(x) - F(a) .$$

Geometrical meaning: If $f(x)\geq 0$ at $[a, +\infty)$, the improper integral $\int_a^{+\infty} f(x)dx$ expresses **the area of unbounded region** with boundary : x =a (x>a) , y = 0 and graph of the function y = f(x)

Harmonic improper integral:

$$\int_1^{\infty}\frac{1}{x^p}dx = \begin{cases} \text{If } p>1 & \Rightarrow \text{convergent} \\ \text{If } p\leq 1 & \Rightarrow \text{divergent} . \end{cases}$$

2) Let a function y = f(x) is continuous at $(-\infty, b]$. **An improper integral from** $-\infty$ **to b** is called the limit :

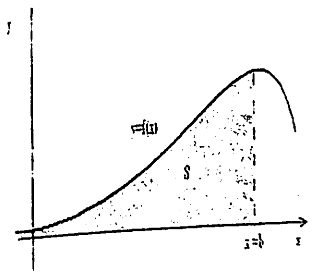
$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx .$$

If the limit exists and equal to finite number, then the improper integral will be called **convergent**, if the limit is infinite or doesn't exist, then the improper integral will be called **divergent**.

Fundamental theorem: (for calculation)
If f(x) is continuous at $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)dx = F(x)\Big|_{-\infty}^b = F(b) - \lim_{x\to-\infty} F(x) .$$

Geometrical meaning: If $f(x)\geq 0$ at $(-\infty, b]$ the improper integral $\int_{-\infty}^b f(x)dx$ expresses **the area of unbounded region** with boundary : x =b (x<b) , y = 0 and graph of the function y = f(x) .



For convergent integral the area is **finite number.**
For divergent integral the area **is infinite.**

3) Let a function y = f(x) be continuous at $(-\infty, +\infty)$. **An improper integral from** $-\infty$ **to** $+\infty$ is called the sum of two limits :

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx .$$

Here c – some arbitrary number.

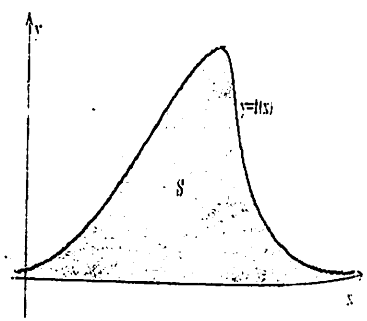
If both integrals $\int_{-\infty}^{c} f(x)dx$ *and* $\int_{c}^{+\infty} f(x)dx$ in the sum are convergent, then the improper integral $\int_{-\infty}^{+\infty} f(x)dx$ will be called **convergent**, if at least one of the integral is divergent, it will be **divergent** too.

Fundamental theorem:

If f(x) is continuous at $(-\infty, +\infty)$, then

$$\int_{-\infty}^{+\infty} f(x)dx = F(x)\Big|_{-\infty}^{+\infty} = \lim_{x\to+\infty} F(x) - \lim_{x\to-\infty} F(x).$$

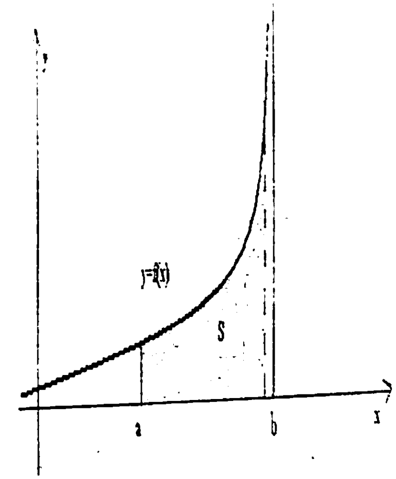
Geometrical meaning: If $f(x)\geq 0$ at $(-\infty, -\infty)$ the improper integral $\int_{-\infty}^{+\infty} f(x)dx$ expresses **the area of unbounded region** with boundary : $y = 0$ $(-\infty < x < +\infty)$ and graph of the function y = f(x) .



15.2. Improper integrals of discontinuous function.

1) Let the function y = f(x) be continuous at [a.b) and has the discontinuous point of the second kind at x=b, The left-hand improper integral from a to b is called the left-hand limit

$$\int_{a}^{b-} f(x)dx = \lim_{t\to b-} \int_{a}^{t} f(x)dx .$$

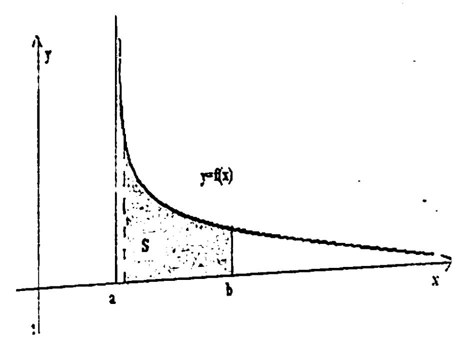


If the limit exists and equal to finite number, then the improper integral will be called **convergent**, if the limit is iqual to infinite or doesn't exist, then the improper integral will be called **divergent.**

Fundamental theorem:

$$\int_{a}^{b-} f(x)dx = F(x)\Big|_{a}^{b-} = \lim_{x\to b-} F(x) - F(a) .$$

2) Let the function y = f(x) is continuous at (a,b] and has the discontinuous point of the second kind at x=a, The riht-hand improper integral from a to b is called the right-hand limit



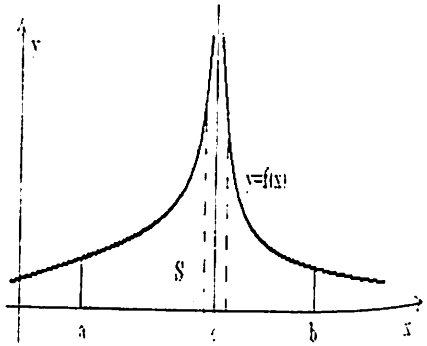
$$\int_{a+}^{b} f(x)dx = \lim_{t\to a+} \int_{t}^{b} f(x)dx$$

If the limit exists and equal to finite number, then the improper integral will be called **convergent**, if the limit is infinite or doesn't exist, then the improper integral will be called **divergent.**

Fundamental theorem:

$$\int_{a+}^{b} f(x)dx = F(x)\Big|_{a+}^{b} = F(b) - \lim_{x\to a+} F(x).$$

3) Let the function y = f(x) be continuous at [a,b] excluding the point **x=c** from interval [a,b] where the function has the point of discontinuity of the second kind. **The improper integral from a to b** is called the sum of two improper integrals.



$$\int_{a}^{b} f(x)dx = \int_{a}^{c-} f(x)dx + \int_{c+}^{b} f(x)dx$$

If both integrals $\int_{a}^{c-} f(x)dx$ *and* $\int_{c+}^{b} f(x)dx$ in the sum are convergent, then the improper integral $\int_{a}^{b} f(x)dx$ will be called **convergent**, if at least one of the integral is divergent, it will be **divergent** too.

15.3. The signs of convergence of improper integrals.

15.3.1. First comparison test.

Let two functions y = f(x) and y = g(x) are continuous at $[a, +\infty)$ and at the same interval the condition is hold : $\mathbf{0 \le f(x) \le g(x)}$, then the following statements are true:

1) If $\int_{a}^{+\infty} g(x)dx$ - is **convergent** , then $\int_{a}^{+\infty} f(x)dx$ is **convergent too** and $\int_{a}^{+\infty} f(x)dx \le \int_{a}^{+\infty} g(x)dx$.

2) If $\int_{a}^{+\infty} f(x)dx$ - is **divergent** , then $\int_{a}^{+\infty} g(x)dx$ is **divergent too.**

15.3.2. Second comparison test.

Let for two functions y=f(x) and y=g(x), which are nonnegative and continuous at $[a, +\infty)$, the limit $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = A$ exists and it is **not equal to 0 or** ∞. Then the improper integrals $\int_{a}^{+\infty} f(x)dx$ and $\int_{a}^{+\infty} g(x)dx$ are **convergent (or divergent) simultaneously**.

EXERCISE XV

1

Evaluate the improper integral or prove its divergence:

1. $\int_{0}^{\infty} \frac{xdx}{16x^4+1}$, $\int_{0}^{1} \frac{dx}{\sqrt[3]{2-4x}}$.
2. $\int_{1}^{\infty} \frac{16xdx}{16x^4-1}$; $\int_{1}^{3} \frac{dx}{\sqrt{x^2-6x+9}}$;
3. $\int_{0}^{\infty} \frac{x^3dx}{\sqrt{16x^4+1}}$; $\int_{0}^{1/3} \frac{e^{3+\frac{1}{x}}}{x^2}dx$.

4. $\int\limits_1^\infty \frac{x\,dx}{\sqrt{16x^4-1}}$; $\int\limits_1^3 \frac{dx}{\sqrt[3]{(3-x)^5}}$

5. $\int\limits_{-\infty}^0 \frac{x\,dx}{\sqrt{(x^2+4)^3}}$; $\int\limits_{1/3}^1 \frac{\ln(3x-1)}{3x-1}\,dx$.

6. $\int\limits_0^\infty \frac{x^2dx}{\sqrt[3]{(x^3+8)^4}}$; $\int\limits_{1/4}^1 \frac{dx}{20x^2-9x+1}$

7. $\int\limits_0^\infty \frac{x\,dx}{\sqrt[4]{(16+x^2)^5}}$; $\int\limits_{1/2}^1 \frac{\ln 2\,dx}{(1-x)\ln^2(1-x)}$

8. $\int\limits_4^\infty \frac{x\,dx}{\sqrt{x^2-4x+1}}$; $\int\limits_0^{2/3} \frac{\sqrt[3]{\ln(2-3x)}}{2-3x}\,dx$

9. $\int\limits_{-1}^\infty \frac{dx}{\pi(x^2+4x+5)}$; $\int\limits_0^1 \frac{x\,dx}{1-x^4}$

10. $\int\limits_{-1}^\infty \frac{x\,dx}{x^2+4x+5}$; $\int\limits_0^{\pi/6} \frac{\cos 3x}{\sqrt[6]{(1-\sin 3x)^5}}\,dx$

11. $\int\limits_0^\infty \frac{\operatorname{arctg} 2x}{\pi(1+4x^2)}\,dx$; $\int\limits_0^1 \frac{2x\,dx}{\sqrt{1-x^4}}$

12. $\int\limits_{1/2}^\infty \frac{16\,dx}{\pi(4x^2+4x+5)}$; $\int\limits_{-1/3}^0 \frac{dx}{\sqrt[3]{1+3x}}$

13. $\int\limits_0^\infty \frac{x\,dx}{4x^2+4x+5}$; $\int\limits_{3/4}^1 \frac{dx}{\sqrt[5]{3-4x}}$

14. $\int\limits_0^\infty \frac{(x+2)\,dx}{\sqrt[3]{(x^2+4x+1)^4}}$; $\int\limits_0^{\pi/2} \frac{e^{\operatorname{tg} x}}{\cos 2x}\,dx$

15. $\int\limits_0^\infty \frac{3-x^2}{x^2+4}\,dx$; $\int\limits_0^1 \frac{2e^{1-\frac{2}{\pi}\arcsin x}}{\pi\sqrt{1-x^2}}\,dx$.

16. $\int\limits_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\operatorname{arctg} 2x}}{1+4x^2}\,dx$; $\int\limits_1^2 \frac{dx}{\sqrt[5]{4x-x^2-4}}$

17. $\int\limits_1^\infty \frac{4\,dx}{x(1+\ln^2 x)}$; $\int\limits_{\pi/2}^\pi \frac{\sin x\,dx}{\sqrt[7]{\cos^2 x}}$

18. $\int\limits_0^\infty x\sin x\,dx$; $\int\limits_{-3/4}^0 \frac{dx}{\sqrt{4x+3}}$

19. $\int\limits_{-\infty}^{-1} \frac{7\,dx}{(x^2-4x)\ln 5}$; $\int\limits_1^2 \frac{x\,dx}{\sqrt{(x^2-1)^3}\ln 2}$

20. $\int\limits_{1/3}^\infty \frac{\pi\,dx}{(1+9x^2)\operatorname{arctg}^2 3x}$; $\int\limits_0^{1/3} \frac{dx}{9x^2-9x+2}$

21. $\int\limits_2^\infty \frac{dx}{(4+x^2)\sqrt{\pi\operatorname{arctg}\frac{x}{2}}}$; $\int\limits_0^{\pi/2} \frac{3\sin^3 x\,dx}{\sqrt{\cos x}}$

22. $\int\limits_1^\infty \frac{dx}{(x^2+2x)\ln 3}$; $\int\limits_0^3 \frac{\sqrt[3]{9}x\,dx}{\sqrt[3]{9-x^2}}$

23. $\int\limits_0^\infty e^{-3x}x\,dx$; $\int\limits_0^1 \frac{x^4dx}{\sqrt[3]{1-x^5}}$

24. $\int\limits_{-\infty}^0 \left(\frac{x^2}{x^3-1}-\frac{x}{1+x^2}\right)dx$; $\int\limits_0^2 \frac{x^2dx}{\sqrt{64-x^6}}$

25. $\int\limits_0 \frac{dx}{2x^2-2x+1}$; $\int\limits_{1/2}^1 \frac{dx}{\sqrt[9]{1-2x}}$

26. $\int\limits_1^\infty \frac{dx}{x^2(x+1)}$; $\int\limits_1^5 \frac{x^2dx}{\sqrt{31(x^3-1)}}$

27. $\int\limits_{e^2}^{\infty}\frac{dx}{x(\ln x-1)^2}$; $\int\limits_{1}^{3/2}\frac{dx}{\sqrt{3x-x^2-2}}$

28. $\int\limits_{1}^{\infty}\frac{dx}{(6x^2-5x+1)\ln\frac{3}{4}}$; $\int\limits_{0}^{4}\frac{10x\,dx}{\sqrt[4]{(16-x^2)^3}}$

29. $\int\limits_{1}^{\infty}\frac{dx}{9x^2-9x+2}$; $\int\limits_{0}^{1/4}\frac{dx}{\sqrt[3]{1-4x}}$

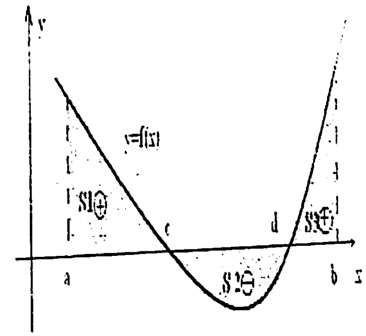
30. $\int\limits_{3}^{\infty}\frac{dx}{x^2-3x+2}$; $\int\limits_{0}^{1/2}\frac{dx}{(2x-1)^2}$

Topic 16. Applications of definite integrals.

16.1. Areas of regions between curves. 16.2. Length of curves in the plane. 16.3. Volumes of solids of revolution. 16.4. Areas of surfaces of revolution. 16.5. Physical applications.

16.1. Finding the areas of regions between curves.

16.1.1. The area below Ox axis.



As is shown above

If $f(x) \geq 0$ at $[a,b]$, then

$\int\limits_a^b f(x)dx = S$ is the **area under curve** which is bounded by lines $x=a$, $x=b$, $y=0$ and graph $y=f(x)$.

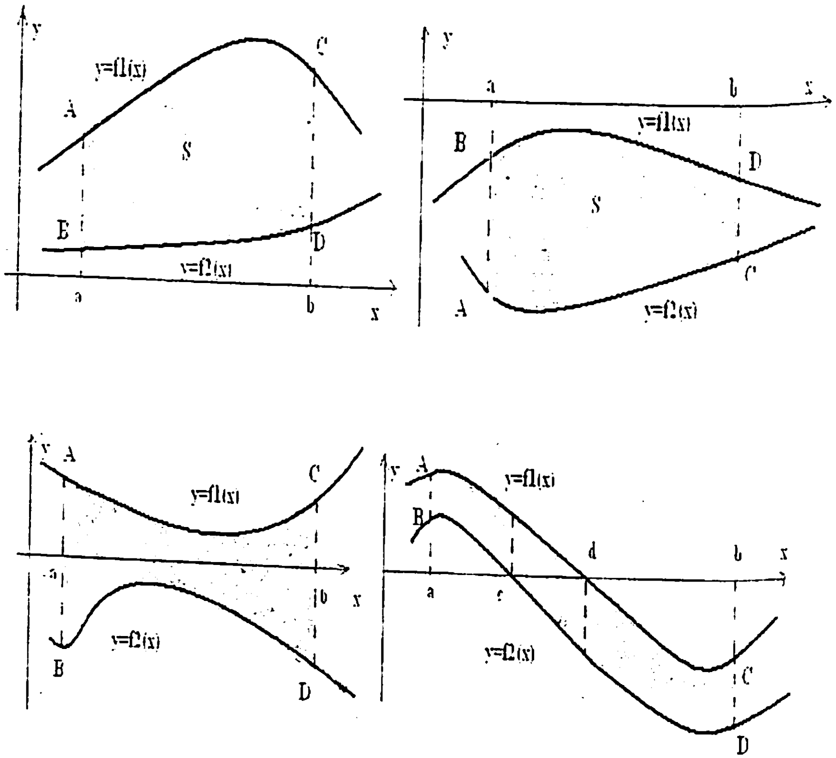
If $f(x) \leq 0$ at $[a,b]$ then the area is disposed below Ox axis and $\int\limits_a^b f(x)dx \leq 0$. That integral expresses the area with negative sign.

In general, if the function gets different signs at $[a,b]$, then $\int\limits_a^b f(x)dx$ is equal to areas of parts of area under curve with positive sign if they lie above of Ox axis and with negative sign if they lie below of Ox axis

and common area is equal : $S = S1 - S2 + S3$.

16.1.2. The area of the region between the curves.
If $f_1(x)$ and $f_2(x)$ are continuous functions with $f_1(x) \geq f_2(x)$ throughout the interval [a,b] , then the area of the region between the curves $y=f_1(x)$ and $y=f_2(x)$ from a to b is the integral of $(f_1 - f_2)$ from a to b.

$$S = \int_a^b (f_1(x) - f_2(x))dx .$$



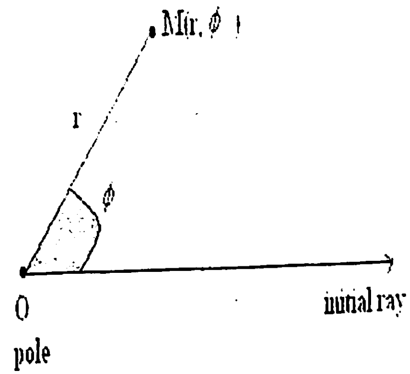
16.1.3. The area under curve given in parametric form.
If graph of the function given in parametric form:

$$\begin{cases} x=x(t) \\ y=y(t), \end{cases} \quad \text{where } \alpha < t < \beta$$

Here $y(t) \geq 0$ is continuous and $x=x(t)$ is monotonic and continuously differentiable function at $[\alpha, \beta]$. Is known that $\phi(\alpha) = a$ and $\phi(\beta) = b$, then the area under curve is defined by formula:

$$S = \int_\alpha^\beta y(t) \cdot x'(t) \cdot dt .$$

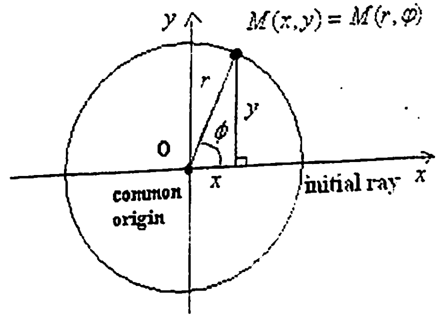
16.1.4. The area under curve given in polar coordinates.
To define polar coordinates, let's first fix an origin O (called the **pole**) and an **initial ray** from O. Then each point M can be located by assigning to it a polar coordinate pair (r, ϕ), in which the first number r, gives the directed angle from the initial ray to the segment OM. Suppose that the angle ϕ is positive when measured anticlockwise, and negative when measured clockwise. The values of r and ϕ are constrained:



$$\begin{cases} r \geq 0 \\ 0 \leq \phi \leq 2\pi \end{cases}$$

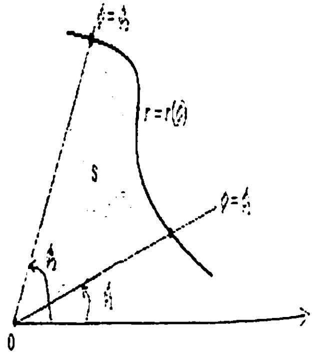
:

The correlation between Cartesian coordinate system and Polar coordinate system is connected by following equations:



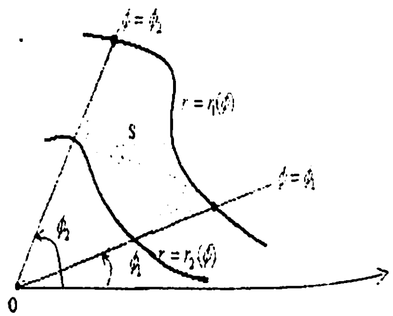
$$\begin{cases} x = r \cdot \cos(\phi) \\ y = r \cdot \sin(\phi). \end{cases}$$

The plane region G, which is constrained by rays $\phi=\phi_1$ *and* $\phi=\varphi_2$ $(\phi_1<\varphi_2)$ issued from the pole O and graph of a function r = r(ϕ), which is continuous and nonnegative at $[\phi_1,\varphi_2]$ is called a **curvilinear triangle**. The area of a curvilinear triangle is equal:



$$S=\frac{1}{2}\int_{\phi_1}^{\phi_2} r^2(\phi)\cdot d\varphi\ .$$

The area of a region constrained by two curvilinear triangles with equations r = $r_1(\phi)$ and r = $r_2(\phi)$, is equal:



$$S=\frac{1}{2}\int_{\phi_1}^{\phi_2}(r_1^2(\phi)-r_1^2(\phi))\cdot d\varphi$$

16.2. Length of curves in the plane.

16.2.1. Length of curves in the plane in Cartesian coordinates.
Let the plane curve L with ends A and B is given by graph of continuously differentiable function y=f(x), where $x\in[a,b]$. Divide the curve on n-parts by points $M_1, M_2, \ldots, M_i, \ldots, M_{n-1}$. Point M_i has coordinates (x_i, y_i), where $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$. The length of **broken line** which is inscribed in L with vertexes in chosen points A and B denote by l_n .It is shown at picture:

$$l_n=\sum_{i=1}^{n}\sqrt{\Delta x_i^2+\Delta y_i^2}\ .$$

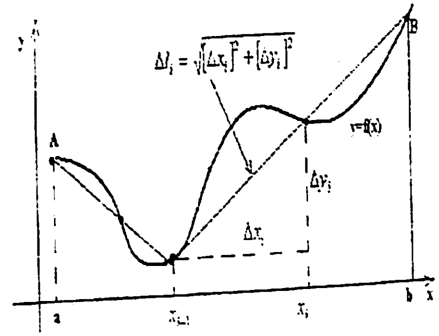
The **length of the plane curve L** is called the limit as max Δx_i approaches zero of the lengths of inscribed in L broken lines.

$$L=\lim_{\max\Delta x_i\to 0} l_n=\lim_{\max\Delta x_i\to 0}\sum_{i=1}^{n}\sqrt{\Delta x_i^2+\Delta y_i^2}=\lim_{\max\Delta x_i\to 0}\sum_{i=1}^{n}\sqrt{1+\frac{\Delta y_i^2}{\Delta x_i^2}}\cdot\Delta x_i$$

Thus, the Riemann sum regarding the function

$$f(x)=\sqrt{1+\frac{\Delta y_i^2}{\Delta x_i^2}}$$

is obtained. With respect the definition of the definite integral the next definition can be introduced:



Definition: If the function f(x) is smooth on $[a,b]$, the **length of the plane curve** y=f(x) **from a to b** is the number : L =

$$\int_a^b\sqrt{1+[f'(x)]^2}\,dx\ .$$

16.2.2. Parametric formula for length of the curve.
1) Let the **plane curve** is given by continuously differentiable functions in a parametric form:

$$\begin{cases} x=x(t)\\ y=y(t),\end{cases}$$ where $\alpha<t<\beta$, the **length of the plane curve**:

$$L=\int_\alpha^\beta\sqrt{[x'(t)]^2+[y'(t)]^2}\cdot dt\ .$$

2) Let the **space curve** is given by continuously differentiable functions in a parametric form:

$$\begin{cases} x=x(t)\\ y=y(t)\\ z=z(t),\end{cases}$$ where $\alpha<t<\beta$, the **length of the space curve**:

$$L = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \cdot dt .$$

16.2.3. Length of the curve in polar coordinates.

Let the **plane curve** is given by continuously differentiable functions in a polar form:

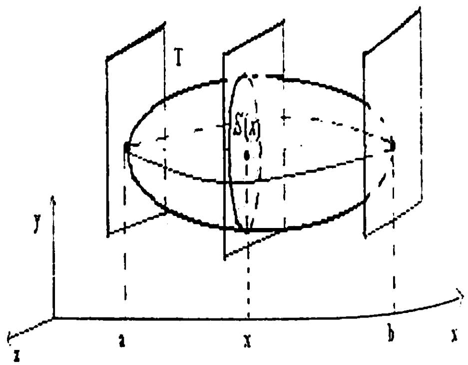
$r = r(\phi)$, where $\phi \in [\alpha, \beta]$. Then the length of the plane curve is the number:

$$L = \int_{\alpha}^{\beta} \sqrt{[r(\phi)]^2 + [r'(\phi)]^2} \cdot d\phi$$

16.3. Volumes of solids of revolution.

16.3.1. Volumes of solid.

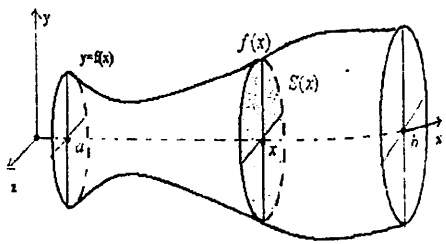
Let a space solid T is given, which is projected on the axis Ox as segment [a, b]. Any plane perpendicular to axis Ox and going trough the point $x \in [a, b]$ forms a plane figure as intersection with the solid. The area of the plane figure is **S(x).** Then the volume of this solid is defined by formula: **V(T)** $= \int_a^b S(x)dx$.



16.3.2. Volumes of solids of revolution.

1) Revolving about the Ox-axis.

Volume of a solid generated by revolving about the Ox-axis the region between the graph of the continuous function y=f(x) and the Ox-axis from x=a and x=b is



$$\mathbf{V(T)} = \pi \int_a^b [f(x)]^2 dx .$$

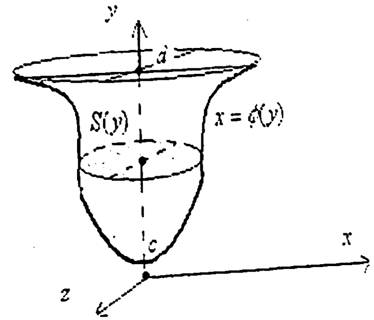
Here $S(x) = \pi \cdot R^2 = |R = f(x)| = \pi \cdot f^2(x)$.

2) Revolving about the Oy-axis.

Volume of a solid generated by revolving about the Oy-axis the region between the graph of the continuous function $x = \phi(y)$ and the Oy-axis from y=c and y=d is

$$\mathbf{V(T)} = \pi \int_c^d [\phi(y)]^2 dy .$$

Here S(y) = $\pi \cdot R^2 = |R = \phi(y)| = \pi \cdot \varphi^2(y)$.



3) Parametric formula for volume of solid of revolution.

Let the constrained curve given in parametric form:

$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, where $\alpha < t < \beta$,then the volume of solid of revolution:

$$\mathbf{V(T)} = \pi \int_{\alpha}^{\beta} [y(x)]^2 x'(t) \cdot dt .$$

16.4. Areas of surfaces of revolution.

1) Let the graph of continuously differentiable function y = f(x), where $x \in [a, b]$ and $f(x) \geq 0$, rotates about Ox- axis. The areas of surfaces H of revolution is equal:

$$S(H) = 2\pi \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2} \cdot dx .$$

2) Parametric formula for area of surface of revolution.

$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, where $\alpha < t < \beta$.

$$S(H) = 2\pi \cdot \int_{\alpha}^{\beta} y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt .$$

3) An area of surface of revolution in polar coordinates.
Graph of function is given by $r = r(\phi)$, where $\alpha \le \phi \le \beta$

$$S(H) = 2\pi \cdot \int_{\alpha}^{\beta} r(\phi) \cdot \sin\phi \cdot \sqrt{[r(\phi)]^2 + [r'(\phi)]^2}\, d\phi .$$

16.5. Physical applications.

16.5.1. The variable force integral formula for work.
The work done by a continuous force F(x) directed along the Ox-axis from a to b is

$$W = \int_a^b F(x)dx .$$

16.5.2. Pumping liquids from containers.
Let a tank is filled up of liquid with specific gravity γ . To find how much work it takes to pump all or part of the liquid from level z=b to lever z=a on lever z=h (a<b<h) from a container can be used the next formula:

$$W = \gamma \int_a^b S(z) \cdot (h - z) \cdot dz$$

Where S(z) is the area of the surface of liquid on the lever z in container.

EXERCISE XVI

1

Determine the area of figure bounded by the indicated curves (with accuracy within two significance figures after point).

1. $\rho = 3\sqrt{\cos 2\varphi}$.
2. $y = x^2,\ y = 3 - x$.
3. $y = \sqrt{x},\ y = x^3$.
4. $x = 7\cos^3 t,\ y = 7\sin^3 t$.
5. $\rho = 4\cos 3\varphi$.
6. $\rho = 3\cos 2\varphi$.
7. $\rho = 2(1 - \cos\varphi)$.
8. $\rho^2 = 2\sin 2\varphi$.
9. $x = 4(t - \sin t),\quad y = 4(1 - \cos t)$.
10. $\rho = 2(1 + \cos\varphi)$.
11. $\rho = 2\sin 3\varphi$.
12. $\rho = 2 + \cos\varphi$.
13. $y = 1/(1 + x^2),\ y = x^2/2$.
14. $y^2 = x + 1,\ y^2 = 9 - x$.
15. $y^2 = x^3,\ x = 0,\ y = 4$.
16. $\rho = 4\sin^2\varphi$.
17. $x = 3\cos t,\ y = 2\sin t$.
18. $y^2 = 9x.\ \ y = 3x$.
1.19. $x = 3(\cos t + t\sin t),\quad y = 3(\sin t - t\cos t),\quad y = 0$ $(0 \leqslant t \leqslant \pi)$.
20. $y^2 = 4x,\ x^2 = 4y$.
21. $y^2 = x^3,\ x = 2$.
22. $y = x^2,\ y = 2 - x^2$.
23. $y^2 = (4 - x^3),\ x = 0$
24. $\rho = 3\sin 4\varphi$.
25. $y = x^3,\ y = 1,\ x = 0$.
26. $xy = 6,\ x + y - 7 = 0$.
27. $y = 2^x,\ \ y = 2x - x^2,\ \ x = 0,\ \ x = 2$.
28. $x^2 = 4y,\ y = 8/(x^2 + 4)$.
29. $y = x + 1,\ y = \cos x,\ y = 0$.
30. $x = 2\cos^3 t,\ y = 2\sin^3 t$.

2

Determine the length of the curve (with accuracy within two significance figures after point).

1. $x = 2\cos^3 t,\ y = 2\sin^3 t$

2.2. $x = 2(\cos t + t\sin t),\quad y = 2(\sin t - t\cos t) \quad (0 \leqslant t \leqslant \pi)$.

3. $\rho = \sin^3(\varphi/3)\ (0 \leqslant \varphi \leqslant \pi/2)$.
4. $\rho = 2\sin^3(\varphi/3)\ (0 \leqslant \varphi \leqslant \pi/2)$.
5. $\sqrt[3]{x^2} + \sqrt[3]{y^2} = \sqrt[3]{9}$.
6. $x^{2/3} + y^{2/3} = 4^{2/3}$
7. $y^2 = (x+1)^3$,
8. $y = 1 - \ln\cos x\ (0 \leqslant x \leqslant \pi/6)$
9. $\rho = 6\cos^3(\varphi/3)\ (0 \leqslant \varphi \leqslant \pi/2)$.
10. $x = 4\cos^3 t,\ y = 4\sin^4 t$.
11. $u^2 = (x-1)^3$
12. $y^2 = x^5$
13. $\rho = 3\cos\varphi$.
14. $\rho = 3(1 - \cos\varphi)$.
15. $\rho = 2\cos^3(\varphi/3)$.
16. $x = 5\cos^2 t,\quad y = 5\sin^2 t \quad (0 \leqslant t \leqslant \pi/2)$.
17. $9y^2 = 4(3-x)^3$
18. $\rho = 3\sin\varphi$.
19. $y = \ln\sin x\ (\pi/3 \leqslant x \leqslant \pi/2)$.
20. $x = 9(t - \sin t),\ y = 9(1 - \cos t)\ (0 \leqslant t \leqslant 2\pi)$
21. $\rho = 2(1 - \cos\varphi)$
22. $y^2 = (x-1)^3$
23. $x = 7(t - \sin t),\quad y = 7(1 - \cos t) \quad (2\pi \leqslant t \leqslant 4\pi)$
24. $y = e^{x/2} + e^{-x/2}\ (0 \leqslant x \leqslant 2)$
25. $x = 4\cos^3 t,\ y = 4\sin^3 t$.
26. $x = \sqrt{3}t^2,\ y = t - t^3$
27. $\rho = 5\sin\varphi$.
28. $\rho = 4\cos\varphi$.
29. $\rho = 5(1 + \cos\varphi.)$
30. $y^2 = x^3$

3

Determine the volume of the solid obtained by revolution the figure Φ about the indicated coordinate axis (with accuracy within two significance figures after point).

1. Φ: $y^2 = 4 - x,\ x = 0,\ Oy$.
2. Φ: $\sqrt{x} + \sqrt{y} = \sqrt{2},\ x = 0,\ y = 0;\ Ox$.
3. Φ: $x^2/9 + y^2/4 = 1,\ Oy$.
4. Φ: $y^3 = x^2,\ y = 1,\ Ox$.
5. Φ: $x = 6(t - \sin t),\quad y = 6(1 - \cos t)$,

16. Φ: $y = 2x - x^2,\ y = 0,\ Ox$.
17. Φ: $\rho = 2(1 + \cos\varphi)$,
18. Φ: $x = 7\cos^3 t,\ y = 7\sin^3 t,\ Oy$
19. Φ: $x^2/16 + y^2/1 = 1,\ Ox$.
20. Φ: $x^3 = (y-1)^2,\ x = 0,\ y = 0,\ Ox$
21. Φ: $xy = 4,\ 2x + y - 6 = 0,\ Ox$.
22. Φ: $x = \sqrt{3}\cos t,\ y = 2\sin t,\ Oy$.
23. Φ: $y = 2 - x^2,\ y = x^2,\ Ox$.
24. Φ: $y = -x^2 + 8,\ y = x^2,\ Ox$.
25. Φ: $y^2 = (x+4)^3,\ x = 0,\ Ox$.
26. Φ: $y = x^3,\ x = 0,\ y = 8,\ Oy$.
27. Φ: $x = \cos^3 t,\ y = \sin^3 t,\ Ox$.
28. Φ: $2y = x^2,\ 2x + 2y - 3 = 0$,
29. Φ: $y = x - x^2,\ y = 0,\ Ox$.
30. Φ: $y = 2 - x^2/2,\quad x + y = 2,\ Oy$

4

Determine the area of the surface obtained by revolution the arc of the curve L about indicated coordinate axis (with accuracy within two significance figures after point):

1. L: $y = x^3/3\ (-1/2 \leqslant x \leqslant 1/2),\ Ox$.
2. L: $\rho = 2\cos\varphi$,

4.3. L: $x = 10(t - \sin t),\ y = 10(1 - \cos t)\ (0 \leqslant t \leqslant 2\pi)$, Ox.

4. L: $y = x^2/2$

5. L: $3y = x^2$ $(0 \leqslant x \leqslant 2)$, Ox

6. L: $y = \sqrt{x}$,

4.7. L: $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$ $(0 \leqslant t \leqslant 2\pi)$, Ox.

8. L: $x = \cos t$, $y = 3 + \sin t$, Ox.

9. L: $3x = y^3$ $(0 \leqslant y \leqslant 2)$, Oy.

10. L: $y = x^3/3$ $(-1 \leqslant x \leqslant 1)$, Ox.

11. L: $x = \cos t$, $y = 1 + \sin t$, Ox

12. L: $x^2 = 4 + y$,

4.13. L: $x = 3(t - \sin t)$, $y = 3(1 - \cos t)$ $(0 \leqslant t \leqslant 2\pi)$, Ox.

14. L: $x = \cos^3 t$, $y = \sin^3 t$, Ox.

15. L: $\rho = \sqrt{\cos 2\varphi}$,

16. L: $y^2 = 4 + x$,

17. L: $y^2 = 2x$.

18. L: $3y = x^3$ $(0 \leqslant x \leqslant 1)$, Ox.

19. L: $\rho^2 = 4 \cos 2\varphi$,

20. L: $\rho = 6 \sin \varphi$,

21. L: $x = t - \sin t$, $y = 1 - \cos t$

22. L: $\rho = 2 \sin \varphi$

23. L: $\rho = \frac{2}{3} \cos \varphi$

24. L: $x = 3 \cos^3 t$, $y = 3 \sin^3 t$, Ox.

25. L: $x = 2 \cos t$, $y = 3 + 2 \sin t$, Ox

26. L: $\rho^2 = 9 \cos 2\varphi$,

27. L: $y = x^3$

28. L: $x = 2 \cos^3 t$, $y = 2 \sin^3 t$, Ox

29. L: $x = \cos t$, $y = 2 + \sin t$, Ox.

30. L: $\rho = 4 \sin \varphi$,

Recommended Literature:

1) G. Thomas, R. Finney. Calculus and Geometry. Parts 1,2. Addison-Wesley Publishing Company,
2) R. Barnett, M. Ziegler, K. Byleen. Calculus for business, economics, life science and social science. Pearson Education International.
3) A. Chang. Fundamental Methods of Mathematical Economics. Mcgraw-Hill Company.
4) E. Dowing. Introduction to Mathematical Economics. Third edition.
5) S. Murray R Shaum's Advanced Calculus, Second edition.
6) S. Lipschutz, M. Lipson. Discrete Mathematics, Second editon.

Нартова Диляра Сагыновна

МАТЕМАТИЧЕСКОЕ ИСЧИСЛЕНИЕ

Краткий курс и упражнения

Часть 1.

Nartova Dilyara Saginovna

CALCULUS.

Brief summary and exercises.

Part 1.

Подписано в печать 14.02.2005г.
Формат 60x84 1/16. Печать RISO.
Бумага офсет №1.Усл.п.л. 12,5.
Тираж 250 экз.

Отпечатано в ТОО «Алла прима»
Тел.51 61 77