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STABILITY OF LINEAR SYSTEMS IN $\mathbf{R}^3$

Abstract. In [1],was considered the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, when$x \in \mathbf{R}^2$ and A is a 2×2 matrix. Using the values of the determinant and trace of A allowed us to show the conditions of stability of systems. That is known that two values is enough to suggest that the system is stable or not. In this paper, by dint of the same method for the matrix A, we solved with a dimension 3×3. We use Cardano's theorem and Descartes' rule for define sign. By using some matrix values, you can specify the stability of the system, such as, stable, unstable and saddle. We looked for all possible cases, and drew it in the table. One example has shown for this type of problem.

Keywords: Stability, Cardano's theorem, Descartes' rule.

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Аңдатпа. [1]-де$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$сызықты жүйе қарастырылды, мұнда $x \in \mathbf{R}^2$ және A 2 × 2 көлеміндегіматрицасы болып табылады. Детерминанттардың мәндерімен бірге матрицаның іздерін пайдалана отырып жүйелердің қалыптылығыныңжағдайларын көрсетуге болады. Бізге белгілі болғандай, жүйенің қалыптылығынемесе қалыпсыздығы болжау үшін екі мәннің болуы жеткілікті. Бұл мақалада, жоғарыдағы методты қолдана отырып, біз 3×3 көлемдіматрицамен осы жұмысты жасаймыз. Кардан теоремасын жәнеДекарт ережесін кейбір белгілерді анықтау үшінқолданамыз. Матрицаның кейбір мәндерін пайдалана отырып, біз жүйеніңқалыптылығынкөрсете аламыз, соның ішінде тұрақты, тұрақсыз және седле болып табылады. Барлық жағдайларды карастыра отырып, берілген кестені толтырдық. Көрсетілген шығару жолын қолдана отырып, біз бір үлгі есепті шығардық.

Кілт сөздер: Устойчивость, теорема Кардано, правило Декарта.

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Аннотация. В [1] рассматривалась линейная система x = Ax, когда x∈ R ^ 2 и A - матрица 2 × 2. Использование значений определителя и следа A позволило нам показать условия устойчивости систем. Известно, что двух значений достаточно, чтобы предположить, что система стабильна или нет. В этой статье с помощью того же метода для матрицы A мы решили с размерностью 3 × 3. Мы используем теорему

Кардано и правило Декарта для определения знака. Используя некоторые значения матрицы, вы можете указать стабильность системы, например, стабильную, нестабильную и седловую. Мы просмотрели все возможные случаи и нарисовали его в таблице. Один пример показан для этого типа проблемы.

Ключевые слова: Устойчивость, теорема Кардано, правило Декарта.

Introduction

Let $b = det\ A$ and $r = trace\ A$ and consider the linear system

$\dot{x} = Ax$ (*)

(a) If $b < 0$ then (*) has a saddle at the origin.

(b) If $b > 0$ and $r^2 - 4b \geq 0$ then (*) has a node at the origin; it is stable if $r < 0$ and unstable if $r > 0$.

(c) If $b > 0$, $r^2 - 4b < 0$, and $r \neq 0$ then (*) has a focus at the origin; it is stable if $r < 0$ and unstable if $r > 0$.

(d) If $b > 0$ and $r = 0$ then (*) has a center at the origin.

Note that in case (b), $r^2 \geq 4|b| > 0$; i.e., $r \neq 0$. The eigenvalues of the matrix A are given by

$$\alpha = \frac{r \pm \sqrt{r^2 - 4b}}{2}$$

It can be seen by this way: (a) if $b < 0$ there are two real eigenvalues of opposite sign. Result is saddle.

(b) If $b > 0$ and $r^2 - 4b > 0$ then there are two real eigenvalues of the same sign as r, if they are positive then it is unstable, if they are negative it is stable.

(c) if b > 0, $r^2 - 4b < 0$ and $r \neq 0$then there are two complex conjugate eigenvalues

$\alpha = a \pm ib$.

(d) if $b > 0$ and $r = 0$ then there are two pure imaginary complex conjugate eigenvalues.

For cases (c) and (d) you can read in [1].

Main part

Let's define itmatrix A, from $\dot{x} = Ax$,

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Stability depends on the eigenvalues of the matrix. The system is stable if $Re(\lambda_{1,2,3}) < 0$, unstable if $Re(\lambda_{1,2,3}) > 0$. In other cases, the system is saddle.

So, let's find eigenvalues:

$\begin{bmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{bmatrix} = -\lambda^3 + (a+e+i)\lambda^2 + (-ae - ei - ai + cg +$
$hf + +bd)\lambda + (aei + dhc + bfd - cge - hfa - bdi) = 0$
(*)
Let's define $rM = M_{11} + M_{22} + M_{33}$, where M_{ij}minors of matrix A.
Now, we can write equation (*) in the form:
$-\lambda^3 + trA\lambda^2 - trM\lambda + detA = 0$

So, stability of the systems depends on solution of equation:
$\lambda^3 - trA\lambda^2 + trM\lambda - detA = 0$ (1)
For define stability, it does not need to know the exact roots of the equation. It is enough to know the signs of the real parts of the roots (positive, negative).
For equation,
$\Delta = -27(-\det A - \frac{2}{27} tr^3 A + \frac{1}{3} trA\ trM)^2 - 4(tr\ M - \frac{1}{3} tr^2 A)^3$ (2)
is called discriminant of cubic equation. Using Cardano's formula, we obtain conclusions below.

Conclusion 1
1) If $\Delta > 0$, there are three different real roots;
2) If $\Delta = 0$, there are three real roots and at least two of them equal to each other;
3) If $\Delta < 0$, there one real root and two conjugate complex roots

Naturally, this does not show the sign of the roots. But nevertheless, it will be useful for us.

We need to use Descartes' rule of signs and Cardano's formula to define it is signs.

Descartes' rule of sign is used to determine the number of real zeros of a polynomial function.

It gives for us that the number of positive real zeroes in a polynomial function f(x) is the same or less than by an even numbers as the number of changes in the sign of the coefficients. The number of negative real zeroes of the f(x) is the same as the number of changes in sign of the coefficients of the terms of f(-x) or less than this by an even number. [2]

Example:
$f(x) = 2x^4 - x^3 + 4x^2 - 5x + 3$ (3)
Firstly, look at $f\ (x)$:
$f(x) = +2x^4 - x^3 + 4x^2 - 5x + 3$
There are four sign changes, so there are 4, 2, or 0 positive roots. Now, look at $f(-x)$:
$f(-x) = +2(-x)^4 - (-x)^3 + 4(-x)^2 - 5(-x) + 3$
$= +2x^4 + x^3 + 4x^2 + 5x + 3$

There are no sign changes, so there are no negative roots. Then answer is:
There are four, two, or zero positive roots, and zero negative roots.

+	4	2	0
-	0	0	0
i	0	2	4

If we solve (3), we get 4 complex roots.
We can use this rule to solve the equation (1):
where
$f(\lambda) = \lambda^3 - trA\lambda^2 + trM\lambda - detA$
and
$f(-\lambda) = -\lambda^3 - trA\lambda^2 - trM\lambda - detA$

So, for our problem number of sign roots depends on value of $trA, trM, \det A$;

Example: Lets fix that $tr\ A > 0, tr\ M > 0, \det A > 0$

λ^3	λ^2	λ^1	λ^0
+1	$-tr\ A$	$tr\ M$	$-\det A$
-1	$-tr\ A$	$-tr\ M$	$-\det A$

	Real root	Complex root
+	3	1
-	0	0
i	0	2

In the table which is given below, we looked at all the possible cases.

Table 1

#		Real roots		Complex roots		
		+	-	+	-	i
1	$tr\ A > 0, tr\ M > 0, \det A > 0$	3	0	1	0	2
2	$tr\ A < 0, tr\ M > 0, \det A > 0$	1	2	1	0	2
3	$tr\ A > 0, tr\ M < 0, \det A > 0$	1	2	1	0	2
4	$tr\ A > 0, tr\ M > 0, \det A < 0$	2	1	0	1	2
5	$tr\ A < 0, tr\ M < 0, \det A > 0$	1	2	1	0	2
6	$tr\ A < 0, tr\ M > 0, \det A < 0$	0	3	0	1	2
7	$tr\ A > 0, tr\ M < 0, \det A < 0$	2	1	0	1	2

8	$tr\ A < 0, tr\ M < 0, \det A < 0$	2	1	0	1	2
9	$tr\ A = 0, tr\ M > 0, \det A > 0$	-	-	1	0	2
10	$tr\ A = 0, tr\ M < 0, \det A > 0$	1	2	1	0	2
11	$tr\ A = 0, tr\ M > 0, \det A < 0$	-	-	0	1	2
12	$tr\ A = 0, tr\ M < 0, \det A < 0$	2	1	0	1	2
13	$tr\ A > 0, tr\ M = 0, \det A > 0$	-	-	1	0	2
14	$tr\ A < 0, tr\ M = 0, \det A > 0$	1	2	-	-	-
15	$tr\ A > 0, tr\ M = 0, \det A < 0$	2	1	-	-	-
16	$tr\ A < 0, tr\ M = 0, \det A > 0$	-	-	0	1	2
17	$tr\ A = 0, tr\ M = 0, \det A > 0$	-	-	1	0	2
18	$tr\ A = 0, tr\ M = 0, \det A < 0$	-	-	0	1	2

In the table 1, we cannot consider the case $\det A = 0$. Because, it means that one of the eigenvalues is zero. We cannot consider this case by [1]. You may notice that our table not full. Since we do not know the signs of the real parts of complex roots, we cannot say anything for the case of complex roots. But, if $\Delta \geq 0$, it may give some results.

Conclusion 2

By 1) 2) of Conclusion 1 and table 1, if $\Delta \geq 0$, then

1) $tr\ A > 0, tr\ M > 0, \det A > 0$, thensystem is unstable;
2) $tr\ A < 0, tr\ M > 0, \det A < 0$, then system is stable;
3) In other cases of $r\ A, tr\ M, \det A$, system is saddle.

For the case $\Delta < 0$, we will have to study the Cardano Formula.

In complex cases, the roots will be as follows:

$$\lambda_1 = \alpha + \beta + \frac{tr\ A}{3} \qquad (4)$$

$$\lambda_{2,3} = -\frac{\alpha+\beta}{2} + \frac{tr\ A}{3} = \frac{-3(\alpha+\beta)+2trA}{6} \qquad (5)$$

Where

$$\alpha = \sqrt[3]{-\frac{q}{2} + \sqrt{Q}}, \qquad \beta = \sqrt[3]{-\frac{q}{2} - \sqrt{Q}},$$

$$\Delta = -108Q$$

$$Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2, \text{ and } Q > 0$$

$$q = -\det A - \frac{2}{27} tr^3 A + \frac{1}{3} trA\ trM$$

$$p = tr\ M - \frac{1}{3} tr^2 A;$$

It is worth noting that when we consider $\lambda_{2,3}$, we do not take into account the imaginary part. Because we are interested in the real part.
As we said, if for any function of degree 3 there exist complex roots then it means that there must be one real and two conjugate complex roots. So, assume that, real root $\lambda_1 > 0$. Then, by (4), we get $\alpha + \beta + \frac{tr\,A}{3} > 0$, $\rightarrow 3(\alpha + \beta) > -trA, \rightarrow -3(\alpha + \beta) < trA$;
If we use last inequality and (5), then we obtain

$$\lambda_{2,3} = \frac{-3(\alpha+\beta)+2trA}{6} < \frac{trA+2trA}{6} = \frac{trA}{2} \qquad (6)$$

If we assume that real root is $\lambda_1 < 0$, in a similar way, we get

$$\lambda_{2,3} > \frac{trA}{2} \qquad (7)$$

Conclusion 3

When $\Delta < 0$,
1) By (6), if $\lambda_1 > 0$ and $tr\ A \leq 0 \rightarrow \lambda_{2,3} < 0 \rightarrow$ saddle
2) By (7), if $\lambda_1 < 0$ and $tr\ A \geq 0 \rightarrow \lambda_{2,3} > 0 \rightarrow$ saddle
In the table 1, we have 10 cases of possible roots, which correspond to conclusion 3. It is #2,4-5,7,9-12,17-18.

Result

Based on Conclusion 2 and Conclusion 3, we can construct the following table 2:

Table 2

#		**Real roots**		**Complex roots**		
		+	-	+	-	i
1	$tr\ A > 0, tr\ M > 0, \det A > 0$	3	0	1	0	2
2	$tr\ A < 0, tr\ M > 0, \det A > 0$	1	2	1	0	2
3	$tr\ A > 0, tr\ M < 0, \det A > 0$	1	2	1	0	2
4	$tr\ A > 0, tr\ M > 0, \det A < 0$	2	1	0	1	2
5	$tr\ A < 0, tr\ M < 0, \det A > 0$	1	2	1	0	2
6	$tr\ A < 0, tr\ M > 0, \det A < 0$	0	3	0	1	2
7	$tr\ A > 0, tr\ M < 0, \det A < 0$	2	1	0	1	2
8	$tr\ A < 0, tr\ M < 0, \det A < 0$	2	1	0	1	2
9	$tr\ A = 0, tr\ M > 0, \det A > 0$	-	-	1	0	2
10	$tr\ A = 0, tr\ M < 0, \det A > 0$	1	2	1	0	2
11	$tr\ A = 0, tr\ M > 0, \det A < 0$	-	-	0	1	2
12	$tr\ A = 0, tr\ M < 0, \det A < 0$	2	1	0	1	2
13	$tr\ A > 0, tr\ M = 0, \det A > 0$	-	-	1	0	2

14	$tr\ A < 0, tr\ M = 0, \det A > 0$	1	2	-	-	-
15	$tr\ A > 0, tr\ M = 0, \det A < 0$	2	1	-	-	-
16	$tr\ A < 0, tr\ M = 0, \det A > 0$	-	-	0	1	2
17	$tr\ A = 0, tr\ M = 0, \det A > 0$	-	-	1	0	2
18	$tr\ A = 0, tr\ M = 0, \det A < 0$	-	-	0	1	2

-unstable
-stable
saddle
undefined
empty

Examples

1)Let's consider linear system and stability of system

$$\begin{cases} \dot{x}_1 = -3x_1 + 2x_2 + 4x_3 \\ \dot{x}_2 = x_2 + 3x_3 \\ \dot{x}_3 = x_1 + x_2 + 2x_3 \end{cases}$$

It can be written as

$\dot{x} = Ax$

where

$$A = \begin{bmatrix} -3 & 2 & 4 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

$detA = 5 > 0,\ trA = -1 < 0,\ trM = -14 < 0; \Delta = 1.1777e + 04 > 0$

By the #5 of table 1., linear system is saddle

#		**Real roots**		**Complex roots**		
		+	-	+	-	i
5	$tr\ A < 0, tr\ M < 0, \det A > 0$	1	2	1	0	2

Exact solution: $\lambda_{1,2,3} = -4.1102;\ -0.3514;\ 3.4616.$

2)$(A - \lambda I) = \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$trA = 11 > 0; trM = 36 > 0; detA = 36 > 0; \rightarrow$ unstable

Exact solution: $\lambda_{1,2,3} = 2; 3; 6 > 0$

$\lambda^3 - trA\lambda^2 + trM\lambda - detA = 0$

3)$(A - \lambda I) = \lambda^3 - 8 = 0$

$trA = 0; trM = 0; detA = 8 > 0,\ \Delta = -1728 < 0;$

If system is saddle, there is two positive and negative eigenvalues.

Exact solution: $2; -1 + 1.7321i, -1 - 1.7321i$

#		Real roots		Complex roots		
		+	-	+	-	i
17	$tr\ A = 0, tr\ M = 0, \det A > 0$	-	-	1	0	2

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