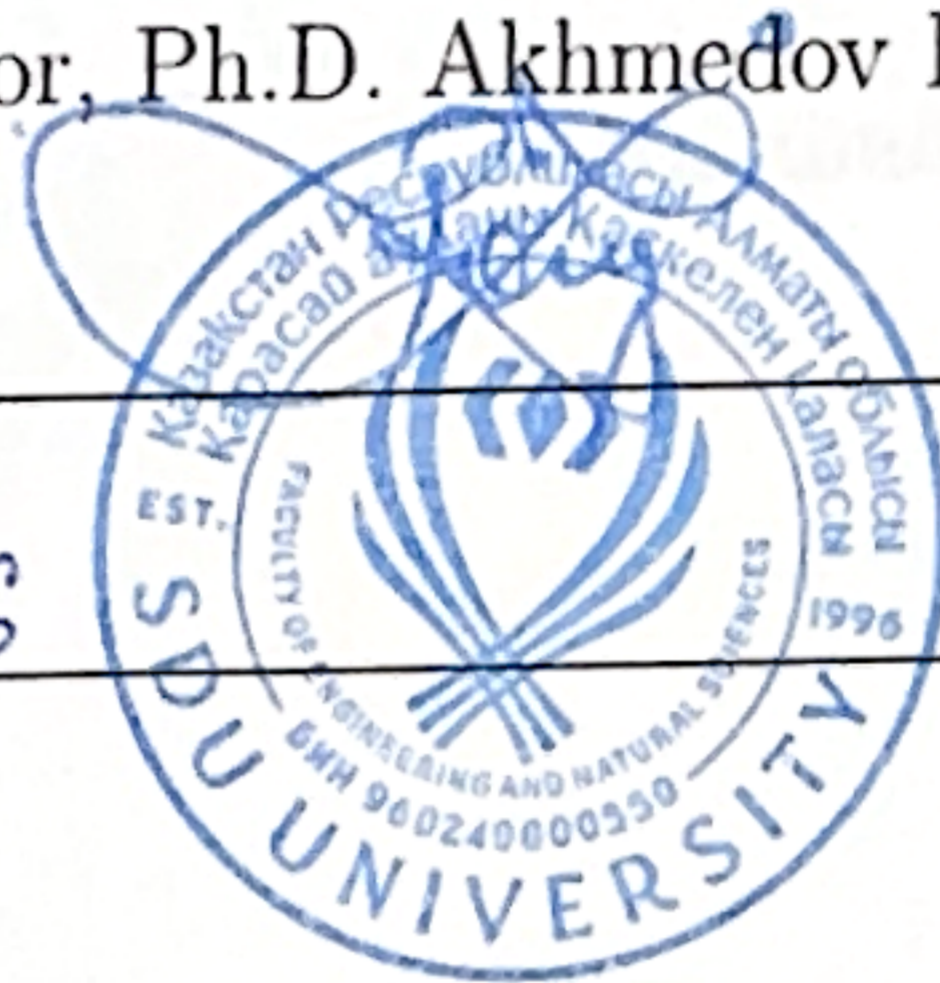


SDU University  
Faculty of Engineering and Natural Sciences  
Department of Mathematics and Natural Sciences

Dean of Faculty of Engineering and Natural Sciences

Assistant Professor, Ph.D. Akhmedov Ramis


« 18 » 06 2025





Topic of the thesis:

Conservative extensions of NIP non dp-minimal theories

Thesis submitted as part of the requirements for the award of the MSc in  
"7M05401-Mathematics"

Head of Department Bekbolat Bayan, PhD 

Academic Supervisor Baizhanov Bektur, Dr.Sc., Prof. 

Master student Rassayeva Nurzhainar. 

Kaskelen, 2025

Ministry of Science and Higher Education of the Republic of  
Kazakhstan  
SDU University



Nurzhainar Rassayeva

**Conservative extensions of NIP non dp-minimal  
theories**  
THESIS

Presented in Partial Fulfilment for the  
*Degree of Master of Science in Mathematics*  
(degree code: 7M05401)  
Department of Mathematics and Natural Sciences  
Faculty of Engineering and Natural Sciences

Supervisor: **Dr.Sc., Prof. Baizhanov B.S.**

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Master student \_\_\_\_\_

Kaskelen, 2025

# Declaration

I confirm that this work is entirely my own, and that all sources and materials used from other authors have been fully and properly acknowledged.

Nurzhainar Rassayeva

June 2025

# Acknowledgements

I would like to express my sincere gratitude to my supervisor, Professor B.S. Baizhanov, for his invaluable guidance, encouragement, and patience throughout the course of this research. His deep knowledge of model theory and his thoughtful insights have played a crucial role in shaping both the direction and quality of this work.

I am especially grateful for his support during moments of confusion and uncertainty. Despite the many times I asked seemingly simple or even silly questions, he never lost patience-thankfully sparing me and my fingers-allowing me to complete this thesis with all ten intact.

# Dedication

I would like to dedicate this diploma work to my dear family and teachers. Your support, encouragement, and belief in me helped me reach this important moment. Thank you for always being there for me.

# Abstract

This dissertation explores conservative extensions in the context of dependent theories (NIP) that are not dp-minimal. We study the model-theoretic properties of the special Cartesian product of ordered structures, focusing on how the dp-rank and definability of types behave under such constructions. It is shown that the product of two o-minimal or two weakly o-minimal structures yields a theory of dp-rank 2, which remains NIP but is no longer dp-minimal.

Further, we analyze the behavior of 1-conservative and n-conservative extensions in these theories. For o-minimal structures, 1-conservativity implies n-conservativity for all finite n, ensuring strong definability of types. However, for weakly o-minimal structures, this implication fails; we construct an explicit example where a 1-conservative extension does not extend to a 2-conservative one.

The results provide new insights into how model-theoretic complexity measured by dp-rank affects definability and extendability in NIP theories, contributing to the classification and understanding of dependent but non-dp-minimal structures.

# Аңдатпа

Бұл диссертацияда тәуелді (NIP) және  $dr$ -миинималды емес теориялар контекстінде консервативтік кеңейтулер зерттеледі. Біз реттелген құрылымдардың арнайы декарттық көбейтіндісінің модель-теориялық қасиеттерін зерттейміз, атап айтқанда, мұндай конструкциялар кезінде  $dr$ -дәреженің және типтердің анықталуының қалай өзгеретінін қарастырамыз. Екі  $o$ -минималды немесе екі әлсіз  $o$ -минималды құрылымның көбейтіндісі  $dr$ -дәрежесі екіге тең болатын теорияны береді, ол әлі де NIP болып қала береді, бірақ енді  $dr$ -миинималды емес.

Сонымен қатар, бұл теориялардағы 1-консервативтік және  $n$ -консервативті кеңейтулердің мінез-құлқы талданады.  $o$ -минималды құрылымдар үшін 1-консервативтілік барлық шекті  $n$  үшін  $n$ -консервативтілікті білдіреді, бұл типтердің күшті анықталуын қамтамасыз етеді. Алайда әлсіз  $o$ -минималды құрылымдарда бұл тұжырым орындалмайды; біз 1-консервативті кеңейту 2-консервативтіге жалғаспайтын нақты мысал келтіреміз.

Бұл нәтижелер  $dr$ -дәрежемен өлшенетін модель-теориялық күрделіліктің NIP теорияларында анықталу мен кеңейтілу мүмкіндігіне қалай әсер ететінін түсінуге жаңа көзқарас береді және тәуелді, бірақ  $dr$ -миинималды емес құрылымдарды классификациялауға өз үлесін қосады.

# Аннотация

В этой диссертации рассматриваются консервативные расширения в рамках зависимых теорий (NIP), которые не являются  $\text{dp}$ -минимальными. В данной работе исследуем модельно-теоретические свойства специального декартова произведения упорядоченных структур, сосредотачиваясь на поведении  $\text{dp}$ -ранга и определимости типов в таких конструкциях. Показано, что произведение двух  $\text{o}$ -минимальных или двух слабо  $\text{o}$ -минимальных структур приводит к теории с  $\text{dp}$ -рангом 2, которая остаётся в рамках зависимых теорий (NIP), но больше не является  $\text{dp}$ -минимальной.

Далее анализируется поведение 1-консервативных и  $n$ -консервативных расширений в этих теориях. Для  $\text{o}$ -минимальных структур 1-консервативность влечёт за собой  $n$ -консервативность для всех конечных  $n$ , что обеспечивает сильную определимость типов. Однако для слабо  $\text{o}$ -минимальных структур это не выполняется; мы приводим явный пример, в котором 1-консервативное расширение не продолжается до 2-консервативного.

Полученные результаты дают новое понимание того, как модельно-теоретическая сложность, измеряемая  $\text{dp}$ -рангом, влияет на определимость и возможность расширения в NIP-теориях, внося вклад в классификацию и изучение зависимых, но не  $\text{dp}$ -минимальных структур.

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# 1. Introduction

Model theory is a branch of mathematical logic that investigates the relationship between formal languages and the structures, or models, that interpret them. It provides powerful tools to analyze mathematical structures by focusing on definability, types, and classification of theories. Among the central topics in contemporary model theory are classification-theoretic properties such as stability, NIP (Not the Independence Property), and dp minimality, which provide a refined understanding of the complexity of theories out of classical notions of completeness and categoricity.

The class of NIP theories, introduced to capture tame combinatorial behavior, generalizes stable theories while excluding those with wild independence properties. Within this class, dp-minimal theories occupy a particularly important position, offering a framework in which definability behaves in a simple and well-structured manner. However, not all NIP theories are dp-minimal. Theories that are NIP but not dp-minimal particularly those with dp-rank greater than one present richer structural behavior and pose interesting questions about definability, types, and extensions.

A key model-theoretic concept explored in this dissertation is that conservative extensions. Given a model  $M$ , an extension  $N \succ M$  is called conservative if it does not introduce new definable sets over  $M$ . Studying the existence and structure of conservative extensions in unstable NIP theories that are not dp-minimal provides deep insight into how types behave in more complex logical settings. In particular, the definability of types and classification via cuts in linearly ordered structures play a central role in the construction and analysis of such extensions.

This dissertation focuses on the construction and analysis of conservative extensions of NIP theories that are not stable and not dp-minimal. We investigate how higher dp-rank interacts with definability, and we present explicit examples based on products of o-minimal and weakly o-minimal structures. We study the behavior of 1-types through the geometry of their realization sets, distinguishing between rational, quasi-rational, and irrational types, and analyze how these impact the existence of conservative extensions.

Through a combination of theoretical results and structured examples, this

work contributes to the broader understanding of how logical complexity, measured through stability and dp-rank, influences model-theoretic extensions. It provides a foundation for future research in the classification and manipulation of complex NIP structures beyond the dp-minimal case.

## 1.1 Model Theory in Kazakhstan

Together with set theory, algorithm theory, and proof theory, model theory has been one of the most rapidly developing areas of mathematical logic since the mid-20th century. A central challenge in this field is determining whether the fundamental theories of mathematical structures are decidable, and if so, formulating explicit procedures for such decision-making.

The history of model theory in Kazakhstan can be understood through the lens of the core problems, assumptions, and conceptual frameworks that have shaped the search for solutions. In this context, we outline the main objectives and important theorems and methodologies developed by leading mathematicians globally, alongside significant contributions made by Kazakhstani scholars.

We pay particular attention to the periods during which the field experienced the most rapid growth. At the same time, we acknowledge that there remain actively evolving areas of model theory beyond the scope of this review where Kazakhstani researchers have been underrepresented.

**Asan Dabsovich Taimanov** was a prominent Kazakh mathematician who significantly contributed to the fields of mathematical logic and model theory. Born in the Urda region of West Kazakhstan, overcame early hardships, including growing up in an orphanage, to become a leading figure in mathematics. He studied at the Ural State Pedagogical Institute before enrolling at Moscow State University, where he was mentored by notable mathematicians such as A.Y. Khinchin, P.S. Aleksandrov, and A.A. Lyapunov[1].

He was instrumental in the development of model theory in Kazakhstan, creating a strong mathematical school that influenced generations of scholars. He was recognized with several prestigious awards, including the Order of the Red Banner of Labour and the Order of the Patriotic War of 1st Class.

A.D.Taimanov mentored and influenced many students who went on to make significant contributions in mathematical logic and model theory. Their biographies and research are detailed below:

**A.I.Omarov:** One of A.D.Taimanov's students, Omarov worked extensively on filtered products, ultraproducts, and Boolean constructions in model theory and universal algebra. His research provided insights into axiomatizable classes and algebraic logic.

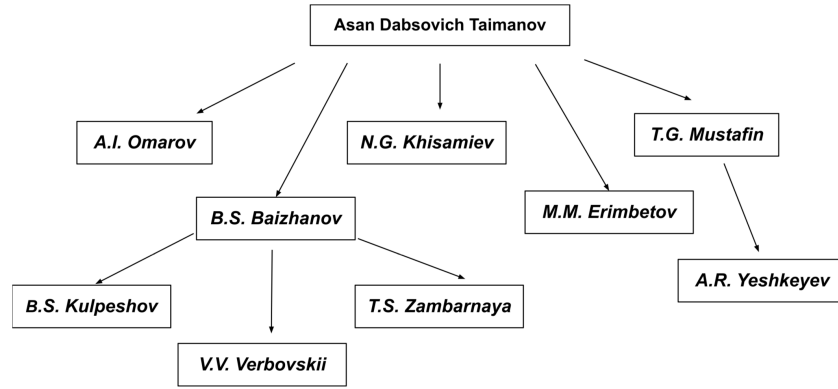


Figure 1.1: A.D.Taimanov: his school and its history

*A.I. Omarov, “Some applications of filtered products in model theory”, 1967 (supervisor A.D. Taimanov);*

*R.T. Kel’tenova, “Equational compactness of some algebras”, 1975 (supervisors T.I. Amanov and A.I. Omarov);*

*K.A. Nauryzbaev “Equational compactness of complete continuous distributive lattices”, 1987 (supervisor A.I. Omarov);*

Doctor of science dissertations:

*A.I. Omarov, “P-formulas and Boolean constructions in model theory and universal algebra”, 1992.*

**N.G. Khisamiev:** Closely associated with A.D.Taimanov through research, Khisamiev focused on constructive models and algorithmic problems in algebraic structures, particularly in constructive abelian groups. His contributions to the computable structure theory and recursive algebra were notable.

*N.G. Khisamiev, “Questions of elementary theory of lattice-ordered algebraic structures”, 1968 (supervisor A.D.Taimanov and co-supervisor A.I. Kokorin);*

*A.A. Konyrkhanova, “Computable decidable groups”, 2017 (supervisors N.G. Khisamiev and V. Roman’kov).*

Doctor of science dissertations:

*N.G. Khisamiev, “Constructive abelian groups”, 1990;*

**T.G. Mustafin:** A student or close colleague of A.D.Taimanov, Mustafin studied complete theories, their similarities, and axiomatizable classes of models. His work included foundational results in semantic similarity of structures and Johnson theories.

*T.G. Mustafin, “Some questions of axiomatizable classes of models”, 1971 (super-*

visor A.D. Taimanov);

K. Zhetpisov, “The number of automorphisms of models of superstable theories”, 1992 (supervisor T.G. Mustafin);

A.R. Yeshkeyev, “Johnson theories”, 1995 (supervisor T.G. Mustafin);

Doctor of science dissertations:

T.G. Mustafin, “Stable theories”, 1990;

**M.M. Erimbetov:** Likely a scientific successor or colleague of A.D.Taimanov, Erimbetov conducted research on software logic, definability of subset, and stability of formulas. His work contributed to the interaction between logic and computation.

M.M. Erimbetov, “On some questions of countable complete theories related to the concept of rank”, 1975 (supervisor A.D. Taimanov);

**B.S. Baizhanov:** A key figure with a broader academic network and influenced B.Sh.Kulpeshov, T.S. Zambarnaya, and V.V.Verbovskiy. Specialized in o-minimal structures, weakly o-minimal theories, and stability theory, expanding the understanding of definable sets and type spaces.

B.S. Baizhanov, “Spectral questions of totally transcendental theories of finite rank”, 1981 (supervisor A.D. Taimanov);

T.S. Zambarnaya, “Countable models of small dependent theories”, 2019 (supervisors B.S. Baizhanov and J.T. Baldwin).

B.Sh. Kulpeshov, “Quasiurbanik minimal structures”, 1997 (supervisor B.S. Baizhanov);

V.V. Verbovskiy, “Properties of functions definable in structures with the condition of minimality on the family of definable sets”, 2002 (supervisor B.S.Baizhanov)

Doctor of science dissertations:

B.S. Baizhanov, “Expansion of models of weakly o-minimal and stable theories”, 2008 (supervisor E.A. Palyutin);

**B.Sh. Kulpeshov:** A direct academic descendant of B.S.Baizhanov, Kulpeshov explored weakly o-minimal theories and the behavior of formulas in model-theoretic structures. His research analyzed 2-formulas and studied properties of ordered structures.

B.Sh. Kulpeshov, “Quasiurbanik minimal structures”, 1997 (supervisor B.S. Baizhanov);

**T.S. Zambarnaya:** Another scholar following the academic lineage of B.S. Baizhanov, Zambarnaya investigated weakly o-minimal theories and their applications in mathematical logic, focusing on stability and spectral properties.

*T.S. Zambarnaya, “Countable models of small dependent theories”, 2019 (supervisors B.S. Baizhanov and J.T. Baldwin).*

**V.V. Verbovskiy:** Also a student of B.S. Baizhanov, Verbovskiy studied ordered theories, weakly o-minimal groups. His research contributed to the classification of ordered groups in model theory.

*V.V. Verbovskiy, “Properties of functions definable in structures with the condition of minimality on the family of definable sets”, 2002 (supervisor B.S. Baizhanov)*

Doctor of science dissertations:

*V.V. Verbovskiy, “Methods of stability theory in the study of ordered structures”, 2010 (supervisor B.S. Baizhanov).*

**A.R. Yeshkeyev:** A continuation of T.G. Mustafin’s research lineage, Yeshkeyev focused on Johnson theories and logical structures related to positive model completeness. He contributed to the development of companion theories in mathematical logic.

*A.R. Yeshkeyev, “Johnson theories”, 1995 (supervisor T.G. Mustafin);*

*N.K. Shamatayeva, “The structure of convex existential prime Jonsson theories and their classes of models”, 2019 (supervisors A.R. Yeshkeyev and B. Poizat);*

Doctor of science dissertations:

*A.R. Yeshkeyev, “Structure of perfect positive Johnson theories”, 2010 (supervisor S.S. Goncharov);*

Naturally, a researcher would remain stationary and his work would not always reach a large number of specialists without continuous communication with a diversified team. The work of the researcher has to be acknowledged. Collaboration with mathematicians from other nations is crucial in this area. Additionally, we international relationships ties with Russia, the United States, France, and Great Britain.

As a conclusion, we point out that Kazakh mathematics has, to varied degrees, contributed to the quest for solutions to the major problems that mathematical logic professionals faced at the time.

## 2. Quantification Theory

### 2.1 First-order logic

First-order logic (FOL) is a fundamental concept in mathematics that provides a formal vocabulary for expressing and analyzing relationships between objects in the real world. It allows us to formulate mathematical statements using logical symbols, predicates, variables, and quantifiers. First-order logic is widely used in mathematics, computer science, linguistics, and artificial intelligence due to its expressive power and precise semantics.

Now let us define its components in detail.

The alphabet of first-order logic consists of seven main types of symbols, each serving a distinct role in the formation of logical expressions:

#### 1. Variables

Variables represent mathematical objects or elements within a domain. They do not have fixed values and can be quantified using quantifiers. They act as placeholders that can take on any value from the domain of discourse.

Examples:  $x, y, z, u, v, w$ .

#### 2. Constants

Constants refer to specific, fixed objects in the domain. They provide direct references to elements without ambiguity and serve as names for individual entities.

Examples:  $a, b, c$ , Mary, Green.

#### 3. Function Symbols

Function symbols describe mappings between objects in the domain. Each function symbol has an associated *arity*, indicating the number of arguments it takes. Functions return a single object from the domain as output.

Examples:

- $\text{fatherof}(\text{Mary})$  – returns the father of Mary.

- `isBlue(sky)` – returns an object indicating the color of the sky.

#### 4. Logical Connectives

Connectives allow the construction of complex logical expressions.

Common connectives include:

- Negation ( $\neg$  or NOT) – reverses the truth value of a statement.
- Conjunction ( $\wedge$  or AND) – true if both statements are true.
- Disjunction ( $\vee$  or OR) – true if at least one statement is true.
- Implication ( $\rightarrow$  or IMPLIES) – true if the first statement implies the second.
- Biconditional ( $\leftrightarrow$  or IF AND ONLY IF) – true if both statements have the same truth value.

#### 5. Punctuation Symbols

These symbols are used to structure logical expressions and clarify their syntactic structure.

Examples:

- Parentheses ( , ) – group expressions for clarity and determine the order of operations.
- Comma , – separates arguments in function and predicate applications.

#### 6. Predicate Symbols

Predicate symbols are used to express properties of objects or relationships between objects. A predicate applied to a term forms an atomic formula, which is either true or false depending on the interpretation.

Each predicate also has an arity (the number of arguments it takes).

Examples:

- `isHuman( $x$ )` – expresses that  $x$  is a human.
- `loves( $x, y$ )` – expresses that  $x$  loves  $y$ .
- `GreaterThan( $x, y$ )` – expresses that  $x > y$ .

#### 7. Quantifiers

Quantifiers define the scope of variables in a logical statement. They define whether a statement refers to all elements in the domain or merely to some.

## 2.2 Theories. Models

### Atomic Formulas

An **atomic formula** is a basic building block in logic. It's just a predicate applied to variables. For example:

$$P^n(x_1, \dots, x_n)$$

is an atomic formula if  $P^n$  is an  $n$ -ary predicate symbol in our signature  $\Sigma$ .

Think of it as a single fact like  $P(a, b)$  which says “ $a$  is related to  $b$  by  $P$ ”.

Let  $t_1, \dots, t_n$  be terms, and let  $P$  be an  $n$ -ary predicate symbol.

1. **Atomic formulas:** If  $t_1, \dots, t_n$  are terms, then

$$P(t_1, \dots, t_n)$$

is a formula (also called an atomic formula).

2. **Closure under logical connectives:** If  $\varphi$  and  $\theta$  are formulas, then the following are also formulas:

$$(\varphi \wedge \theta), \quad (\varphi \vee \theta), \quad (\varphi \rightarrow \theta), \quad \neg\varphi$$

3. **Closure under quantifiers:** If  $\varphi(x)$  is a formula with free variable  $x$ , then the following are also formulas:

$$\exists x \varphi(x), \quad \forall x \varphi(x)$$

**Definition 2.2.1.** A **structure**  $\mathfrak{M} = \langle M, \Sigma \rangle$  gives meaning to symbols in logic. It has:

- A domain  $M$  — the set of objects we're talking about
- An interpretation of each  $P^n \in \Sigma$  as a relation:

$$P^n(\mathfrak{M}) \subseteq M^n$$

### Truth in a Structure

Let  $M$  be a structure, and  $\bar{a} \in M^n$ . Let  $P^n$  be an  $n$ -ary predicate symbol. We define the satisfaction relation  $M \models \varphi$  inductively:

- $M \models P^n(\bar{a}) \iff \bar{a} \in P^n(M)$
- $M \models \varphi \wedge \theta \iff M \models \varphi$  and  $M \models \theta$

- $M \models \varphi \vee \theta \iff M \models \varphi \text{ or } M \models \theta$
- $M \models \neg\varphi \iff M \not\models \varphi$
- $M \models \varphi \rightarrow \theta \iff M \not\models \varphi \text{ or } M \models \theta$

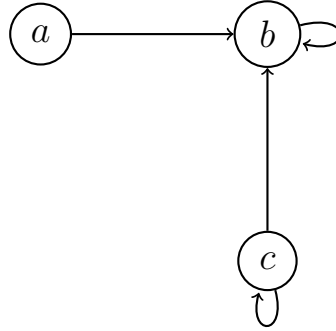
Now suppose  $\varphi(x, \bar{a})$  is a formula with free variable  $x$  and parameters  $\bar{a} \in M^k$ .

- $M \models \exists x \varphi(x, \bar{a})$  if there exists  $b \in M$  such that  $M \models \varphi(b, \bar{a})$
- $M \models \forall x \varphi(x, \bar{a})$  if for all  $b \in M$ ,  $M \models \varphi(b, \bar{a})$

**Example 2.2.1.** Let  $M = \{a, b, c\}$  and define one binary predicate  $P$  as:

$$P(\mathfrak{M}_1) = \{(a, b), (b, b), (c, b), (c, c)\}$$

We can draw this like a directed graph:



### Truth in a Model

We write  $\mathfrak{M} \models \varphi$  if the formula  $\varphi$  is true in  $\mathfrak{M}$ .

- $\mathfrak{M}_1 \models P(a, b)$  (true, since  $(a, b) \in P$ )
- $\mathfrak{M}_1 \not\models P(a, a)$  (false,  $(a, a) \notin P$ )

Now consider more complex formulas:

- $\forall x P(x, x)$  means “everyone relates to themselves” — false here.
- $\exists y \forall x P(x, y)$  means “there’s a universal receiver of arrows”.

Let us test the second one:

- Try  $y = b$ . Then  $P(a, b), P(b, b), P(c, b)$  — all true.
- So:  $\mathfrak{M}_1 \models \exists y \forall x P(x, y)$

A **substructure** has a smaller domain, keeping the relations restricted.

Let  $\mathfrak{M}_2$  have domain  $\{b, c\}$  and  $P(\mathfrak{M}_2) = \{(b, b), (c, b), (c, c)\}$ .

Some formulas that were true in  $\mathfrak{M}_1$  may no longer be true in  $\mathfrak{M}_2$ . This is where logical preservation becomes important — a major theme in model theory.

We have explored what atomic formulas are, how models interpret them, and how truth works in a structure. These concepts form the base of deeper topics like definability, types, and model comparison.

**Definition 2.2.2.** *A set of first-order sentences is considered a **theory**.*

A theory is a set of first-order logical sentences that describe a certain structure or system. These sentences are written in a formal language and define relationships between elements according to specific axioms.

**Definition 2.2.3.** *A **sentence** is a formula in first-order logic that contains no free variables.*

In other words, all variables in a sentence are bound by quantifiers such as  $\forall$  (for all) or  $\exists$  (there exists). Sentences express complete logical assertions that can be evaluated as true or false within a given structure.

**Definition 2.2.4.** *A theory  $T$  is said to be **countable** if it consists of only countably many sentences.*

This typically means that the set of axioms in the theory is either finite or can be put into a one-to-one correspondence with the natural numbers.

**Definition 2.2.5.** *A theory  $T$  is referred to as **categorical** if it's consistent and all of its models are isomorphic.*

This means that any two models of the theory are structurally identical (there exists a bijection preserving the structure).

**Definition 2.2.6.** *If any two models of a theory  $T$  for  $\mathfrak{M}$  have elementarily equivalent, then  $T$  is considered **complete theory**.*

A theory  $T$  is complete if for any two models  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$ , they are elementarily equivalent (they satisfy the same first-order sentences).

## 2.3 Predicates and Quantifiers

Predicates and quantifiers are fundamental tools in first-order logic for expressing propositions involving objects from a given domain.

**Predicates** can be viewed as functions that assign a truth value—either true or false—to a tuple of objects. They are used to state properties of objects or relationships among them.

In the language of first-order logic, a predicate is combined with terms to form a *formula*. For instance, if "P" is a unary predicate and "a" is a term, then "P(a)" is a formula that may be true or false depending on the interpretation.

**Quantifiers** allow us to express propositions involving all or some elements of the domain. They bind variables in formulas and are essential for constructing general statements. There are two standard types of quantifiers:

**Definition 2.3.1.** A *universal quantification* uses the symbol  $\forall$  and asserts that a given property holds for all elements of the domain.

The expression  $\forall x \varphi(x)$  means “for all  $x$  in the domain, the formula  $\varphi(x)$  is true.” Universal quantification is used when a statement is intended to be valid for every possible instance.

**Definition 2.3.2.** An *existential quantification* uses the symbol  $\exists$  and asserts that there is at least one element in the domain for which the property holds.

The expression  $\exists x \varphi(x)$  means “there exists an  $x$  in the domain such that  $\varphi(x)$  is true.” It indicates that the formula is satisfied by at least one object in the domain.

## 2.4 Quantifier Elimination

**Definition 2.4.1.**  $\mathfrak{M} = \langle M; \Sigma \rangle$ .  $\mathfrak{M}$  admits *Quantifier Elimination (QE)*, if for  $\psi(\bar{x})$  there exists *Quantifier Free* - formula  $\Theta(\bar{x})$ , such that  $\mathfrak{M} \models \forall \bar{x} [\psi(\bar{x}) \leftrightarrow \Theta(\bar{x})]$ .

**Theorem 2.4.1 (Tarski).** Let  $\mathfrak{M}$  be a structure of  $\Sigma$ .  $\mathfrak{M}$  admits *Quantifier Elimination*, if and only if when for any  $\psi_0(\bar{x}, y)$  *Quantifier Free* formula, there exists  $\Theta_1(\bar{x})$ , such that

$$\mathfrak{M} \models \forall \bar{x} [\exists y \psi_0(\bar{x}, y) \leftrightarrow \Theta_1(\bar{x})].$$

This means that if I understand how to eliminate a quantifier from a specific type of formula where the quantifier plays a crucial role, then I effectively know how to carry out quantifier elimination in such cases.

*Proof.* ( $\Rightarrow$ ) This is a particular case, because if it is available allowing for the elimination of quantifiers then we can take  $[\exists y \psi(\bar{x}, y)]$  as a formula  $\psi(\bar{x})$ [2].

( $\Leftarrow$ ) Suppose we know how to remove a quantor from formulas of a special kind. Take an arbitrary formula, but we know that any formula is equivalent in prenex normal form (PNF):

$$H \sim K \rightarrow \forall \mathfrak{M}; \Sigma \models H \Leftrightarrow \mathfrak{M} = \langle M; \Sigma \rangle \models K.$$

PNF is when all the quantifiers go into the head of the formulas.

$\psi(\bar{x}) \equiv Q_1y_1, Q_2y_2, \dots, Q_ny_n \quad \psi(\bar{x}, y_1, \dots, y_{n-1}, y_n)$ .  $\psi_0(\bar{x}, y)$  is Quantifier Free.

$\psi_{n-1}(\bar{x}, y_1, \dots, y_{n-1}) := Q_ny_n \quad \psi(\bar{x}, y_1, y_2, \dots, y_{n-1}, y_n)$ . In fact, there are two cases in the PNF, its  $Q_i = \{\exists, \forall\}$ . So *first case* is  $\exists y_n \psi(\bar{x}, y_{n-1}, y_n)$ , and *second case* is  $\forall y_n \psi(\bar{x}, y_{n-1}, y_n)$ . For instance:

1.  $\exists x(\psi(x) \vee \Theta(x)) \sim \exists x\psi(x) \vee \exists y\Theta(y)$ ;
2.  $\exists x\exists y(\psi(x) \wedge \Theta(y)) \sim \exists x\psi(x) \wedge \exists y\Theta(y)$ ;
3.  $\forall x(\psi(x) \wedge \Theta(x)) \sim \forall x\psi(x) \wedge \forall y\Theta(y)$ ;
4.  $\forall x\forall y(\psi(x) \vee \Theta(x)) \sim \forall x\psi(x) \vee \forall y\Theta(y)$ ;
5.  $\neg\forall\psi(x) \sim \exists x\neg\psi(x)$ ;
6.  $\neg\exists\psi(x) \sim \forall x\neg\psi(x)$ .

$\psi_i(\bar{x}, y_{n-1})$  For, in fact, the negation of the quantifiers is itself a non - quantifiers:  $\neg\exists y_n\neg\psi_0(\bar{x}, y_{n-1}, y_n) \sim \neg\exists y_n\neg\psi_1(\bar{x}, y_{n-1}) \sim \neg\exists y_n\neg\psi_1(\bar{x}, y_{n-1})$ . It turns out that to make it  $Q$  Free it is necessary step by step to remove the quantizer in each formula. And if it has this property, using this method step by step it is possible to reach the  $Q$  Free -formula  $\psi(\bar{x}) \equiv \Theta(\bar{x})$ .

□

#### Example 2.4.1. $\langle Q; =, < \rangle$ admits $QE$

Let us consider a formula  $\Theta(x_1, x_2, y)$  and show that any such formula can be removed from a quantor. Here we have to say right away that our atomic formula consists of an equal sign and a lesser sign.

$$\begin{aligned} x_1 = x_1, \quad x_2 = x_2, \quad y = y, \quad x_1 = x_2 \\ x_1 < x_2, \quad x_2 < x_1, \quad x_1 < y, \quad x_1 = y \\ x_2 = y, \quad x_2 < y, \quad y < x_1, \quad y < x_2 \end{aligned}$$

We get a formula of logic of statements. And we know that any formula of statement logic can be reduced to disjunctive normal form.

$\Theta(p_1, p_2, \dots, p_{12}) \equiv \text{DNF}$ . It is possible to normalize a logical formula by using the disjunctive normal form (DNF). To put it another way, a logical formula is said to be in disjunctive normal form if all its members are disjunctions of elementary conjunctions or their negations. Here, by changing the formula  $\neg(x = y)$  to  $((x < y) \vee (x > y))$  and the formula  $\neg(x < y)$  to  $((x = y) \vee (x > y))$  we may eliminate negations. After that, we must use distributivity to return to the disjunctive normal form with extra terms but no negations. At this point, we

must benefit from the fact that the existence quantor can be reconfigured using the standard disjunction.

$$\begin{aligned} \exists x\varphi(x) \vee \exists y\Theta(y) &\sim \exists x(\varphi(x) \vee \Theta(x)) \\ \forall x_1\forall x_2 [\exists y (x_1 < y \wedge y < x_2) \rightarrow x_1 < x_2] &\sim \\ \forall x_1\forall x_2 [x_1 < x_2 \rightarrow \exists y (x_1 < y \wedge y < x_2)] & \end{aligned}$$

It turns out that any formula  $\Theta(x_1, x_2, y)$  is equivalent to some  $\Theta_1$  only from  $x_1, x_2$ . The conclusion is that according to Tarski's criterion  $Q$  admits Quantifier Elimination. When we have a structure  $\mathfrak{M} = \langle M; \Sigma \rangle$  and if it is  $|\Sigma|$  - infinite, we will say that

$\mathfrak{M}$  - admits  $QE$ , if and only if for any  $\Sigma_0 \subset_{\text{finite}} \Sigma$ ,  $\langle \mathfrak{M}; \Sigma_0 \rangle$  admits Quantifier Elimination.

*Proof.*  $\mathfrak{M}$  - admits Quantifier Elimination,  $\mathfrak{M} = \langle \mathfrak{M}; \Sigma_0 \rangle$  fix  $\Sigma_0 \subset \Sigma$ . We take the formula  $\psi(\bar{x})$  of the signature  $\Sigma_0$ , where  $\mathfrak{M} \models \psi(\bar{x})$ , but the prenex formulation does not add any new predicates, we just transfer the equivalent formula to it.

$$\mathfrak{M} = \langle \mathfrak{M}; \Sigma_0 \rangle \models \forall \bar{x} \psi(\bar{x}) \equiv Q_1 y_1, Q_2 y_2, \dots, Q_n y_n \Theta(\bar{x}, \bar{y})$$

The idea is that if it is infinite, and for every finite it will allow the elimination of quantifiers, then it will also allow it in general. Because we use formulas and can always find an equivalent replacement for them in this form.

□

# 3. Properties of classes of theories

## 3.1 IP and NIP

**Definition 3.1.1.** Let  $\varphi(x, y)$  has IP (the Independence Property) if there are  $(a_i : i < \omega)$  for any  $\forall n < \omega$  and for any sequences  $\forall \epsilon_1, \epsilon_2, \dots, \epsilon_n$  such that  $\epsilon_i \in \{\neg, \emptyset\}$   $\exists x \wedge \epsilon_i \varphi(x, a_i)$ .

That is, the presence of a tree implies that the property is **independent**. To gain a clearer understanding of this concept, we now turn to Figure(3.1).

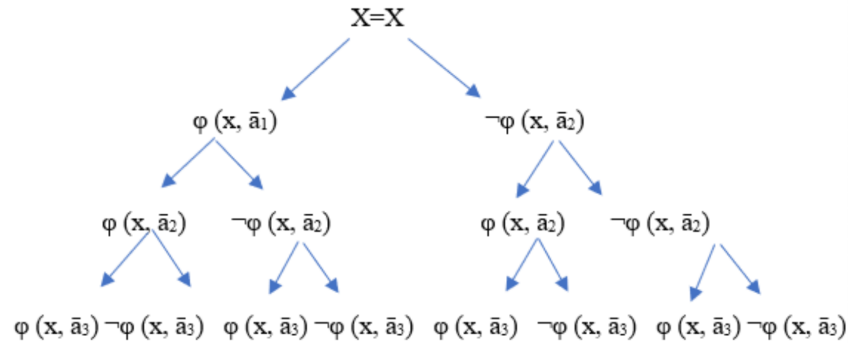


Figure 3.1: Tree for the independence property

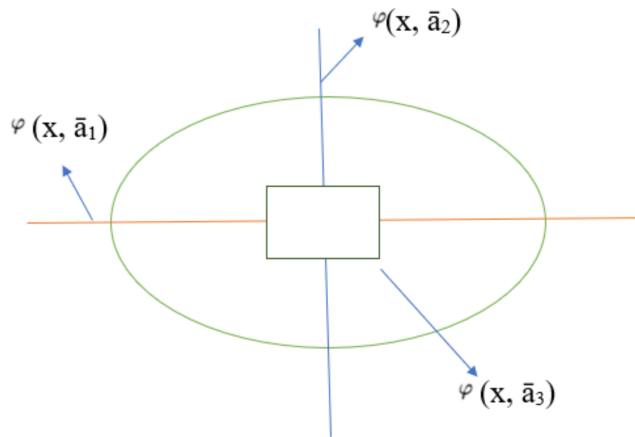


Figure 3.2: How trees are built

A tree of formulas begins with a simple statement such as  $x=x$ , which is then split into two non-overlapping parts. Each formula in the tree undergoes a similar

process. But what exactly does “split” mean in this context? To understand this, we can refer to Figure(3.2).

When we split the original formula  $x=x$ , we create two branches: one where the formula is considered true and another where it is considered false. If we continue this process splitting each resulting formula again we end up with four branches, as shown in Figure(3.1). This process can be repeated indefinitely, generating an infinite number of branches that are independent from one another.

A broader, more general structure of this branching process can be seen in Figure(3.3). The final outcome depends on the particular path we choose to follow through the tree.

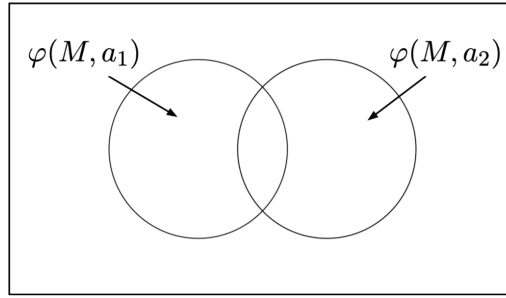


Figure 3.3: Structure of the model

Each line finally has an element satisfying the line

$$\models \exists x (\varphi(x, \bar{a}_1) \cap \varphi(x, \bar{a}_2)) \quad \models \exists x (\varphi(x, \bar{a}_1) \cap \neg\varphi(x, \bar{a}_2))$$

**Example 3.1.1.** Let us consider the structure  $\langle \mathbb{N}, =, \cdot \rangle$  where we want to display the existence of a formula that identifies when a number is divisible by 2. Define the formula:

$$\varphi(x, y) := \exists z(z \times y = x, x \neq y)$$

This formula says that  $x$  is divisible by  $y$ , and  $x$  is not equal to  $y$  ( $x \div y = z$  for some  $z$ ).

Now, consider the prime numbers  $(2, 3, 5, 7, \dots, n)$ . We know that:

1.  $\varphi(4, 2)$  is true, since  $2 \cdot 2 = 4$ ,
2.  $\varphi(5, 2)$  is false, since 5 is not divisible by 2.

As an example,  $\varphi(8, 2)$  is also true. In fact,  $\varphi(8, 2) \Rightarrow \exists z(\varphi(4, 2)(4 \cdot 2 = 8))$ .

This creates an infinite tree we can continue branching forever, since we have an unlimited number of constants to build new formulas. This illustrates the concept of independence.

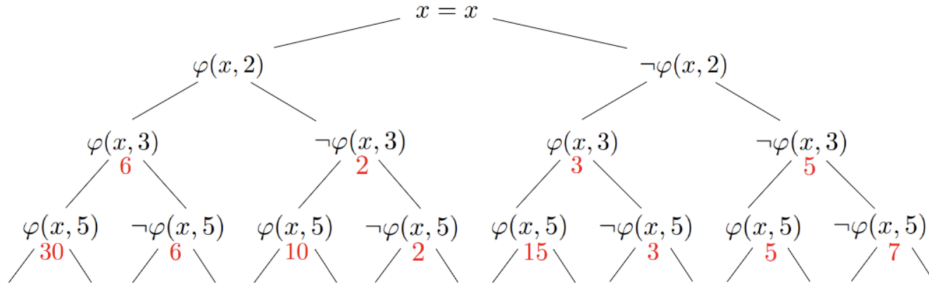


Figure 3.4: Tree for the independence property

In mathematical logic and model theory, the term NIP (Not the Independence Property) [3, 4] refers to structures where certain formulas do not display this kind of independence. In other words, NIP structures prevent formulas from generating such infinitely branching, independent behavior.

**Definition 3.1.2.** A theory is said to be **NIP** (Not the Independence Property, also known as dependent) if no formula within the theory has the Independence Property.

A formula is said to have the IP if there exists a sequence of elements such that the truth value of the formula can vary depending on the membership of indices in a subset. In simpler terms, the formula is capable of encoding arbitrary subsets of natural numbers essentially "choosing" freely which combinations are true or false based on the structure of the subset.

A theory is called NIP (Not the Independence Property) if no formula within the theory behaves this way-that is, none of the formulas allow such arbitrary patterns of truth and falsehood.

## 3.2 OP on basic facts

**Definition 3.2.1.** If no formula possesses the order property, a theory is called stable.

**Definition 3.2.2.** A formula  $\varphi(x, y)$  is said to have the **order property** if there exist sequences  $(a_i : i < \omega)$  and  $(b_j : j < \omega)$  such that

$$\models \varphi(a_i, b_j) \Leftrightarrow i < j.$$

Let us now take a closer look at Figure(3.5). In this diagram, the green arrows represent paths where the formula is true, while the red arrows indicate false outcomes. Based on these branches, we can construct general patterns or "views" of how the formulas behave. Below are a few examples that illustrate this idea more clearly:

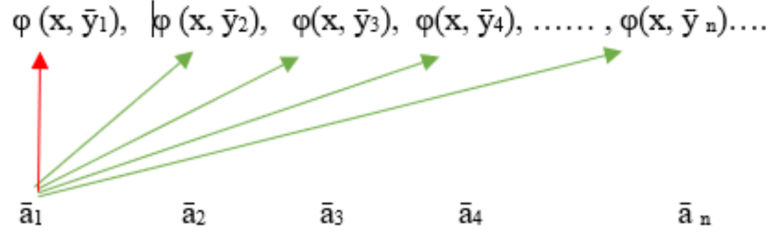


Figure 3.5: The concept of an ordered property

1.  $\neg\varphi(\bar{a}_1, y_1) \cap \neg\varphi(\bar{a}_2, y_1) \cap \neg\varphi(\bar{a}_2, y_2)$
2.  $\neg\varphi(\bar{a}_3, y_1) \cap \neg\varphi(\bar{a}_3, y_2) \cap \neg\varphi(\bar{a}_3, y_3)$ .

How does the definition we introduced at the beginning connect to all of this? Figure(3.6) illustrates exactly what we've been discussing throughout it brings together the key ideas we've been building up to.

+	+
<b>SOP+NIP</b>	<b>SOP+IP</b>
<b>NSOP+NIP</b>	<b>NSOP+IP</b>
-	+

Figure 3.6: Concept of the classes

*Proof.* We aim to clarify the relationship between the Order Property (OP), the Strict Order Property (SOP), and the Independence Property (IP).

We have the following equivalences and implications:

$$OP \iff SOP \cup IP, \quad \neg OP \iff NSOP \cap NIP$$

That is, a theory has the OP if and only if it has either the SOP or the IP. Conversely, a theory is NOP (Not the Order Property) if and only if it is both NSOP (Not the Strict Order Property) and NIP (Not the Independence Property).

Moreover, we know the following implications:

$$SOP \Rightarrow OP, \quad IP \Rightarrow OP$$

Additionally, we have these characterizations:

$$OP \cap NIP \iff SOP, \quad OP \cap NSOP \iff IP$$

These equivalences help classify the combinatorial behavior of formulas in a theory, depending on which dividing lines (OP, IP, SOP) they satisfy.

□

### 3.3 SOP and NSOP properties and examples

Let us now turn our attention to the next class SOP (Strict Order Property) and examine it more closely. This concept plays a significant role in various theoretical frameworks and can help us understand deeper structural properties of models.

**Definition 3.3.1.** A formula  $\varphi(x, y)$  has strict order property if there are  $(\bar{a}_1, \dots, \bar{a}_n)$  where  $n > \omega$ ,

$$\neg\varphi(M, \bar{a}_n) \subset \neg\varphi(M, \bar{a}_{n+1}).$$

A formula  $\varphi$  is said to have the SOP if there exist a sequence of elements such that the set of instances where  $\varphi$  is not true for one part of the sequence is a proper subset of the set where  $\varphi$  is not true for the next part. In other words, the truth of the formula creates a strict, growing hierarchy across the sequence.

**Definition 3.3.2.**  $SOP \iff$  there exists  $\varphi(x, y) \exists (\bar{a}_1, \dots, \bar{a}_n \dots)$  then

$$\varphi(M, \bar{a}_n) \subset \varphi(M, \bar{a}_{n+1}).$$

In this definition, the structure is presented in an increasing form—that is, each step in the sequence builds on the previous one, creating a strictly expanding pattern.

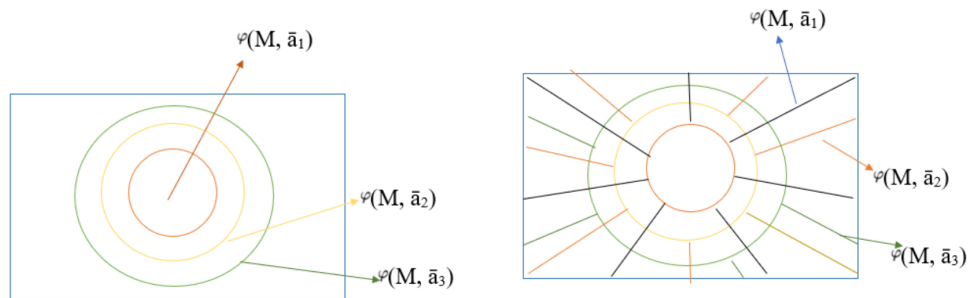


Figure 3.7: Visual representation of the Strict Order Property (SOP)

By examining Figure(3.7), we observe an infinitely increasing chain of definable sets, each one contained within  $\varphi(M, \bar{a}_{n+1})$ . This illustrates the growing nature of the sets as the sequence progresses. However, we can also consider the opposite direction where the chain decreases instead of increases. Let us now explore this decreasing form.

**Example 3.3.1.** Consider the structure  $\langle Q, =, < \rangle$ , and let the formula be  $\varphi(x, y); = x < y$ . We can express this as:

$$\begin{aligned}\varphi(Q, 1) &= \{x \in Q \mid x < 1\} = (-\infty, 1) \\ \varphi(Q, 2) &= \{x \in Q \mid x < 2\} = (-\infty, 2) \\ &\vdots \\ \varphi(Q, n+1) &= \{x \in Q \mid x < n+1\} = (-\infty, n+1)\end{aligned}$$

This forms an increasing chain of definable sets, illustrating how the formula  $\varphi(x, y)$  defines initial segments of  $Q$  based on the parameter  $y$ .

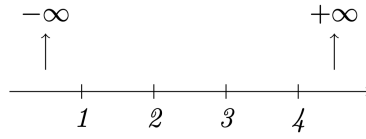


Figure 3.8: Initial segments in  $(Q, <)$  via  $x < y$ .

**Definition 3.3.3.** A theory is called *NSOP* (Not the Strict Order Property) if none of its formulas exhibit the strict order property.

## 3.4 Stability

A field of mathematical logic that studies the properties and structure of stable theories.[\[3\]](#)

**Definition 3.4.1.** The theory  $T$  is stable if and only if all types are definable.

**Definition 3.4.2.** A formula  $\varphi(x, y)$  has the order property if there are elements  $a_0, a_1, \dots$  and  $b_0, b_1, \dots$  such that for all  $i, j \in \omega$

$$\models \varphi(a_i, b_j) \quad \text{if and only if} \quad i < j.$$

**Theorem 3.4.1.** A theory is stable if no formula has the order property  $\models \varphi(a_i, b_j) \Leftrightarrow i < j$ . Also here doesn't exist  $2^\omega$  branching tree.

$$\neg OP \iff (\neg OP \wedge \neg IP) \iff \text{stable}$$

**Fact 3.4.1.** Stable formulas are NIP and NSOP.

# 4. DP-minimal properties and examples

First let us look at what is contained in DP-minimal:

1. o-minimal
2. weakly o-minimal

## 4.1 O-minimal

The concept of *o-minimality* was introduced and developed by L. van den Dries, A. Pillay, C. Steinhorn [5].

**Definition 4.1.1.** *A linearly ordered structure  $\mathfrak{M} = \langle M, <, \dots \rangle$  is said to be o-minimal (here, 'o' stands for ordered) if the realization of any formula of  $M$  is one free variable is a disjoint union of finitely many open intervals and points.*

In everyday terms, o-minimality says that no matter how you slice the universe  $M$  with a single-variable definable condition, you can only carve out simple “chunks” intervals or isolated points with no wild oscillations or fractal-like behavior. Also, every definable subset of the line is a finite union of sets of the form  $(a, b)$ ,  $(a, \infty)$ ,  $(-\infty, b)$ , or  $\{c\}$ .

**Example 4.1.1** (Real closed fields). *The field of real numbers  $R$  equipped with its usual order, addition and multiplication,*

$$(R, +, \cdot, 0, 1, <),$$

*is o-minimal. Indeed, any real-polynomial inequality*

$$f(x, \bar{a}) > 0; \quad f(x, \bar{a}) = a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_k)$$

*cuts  $R$  into finitely many sign-constant intervals separated by the finitely many real roots  $\alpha_1, \dots, \alpha_k$  plus any isolated zeros. More generally, any real closed field inherits this property: quantifier elimination in the language of ordered rings*

forces every one-variable definable set to decompose into finitely many intervals and points.

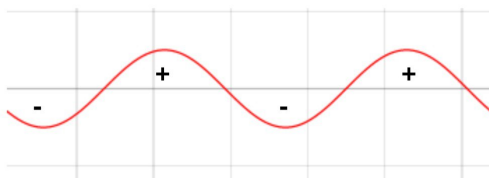


Figure 4.1: Sign changes of a real polynomial.

A key result by D. Marker and C. Steinhorn highlights a powerful consequence of o-minimality: it guarantees a strong form of conservativity under elementary extensions.

**Theorem 4.1.1.** *Let  $\mathfrak{M} \prec \mathfrak{N}$  be two models of o-minimal theory. If  $\mathfrak{M} \prec_{1,c} \mathfrak{N}$ , then  $\forall n < \omega$   $\mathfrak{M} \prec_{n,c} \mathfrak{N}$ ,  $\mathfrak{N}$  is  $n$ -conservative extension of  $\mathfrak{M}$ .*

## 4.2 Weakly o-minimal

Notions of *weakly o-minimal* was presented by M.A.Dickmann, D. Macpherson, D. Marker, and C.Steinhorn[6].

**Definition 4.2.1.** *A linearly ordered structure  $\mathfrak{M} = \langle M, <, \dots \rangle$  is weakly o-minimal if the realization of each formula in one free variable is a disjoint union of finitely many convex subsets.*

Here a subset  $C \subseteq M$  is called convex if whenever  $a, b \in C$  and  $c \in M$  satisfies  $a < c < b$ , then  $c \in C$ . Equivalently, a convex definable piece looks like an interval (perhaps closed at one or both ends), but a single definable set may glue together several such convex chunks interval-like pieces that need not be open. Every o-minimal structure is weakly o-minimal, but the converse fails dramatically:

**Theorem 4.2.1** (B.Baizhanov). *There is pair of models  $\mathfrak{M}, \mathfrak{N}$  for some weakly o-minimal theory  $T$ , such that  $\mathfrak{M} \prec_{1,c} \mathfrak{N}$ , but  $\mathfrak{M} \not\prec_{2,c} \mathfrak{N}$ .*

## 4.3 DP-rank has more than one

**Definition 4.3.1.** *A theory has dp-rank  $\geq n$  if there are formulas  $\varphi_1(x, y), \dots, \varphi_n(x, y)$  and mutually indiscernible sequences  $(a_i^1)_{i < \omega}, \dots, (a_i^n)_{i < \omega}$  such that for*

any function  $\sigma : \{1, \dots, n\} \rightarrow \omega$ , the type

$$\left\{ \varphi_k \left( x, a_{\sigma(k)}^k \right) : k \leq n \right\} \cup \left\{ \neg \varphi_k \left( x, a_i^k \right) : i \neq \sigma(k), k \leq n \right\}$$

is consistent. A theory is **dp-minimal** if it has **dp-rank=1** [7, 8].

Let us consider Definition 4.3.1, which introduces the concept of *dp-rank* a measure of complexity in model theory, particularly in the study of dependent theories. Informally, the dp-rank of a theory tells us how many independent sequences are needed to describe the dividing behavior of formulas in that theory.

If a theory has **dp-rank=1**, this means that its complexity can be captured using a single formula and a single sequence of mutually indiscernible elements. Such theories are called **dp-minimal**. In other words, a dp-minimal theory is one that requires the minimal amount of structure to describe dividing.

Now, let us consider a theory of **dp-rank=2**. In this case, there exist formulas  $\varphi_1(x, \bar{y})$  and  $\varphi_2(x, \bar{z})$ , along with two sequences:

$$(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n, \dots), \quad n \in \mathbb{N}, \quad \text{and} \quad (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_m, \dots), \quad m \in \mathbb{N},$$

which are mutually indiscernible and demonstrate independence in two dimensions.

Let us visualize this in geometric terms (see Figure 4.1). Suppose we define a function

$$\sigma : \{1, 2\} \rightarrow \mathbb{N}$$

with values  $\sigma_1 = 3$  and  $\sigma_2 = 2$ . This assignment gives rise to the following consistent conjunction:

$$\left( \varphi_1(x, \bar{a}_3) \wedge \varphi_2(x, \bar{b}_2) \right) \wedge \bigwedge_{i \neq 3} \neg \varphi_1(x, \bar{a}_i) \wedge \bigwedge_{j \neq 2} \neg \varphi_2(x, \bar{b}_j)$$

This formula selects exactly one element from each sequence (indexed by  $\sigma$ ), while excluding all others. Such a configuration illustrates the need for two independent sequences to define the behavior of formulas in the theory. Therefore, the theory has **dp-rank=2**, and is **not dp-minimal**.

We now consider the case where a theory has **dp-rank=3**. In this setting, there exist three formulas

$$\varphi_1(x, \bar{y}), \quad \varphi_2(x, \bar{y}), \quad \varphi_3(x, \bar{y}),$$

each depending on the variables  $x$  and  $\bar{y}$ , and satisfying the necessary conditions for defining dp-rank.

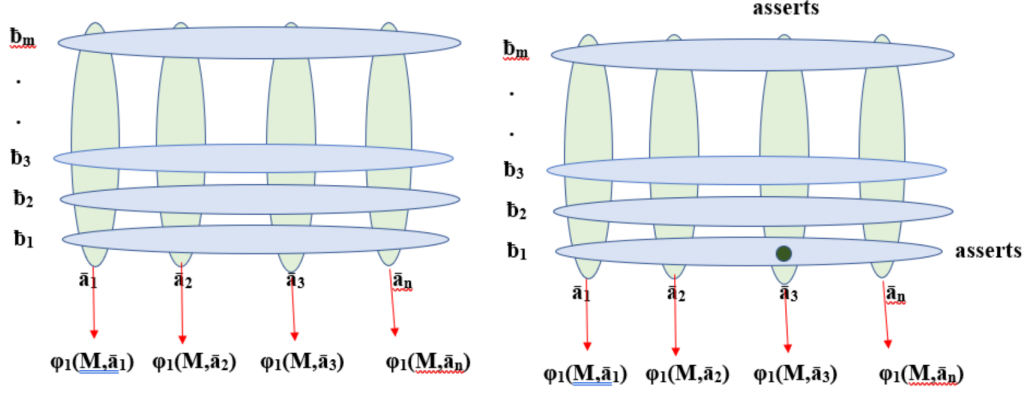


Figure 4.2: Dimension two

Additionally, we are given three sequences:

$$(a_{1i})_{i < \omega}, \quad (a_{2i})_{i < \omega}, \quad (a_{3i})_{i < \omega},$$

which are **mutually indiscernible**.

But what does it mean for sequences to be **mutually indiscernible**?

**Definition 4.3.2.** A sequence  $(a_1, a_2, a_3, \dots)$  is called *indiscernible* if it always looks the same to the model.

This means that for any number  $k$ , if you pick  $k$  elements from the sequence in increasing order, then for any formula  $\varphi(x_1, \dots, x_k)$ , the truth of the formula does not change.

Formally, for any increasing sequences of indices

$$i_1 < i_2 < \dots < i_k \quad \text{and} \quad j_1 < j_2 < \dots < j_k,$$

we have:

$$M \models \varphi(a_{i_1}, \dots, a_{i_k}) \iff M \models \varphi(a_{j_1}, \dots, a_{j_k})$$

**Example 4.3.1.** Let  $\varphi(x, y)$  be a formula.

If:

$$M \models \varphi(a_3, a_5),$$

then by indiscernibility, we also have:

$$M \models \varphi(a_1, a_2) \quad \text{and} \quad M \models \varphi(a_{100}, a_{1000000}).$$

As long as the order is the same (the first index is smaller than the second), the formula gives the same truth value.

This means that the formulas  $\varphi_1, \varphi_2, \varphi_3$  do not distinguish between rearrange-

ments within each sequence when evaluated in the presence of the others. That is, the truth of any instance of a formula remains unchanged regardless of how we reorder the elements in the sequences, as long as mutual indiscernibility is preserved.

Now, for any function

$$\sigma : \{1, 2, 3\} \rightarrow \omega,$$

we can construct a consistent set of formulas of the form:

$$\{\varphi_1(x, a_{1\sigma(1)}) \wedge \varphi_2(x, a_{2\sigma(2)}) \wedge \varphi_3(x, a_{3\sigma(3)})\} \cup \{\neg\varphi_k(x, a_{ki}) \mid i \neq \sigma(k), k = 1, 2, 3\}$$

In conclusion, referring back to Definition 4.3.1, we see that if we can construct such formulas based on sequence and save consistency under arbitrary assignments of indices via the function  $\sigma$ , then the theory is said to have **dp-rank=3**.

This idea can be visualized geometrically as a three-dimensional space, where each dimension corresponds to one of the sequences, and the height in each direction reflects the specific choice of index. Each consistent type represents a distinct point in this 3D configuration.

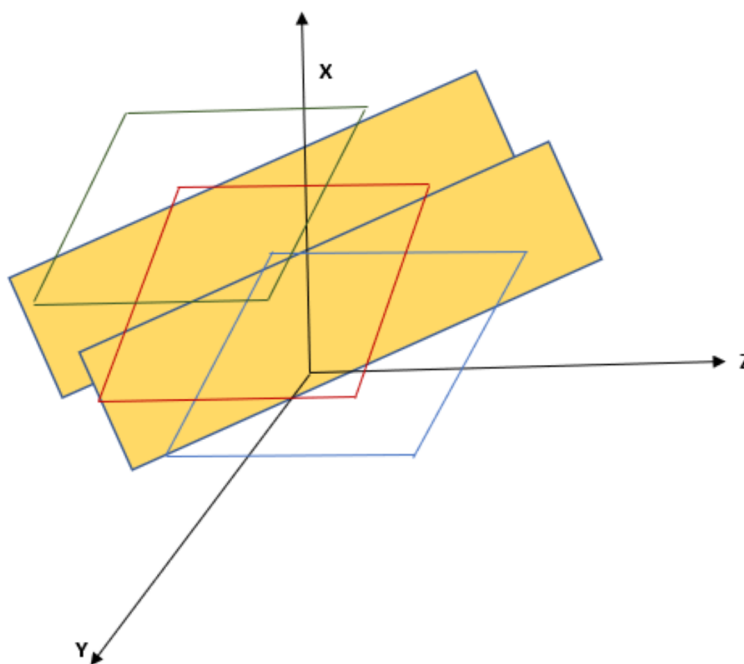


Figure 4.3: Dimension three

We consider the relation  $E_3(a; b, c)$ , which expresses that for tuples  $(a_1, b_1, c_1)$ , the relation implies  $c = c_1$ . This means that the elements lie on the same plane.

Let us work within the basic space  $\mathbb{R}^3$ , where each element is described by three coordinates. In this context, we take planes represented in Figure(4.2) as yellow surfaces that run parallel to the  $Y$ -axis. These planes represent equivalence classes based on the relation  $E_3$ , and each such class corresponds to a distinct "slice" of

the space.

To illustrate this with a simple example, imagine the walls of a house:

- Walls running from **north to south** (aligned along the Y-axis) form one equivalence class.
- **Ceilings and floors** constitute a second, distinct equivalence class.
- Walls running from **west to east** (aligned along the X-axis) make up a third class.

Any two walls from the same class will intersect the floor and ceiling along the same lines. Furthermore, each wall shares common intersections with both the floor and ceiling. These mutual intersections highlight that we are working in a space with three dimensions, where each equivalence class captures alignment along one of the three coordinate directions.

This visualization helps us understand how equivalence relations can be used to partition  $\mathbb{R}^3$  into geometrically meaningful classes.

# 5. Definability types

## 5.1 Cuts and Classification of 1-types in linearly ordered structures

**Definition 5.1.1.** A *type* is a locally consistent set of formulas.

**Definition 5.1.2.** It is *consistent* if every finite part of it is satisfiable.

**Definition 5.1.3.** If there exists an element  $x$  that satisfies all formulas in the complete type, then we say that the type is *realized*.

Let  $\mathfrak{M} = \langle M; =, <_1, <_2, \dots \rangle$  be a structure with possibly multiple linear orders or other relations.

A **1-type** over the model  $\mathfrak{M}$ , denoted  $p \in S_1(M)$ , is a set of formulas

$$p = \{\varphi_i(x, \bar{a}) \mid i \in I, \bar{a} \in M\}$$

such that for every finite subset  $\{\varphi_1(x), \dots, \varphi_n(x)\} \subseteq p$ , the conjunction is satisfiable in  $\mathfrak{M}[9]$ :

$$\mathfrak{M} \models \exists x (\varphi_1(x) \wedge \varphi_2(x) \wedge \dots \wedge \varphi_n(x)).$$

We say that the type  $p$  is **realized in  $\mathfrak{M}$**  if there exists an element  $b \in M$  such that for every formula  $\varphi(x, \bar{a}) \in p$ , we have:

$$\mathfrak{M} \models \varphi(b, \bar{a}).$$

### Kinds of 1 – types in ordered structures.

We now focus on the special case when the structure  $\mathfrak{M}$  is linearly ordered.

In such conditions, the geometry of types becomes particularly meaningful. Specifically, it is well known that in linearly ordered theories including weakly o-minimal or more generally NIP theories the realization set of a non-algebraic 1-type over a set  $A$  forms a **convex subset** of the domain  $M$ .

This observation allows us to connect the notion of types with the classical

model-theoretic concept of a *cut*.

This convex set defines a **cut**-a partition of  $M$  into two convex subsets  $C$  and  $D$ , such that:

- Every element of  $C$  is less than every element of  $D$ ,
- $C \cup D = M$ ,
- The type  $p(x)$  is realized either fully within one of them or precisely at the boundary between them.

Based on the nature of this boundary, we distinguish 1-types by how their cuts “sit” relative to the elements of  $M$ :

- **Rational type:** One side of the cut has an actual endpoint in  $M$ :

Either  $\max C$  exists in  $M$ , or  $\min D$  exists in  $M$ .

Equivalently,  $p$  is defined by a simple formula such as  $x > a$  or  $x < b$  for some  $a, b \in M$ . These types are both realized and definable.



- **Quasi-rational type:** Neither side contains an element of  $M$ , but exactly one boundary (either the “upper edge” of  $C$  or the “lower edge” of  $D$ ) is definable by a parameterized formula. Such a type can be described internally (hence is definable), yet remains unrealized in  $M$ .



- **Irrational type:** Neither boundary of the cut is definable over  $M$ . No formula with parameters in  $M$  isolates the gap from above or below. These types are non-definable and cannot be realized in any small (1-conservative) extension of  $M$ .



This classification of types via cuts plays a central role in the analysis of **definability**, and the construction of **conservative extensions**, especially in *unstable*, *non-dp-minimal* NIP theories, which are the focus of this thesis.

## 5.2 Conservative extension of NIP non dp-minimal theories

**Definition 5.2.1.**  $q \in S(A)$ ,  $q(\bar{x})$  is definable, if for any  $\varphi(\bar{x}, \bar{y})$  there is  $d_\varphi(\bar{y}, \bar{a})$ ,  $\bar{a} \in A$  such that for any  $\bar{b} \in A$ ,  $\varphi(\bar{x}, \bar{b}) \in q$  iff  $\mathfrak{M} \models d_\varphi(\bar{a}, \bar{b})$ .

In other words, for each formula  $\varphi(\bar{x}, \bar{y})$ , there exists a *control formula*  $d_\varphi$  that determines whether  $\varphi(\bar{x}, \bar{b})$  belongs to the type  $q$  - this happens exactly when  $d_\varphi(\bar{a}, \bar{b})$  is true in the model. If we take a new tuple  $\bar{c}$  and want to know whether  $\varphi(\bar{x}, \bar{c}) \in q$ , we just check whether the corresponding control formula holds.

In a **conservative extension**, such control formulas exist for every type and every formula [10].

**Definition 5.2.2.** Let  $\mathfrak{M}$  be a model of a theory  $T$ , and suppose  $\mathfrak{M} \prec \mathfrak{N}$  is an elementary extension. Then  $\mathfrak{N}$  is called an **n-conservative extension** of  $\mathfrak{M}$  if for every tuple  $\bar{\alpha} \in N \setminus M$ , the type  $\text{tp}(\bar{\alpha}/M)$  is definable.

Let  $\mathfrak{M} = \langle M; E \rangle$  and  $\mathfrak{N} = \langle N; E \rangle$ . We say that  $\mathfrak{M} \prec \mathfrak{N}$  is an elementary extension if:

1.  $M \subset N$ ,
2. For every formula  $\varphi(\bar{\alpha})$  with parameters  $\bar{\alpha} \in M$ , we have:

$$\mathfrak{M} \models \varphi(\bar{\alpha}) \iff \mathfrak{N} \models \varphi(\bar{\alpha}).$$

**Theorem 5.2.1.**  $T$  - stable  $\Leftrightarrow \forall p \in S(N), \forall \varphi(x, \bar{y}) \exists d_\varphi(\bar{y}, \bar{a}) \bar{a} \in M$  such that  $\forall \bar{b} \in M [\varphi(x, \bar{b}) \in p \Leftrightarrow M \models d_\varphi(\bar{a}, \bar{b})]$ .

There is also a control formula where T-stable only if for any p here from N, for any formula  $\varphi(\bar{x}, \bar{y})$  exists a control formula  $d_\varphi(\bar{y}, \bar{a})$  here from M such that for any b of M  $\varphi(x, \bar{b})$  belongs to p only if for M truth  $d_\varphi(\bar{a}, \bar{b})$ .

In other words, in conservative extension each type is definable.

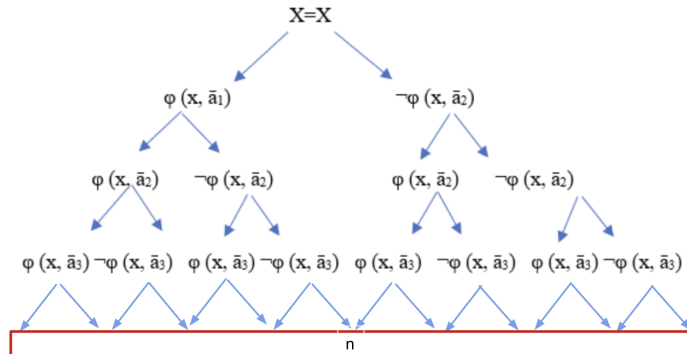


Figure 5.1: Tree for stable theory

A theory is stable if no formula has the order property  $\models \varphi(a_i, b_j) \Leftrightarrow i < j$ . Also here doesn't exist  $2^\varphi(x = x)$  infinitely branching tree. As we can see in Figure 5.1, there is a  $n$  tree where it doesn't go any further. It means that there is no complete  $n + 1$  tree.

The formula for that will be  $r_\varphi(x = x) = n$  rank is maximum depth of the tree.

As we said before any pair models of conservative extension  $\mathfrak{M} \prec \mathfrak{N}$  and each type  $q \in S(A)$  in stable theory is definable.

In Figure (5.2) NIP+NSOP is always stable and it divides into dp-minimal and non dp-minimal. Now our task is to consider for these structures when there is dp-minimal and when there is non dp-minimal.

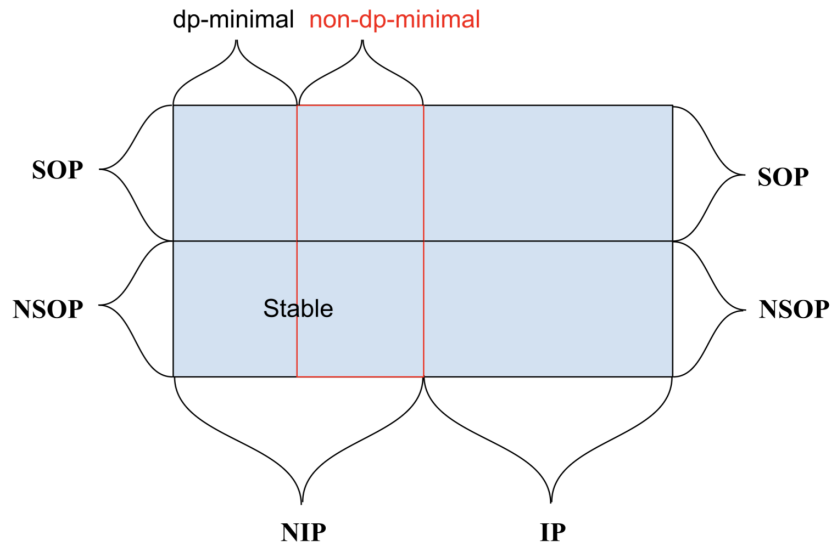


Figure 5.2: The region of interest: non dp-minimal NIP theories.

## 6. Main results

### 6.1 Special Cartesian product of two ordered structures

Let  $\mathfrak{M}_1 = (M_1; \Sigma_1)$ ,  $\mathfrak{M}_2 = (M_2; \Sigma_2)$  be two ordered structures, denote by " $<_1$ " and " $<_2$ " two linear order such that  $<_1 \in \Sigma_1$  and  $<_2 \in \Sigma_2$  and  $\Sigma_1 \cap \Sigma_2 = \{=\}$ .

**Definition 6.1.1.**  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2$  - special Cartesian product of 2 models.

Let:

$$\begin{aligned}\mathfrak{M}_1 &= \langle M_1, \Sigma_1 \rangle, & \mathfrak{M}_2 &= \langle M_2, \Sigma_2 \rangle \\ \mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 &= \langle M_1 \times M_2, \Sigma_1 \cup \Sigma_2 \cup \{<\} \rangle\end{aligned}$$

Let  $P \in \Sigma_1$ ,  $P' \in \Sigma_2$ .

$$\begin{aligned}\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \models P((a_1, b_1), \dots, (a_n, b_n)) &\iff \\ \mathfrak{M}_1 \models P(a_1, \dots, a_n) &\end{aligned}$$

and

$$\begin{aligned}\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \models P'((a_1, b_1), \dots, (a_m, b_m)) &\iff \\ \mathfrak{M}_2 \models P'(b_1, \dots, b_m) & \\ \mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \models (a_1, b_1) < (a_2, b_2) &\iff \\ \mathfrak{M}_1 \models a_1 < a_2 \quad \wedge \quad (\mathfrak{M}_1 \models a_1 = a_2 \Rightarrow \mathfrak{M}_2 \models b_1 < b_2) &\end{aligned}$$

Notice, that for any pair elements  $(a_1, b_1), (a_2, b_2) \in M_1 \times M_2$

$$\begin{aligned}\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \models (a_1, b_1) <_1 (a_2, b_2) &\iff \mathfrak{M}_1 \models a_1 < a_2 \\ \mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \models (a_1, b_1) <_2 (a_2, b_2) &\iff \mathfrak{M}_2 \models b_1 < b_2\end{aligned}$$

#### 6.1.1 Formulas of special Cartesian product

Introduce two binary formulas of  $\Sigma_{scp} := \Sigma_1 \cup \Sigma_2 \cup \{<\}$ .

$E_1(x, y) := \neg(x <_1 y) \wedge \neg(y <_1 x)$  and  $E_2(x, y) := \neg(x <_2 y) \wedge \neg(y <_2 x)$ .

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \models E_1((a_1, b_1), (a_2, b_2)) \iff \mathfrak{M}_1 \models a_1 = a_2$$

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \models E_2((a_1, b_1), (a_2, b_2)) \iff \mathfrak{M}_2 \models b_1 = b_2$$

It follows from the definition that  $E_1$  and  $E_2$  are relations of equivalence.

### 6.1.2 NIP for special Cartesian product

We assume that the structures  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are NIP.

Assume that our structure admits a formula  $\varphi(x, \bar{y})$  that has the Independence Property (IP). This means there exists an infinite sequence of parameter tuples from our model, denoted

$$(\bar{b}_1, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_n, \dots)$$

such that the formula behaves like a binary tree of arbitrary depth.

More precisely, for every natural number  $n \in \omega$  and each binary sequence

$$\tau = (\tau_1, \tau_2, \dots, \tau_n) \in 2^n, \quad \text{where } \tau_i \in \{0, 1\},$$

we define:

$$\tau_i \varphi(x, \bar{b}_i) := \begin{cases} \varphi(x, \bar{b}_i), & \text{if } \tau_i = 1 \\ \neg \varphi(x, \bar{b}_i), & \text{if } \tau_i = 0 \end{cases}$$

Then the IP property implies that for every  $\tau \in 2^n$ , there exists an element

$$x \in \mathfrak{M}_1 \times_{scp} \mathfrak{M}_2$$

such that

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \models \exists x \bigwedge_{i=1}^n \tau_i \varphi(x, \bar{b}_i).$$

In simple terms, this means that for every possible combination of truth values applied to the formula  $\varphi(x, \bar{b}_i)$ , there exists some element in our special Cartesian product structure that realizes that pattern.

If such realizations exist for all binary patterns, then the formula  $\varphi$  has the IP. But in **NIP** structures, this cannot happen.

Therefore, if both  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are NIP, and such a full binary pattern is realized in their SCP product, this leads to a contradiction. Hence, the **special Cartesian product**

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2$$

must also be NIP.

Denote

$$\bigwedge_{i=1}^n \tau_i \varphi(x, \bar{b}_i) = \varphi(x, \bar{b}_n).$$

This means that there is a 2-tree of formulas, reflecting the IP property of  $\varphi(x, \bar{y})$ .

Since for any  $\tau \in 2^n$ , there is  $a_\tau \in M_1 \times M_2$  such that

$$\mathfrak{M} \models \bigwedge_i \tau_i \varphi(a_\tau, \bar{b}_i),$$

it is necessary to understand the structure of:

$$E(x, a_i, b_j).$$

If there is  $a_i \in M_1$  or  $b_j \in M_2$  such that

$$E_1(\mathfrak{M}_1 \times_{\text{scp}} \mathfrak{M}_2, (a_i, b_j)) \wedge \bigwedge_{n \in 2^n} \varphi_\tau(\mathfrak{M}_1 \times_{\text{scp}} \mathfrak{M}_2, \bar{b}_n),$$

or

$$E_2(\mathfrak{M}_1 \times_{\text{scp}} \mathfrak{M}_2, (a_i, b_j)) \vee \bigcup_{n \in 2^n} \varphi_\tau(\mathfrak{M}_1 \times_{\text{scp}} \mathfrak{M}_2, \bar{b}_n),$$

then we distinguish two possibilities.

- If the first definable set contains an infinite 2-tree, then  $M_2$  has IP.
- If the second set contains an infinite 2-tree, then  $M_1$  has IP.

In both cases, we obtain a contradiction, since both  $M_1$  and  $M_2$  are assumed to be NIP. Therefore,

$$\mathfrak{M}_1 \times_{\text{scp}} \mathfrak{M}_2 \text{ is NIP.}$$

### 6.1.3 dp-rank of special Cartesian product

We show that the dp-rank of the special Cartesian product  $\mathfrak{M}_1 \times_{\text{scp}} \mathfrak{M}_2$  is equal to 2, under the assumption that both  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are linearly ordered structures and that  $E_1$  and  $E_2$  denote the respective coordinate projections.

Let  $(a_n)_{n < \omega}$  be a strictly increasing sequence in  $M_1$  such that

$$\mathfrak{M}_1 \models a_n < a_{n+1}, \quad \text{for all } n < \omega.$$

Likewise, let  $(b_m)_{m < \omega}$  be a strictly increasing sequence in  $M_2$  such that

$$\mathfrak{M}_2 \models b_m < b_{m+1}, \quad \text{for all } m < \omega.$$

Consider two sequences in the product structure  $M_1 \times M_2$ :

$$\begin{aligned} & \{(a_1, b_1), (a_2, b_1), \dots, (a_n, b_1), \dots\}, \\ & \{(a_1, b_1), (a_1, b_2), \dots, (a_1, b_m), \dots\}. \end{aligned}$$

Denote:

$$c_n := (a_n, b_1), \quad d_m := (a_1, b_m).$$

Let  $x$  be a variable ranging over elements of  $M_1 \times_{\text{scp}} M_2$ . Define the formulas  $E_1(x, y)$  and  $E_2(x, y)$  to correspond to the projections onto the first and second coordinates respectively,

$$E_1(x, c_n) := \pi_1(x) = a_n, \quad E_2(x, d_m) := \pi_2(x) = b_m.$$

Now fix any pair  $(i, j) \in \omega \times \omega$ . Then consider the following set of formulas:

$$\Delta_{i,j} := \{E_1(x, c_i) \wedge E_2(x, d_j)\} \cup \{\neg E_1(x, c_n) \mid n \neq i\} \cup \{\neg E_2(x, d_m) \mid m \neq j\}.$$

We claim that this set  $\Delta_{i,j}$  is consistent. Intuitively, this means that there exists an element  $x \in M_1 \times_{\text{scp}} M_2$  such that:

- the first coordinate of  $x$  is exactly  $a_i$  and no other  $a_n$ ,
- the second coordinate of  $x$  is exactly  $b_j$  and no other  $b_m$ .

Such an  $x = (a_i, b_j)$  clearly satisfies all formulas in  $\Delta_{i,j}$ , so the set is consistent.

This setup defines an indiscernible array  $\{(c_n, d_m)\}_{n,m < \omega}$  that witnesses  $\text{dp-rank} \geq 2$ . Since  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are both NIP (e.g., o-minimal or weakly o-minimal), their product in the special Cartesian sense remains NIP, and no array of this kind can witness a rank greater than 2.

The last statement means that  $\text{dp-rank}$  of  $\mathfrak{M}_1 \times_{\text{scp}} \mathfrak{M}_2$  is equal to 2.

#### 6.1.4 1-Types over a model of special Cartesian product of two dp-minimal ordered structures

We will consider 1-types over model  $\mathfrak{M}$  of theory of special Cartesian product.

Let  $p \in S_1(M)$ . Consider the following cases:

1. (a) Consider the irrational cut  $(C, D)$  in  $\mathfrak{M}_1$  and let

$$p_{1,1}(x) := \{E_1(x, (a_1, b_j))\} \cup \{(a, b_j) <_1 x <_1 (a', b_j) | a \in C, a' \in D\}$$

- (b) Consider the (quasi)rational cut  $(C, D)$  in  $\mathfrak{M}_1$  and let

$$p_{1,2}(x) := \{E_1(x, (a_1, b_j))\} \cup \{(a, b_j) <_1 x <_1 (a_0, b_j) | a \in C,$$

$a_0$  minimal in  $D\}$

2. (a) Consider the irrational cut  $(C, D)$  in  $\mathfrak{M}_1$  and let

$$p_{2,1}(x) := \{E_2(x, (a_i, b_1))\} \cup \{(a_i, b) <_2 x <_2 (a_i, b') | b \in C, b' \in D\}$$

- (b) Consider the (quasi)rational cut  $(C, D)$  in  $\mathfrak{M}_1$  and let

$$p_{2,2}(x) := \{E_2(x, (a_i, b_1))\} \cup \{(a_i, b) <_2 x <_2 (a_i, b_0) | b \in C,$$

$b_0$  minimal in  $D\}$

3.  $p_3(x) = \{\neg E_1(x, a) \wedge \neg E_2(x, b) \wedge \forall y(E_1(y, a) \rightarrow y < x) \wedge \forall z(E_2(z, a) \rightarrow z < x) | a, b \in M \times M_2\}$

4.  $p_4(x) = \{\neg E_1(x, a) \wedge \neg E_2(x, b) \wedge \forall y(E_1(y, a) \rightarrow x < y) \wedge \forall z(E_2(z, a) \rightarrow x < z) | a, b \in M \times M_2\}$

5. Consider two irrational cuts: first  $(C_1, D_1)$  in  $\mathfrak{M}_1$  and second  $(C_2, D_2)$  in  $\mathfrak{M}_2$

$$p_5(x) = \{(a, b) < (c, d) | a \in C_1, b \in C_2, c \in D_1, d \in D_2\}$$

### 6.1.5 Conclusion on classes o-minimal and weakly o-minimal theories

So, we have the definition of the special Cartesian product of two structures, denoted by  $\mathfrak{M}_1 \times_{\text{scp}} \mathfrak{M}_2$ .

We focus on the case where  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are linearly ordered structures without the independence property (IP). By Subsection 6.1.2, it was established that the theory of  $\mathfrak{M}_1 \times_{\text{scp}} \mathfrak{M}_2$  is dependent (NIP). Furthermore, by Subsection 6.1.3, the dp-rank of this product is equals to 2.

Since models of weakly o-minimal theories are known to be dp-minimal, and since the class of o-minimal theories is a subclass of weakly o-minimal theories, it follows that:

- Any model of a weakly o-minimal theory is NIP.

- The special Cartesian product of weakly o-minimal models is therefore NIP.
- Consequently, the theory of o-minimal structures, being a subclass of weakly o-minimal theories, is NIP as well.

**Theorem 6.1.1.** *Let  $\mathfrak{M}_1, \mathfrak{M}_2$  be two o-minimal structures, then  $\text{dp-rank}(\mathfrak{M}_1 \times \mathfrak{M}_2)=2$  and theory of  $\mathfrak{M}_1 \times_{\text{sco}} \mathfrak{M}_2$  is dependent (NIP).*

**Theorem 6.1.2.** *Let  $\mathfrak{M}_1, \mathfrak{M}_2$  be two weakly o-minimal structures, then  $\text{dp-rank}(\mathfrak{M}_1 \times \mathfrak{M}_2)=2$  and theory of  $\mathfrak{M}_1 \times_{\text{scp}} \mathfrak{M}_2$  is dependent (NIP).*

## 6.2 Conservative extensions on special Cartesian product of two ordered structures

### O-minimal structures have dp-rank 1

An o-minimal structure is a model-theoretic structure over a linear order where every definable set in one variable is a finite union of points and intervals. It is known that any o-minimal structure has dp-rank equal to 1

$\mathfrak{M}$  is called to be weakly o-minimal if for any 1-formula  $\varphi(x)$  definable set  $\varphi(\mathfrak{M})$  is finite union convex sets

### 6.2.1 Definability of 1-type in ordered structures

Let  $\mathfrak{M}$  be an ordered structure and let  $\mathfrak{N} \succ \mathfrak{M}$  be a saturated elementary extension. Let  $\varphi(x)$  be an  $\mathfrak{M}$ -formula. The *convex closure* of this formula is defined as:

$$\varphi^c(x) := \exists y \exists z (\varphi(z) \wedge \varphi(y) \wedge z \leq x \leq y).$$

The *convex closure* of a 1-type  $p \in S_1(\mathfrak{M})$  is defined as the set:

$$p^c(x) := \{\varphi^c(x) \mid \varphi(x) \in p\}.$$

If the complete theory  $T = \text{Th}(\mathfrak{M})$  is o-minimal or weakly o-minimal, then for any  $p \in S_1(\mathfrak{M})$ , the following equality holds:

$$p(\mathfrak{N}) = p^c(\mathfrak{N}),$$

i.e.,  $p^c$  is a convex closure of  $p$ , and its realization set in  $\mathfrak{N}$  coincides with that of  $p$ .

In this situation there just two cases on definability of 1-type.

**Theorem 6.2.1.** *Let  $\mathfrak{M}$  be a model of ordered theory  $T$ ,  $p \in S_1(\mathfrak{M})$ . Then*

(i) if  $T$  is o-minimal,  $p$  is definable iff  $p$  is rational. Or equivalently,  $p$  is non-definable iff  $p$  is irrational.

(ii) If  $T$  is weakly o-minimal then  $p$  is definable iff  $p$  is quasi-rational. Or equivalently,  $p$  is non-definable iff  $p$  is irrational.

Since  $p(\mathfrak{M}) = p^c(\mathfrak{N})$ , the set of realization of 1-type coincide with the set of realization of the cut if -type is irrational. The last means non definability of cut and 1-type.

Thus the definable 1-types over  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2$  are  $p_{1,2}, p_{2,2}, p_4, p_5$ .

## 6.2.2 1-Conservative extension in $SCP_2$ based on two o-minimal models

**Theorem 6.2.2.** *Let  $T$  be a theory of a model  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2$  from Theorem 6.1.1. Then for any pair of models  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \prec \mathfrak{N}$  of theory  $T$ . If  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \prec_{1,c} \mathfrak{N}$ , then  $\forall n < \omega$ , we have  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \prec_{n,c} \mathfrak{N}$ . That is  $\mathfrak{N}$  is  $n$ -conservative extension of  $\mathfrak{M}$ .*

In this setting, the model  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2$  is built from two o-minimal structures. We know that types  $p_{1,2}$  and  $p_{2,2}$  are conservative. In fact:

- The type  $p_{1,2}$  corresponds to a rational type in the o-minimal structure  $\mathfrak{M}_2$ .
- The type  $p_{2,2}$  corresponds to a rational type in the o-minimal structure  $\mathfrak{M}_1$ .

So, any elementary extension that contains elements realizing  $p_1$  and  $p_2$  will consist of rational types from o-minimal structures. If we assume a 1-conservative extension, then by the Marker-Steinhorn Theorem, any difference between the models is definable, and every tuple in the difference realizes a definable type.

Thus, if a 1-type over  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2$  is of the form  $p_{1,2}$  and  $p_{2,2}$ , then:

- $p_{1,2}$  is a 1-type of  $\mathfrak{M}_2$ ,
- $p_{2,2}$  is a 1-type of  $\mathfrak{M}_1$ .

Since both types are rational, we can conclude that they are definable. This implies that any pair of elements, each realizing a definable type, will together also realize a definable type. Hence, 1-conservativity implies full  $n$ -conservativity.

## 6.2.3 1-Conservative extension in $SCP_2$ based on two weakly o-minimal models

**Theorem 6.2.3.** *There exists theory  $T$  such that  $T$  is theory of model  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2$  from Theorem 6.1.2 and there is pair of models  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \prec \mathfrak{N}$  of theory  $T$  with  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \prec_{1,c} \mathfrak{N}$ , but  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \not\prec_{2,c} \mathfrak{N}$ .*

In contrast to the o-minimal case, weakly o-minimal structures allow more complex definable sets. Here again, we consider types  $p_{1.2}$  and  $p_{2.2}$ :

- $p_{1.2}$  corresponds to a type in  $\mathfrak{M}_2$ ,
- $p_{2.2}$  corresponds to a type in  $\mathfrak{M}_1$ .

Individually, these types are definable. However, we can construct a pair of elements, each realizing a definable type, such that their combined type is not definable. This failure of definability for tuples indicates that while  $\mathfrak{N}$  is a 1-conservative extension, it is not 2-conservative.

The proof of this theorem relies on an analysis of the B.Baizhanov theorem, which shows that even in NIP theories with weakly o-minimal components, the hierarchy of definability does not crash. Unlike the o-minimal case, 1-conservativity does not guarantee higher levels of conservativity.

This contrast highlights the structural differences between SCPs built from o-minimal versus weakly o-minimal structures and the role played by rationality and definability in determining conservative hierarchies.

## 6.3 Special Cartesian product of n ordered structures

### 6.3.1 Definition of special Cartesian product of n-models

Let:

$$\mathfrak{M}_1 = \langle M_1, \Sigma_1 \rangle, \quad \mathfrak{M}_2 = \langle M_2, \Sigma_2 \rangle, \dots, \mathfrak{M}_n = \langle M_n, \Sigma_n \rangle$$

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n = \langle M_1 \times M_2 \times \dots \times M_n, \Sigma_1 \cup \Sigma_2 \cup \dots \cup \Sigma_n \cup \{<\} \rangle$$

Let  $P^m \in \Sigma_i$ , where  $1 \leq i \leq n$ , also let  $\bar{a}^1, \bar{a}^2, \dots, \bar{a}^m \in M_1 \times M_2 \times \dots \times M_n$  such that  $\bar{a}^1 = (a_1^1, a_2^1, \dots, a_n^1), \dots, \bar{a}^m = (a_1^m, a_2^m, \dots, a_n^m)$

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n \models P^m(\bar{a}^1, \bar{a}^2, \dots, \bar{a}^m) \iff \mathfrak{M}_i \models P^m(a_i^1, a_i^2, \dots, a_i^m)$$

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n \models (a_1, a_2, \dots, a_i, \dots, a_n) <_i (b_1, b_2, \dots, b_i, \dots, b_n) \iff \mathfrak{M}_i \models a_i < b_i$$

Notice, that for any pair elements

$$(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in M_1 \times M_2 \times \dots \times M_n$$

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n \models (a_1, a_2, \dots, a_i, \dots, a_n) <_1 (b_1, b_2, \dots, b_i, \dots, b_n) \iff \mathfrak{M}_1 \models a_1 < b_1$$

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n \models (a_1, a_2, \dots, a_i, \dots, a_n) <_2 (b_1, b_2, \dots, b_i, \dots, b_n) \iff \\ \mathfrak{M}_2 \models a_2 < b_2$$

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n \models (a_1, a_2, \dots, a_i, \dots, a_n) <_n (b_1, b_2, \dots, b_i, \dots, b_n) \iff \\ \mathfrak{M}_n \models a_n < b_n$$

### 6.3.2 Formulas of special Cartesian product

Introduced  $n$  binary formulas of  $\Sigma_{scp} := \Sigma_1 \cup \Sigma_2 \cup \dots \cup \Sigma_n \cup \{<\}$ .

$$E_i(x, y) := \neg(x <_i y) \wedge \neg(y <_i x), \quad \text{for } 1 \leq i \leq n$$

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n \models E_i((a_i), (b_i)) \iff \mathfrak{M}_i \models a_i = b_i$$

It follows from definition that  $E_1$  and  $E_2$  are relations of equivalence.

### 6.3.3 NIP for special Cartesian product of $n$ ordered NIP structures

Consider  $\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n$ . As in the case  $n = 2$ , the consideration of proof leads to one of the basic structures  $\mathfrak{M}_i$ . We assume that the structures  $\mathfrak{M}_i$  ( $1 \leq i \leq n$ ) are NIP.

Assume that our structure admits a formula  $\varphi(x, \bar{y})$  that has the Independence Property (IP). This means there exists an infinite sequence of parameter tuples from our model, denoted

$$(\bar{b}_1, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_n, \dots)$$

such that the formula behaves like a binary tree of arbitrary depth.

More precisely, for every natural number  $n \in \omega$  and each binary sequence

$$\tau = (\tau_1, \tau_2, \dots, \tau_n) \in 2^n, \quad \text{where } \tau_i \in \{0, 1\},$$

we define:

$$\tau_i \varphi(x, \bar{b}_i) := \begin{cases} \varphi(x, \bar{b}_i), & \text{if } \tau_i = 1 \\ \neg \varphi(x, \bar{b}_i), & \text{if } \tau_i = 0 \end{cases}$$

Then the IP property implies that for every  $\tau \in 2^n$ , there exists an element

$$x \in \mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n$$

such that

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n \models \exists x \bigwedge_{i=1}^n \tau_i \varphi(x, \bar{b}_i).$$

In simple terms, this means that for every possible combination of truth values applied to the formula  $\varphi(x, \bar{b}_i)$ , there exists some element in our special Cartesian product structure that realizes that pattern.

If such realizations exist for all binary patterns, then the formula  $\varphi$  has the IP. But in **NIP** structures, this cannot happen.

Therefore, if both  $\mathfrak{M}_i$  ( $1 \leq i \leq n$ ) are NIP, and such a full binary pattern is realized in their SCP product, this leads to a contradiction. Hence, the **special Cartesian product**

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n$$

must also be NIP.

Denote

$$\bigwedge_{i=1}^n \tau_i \varphi(x, \bar{b}_i) = \varphi(x, \bar{b}_n).$$

This means that there is a 2-tree of formulas, reflecting the IP property of  $\varphi(x, \bar{y})$ .

Since for any  $\tau \in 2^n$ , there is  $a_\tau \in M_1 \times M_2 \times \dots \times M_n$  such that

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n \models \bigwedge_i \tau_i \varphi(a_\tau, \bar{b}_i),$$

it is necessary to understand the structure of:

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n$$

If there is  $(a_1, a_2, \dots, a_i, \dots, a_n)$  such that

$$E_i(\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n, (a_i, b_j)) \wedge \bigwedge_{n \in 2^n} \varphi_\tau(M_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n, \bar{b}_n),$$

then we distinguish n possibilities.

- If the i-th definable set contains an infinite 2-tree, then  $M_i$  has IP.
- If the j-th set contains an infinite 2-tree, then  $M_j$  has IP.

In n cases, we obtain a contradiction, since both  $M_i$  ( $i \leq i \leq n$ ) are assumed to be NIP. Therefore,

$$\mathfrak{M}_1 \times_{scp} \mathfrak{M}_2 \times_{scp} \dots \times_{scp} \mathfrak{M}_n \text{ is NIP.}$$

## 7. Conclusion

In this dissertation, we studied the behavior of conservative extensions in NIP theories that are not dp-minimal. By constructing special Cartesian products of two o-minimal and two weakly o-minimal structures, we proved that the resulting theories have dp-rank 2 and preserve the NIP property. This shows that even when the components are dp-minimal, their product may have strictly higher complexity while still remaining dependent.

We further investigated the definability of types and the existence of conservative extensions. For Cartesian products of o-minimal structures, every 1-conservative extension is also an  $n$ -conservative extension for all finite  $n$ , confirming strong definability properties. However, for weakly o-minimal structures, we constructed an example where 1-conservativity does not imply 2-conservativity. This distinction highlights a crucial limitation in transferring definability between different classes of ordered structures.

Our results contribute to the structural analysis of NIP theories beyond dp-minimality and open further questions on the interaction between definability, type classification, and model-theoretic extensions in higher dp-rank settings.

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