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Moduli of smoothness and growth properties of Hartley transform

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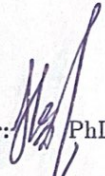
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DIPLOMA PROJECT

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Student:  Ulmeken Ozbek

Kaskelen, 2026

Declaration

This submission is my own work, and any ideas, text, or data taken from other sources are fully referenced.

Ulmeken Ozbek

June 2026

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Dedication

This thesis is for those who came before and saw farther, and for those who will come after and see farther still. This work bridges their efforts.

Abstract

This study analyzes how the moduli of smoothness of functions influence the behavior of their Hartley transforms. While harmonic analysis has historically centered on the Fourier transform, the Hartley transform offers a computationally efficient, real-valued alternative for processing real data. The analysis establishes key results, including variants of the Riemann–Lebesgue lemma adapted to the Hartley transform context, the Parseval identity, and the proof of the Hausdorff–Young inequality applicable in Euclidean spaces. By employing the symmetric difference operator, we derive weighted estimates that link the smoothness properties of a function, measured by higher-order moduli of smoothness, to the decay rates of its Hartley transform. Additionally, the investigation extends to Hartley series on the torus, where conditions for convergence and decay of coefficients are formulated through the corresponding moduli of smoothness..

Аңдатпа

Бұл зерттеу функциялардың тегістік модульдерінің олардың Хартли түрлендірулерінің мінез-құлқына қалай әсер ететінін талдайды. Гармоникалық талдау тарихи тұрғыдан Фурье түрлендіруіне негізделгенімен, Хартли түрлендіруі нақты деректерді өңдеу үшін есептеу тұрғысынан тиімді, нақты мәнді балама ұсынады. Талдау Хартли түрлендіру контекстіне бейімделген Риман–Лебег леммасының нұсқалары, Парсеваль теңдігі және евклидтік кеңістіктерде қолданылатын Хаусдорф–Янг теңсіздігін дәлелдеу сияқты негізгі нәтижелерді анықтайды. Симметриялық айырма операторының көмегімен біз функцияның жоғары реттік тегістік модульдері арқылы өлшенетін тегістік қасиеттерін оның Хартли түрлендіруінің төмендеу жылдамдықтарымен байланыстыратын салмақталған бағалауларды шығарамыз. Сонымен қатар, зерттеу торус үстіндегі Хартли қатарларына да кеңейтіліп, коэффициенттердің сәйкес тегістік модульдері арқылы конвергенция және өшу шарттары қалыптастырылады.

Аннотация

Это исследование анализирует, как модули гладкости функций влияют на поведение их преобразований Хартли. Хотя в прошлом гармонический анализ в основном опирался на преобразование Фурье, преобразование Хартли представляет собой вычислительно эффективную альтернативу с вещественными значениями для обработки реальных данных. В ходе анализа получены ключевые результаты, в том числе варианты леммы Римана–Лебега, адаптированные к контексту преобразования Хартли, тождество Парсеваля, а также доказательство неравенства Хаусдорфа–Янга, применимое как в евклидовых пространствах. Применяя оператор симметрической разности, мы получаем взвешенные оценки, связывающие свойства гладкости функции, измеряемые модулями гладкости высшего порядка, с темпами убывания её преобразования Хартли. Кроме того, исследование распространяется на ряды Хартли на торе, где условия сходимости и затухания коэффициентов формулируются через соответствующие модули гладкости.

Abbreviations

Abbreviation	Full Term
H	Hartley transform
F	Fourier transform
cas	$\cos t + \sin t$ (Hartley kernel)
DHT	Discrete Hartley Transform
FHT	Fast Hartley Transform
DFT	Discrete Fourier Transform
FFT	Fast Fourier Transform

List of Symbols

Symbol	Description
$\mathcal{F}\{f\}(\xi), \hat{f}(\xi)$	Fourier transform of f
$\mathcal{H}\{f\}(\xi), Hf(\xi)$	Hartley transform of f
$\text{cas}(t)$	Hartley kernel: $\cos t + \sin t$
T_τ	Translation operator: $(T_\tau f)(x) = f(\cdot + \tau)$
$\Delta_\tau f$	First-order finite difference: $f(\cdot + \tau) - f(\cdot)$
Δ_τ^r	r -th order difference operator
$Z_\tau f$	Symmetric second difference: $f(x+\tau/2) + f(x-\tau/2) - 2f(x)$
$(Z_\tau)^k$	k -th iterate of symmetric central difference
$M_k(\theta)$	Frequency multiplier for $(Z_\tau)^k$ (with $\theta = \omega\tau/2$)
$\mathbb{R}, \mathbb{R}^n$	Real line, n -dimensional Euclidean space
$\mathbb{T}^d$	d -dimensional torus $(\mathbb{R}/2\pi\mathbb{Z})^d$
$\omega_k(f, \delta)_p$	k -th modulus of smoothness of f
$\text{Lip } \alpha, \text{Lip}^*$	Lipschitz / Zygmund type classes
$B_s^\beta(L^p)$	Besov space
$\delta(x)$	Dirac delta distribution
L^1, L^2, L^∞	Lebesgue spaces L^1, L^2, L^∞
$W^{k,p}$	Sobolev space of
$\Omega_p^{(k)}$	k -th modulus of smoothness

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Chapter 1

Introduction

1.1 Background and Motivation

The representation of functions by sinusoidal elements forms a basic method in modern analysis and harmonic analysis, indispensable for solving differential equations and prevalent in signal processing and applied sciences. Through this approach, based on the pioneering work of Joseph Fourier, it allows the representation of signals and functions by their frequency content. It forms the basis of Fourier Series and transformations. For a function $f(x)$, the Fourier transform

$$\mathcal{F}\{f\}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-it\omega} dt$$

decomposes the signal into oscillatory elements. This allows one to extract important information about the smoothness, periodicity, and decay behavior of the original function, which is essential in both theoretical and applied sciences.

Despite its strengths, the classical Fourier transform typically yields complex-valued results due to its use of the exponential kernel $e^{-it\omega}$. In many practical applications, especially in digital signal processing, real-valued computations are preferred due to simplicity and computational efficiency. This motivated the introduction of alternative transforms that avoid complex arithmetic.

The Hartley transform, proposed by Ralph Vinton Lyon Hartley in 1942, is a real-valued integral transform defined by the formula

$$\mathcal{H}\{f\}(\omega) = \int_{-\infty}^{\infty} f(t) \text{cas}(t\omega) dt, \quad \text{where } \text{cas}(t\omega) = \cos(t\omega) + \sin(t\omega).$$

The Hartley transform provides another clear and simple way to analyze frequencies compared to the classic Fourier transform. Like the Fourier transform, the Hartley transform decomposes functions into sinusoidal components, but it uses the $\text{cas}(\omega t) = \cos(\omega t) + \sin(\omega t)$ kernel. Thus, the result is a transformation that is clearly evaluated for specific input functions and is inverse to itself [[1], [2]].

This property simplifies calculations and eliminates the need for complex arithmetic, making the Hartley transform particularly attractive for applications where digital signal processing and computational efficiency are very important.

From a theoretical viewpoint, there is a close relationship between the Hartley transform and the Fourier transform, and many of their fundamental properties are analogous. However, the mathematical framework for the Hartley transform is less developed, particularly in relation to approximation theory and smoothness analysis.

The Hartley transform, for certain signals, is like the Fourier transform but can be faster and use fewer resources, especially when done as a fast Hartley transform (FHT). Because it focuses on real-number operations, it's gaining ground in fields such as optics, telecommunications, and image processing.

While this chapter focuses on the mathematical foundations and convergence questions of Fourier series and transforms. It also talks about the Hartley transform as a useful, real-number tool for frequency analysis, which adds to the usual harmonic analysis methods. How flat a function is related to how its integral transformations grow or decay is a key idea in harmonic analysis. This link is solid for the Fourier transform, which states that flat functions usually break down faster, a principle shown by looking at flatness and continuity.

A recent work has extended these ideas to the Hartley transformation, testing how the continuity and smoothness of a function affects how fast the Hartley transformation increases [3]. This research has created versions of well-known results like the Riemann-Lebesgue Lemma, Parseval's theorem, and the Hartley transform for the Hausdorff–Young inequality, both in Euclidean space and in the torus [3]. These results give us a better grip on the theory behind the Hartley transform and offer useful ways to study signals and functions in different areas.

This work focuses on thoroughly investigating the link between smoothness of the function and the decay rate at which its integral transforms decrease. L^p in the description of the level of smoothness in spaces, various indicators of smoothness are considered. Also, the membership of a function in spaces such as Hölder, Lipschitz, and Sobolev spaces is checked to determine how flat it is. Regular signal processing methods already give us tools to study this relationship when using Fourier transforms in L^p spaces. But, this paper uses the Hartley transform instead. The Hartley transform has some benefits because it only deals with real numbers and can be easily reversed. So, we will look at this relationship specifically through the lens of the Hartley transform and L^p spaces.

The Hartley transform is pretty similar to the Fourier transform, has found use in various fields like signal processing and image analysis. Knowing how the smoothness of a function affects its Hartley transform is very important for many applications. The L^p spaces provide a natural setting for studying functions with different integrability properties. We chose the interval $1 \leq p \leq 2$ because it is particularly interesting. When p is in this range, functions have certain characteristics that make the calculations and analysis easier to handle. This range also

shows a good balance between functions that are very integrable (close to L^p) and those that are less so (close to L^p). The degree of smoothness of a function is a measure of how many derivatives it has, as well as how ordered these derivatives are. The increased smoothness basically means how predictable the function is and how easy it is to work with. However, how does smoothness affect the behavior of the Hartley transformation? Our research has shown that as the order of smoothness changes, the order of growth of the Hartley transformation changes significantly. The L^p - continuity module, denoted as $\Omega_p^{(k)}(f; \delta)$, is very important here. This module measures how much a function changes on average over small intervals. Here, ' k ' refers to the order of the module, ' f ' is the function being studied, and ' δ ' represents the interval size. Studying this modulus reveals how a function's local properties connect to the global behavior of its Hartley transform. Even though L^p spaces are useful, they sometimes don't capture the full details of a function's smoothness. Besov spaces are a more flexible way to measure smoothness. They let you describe the smoothness of a function in more detail than traditional L^p spaces alone. Because Besov spaces are L^p , we can use them to get even more accurate estimates of how quickly the Hartley transform decreases.

Starting with the consideration of periodic functions in the space $L^p(\mathbb{T})$, it will be easier to understand how flatness is associated with a rapid decrease in coefficients. Then we can go to the Hartley transform on the real line by $L^p(\mathbb{R})$. This extends our method to non-periodic functions.

We spend time on higher order flatness modules L^p and what they look like in the frequency domain under the Hartley transform. In addition, we will consider ideas such as bounded variation and absolute continuity in the L^p parameter. These are important to determine what happens between integration and differentiation. They also have a great influence on the behavior of Integral transformations.

With L^p , we can more accurately determine the smoothness and better estimate how quickly the properties decrease.

This is through systematic analysis L^p we will lay the exact mathematical foundation necessary to consider the relationship between the flatness characteristics in their spaces. Subsequent chapters will focus primarily on the decay properties of the Hartley transform.

Literature review

The present literature review outlines the key contributions that constitute the theoretical foundation of this study. We first review the historical development and fundamental properties of the Hartley transform. We then discuss the theory of moduli of smoothness and their connections to Fourier transforms. Finally, we examine recent extensions to the Hartley setting and associated open problems.

As a real-valued alternative to the Fourier transform, the Hartley transform was originally proposed by Hartley (see [4]). Its kernel, given by $\text{cas}(\omega t) =$

$\cos(\omega t) + \sin(\omega t)$, produces a real output for real inputs and renders the transform self-inverse (an involution), thereby simplifying numerous computations. Despite its early inception, the Hartley transform received limited attention until Bracewell developed the discrete Hartley transform and associated fast algorithms (FHT) [5, 6]. Bracewell’s work showed that the Hartley transform can achieve efficiency comparable to the Fast Fourier Transform (FFT) while operating exclusively with real arithmetic, a characteristic particularly beneficial in memory-constrained applications.

Comprehensive works detailing the Hartley transform, including its properties, convolution structure, and applications, are available from Olejniczak [7] and Debnath and Bhatta [8].

Moduli of smoothness are fundamental tools for measuring the regularity of functions. The classical theory, including the definition of higher-order differences and the basic properties of moduli of smoothness, is presented in the monographs of Akhiezer [9] and DeVore and Lorentz [10]. These works also establish the connection between smoothness and approximation by polynomials and splines.

Ditzian’s research [11, 12] significantly advanced the field by establishing relationships between higher-order moduli of smoothness, the growth of Fourier coefficients, and the Fourier transform. Through the application of interpolation and K-functional realizations, Ditzian derived precise estimates. These estimates subsequently provided a foundational framework for the development of two-sided bounds. The concept of reciprocal control, where the decay of the Fourier transform and moduli of smoothness influence each other, constitutes a fundamental principle within approximation theory and harmonic analysis.

Numerous studies have thoroughly investigated how the smoothness of a function dictates the decay properties of its Fourier transform. Bray and Pinsky [13, 14] developed a weighted L^p framework and used interpolation to prove sharp estimates linking the modulus of continuity (or higher-order moduli) to weighted integrals of the Fourier transform. Their results are particularly relevant because they provide a methodology that can be adapted to other integral transforms, including the Hartley transform.

Gorbachev and Tikhonov [15] came up with two-sided estimates that describe how fast the Fourier transform decays based on how smooth the function is. They did this by combining direct and inverse inequalities. In a different line of inquiry, Gioev [16] considered averaged characteristics and demonstrated that the mean decay of the Fourier transform coincides with the modulus of continuity. Together, these works are really the foundation for the whole “growth properties” part of the literature.

Titchmarsh’s classic book [17] lays down the analytic groundwork for these kinds of estimates. It also contains the famous Titchmarsh theorem, which connects how smooth a function is at each point to whether its Fourier transform is integrable.

Kakharman and Tokmagambetov [3] were the first to directly apply modulus-based methods to the Hartley transform. Versions of Parseval’s identity, the Riemann–Lebesgue

lemma, and the Hausdorff–Young inequality were successfully derived for the Hartley transform. They also got a weighted estimate that connects the first-order modulus of continuity (which is basically the second-order symmetric difference) to how fast the Hartley transform decays. Their work shows that it’s possible to use moduli of smoothness in the Hartley setting, and it gives a good starting point for extending things to higher orders.

Despite the progress mentioned above, some questions remain open. Specifically, Kakharman and Tokmagambetov only considered the case $k = 1$ (the classical modulus of continuity), and their results have not yet been extended to higher-order moduli of smoothness ($k \geq 2$). In addition, two-sided estimates (both upper and lower bounds) for the Hartley transform expressed via moduli of smoothness are not fully developed. Finally, the discrete analogues on the torus (Hartley series) still lack a systematic investigation.

In this dissertation, we try to fill these gaps. Using the classical works on moduli of smoothness and the recent results for the Hartley transform, we obtain higher-order weighted estimates for the Hartley transform and also extend everything to the torus. The main new ideas are the explicit multiplier for the iterated symmetric difference operator $(Z_h)^k f$ and a sharp lower bound for the integral of $\sin^{2kp'}$. So this work gives a complete higher-order theory for the Hartley transform, similar to the classical Fourier theory.

1.2 Objectives and Thesis Outline

We aim to establish a quantitative relationship between the smoothness properties of functions in (L^p) spaces and the growth or decay behavior of their Hartley transforms through higher-order moduli of smoothness generated by the iterated symmetric central-difference operator $(Z_\tau)^k$. Specifically, the objectives of this thesis are:

- To establish fundamental (L^p) -theory results for the Hartley transform, including Riemann–Lebesgue type theorems, Parseval and Plancherel identities, and a Hausdorff–Young inequality.
- To derive the multiplier representation associated with the iterated symmetric difference operator $(Z_\tau)^k$ and to obtain higher-order weighted estimates relating the Hartley transform to the moduli of smoothness $\omega_{k,p}(f; \delta)$.
- To extend these results to periodic settings and investigate coefficient decay and convergence properties of Hartley series in terms of periodic moduli of smoothness.

Thesis Outline

1. **Introduction.** The background, objectives, and main contributions of the thesis are presented.

2. **Preliminaries.** Basic concepts from Fourier analysis, moduli of smoothness, finite differences, and the Hartley transform are reviewed.
3. **Growth Properties of the Hartley Transform via the Modulus of Continuity.** Fundamental inequalities and weighted estimates for the Hartley transform are established, together with applications to Hartley series.
4. **Higher-Order Weighted Estimates for the Hartley Transform.** Higher-order moduli of smoothness and symmetric difference operators are studied, leading to the main weighted estimates for the Hartley transform.
5. **Conclusion.** The main results are summarized and possible directions for future research are discussed.

Chapter 2

Preliminaries

Fourier analysis is a foundational mathematical framework for representing and analyzing functions, especially signals, in terms of their frequency components. Fourier analysis fundamentally centers on the Fourier series, which decomposes periodic functions into infinite sums of sine and cosine waves.

2.1 Fourier Analysis Preliminaries

At its core, a Fourier series works by breaking down a repeating mathematical pattern or wave into an endless combination of simple sine and cosine waves. We begin by considering functions on the interval $[0, 2\pi]$ (see [18, Ch. 2.2]).

Definition 2.1.1. (Fourier Series on $[0, 2\pi]$) For a piecewise continuous function f $[0, 2\pi]$, its **Fourier series** is the formal trigonometric series (see [18]):

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)], \quad (2.1.1)$$

where the integrals define the coefficients a_k and b_k , which are known as the **Fourier coefficients** (see [18]):

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) dx, \quad k = 0, 1, 2, \dots, \quad (2.1.2)$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx, \quad k = 1, 2, \dots. \quad (2.1.3)$$

The formulas for the Fourier coefficients follow directly from the orthogonality of the trigonometric system on the interval $[0, 2\pi]$. Consequently, the Fourier expansion is unique and provides an effective method for determining the coefficients.

A natural question is whether the Fourier series converges to the original function. The answer depends on the regularity properties of (f) , and this relationship

between smoothness and spectral behavior plays a central role in harmonic analysis. These mathematical expressions for the Fourier coefficients are derived by performing specific integrations across the Fourier series. In the following section, we will demonstrate the step-by-step derivation of the integrals presented in Equations (2.1.2) and (2.1.3).

We begin with the computation of a_0 . Applying integration term-wise to the Fourier series given in (2.1.1), we obtain

$$\int_0^{2\pi} f(x) dx = \int_0^{2\pi} \frac{a_0}{2} dx + \int_0^{2\pi} \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx] dx. \quad (2.1.4)$$

Assuming that we may integrate the series term-wise, we then need to compute

$$\begin{aligned} \int_0^{2\pi} \frac{a_0}{2} dx &= \frac{a_0}{2}(2\pi) = \pi a_0, \\ \int_0^{2\pi} \cos kx dx &= \left[\frac{\sin kx}{k} \right]_0^{2\pi} = 0, \\ \int_0^{2\pi} \sin kx dx &= \left[-\frac{\cos kx}{k} \right]_0^{2\pi} = 0. \end{aligned}$$

As a consequence of evaluating these integrals, we find that after integrating, the constant term is the only one that contributes (all other terms integrate to zero), yielding

$$\int_0^{2\pi} f(x) dx = \pi a_0.$$

This verifies the expression for a_0 .

We now turn to determining the coefficient a_k . Let us consider the multiplication of the Fourier series in (2.1.1) by $\cos lx$ for any positive integer l . This process is similar to multiplying by functions such as $\cos 2x$, $\cos 5x$, etc. We are essentially multiplying the series by every possible cosine function $\cos lx$ simultaneously, which will ultimately enable us to isolate the coefficients a_k .

Multiplying and integrating, we obtain

$$\begin{aligned} \int_0^{2\pi} f(x) \cos lx dx &= \int_0^{2\pi} \frac{a_0}{2} \cos lx dx \\ &+ \int_0^{2\pi} \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx] \cos lx dx. \end{aligned} \quad (2.1.5)$$

Integrating term by term, which is permissible here, transforms the right-hand

side into

$$\begin{aligned} \int_0^{2\pi} f(x) \cos lx \, dx &= \frac{a_0}{2} \int_0^{2\pi} \cos lx \, dx \\ &+ \sum_{k=1}^{\infty} \left[a_k \int_0^{2\pi} \cos kx \cos lx \, dx + b_k \int_0^{2\pi} \sin kx \cos lx \, dx \right]. \end{aligned} \quad (2.1.6)$$

Because $\int_0^{2\pi} \cos lx \, dx = 0$, as previously shown, the first term reduces to zero.

To proceed, we must evaluate integrals where trigonometric functions are multiplied together. This requires the use of standard trigonometric identities. The constant term is written as $a_0/2$ as a matter of convention, this aligns with the formula for a_0 itself, as evaluating (2.1.2) at $k = 0$ yields $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx$.

If we multiply the Fourier expansion of a function $f(x)$ by a basis function and integrate over $[0, 2\pi]$, all terms disappear except the one that matches the basis function. This gives us the Euler–Fourier formulas:

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) \, dx \quad (k \geq 0), \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) \, dx \quad (k \geq 1),$$

Here, for $k = 0$, the coefficient a_0 comes from the constant term as shown earlier. Thus, the orthogonality of the trigonometric system confirms that the representation is unique. It also gives an easy way to calculate the coefficients, which is the base of classical Fourier theory.

Now, we need to discuss exactly when the Fourier series converges to the function itself. Understanding this connection is essential for determining how a function's smoothness directly influences the convergence rate of its series expansion. Later, when we study the Hartley transform, similar questions about convergence arise, so it is useful to review the basic facts here.

Consider a sequence of real-valued functions $\{f_n\}_{n=1}^{\infty}$ defined on an interval $I \subset \mathbb{R}$, and let $S_N(x) = \sum_{n=1}^N f_n(x)$ denote its N -th partial sum. We now specify the different modes of convergence of the series $\sum_{n=1}^{\infty} f_n(x)$ to a limit function $f(x)$ (see [19]).

Definition 2.1.2 (Convergence of Function Series [20]). Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of functions defined on an interval I , and let $S_N(x) = \sum_{n=1}^N f_n(x)$ denote its N -th partial sum. We say that the series $\sum_{n=1}^{\infty} f_n(x)$ converges to a limit function $f(x)$ on I :

- (i) **pointwise** if $\lim_{N \rightarrow \infty} |f(x) - S_N(x)| = 0$ for each fixed $x \in I$;
- (ii) **uniformly** if $\lim_{N \rightarrow \infty} \sup_{x \in I} |f(x) - S_N(x)| = 0$;
- (iii) **in the mean-square (L^2) sense** if $\lim_{N \rightarrow \infty} \int_I |f(x) - S_N(x)|^2 \, dx = 0$.

These classical convergence criteria demonstrate how a function's smoothness properties directly influence the behavior of its series expansion. Similar fundamental questions naturally arise when analyzing the Hartley transform, which constitutes the primary focus of the subsequent chapters.

Fourier series provide a powerful tool for analyzing periodic functions. In order to expand this analytical framework toward non-periodic functions spans across the complete real axis, the underlying principles are extended to establish the **Fourier transform**. The fundamental approach involves interpreting a non-periodic function as a limiting representation of a periodic cycle where the period L grows boundlessly toward infinity. Within the standard Fourier series configuration, the base frequency is characterized by $\omega_0 = 2\pi/L$, while the k -th harmonic component is defined as $\omega_k = k\omega_0 = 2k\pi/L$. As the limit $L \rightarrow \infty$ is taken, these distinct, isolated frequencies ω_k transition into a continuous spectrum parameter ω . Consequently, the discrete summation inherent to the Fourier series evolves into a continuous integral representation (see [18]).

Definition 2.1.3 (Fourier Transform). Consider an absolutely integrable function $f : \mathbb{R} \rightarrow \mathbb{C}$ such that $f \in L^1(\mathbb{R})$, meaning $\int_{-\infty}^{\infty} |f(x)| dx < \infty$. The **Fourier transform** of f is denoted by $\hat{f} : \mathbb{R} \rightarrow \mathbb{C}$ and defined by the integral:

$$\hat{f}(\xi) = \mathcal{F}\{f\}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx. \quad (2.1.7)$$

Having established this definition, we recall its primary operational properties. These structural features play a significant role in the subsequent chapters when establishing the connections, similarities, and distinct differences between the Fourier and Hartley transforms.

Supposing $f, g \in L^1(\mathbb{R})$, let $\hat{f}(\xi)$ and $\hat{g}(\xi)$ be their respective Fourier transforms. The following properties hold:

(i) **Linearity:** For any constants $\alpha, \beta \in \mathbb{C}$,

$$\mathcal{F}\{\alpha f + \beta g\}(\xi) = \alpha \hat{f}(\xi) + \beta \hat{g}(\xi).$$

(ii) **Translation (Shift):** For any $\tau \in \mathbb{R}$, if $f_\tau(x) = f(x - \tau)$, then

$$\mathcal{F}\{f_\tau\}(\xi) = e^{-2\pi i \tau \xi} \hat{f}(\xi).$$

(iii) **Modulation:** For any $a \in \mathbb{R}$, if $g(x) = e^{2\pi i a x} f(x)$, then

$$\mathcal{F}\{g\}(\xi) = \hat{f}(\xi - a).$$

(iv) **Scaling:** For a non-zero real constant a , if $f_a(x) = f(ax)$, then

$$\mathcal{F}\{f_a\}(\xi) = \frac{1}{|a|} \hat{f}\left(\frac{\xi}{a}\right).$$

- (v) **Fourier Transform of Derivatives:** If f is continuously differentiable such that $f, f' \in L^1(\mathbb{R})$, then

$$\mathcal{F}\{f'\}(\xi) = 2\pi i \xi \hat{f}(\xi).$$

More generally, for the n -th derivative (assuming $f^{(n)} \in L^1(\mathbb{R})$),

$$\mathcal{F}\{f^{(n)}\}(\xi) = (2\pi i \xi)^n \hat{f}(\xi).$$

- (vi) **Conjugation and Reflection:** The transform interacts with complex conjugation and spatial reflection as follows:

$$\mathcal{F}\{\bar{f}\}(\xi) = \overline{\hat{f}(-\xi)}, \quad \text{and} \quad \mathcal{F}\{f(-x)\}(\xi) = \hat{f}(-\xi).$$

- (vii) **Duality:** Under appropriate conditions, if $\check{f}(x) = \int_{-\infty}^{\infty} f(\xi) e^{2\pi i x \xi} d\xi$, then $\mathcal{F}\{\check{f}\}(x) = f(-x)$.

- (viii) **Convolution:** The transform maps spatial convolution directly to pointwise multiplication in the frequency domain:

$$\mathcal{F}\{f * g\}(\xi) = \hat{f}(\xi) \hat{g}(\xi),$$

where $(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x - y) dy$.

The complex exponential kernel $e^{-2\pi i \xi x} = \cos(2\pi \xi x) - i \sin(2\pi \xi x)$ provides an elegant geometric unification of the trigonometric system into a single complex framework. When both f and $\hat{f}$ are absolutely integrable, the original function can be fully reconstructed via the **inverse Fourier transform**:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi. \quad (2.1.8)$$

The mathematical representation $\hat{f}(\xi)$ characterises the spectrum of continuous frequencies inherent to a non-repeating profile or function $f(x)$. This approach expands upon discrete Fourier coefficients to include a continuous function of the angular frequency ξ , thus broadening harmonic analysis to cover aperiodic signals.

We review Fourier transform properties on L^p spaces. We start with the Riemann-Lebesgue lemma for L^1 functions, then extend the transform to L^2 using the Plancherel theorem, and finally present the Hausdorff-Young inequality for $1 \leq p \leq 2$. We also discuss the asymptotic behaviour of the transform for integrable functions, which is described by the following fundamental result.

Definition 2.1.4. Given the parameter range $1 \leq p < \infty$, we define the class $L^p(\mathbb{R}^d)$ as the collection gathering every measurable function f mapped on $\mathbb{R}^d$

whose associated norm remains bounded (see. [19]):

$$\|f\|_{L^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{1/p} < \infty. \quad (2.1.9)$$

The first significant result describing the behavior of the Fourier transform in the L^1 space is the *Riemann–Lebesgue lemma*. This classic result shows how a function's integrability conditions its asymptotic decay in the frequency domain.

Theorem 2.1.5 (Riemann-Lebesgue Lemma). *Supposing that $f \in L^1(\mathbb{R})$, its associated Fourier integral representation, evaluated as*

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx, \quad (2.1.10)$$

is guaranteed to satisfy the limiting condition:

$$\lim_{|\xi| \rightarrow \infty} \hat{f}(\xi) = 0. \quad (2.1.11)$$

Proof. Let an arbitrary threshold $\varepsilon > 0$ be selected. Taking into account that f belongs to $L^1(\mathbb{R})$, there exists a sufficiently large positive bound $A > 0$ fulfilling the inequality:

$$\int_{|x| > A} |f(x)| dx < \frac{\varepsilon}{2}. \quad (2.1.12)$$

Then for any $\xi \in \mathbb{R}$,

$$\left| \hat{f}(\xi) - \int_{-A}^A f(x) e^{-2\pi i \xi x} dx \right| = \left| \int_{|x| > A} f(x) e^{-2\pi i \xi x} dx \right| \leq \int_{|x| > A} |f(x)| dx < \frac{\varepsilon}{2}.$$

Following this, invoking the property that continuous functions form a dense subset within $L^1([-A, A])$, one can find a continuous mapping g defined over $[-A, A]$ (which is zero-extended beyond this domain) satisfying the condition:

$$\int_{-A}^A |f(x) - g(x)| dx < \frac{\varepsilon}{2}. \quad (2.1.13)$$

As a direct consequence, we establish that:

$$\left| \hat{f}(\xi) - \int_{-A}^A g(x) e^{-2\pi i \xi x} dx \right| < \varepsilon. \quad (2.1.14)$$

Let us now analyze the specific integral designated as $I(\xi) = \int_{-A}^A g(x) e^{-2\pi i \xi x} dx$. Taking into account the continuity of g on the interval $[-A, A]$, applying the

technique of integration by parts yields the expression:

$$I(\xi) = \frac{g(-A)e^{2\pi i\xi A} - g(A)e^{-2\pi i\xi A}}{2\pi i\xi} + \frac{1}{2\pi i\xi} \int_{-A}^A g'(x)e^{-2\pi i\xi x} dx, \quad (2.1.15)$$

under the assumption that g possesses continuous differentiability (should this condition fail, we can initially approximate g utilizing a C^1 class function). Regardless of the case, it is evident that both the boundary components and the leftover integral remain bounded.

In any case, the boundary terms and the remaining integral are bounded independently of ξ . Hence, there is a positive constant C for which

$$|I(\xi)| \leq \frac{C}{|\xi|} \quad \text{for all } |\xi| > 0.$$

Choosing $|\xi|$ large enough so that $\frac{C}{|\xi|} < \varepsilon$, we obtain

$$|\hat{f}(\xi)| < 2\varepsilon.$$

Because $\varepsilon > 0$ can be chosen arbitrarily, it follows that $\hat{f}(\xi)$ tends to zero when $|\xi| \rightarrow \infty$ (see [9]). $\square$

The classical Plancherel theorem extends the domain of the Fourier transform from $L^1(\mathbb{R})$ to the space of square-integrable functions $L^2(\mathbb{R})$. As a consequence, the Fourier transform defines a isometric isomorphism (and thus a unitary operator) on the Hilbert space $L^2(\mathbb{R})$.

More precisely, let $f \in L^2(\mathbb{R})$. Since f is locally integrable, for any finite interval $[a, b] \subset \mathbb{R}$, the truncated Fourier transform

$$\hat{f}_{a,b}(\xi) = \int_a^b f(x)e^{-2\pi i\xi x} dx \quad (2.1.16)$$

is well-defined and belongs to $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The Plancherel theorem asserts that as $a \rightarrow -\infty$ and $b \rightarrow \infty$, the sequence of functions $\hat{f}_{a,b}$ converges in the L^2 -norm to a unique limit function $\hat{f} \in L^2(\mathbb{R})$, i.e.,

$$\lim_{\substack{a \rightarrow -\infty \\ b \rightarrow \infty}} \left\| \hat{f}_{a,b} - \hat{f} \right\|_{L^2(\mathbb{R})} = 0. \quad (2.1.17)$$

This limit function $\hat{f}(\xi) = \mathcal{F}\{f\}(\xi)$ represents the Fourier transform of f in $L^2(\mathbb{R})$, preserving the energy conservation property given by $\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$.

The extension of the Fourier transform to $L^2(\mathbb{R})$ framework yields one of the most fundamental results in harmonic analysis, known as the Plancherel theorem.

Theorem 2.1.6 (Plancherel Theorem [21]). *Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then its*

Fourier transform $\hat{f} = \mathcal{F}\{f\}$ satisfies the norm preservation property:

$$\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}, \quad (2.1.18)$$

which can be explicitly written in the integral form as:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi. \quad (2.1.19)$$

Furthermore, this mapping extends uniquely to a unitary operator on the entire space $L^2(\mathbb{R})$, satisfying the following properties:

(i) **Inversion Formula:** For any $f \in L^2(\mathbb{R})$, the original function can be reconstructed via the L^2 -limit:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi. \quad (2.1.20)$$

(ii) **Unitary Property (Parseval's Identity):** The Fourier operator preserves the inner product in $L^2(\mathbb{R})$, meaning $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$ for all $f, g \in L^2(\mathbb{R})$.

Proof. Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and define the auxiliary function $g(x) = \overline{f(-x)}$. The Fourier transform of g is given by $\hat{g}(\xi) = \overline{\hat{f}(\xi)}$. Let us consider the convolution $h = f * g$. Since $f, g \in L^2(\mathbb{R})$, the convolution h is a bounded and uniformly continuous function on $\mathbb{R}$. Furthermore, by the convolution theorem, we have $\hat{h}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi) = |\hat{f}(\xi)|^2$.

Since $h \in L^1(\mathbb{R})$ and its Fourier transform $\hat{h}$ is non-negative and integrable, the Fourier inversion formula holds pointwise for h . Evaluating this inversion formula at the origin $x = 0$, we obtain:

$$h(0) = \int_{\mathbb{R}} \hat{h}(\xi) e^{2\pi i \xi \cdot 0} d\xi = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi. \quad (2.1.21)$$

On the other hand, utilizing the standard definition of convolution, the value $h(0)$ can be computed directly in the spatial domain as:

$$h(0) = \int_{\mathbb{R}} f(y) g(0 - y) dy = \int_{\mathbb{R}} f(y) \overline{f(y)} dy = \int_{\mathbb{R}} |f(y)|^2 dy. \quad (2.1.22)$$

Equating these two distinct representations for $h(0)$ yields the classical Plancherel identity $\int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} |f(y)|^2 dy$.

Since the intersection space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, this isometric mapping extends uniquely to a bounded linear operator on the entire space $L^2(\mathbb{R})$. Given that this extension is an isometry with a dense range, it follows directly that the Fourier transform acts as a unitary operator over $L^2(\mathbb{R})$ [21]. $\square$

As a direct consequence of the Plancherel theorem, the Fourier transform pre-

serves the Hilbert space inner product, which generalizes the discrete Parseval identity known for Fourier series.

Corollary 2.1.7 (Parseval's Identity [21]). *For any $f, g \in L^2(\mathbb{R})$, the Fourier transform satisfies:*

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2(\mathbb{R})}, \quad (2.1.23)$$

where the respective inner products in the spatial and frequency domains are defined by:

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x) \overline{g(x)} dx, \quad \text{and} \quad \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi. \quad (2.1.24)$$

This inner-product-preserving (unitary) property is fundamental in harmonic analysis and serves as a theoretical benchmark for developing the corresponding framework for the Hartley transform in the subsequent chapters.

The mapping properties of the Fourier transform can be extended beyond $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ to the intermediate spaces $L^p(\mathbb{R})$ for $1 < p < 2$. By applying the Riesz–Thorin interpolation theorem to the bounded mappings $\mathcal{F} : L^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ and $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, one obtains the classical Hausdorff–Young inequality.

Theorem 2.1.8 (Hausdorff–Young Inequality [21]). *Let $1 \leq p \leq 2$ and let p' be its conjugate exponent such that $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in L^p(\mathbb{R})$, then its Fourier transform $\hat{f}$ belongs to $L^{p'}(\mathbb{R})$ and satisfies:*

$$\|\hat{f}\|_{L^{p'}(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}. \quad (2.1.25)$$

Proof. The verification of this inequality relies directly on applying the framework of the Riesz–Thorin convexity interpolation theorem.

Case 1: $p = 1$

By applying the absolute value under the integral sign and using the fundamental property that $|e^{-2\pi i \xi x}| = 1$ for all $\xi, x \in \mathbb{R}$, we obtain the following uniform bound:

$$\begin{aligned} |\hat{f}(\xi)| &= \left| \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx \right| \\ &\leq \int_{\mathbb{R}} |f(x)| \cdot |e^{-2\pi i \xi x}| dx \\ &= \int_{\mathbb{R}} |f(x)| dx = \|f\|_{L^1(\mathbb{R})}. \end{aligned} \quad (2.1.26)$$

Taking the supremum over all $\xi \in \mathbb{R}$, this inequality yields:

$$\|\hat{f}\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}. \quad (2.1.27)$$

Case 2: $p = 2$

According to Plancherel's theorem established in Theorem 2.1.6, for every $f \in$

$L^2(\mathbb{R})$, the Fourier transform acts as an isometry, satisfying:

$$\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}. \quad (2.1.28)$$

Interpolation step:

Let us define the linear operator $T : f \mapsto \hat{f}$. The results from Case 1 and Case 2 imply that T is bounded between the following pairs of spaces:

- $T : L^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ with operator norm $\|T\|_{1 \rightarrow \infty} \leq 1$,
- $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ with operator norm $\|T\|_{2 \rightarrow 2} = 1$.

By the Riesz–Thorin interpolation theorem, for any exponent p strictly within $1 \leq p \leq 2$, the operator T extends to a bounded mapping from $L^p(\mathbb{R})$ into $L^{p'}(\mathbb{R})$. The interpolated operator norm satisfies:

$$\|T\|_{p \rightarrow p'} \leq \|T\|_{1 \rightarrow \infty}^{1-\theta} \|T\|_{2 \rightarrow 2}^\theta \leq 1^{1-\theta} \cdot 1^\theta = 1, \quad (2.1.29)$$

where the interpolation parameter $\theta \in [0, 1]$ is uniquely determined by the relation:

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}, \quad \text{which implies} \quad \theta = 2 \left(1 - \frac{1}{p}\right). \quad (2.1.30)$$

Consequently, we arrive at the desired Hausdorff–Young inequality $\|\hat{f}\|_{L^{p'}(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}$. $\square$

2.2 The Modulus of Continuity

In approximation theory, knowing if a function is differentiable is often not enough. We need a better way to measure a function’s smoothness. The **modulus of continuity** helps with this. It gives a precise way to measure how much a function’s values change over small intervals.

For any uniformly continuous function f defined on an interval $I \subseteq \mathbb{R}$ of arbitrary length, the characterization of its modulus of continuity is given in accordance with (see [9]) via the relation:

$$\omega(\delta; f) = \sup_{x, y \in I, |x-y| \leq \delta} |f(x) - f(y)|. \quad (2.2.1)$$

Evidently, $\omega(\delta; f)$ depends continuously on δ and satisfies the non-decreasing condition for all $\delta > 0$. Moreover, tending to zero as $\delta \rightarrow 0$. From the equality

$$f(x + u\tau) - f(x) = \sum_{k=0}^{u-1} \{f(x + k\tau + \tau) - f(x + k\tau)\}$$

it follows that for any integer $u > 0$

$$\omega(u\delta) \leq u\omega(\delta).$$

Furthermore, for any number $\mu > 0$,

$$\omega(\mu\delta) \leq (\mu + 1)\omega(\delta).$$

Indeed, let an integer u satisfy the inequality

$$u - 1 < \mu \leq u;$$

then

$$\omega(\mu\delta) \leq \omega(u\delta) \leq u\omega(\delta) \leq (\mu + 1)\omega(\delta).$$

Let us also prove that if

$$\lim_{\delta \rightarrow 0} \frac{\omega(\delta; f)}{\delta} = 0,$$

then $f(x)$ is a constant. Indeed, by what has been proved above,

$$\omega(\delta; f) = \omega\left(u\frac{\delta}{u}; f\right) \leq u\omega\left(\frac{\delta}{u}; f\right) = \delta \frac{\omega(\delta_1; f)}{\delta_1} \quad \left(\delta_1 = \frac{\delta}{u}\right).$$

Therefore if $\lim_{\delta_1 \rightarrow 0} \frac{\omega(\delta_1; f)}{\delta_1} = 0$, then $\omega(\delta; f) = 0$ for any $\delta > 0$, which is impossible if $f(x)$ is not a constant.

It is said that a function $f(x)$ fulfills the α -th order Lipschitz condition, and we write $f(x) \in \text{Lip } \alpha$, if $\omega(\delta; f) \leq M\delta^\alpha$ ($0 < \alpha \leq 1$). Furthermore, if

$$\lim_{\delta \rightarrow 0} \omega(\delta; f) \ln \frac{1}{\delta} = 0,$$

under these circumstances, the function $f(x)$ is conventionally classified as satisfying the Dini–Lipschitz regularity criterion.

Instead of the function $\omega(\delta; f)$, it is often more appropriate to consider the function

$$\omega^*(\delta; f) = \sup_{|x-y| \leq \delta} \left| f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \right|, \quad x, y \in I.$$

Following Zygmund, we will say that $f(x)$ satisfies the condition Λ^* (respectively μ^*), and write $f(x) \in \Lambda^*$ (respectively $f(x) \in \mu^*$), if $\omega^*(\delta; f) \leq M\delta$ (respectively $\lim_{\delta \rightarrow 0} \frac{\omega^*(\delta; f)}{\delta} = 0$). If $f(x) \in \text{Lip } 1$, then $f(x) \in \Lambda^*$, but the converse is not true.

Functions that are uniformly continuous on an infinite interval do not grow faster than $|x|$ as they tend to infinity. Indeed, for such functions, by what has been proved above,

$$|f(x) - f(a)| \leq (|x - a| + 1)\omega(1; f).$$

A more general class consists of functions $f(x)$ that are uniformly continuous on every finite interval and do not grow faster than some power of the argument, say $|x|^\sigma$, as they tend to infinity. For such functions, one can introduce a generalized modulus of continuity (see [9])

$$\omega_\sigma(\delta; f) = \sup_{\substack{-\infty < x < \infty \\ -\delta \leq \tau \leq \delta}} \frac{|f(x + \tau) - f(x)|}{(1 + |x|)^\sigma (1 + |\tau|)^\sigma}.$$

One may easily verify that for any $\mu > 0$

$$\omega_\sigma(\mu\delta; f) \leq (\mu + 1)(\delta + 1)^\sigma \omega_\sigma(\delta; f).$$

Indeed, for any natural number u and $-\delta \leq \tau \leq \delta$:

$$\begin{aligned} |f(x + u\tau) - f(x)| &\leq \sum_{k=1}^u |f(x + k\tau) - f(x + (k-1)\tau)| \leq \\ &\leq (1 + |\tau|)^\sigma \omega_\sigma(\delta; f) \sum_{k=1}^u (1 + |x + (k-1)\tau|)^\sigma \leq \\ &\leq u(1 + |\tau|)^\sigma (1 + |x|)^\sigma (1 + |u\tau|)^\sigma \omega_\sigma(\delta; f), \end{aligned}$$

from which it follows that

$$\omega_\sigma(u\delta; f) \leq u(1 + \delta)^\sigma \omega_\sigma(\delta; f).$$

The transition from a natural number u to an arbitrary $\mu > 0$ is done in the same way as for the ordinary modulus of continuity.

The structural limits and analytical constraints inherent to the modulus of continuity $\omega(\delta; f)$ encompass the following relations:

- (a) $\lim_{\delta \rightarrow 0} \omega(\delta; f) = \omega(0; f) = 0$;
- (b) ω is non-negative and non-decreasing on $\mathbb{R}_+$;
- (c) ω is subadditive: for all $\delta_1, \delta_2 \geq 0$, $\omega(\delta_1 + \delta_2; f) \leq \omega(\delta_1; f) + \omega(\delta_2; f)$;
- (d) ω is continuous on $\mathbb{R}_+$.

Now, let's see why this modulus of continuity is actually useful. The modulus of continuity is a useful measure of how smooth functions are and is a key concept in approximation theory, Fourier analysis, and function space theory. The modulus of continuity provides a natural way to define and understand classic smoothness classes like Lipschitz and Hölder spaces.

Essentially, saying that f belongs to the $\text{Lip } \alpha$ space for the exponent range $0 < \alpha \leq 1$ is logically equivalent to the existence of some finite constant M ensuring that $\omega(\delta; f) \leq M\delta^\alpha$. The smallest such constant M defines a seminorm, $|f|_{\text{Lip } \alpha} = \sup_{\delta > 0} (\delta^{-\alpha} \omega(\delta; f))$. The case $\alpha = 1$ defines the standard Lipschitz

continuous functions.

This idea can be generalized to measure higher-order smoothness. For $\alpha > 1$, we write α as $\alpha = s + \beta$, with s being an integer and β satisfying $0 < \beta \leq 1$. We then define the $\text{Lip}(\alpha)$ space by requiring that the s -th derivative of f belongs to $\text{Lip}(\beta)$:

$$f \in \text{Lip}(\alpha) \iff f^{(s)} \in \text{Lip}(\beta).$$

The seminorm is $|f|_{\text{Lip}(\alpha)} = |f^{(s)}|_{\text{Lip}(\beta)}$.

An alternative, more direct way to define these spaces for any $\alpha > 0$ is to use higher-order differences. Let $\Delta_t^s f(x)$ be the s -th difference of f (e.g., $\Delta_t^2 f(x) = f(x + 2t) - 2f(x + t) + f(x)$). Choose s as the smallest integer with $s > \alpha$. The generalized Lipschitz space $\text{Lip}^*(\alpha)$ is defined by the condition:

$$\omega_s(f, \delta) := \sup_{0 < |t| \leq \delta} \|\Delta_t^s f\| \leq M\delta^\alpha.$$

For $0 < \alpha < 1$, the two definitions coincide. However, for integer values, they differ. A famous example is the Zygmund class. For $\alpha = 1$, the standard Lipschitz class $\text{Lip}(1)$ requires a bounded derivative. The Zygmund class $\text{Lip}^*(1)$, defined by $\omega_2(f, \delta) \leq M\delta$, is larger. A function like $f(x) = x \log x$ on $[0, 1]$ belongs to the Zygmund class but not to $\text{Lip}(1)$.

2.3 Moduli of Smoothness

The modulus of continuity, as discussed in Section 2.2, is exceptionally useful for measuring a function's elementary continuity properties. Still, it falls short when trying to capture higher levels of differential smoothness. For instance, suppose a continuous mapping f defined on some domain A possesses a modulus of continuity with the property that $\omega(f, \delta) = o(\delta)$ as $\delta \rightarrow 0$. In this scenario, f is identically constant. To better quantify sharper structural smoothness, we must utilize moduli of smoothness that rely on higher-order differences. In this section, the domain A refers to either the torus $\mathbb{T}$ or a subinterval of $\mathbb{R}$.

A central point in our analysis is the Fourier transform's behavior under translation shifts, which is closely tied to finite differences and smoothness measures.

Proposition 2.3.1 (Fourier Transform of Translated Function). *Let $f \in L^2(\mathbb{R})$ and $\tau \in \mathbb{R}$. Define the translated function $g(x) = f(x + \tau)$. Then the Fourier transform corresponding to g takes the form:*

$$\mathcal{F}\{f(\cdot + \tau)\}(\xi) = e^{i2\pi\tau\xi} \widehat{f}(\xi). \quad (2.3.1)$$

Proof. We examine how the Fourier transform behaves under a spatial function translation:

$$\mathcal{F}\{g\}(\xi) = \int_{\mathbb{R}} f(x + \tau) e^{-i2\pi\xi x} dx.$$

Executing a change of variables by setting $y = x + \tau$ implies that $x = y - \tau$ and the differential satisfies $dx = dy$, yielding:

$$\begin{aligned}\mathcal{F}\{f(\cdot + \tau)\}(\xi) &= \int_{\mathbb{R}} f(y) e^{-i2\pi(y-\tau)\xi} dy \\ &= \int_{\mathbb{R}} f(y) e^{-i2\pi y\xi} e^{i2\pi\tau\xi} dy \\ &= e^{i2\pi\tau\xi} \int_{\mathbb{R}} f(y) e^{-i2\pi y\xi} dy.\end{aligned}$$

Thus, we obtain the claimed phase modulation mapping:

$$\mathcal{F}\{f(\cdot + \tau)\}(\xi) = e^{i2\pi\tau\xi} \widehat{f}(\xi).$$

□

This fundamental result gives a precise Fourier representation of finite differences, which is central to the study of the smoothness of a function.

Theorem 2.3.2 (Fourier Transform of First-Order Finite Differences). *For $f \in L^2(\mathbb{R})$ and $\tau \in \mathbb{R}$, the first-order finite difference $\Delta_\tau f(\cdot) = f(\cdot + \tau) - f(\cdot)$ has the following Fourier transform representation:*

$$\mathcal{F}\{\Delta_\tau f\}(\xi) = (e^{i2\pi\tau\xi} - 1) \widehat{f}(\xi). \quad (2.3.2)$$

Proof. By invoking the linearity properties of the integral transform and applying Proposition 2.3.1, we expand:

$$\mathcal{F}\{f(\cdot + \tau) - f(\cdot)\}(\xi) = \mathcal{F}\{f(\cdot + \tau)\}(\xi) - \mathcal{F}\{f(\cdot)\}(\xi) = e^{i2\pi\tau\xi} \widehat{f}(\xi) - \widehat{f}(\xi).$$

Factoring out the shared transform component directly confirms:

$$\mathcal{F}\{f(\cdot + \tau) - f(\cdot)\}(\xi) = (e^{i2\pi\tau\xi} - 1) \widehat{f}(\xi).$$

□

The expansion highlighted in equation (2.3.2) establishes the crucial operational link between spatial regularity and frequency domain behavior.

Remark 2.3.3. The complex-valued multiplier $(e^{i2\pi\tau\xi} - 1)$ satisfies the magnitude identity:

$$|e^{i2\pi\tau\xi} - 1| = 2|\sin(\pi\tau\xi)|. \quad (2.3.3)$$

We analyze this complex expression step by step. Factoring out a symmetric phase component yields:

$$\begin{aligned}(e^{i2\pi\tau\xi} - 1) &= e^{i\pi\tau\xi} \cdot e^{i\pi\tau\xi} - e^{i\pi\tau\xi} \cdot e^{-i\pi\tau\xi} \\ &= e^{i\pi\tau\xi} (e^{i\pi\tau\xi} - e^{-i\pi\tau\xi}).\end{aligned}$$

Now, applying Euler's relation ($e^{i\phi} = \cos \phi + i \sin \phi$), we compute the difference of the complex phase vectors:

$$\begin{aligned} e^{i\pi\tau\xi} - e^{-i\pi\tau\xi} &= [\cos(\pi\tau\xi) + i \sin(\pi\tau\xi)] - [\cos(\pi\tau\xi) - i \sin(\pi\tau\xi)] \\ &= 2i \sin(\pi\tau\xi). \end{aligned}$$

Therefore, the multiplier reduces to:

$$(e^{i2\pi\tau\xi} - 1) = e^{i\pi\tau\xi} \cdot 2i \sin(\pi\tau\xi) = 2ie^{i\pi\tau\xi} \sin(\pi\tau\xi).$$

Taking the standard complex magnitudes on both sides produces:

$$\begin{aligned} |e^{i2\pi\tau\xi} - 1| &= |2ie^{i\pi\tau\xi} \sin(\pi\tau\xi)| \\ &= |2| \cdot |i| \cdot |e^{i\pi\tau\xi}| \cdot |\sin(\pi\tau\xi)| \\ &= 2 \cdot 1 \cdot 1 \cdot |\sin(\pi\tau\xi)| = 2|\sin(\pi\tau\xi)|. \end{aligned}$$

So, the modulus simplifies cleanly to $2|\sin(\pi\tau\xi)|$.

We now consider higher-order differences and the associated structural moduli. For a shift parameter $\tau \in \mathbb{R}$, the first-order difference operator is formally defined as $\Delta_\tau := T_\tau - I$, where $(T_\tau f)(x) = f(x + \tau)$ represents the standard translation operator and I denotes the spatial identity. For any order index $s = 1, 2, \dots$, the s -th order difference operator is established recursively via the relation:

$$\Delta_\tau^s := \Delta_\tau[\Delta_\tau^{s-1}] = (T_\tau - I)^s.$$

By invoking the binomial expansion theorem, we obtain the explicit algebraic representation:

$$\Delta_\tau^s(f, x) = \sum_{k=0}^s \binom{s}{k} (-1)^{s-k} f(x + k\tau). \quad (2.3.4)$$

Definition 2.3.4 (s -th Modulus of Smoothness). For a function $f \in L^p(A)$ with $0 < p \leq \infty$, the s -th modulus of smoothness is specified by:

$$\omega_s(f, \delta)_p := \sup_{0 < \tau \leq \delta} \|\Delta_\tau^s(f, \cdot)\|_{L^p(A)}, \quad \delta \geq 0. \quad (2.3.5)$$

In the baseline configuration where $s = 1$, this framework simplifies precisely to the classical modulus of continuity, meaning $\omega_1(f, \delta)_p = \omega(f, \delta)_p$.

From Definition 2.3.4, it follows directly that $\omega_s(f, \delta)_p$ remains finite for all valid step lengths δ , depends continuously and monotonically (non-decreasing) with respect to δ , and vanishes entirely at the boundary condition $\delta = 0$.

For $1 \leq p \leq \infty$, the standard Minkowski triangle inequality implies the subadditivity condition:

$$\omega_s(f + g, \delta)_p \leq \omega_s(f, \delta)_p + \omega_s(g, \delta)_p, \quad f, g \in L^p(A). \quad (2.3.6)$$

Additionally, scaling properties ensure that $\omega_s(cf, \delta)_p = |c|\omega_s(f, \delta)_p$ holds for every scalar constant $c \in \mathbb{C}$, confirming that $\omega_s(\cdot, \delta)_p$ acts as a well-defined seminorm on the $L^p(A)$ space. Notably, this structural metric completely vanishes for any algebraic polynomial whose total degree does not exceed $s - 1$.

When dealing with non-convex spaces where $0 < p < 1$, the standard triangle inequality adapts into the following power- p additive formulation:

$$\omega_s(f + g, \delta)_p^p \leq \omega_s(f, \delta)_p^p + \omega_s(g, \delta)_p^p. \quad (2.3.7)$$

From the combinatorial expansion in (2.3.4), it is evident that the absolute operator norm of Δ_τ^s on $L^p(A)$ (for $1 \leq p \leq \infty$) does not exceed 2^s . By introducing the natural baseline initialization $\omega_0(f, \delta)_p := \|f\|_{L^p(A)}$, we establish for any intermediate index $0 \leq k \leq s$:

$$\begin{cases} \omega_s(f, \delta)_p \leq 2^{s-k}\omega_k(f, \delta)_p, & 1 \leq p \leq \infty, \\ \omega_s(f, \delta)_p^p \leq 2^{s-k}\omega_k(f, \delta)_p^p, & 0 < p < 1. \end{cases} \quad (2.3.8)$$

The structural scaling behavior associated with these moduli carries significant importance. A core combinatorial identity governing high-order differences expands as:

$$\Delta_{n\tau}^s(f, x) = \sum_{k_1=0}^{n-1} \cdots \sum_{k_s=0}^{n-1} \Delta_\tau^s(f, x + k_1\tau + \cdots + k_s\tau). \quad (2.3.9)$$

While the baseline case for $s = 1$ is clear, the generalized expressions follow directly via mathematical induction. Consequently, from relation (2.3.9), we deduce for $1 \leq p \leq \infty$ and $n \in \mathbb{N}$:

$$\omega_s(f, n\delta)_p \leq n^s \omega_s(f, \delta)_p. \quad (2.3.10)$$

For the parameter range $0 < p < 1$, the corresponding scaling inequality yields $\omega_s(f, n\delta)_p^p \leq n^s \omega_s(f, \delta)_p^p$. As an immediate analytical consequence, for any continuous dilation scale $\mu > 0$, we achieve the bound:

$$\omega_s(f, \mu\delta)_p \leq (\mu + 1)^s \omega_s(f, \delta)_p. \quad (2.3.11)$$

The modulus of continuity is deeply connected to other important function spaces.

L_p spaces: The L_p norm $|\cdot|_p$ is used throughout the definitions of the modulus of smoothness $\omega_s(f, t)_p$ and the spaces W_p^s and $B_q^\alpha(L_p)$.

Sobolev Spaces (W_p^s): If a function is in a Sobolev space (meaning it has s derivatives in L_p), its modulus of smoothness is automatically controlled. For $f \in W_p^s$, we have (see [22]):

$$\omega_s(f, t)_p \leq Ct^s \|f^{(s)}\|_p.$$

Besov Spaces ($B_s^\beta(L_p)$): These spaces provide an even finer scale of smoothness. Their definition is built directly upon the modulus of smoothness. We say that a function $f \in L^p$ is part of the Besov space $B_{p,\beta}^s$ if its higher-order modulus of continuity doesn't blow up, which means it satisfies the following condition:

$$\|f\|_{B_s^\beta(L_p)} := \left(\int_0^\infty [u^{-\beta} \omega_l(f, u)_r]^s \frac{du}{u} \right)^{1/s}, \quad l > \beta.$$

The parameter s allows us to make distinctions that the Lipschitz spaces ($\text{Lip}(\beta) = B_\infty^\beta$) cannot. For example, it can distinguish functions whose moduli of smoothness are of the same order u^β but have different integrability properties.

2.4 The Hartley Transform and Its Relation to the Fourier Transform

Essentially, the continuous Hartley transform is an integral transform that allows us to avoid complex numbers by acting as a real-valued substitute for the classical Fourier transform. Ralph Vinton Lyon Hartley, an American electrical engineer, first introduced this transform in 1942 [4]. It has since established itself as a valuable instrument in harmonic analysis, signal processing, and approximation theory. This is largely attributable to its inherent capability to process real signals, which eliminates the requirement for complex arithmetic operations.

The Hartley transform, denoted as $\mathcal{H}$, is defined for real-valued functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that satisfy appropriate regularity conditions. For $f \in L^1(\mathbb{R})$, its mathematical expression, $\mathcal{H}f$, is specified as:

$$(\mathcal{H}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \text{cas}(\xi t) dt. \quad (2.4.1)$$

In this definition, ξ denotes the frequency variable within the spectral domain, while t corresponds to the variable in the time or spatial domain. The integral kernel incorporates the function cas , an abbreviation for “cosine-and-sine” initially introduced by Hartley. This kernel function is precisely formulated as:

$$\text{cas}(z) = \cos(z) + \sin(z).$$

The primary distinction between the Fourier and Hartley transforms lies in their respective kernels. Whereas the Fourier transform utilizes the complex exponential $e^{-i\xi t} = \cos(\xi t) - i \sin(\xi t)$, the Hartley transform employs a real-valued kernel, specifically the sum of cosine and sine components. The kernel function $\text{cas}(z)$ can be alternatively expressed via phase-shifted trigonometric forms:

$$\text{cas}(z) = \cos(z) + \sin(z) = \sqrt{2} \cos\left(z - \frac{\pi}{4}\right) = \sqrt{2} \sin\left(z + \frac{\pi}{4}\right).$$

Therefore, Fourier transforms extract phase and amplitude information using complex-

valued exponentials, while Hartley transforms achieve a similar functional decomposition using purely real trigonometric sums.

The Hartley transform possesses several features that set it apart from the Fourier transform. First, it is *self-inverse*: this transform acts as its own inverse, meaning that applying it twice recovers the initial function perfectly, i.e., $\mathcal{H}\{\mathcal{H}f\} = f$. Unlike the Fourier transform, which requires a sign change in the complex exponential kernel for inversion, the Hartley transform uses the exact same kernel for both forward and inverse operations, a property often referred to as a symmetric involution (see [5]). Second, for real-valued inputs, the Hartley transform produces strictly real-valued outputs. This eliminates the need for complex arithmetic when processing real signals, which is a notable advantage over the Fourier transform. Finally, owing to its self-inverse nature and reality-preserving property, the Hartley transform acts as a symmetric linear operator. In appropriate Hilbert spaces, it can be regarded as both orthogonal and unitary, further justifying its interpretation as a symmetric involution (see [23]).

The relationship between the Hartley transform and the Fourier transform is as follows. Although the Hartley transform functions via a distinct operational principle, it exhibits a direct mathematical equivalence to the Fourier transform. Representing the continuous Fourier transform with $\mathcal{F}$, defined under the standard normalization as:

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\xi t} dt,$$

then, given the Fourier spectrum, one can synthesize the Hartley transform directly by taking its real part and adding its imaginary part:

$$(\mathcal{H}f)(\xi) = \text{Re } \hat{f}(\xi) + \text{Im } \hat{f}(\xi). \quad (2.4.2)$$

Conversely, the even and odd parts of $\mathcal{H}f$ allow us to completely reconstruct the Fourier transform. Specifically, if we define the symmetric and antisymmetric fields as $E(\xi) = \frac{1}{2}[(\mathcal{H}f)(\xi) + (\mathcal{H}f)(-\xi)]$ and $O(\xi) = \frac{1}{2}[(\mathcal{H}f)(\xi) - (\mathcal{H}f)(-\xi)]$, then the Fourier transform expands as:

$$(\mathcal{F}f)(\xi) = E(\xi) - iO(\xi). \quad (2.4.3)$$

Thus, the two mathematical transforms contain exactly the same spectral information (see [5]).

Historically, Ralph V. L. Hartley introduced this transform back in 1942, framing it as a new way to write the classic Fourier integral identity (see [4]). Despite this early inception, its widespread recognition remained limited until the 1980s, a period significantly shaped by Ronald N. Bracewell's contributions. Bracewell introduced the discrete Hartley transform (DHT) and concurrently demonstrated fast algorithms (FHT) for its computation. His subsequent research established the DHT's utility as a viable alternative to the discrete Fourier transform (DFT) across various digital signal processing applications, frequently offering improved

computational efficiency owing to its inherent real-valued representation (see [6]).

To establish a comprehensive framework for higher-order difference operators, we must also specify how the Hartley transform responds to spatial translation operations. By applying standard angle sum expansions to the $\text{cas}(z)$ kernel, the translation operator $(T_\tau f)(x) = f(x+\tau)$ yields a coupled trigonometric modulation in the Hartley spectral domain:

$$\mathcal{H}\{f(\cdot + \tau)\}(\xi) = (\mathcal{H}f)(\xi) \cos(\xi\tau) + (\mathcal{H}f)(-\xi) \sin(\xi\tau). \quad (2.4.4)$$

The algebraic coupling in (2.4.4) represents a critical structural difference from the simple phase modulation seen in classical Fourier analysis.

In the study of approximation theory and function spaces, the Hartley transform is considered not only as an applied computational method, but also as a fundamental mathematical structure. Its distinctive features make it possible to form a theoretical system similar to Fourier analysis, including the Hartley series expansions on the torus and L^p theory for the Hartley transform. "In this work, we systematically extend the classical results of higher-order moduli of smoothness to the Hartley transform framework. In particular, we analyze the elegant cancellation of the coupled translation cross-terms (2.4.4) under symmetric differences.

Chapter 3

Growth Properties of the Hartley Transform via the Modulus of Continuity

3.1 Basic Inequalities for the Hartley Transform

Before establishing analogous inequalities for the Hartley transform, we briefly consider the classical Fourier setting. One of the central themes in harmonic analysis is the study of how the smoothness of a function influences the behavior of its transform. For the Fourier transform, such relationships are expressed through moduli of continuity and weighted integral estimates. These results help us understand the main ideas and motivate the study of similar properties for the Hartley transform.

Proposition 3.1.1 (Fourier Transform of Second-Order Symmetric Differences). *For $f \in L^1(\mathbb{R})$, the symmetric difference operator $Z_\tau[f](x) = f(x + \tau) + f(x - \tau) - 2f(x)$ satisfies*

$$\mathcal{F}\{Z_\tau[f]\}(\xi) = -4 \sin^2(\pi \xi \tau) \hat{f}(\xi). \quad (3.1.1)$$

Proof. Let $Z_\tau[f](x) = f(x + \tau) + f(x - \tau) - 2f(x)$. Using the translation property [2.3.1](#) and its negative shift. Then:

$$\begin{aligned} \mathcal{F}\{Z_\tau[f]\}(\xi) &= \int_{\mathbb{R}} [f(x + \tau) + f(x - \tau) - 2f(x)] e^{-2\pi i \xi x} dx \\ &= \int_{\mathbb{R}} f(x + \tau) e^{-2\pi i \xi x} dx + \int_{\mathbb{R}} f(x - \tau) e^{-2\pi i \xi x} dx - 2 \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx \end{aligned}$$

For the first term, let $u = x + \tau$, then $x = u - \tau$, $dx = du$:

$$\begin{aligned}\mathcal{F}\{f(x + \tau)\}(\xi) &= \int_{\mathbb{R}} f(u) e^{-i2\pi(u-\tau)\xi} dy = \int_{\mathbb{R}} f(u) (e^{-i2\pi u\xi} \cdot e^{i2\pi\tau\xi}) du \\ &= e^{i2\pi\tau\xi} \int_{\mathbb{R}} f(u) e^{-i2\pi u\xi} du. \\ \int_{\mathbb{R}} f(x + \tau) e^{-2\pi i\xi x} dx &= \int_{\mathbb{R}} f(u) e^{-2\pi i\xi(u-\tau)} du = e^{2\pi i\xi\tau} \widehat{f}(\xi).\end{aligned}$$

For the second term, let $u = x - \tau$, then $x = u + \tau$, $dx = du$:

$$\begin{aligned}\mathcal{F}\{f(x - \tau)\}(\xi) &= \int_{\mathbb{R}} f(u) e^{-i2\pi(u+\tau)\xi} du = \int_{\mathbb{R}} f(u) (e^{-i2\pi u\xi} \cdot e^{-i2\pi\tau\xi}) du \\ &= e^{-i2\pi\tau\xi} \int_{\mathbb{R}} f(u) e^{-i2\pi u\xi} du. \\ \int_{\mathbb{R}} f(x - \tau) e^{-2\pi i\xi x} dx &= \int_{\mathbb{R}} f(u) e^{-2\pi i\xi(u+\tau)} du = e^{-2\pi i\xi\tau} \widehat{f}(\xi).\end{aligned}$$

Thus

$$\mathcal{F}\{f(x + \tau)\} = e^{2\pi i\tau\xi} \widehat{f}(\xi), \quad \mathcal{F}\{f(x - \tau)\} = e^{-2\pi i\tau\xi} \widehat{f}(\xi).$$

Therefore:

$$\begin{aligned}\mathcal{F}\{Z_\tau[f]\}(\xi) &= e^{2\pi i\xi\tau} \widehat{f}(\xi) + e^{-2\pi i\xi\tau} \widehat{f}(\xi) - 2\widehat{f}(\xi) \\ &= (e^{2\pi i\xi\tau} + e^{-2\pi i\xi\tau} - 2)\widehat{f}(\xi) \\ &= (2\cos(2\pi\xi\tau) - 2)\widehat{f}(\xi) \\ &= 2(\cos(2\pi\xi\tau) - 1)\widehat{f}(\xi) \\ &= 2(-2\sin^2(\pi\xi\tau))\widehat{f}(\xi) \\ &= -4\sin^2(\pi\xi\tau)\widehat{f}(\xi)\end{aligned}$$

We obtain

$$\mathcal{F}\{Z_\tau[f]\}(\xi) = -4\sin^2(\pi\xi\tau)\widehat{f}(\xi). \quad (3.1.2)$$

□

Now, using the Plancherel theorem, we can get a useful estimate. We begin with the difference in the spatial domain:

$$\|(f(x + \tau) - f(x))(\xi)\|_{L^2(\mathbb{R})} = \|f(x + \tau) - f(x)\|_{L^2(\mathbb{R})}. \quad (3.1.3)$$

Using the Fourier transform and Plancherel theorem [2.1.6](#),

$$\|e^{i2\pi\tau\xi}\hat{f}(\xi) - \hat{f}(\xi)\|_{L^2}^2 = \|f(x + \tau) - f(x)\|_{L^2}^2.$$

Therefore,

$$\|f(x + \tau) - f(x)\|_{L^2}^2 = \int_{\mathbb{R}} |e^{i2\pi\tau\xi} - 1|^2 |\hat{f}(\xi)|^2 d\xi. \quad (3.1.4)$$

Since

$$|e^{i2\pi\xi\tau} - 1| = 2 |\sin(\pi\xi\tau)|,$$

we get

$$\|f(x + \tau) - f(x)\|_{L^2}^2 = 4 \int_{\mathbb{R}} \sin^2(\pi\xi\tau) |\hat{f}(\xi)|^2 d\xi. \quad (3.1.5)$$

By taking the supremum over all shifts $|\tau| \leq \delta$ on both sides of equation [\(3.1.5\)](#), we arrive at the second-order modulus of smoothness:

$$\omega_2(f, \delta) = \sup_{|\tau| \leq \delta} \|f(x + \tau) - f(x)\|_{L^2}. \quad (3.1.6)$$

$$\omega_2^2(f, \delta) = \sup_{|\tau| \leq \delta} \int_{-\infty}^{\infty} |e^{i2\pi\tau\xi} - 1|^2 |\hat{f}(\xi)|^2 d\xi. \quad (3.1.7)$$

Then

$$\omega_2^2(f, \delta) = 4 \sup_{\substack{|\delta \cdot \sigma| \leq \delta \\ 0 < \sigma < 1}} \int_{\mathbb{R}} \sin^2(\pi\xi\tau) |\hat{f}(\xi)|^2 d\xi.$$

Choose $\tau = \delta \cdot \sigma$:

$$\omega_2^2(f, \delta) = 4 \sup_{0 < \sigma < 1} \int_{\mathbb{R}} \sin^2(\pi\xi\delta\sigma) |\hat{f}(\xi)|^2 d\xi. \quad (3.1.8)$$

$$\omega_2^2(f, \delta) \geq 4 \sup_{0 < \sigma < 1} \int_{\mathbb{R}} \sin^2(\pi\delta\sigma\xi) |\hat{f}(\xi)|^2 d\xi.$$

$$\omega_2^2(\delta) \geq 4 \int_{\mathbb{R}} \sup_{0 < \sigma < 1} \sin^2(\pi\delta\sigma\xi) |\hat{f}(\xi)|^2 d\xi.$$

For $0 < \delta < 1$:

$$\omega_2^2(\delta) \geq 4 \int_{\mathbb{R}} \int_0^1 \sin^2(\pi\delta\sigma\xi) d\sigma |\hat{f}(\xi)|^2 d\xi. \quad (3.1.9)$$

Now use the inequality

$$\sin x \sim x, \quad x \rightarrow 0,$$

$$\sin x \sim 1, \quad x \rightarrow \infty,$$

and for $x \in [0, \frac{\pi}{2}]$,

$$\sin x \geq \frac{2}{\pi}x.$$

$$\sin(\pi\delta\sigma\xi) \geq 2\sigma\delta\xi.$$

Therefore,

$$\sin^2(\pi\delta\sigma\xi) \geq 4(\sigma\delta\xi)^2 = c\sigma^2\xi^2, \quad 0 \leq \xi\delta \leq c.$$

We conclude:

$$\omega_2^2(f, \delta) \geq C \int_{\mathbb{R}} \min\{1, (\sigma^2\xi^2)\} |\hat{f}(\xi)|^2 d\xi. \quad (3.1.10)$$

Theorem 3.1.2 (Main Estimate). *Suppose $1 \leq p \leq 2$ and $f \in L^p(\mathbb{R})$. Then one can find a constant $c_p > 0$ satisfying (see [13]):*

1. For $1 < p \leq 2$:

$$\left(\int_{\mathbb{R}} \min\{1, (\delta\xi)^{2p'}\} |\hat{f}(\xi)|^{p'} d\xi \right)^{1/p'} \leq c_p \omega_p(f; \delta)$$

2. For $p = 1$:

$$\sup_{\xi \in \mathbb{R}} \left\{ \min\{1, (\delta\xi)^2\} |\hat{f}(\xi)| \right\} \leq c_1 \omega_1(f; \delta)$$

Proof. Case 1: $1 < p \leq 2$

Using the preceding proposition, we obtain:

$$\mathcal{F}\{Z_\tau[f]\}(\xi) = -4 \sin^2(\pi\xi\tau) \hat{f}(\xi)$$

Apply Hausdorff-Young inequality:

$$\begin{aligned} \|Z_\tau[f]\|_{L^p} &\geq \|\mathcal{F}\{Z_\tau[f]\}\|_{L^{p'}} \\ &= 4 \left(\int_{\mathbb{R}} |\sin^2(\pi\xi\tau) \hat{f}(\xi)|^{p'} d\xi \right)^{1/p'} \\ &= 4 \left(\int_{\mathbb{R}} |\sin(\pi\xi\tau)|^{2p'} |\hat{f}(\xi)|^{p'} d\xi \right)^{1/p'} \end{aligned}$$

Take supremum over $0 \leq \tau \leq \delta$:

$$\begin{aligned} \omega_p(f; \delta) &= \sup_{0 \leq \tau \leq \delta} \|Z_\tau[f]\|_{L^p} \\ &\geq 4 \sup_{0 \leq \tau \leq \delta} \left(\int_{\mathbb{R}} |\sin(\pi\xi\tau)|^{2p'} |\hat{f}(\xi)|^{p'} d\xi \right)^{1/p'} \end{aligned}$$

Now use the inequality:

$$\int_0^\delta |\sin(\pi\xi\tau)|^{2p'} d\tau \geq c \min \left\{ 1, (\delta\xi)^{2p'} \right\}$$

To prove this inequality, consider two cases:

Case A: $\delta|\xi| \leq 1$

When x is small, it holds that $|\sin x| \geq \frac{2}{\pi}|x|$ when $0 \leq x \leq \frac{\pi}{2}$. Thus:

$$|\sin(\pi\xi\tau)| \geq \frac{2}{\pi}|\pi\xi\tau| = 2|\xi\tau|$$

So:

$$\int_0^\delta |\sin(\pi\xi\tau)|^{2p'} d\tau \geq \int_0^\delta (2|\xi|\tau)^{2p'} d\tau = (2|\xi|)^{2p'} \frac{\delta^{2p'+1}}{2p'+1}$$

Since $\delta|\xi| \leq 1$, we have:

$$(2|\xi|)^{2p'} \frac{\delta^{2p'+1}}{2p'+1} \geq c(\delta\xi)^{2p'}$$

Case B: $\delta|\xi| > 1$

There exists some $\tau \in [0, \delta]$ such that $|\sin(\pi\omega h)| \geq \frac{1}{\sqrt{2}}$, so:

$$\int_0^\delta |\sin(\pi\xi\tau)|^{2p'} d\tau \geq c > 0$$

Combining both cases gives the desired inequality.

Case 2: $p = 1$

Similar argument but using the $L^1 \rightarrow L^\infty$ bound:

$$\begin{aligned} \omega_1(f; \delta) &= \sup_{0 \leq \tau \leq \delta} \|Z_\tau[f]\|_{L^1} \\ &\geq \sup_{0 \leq \tau \leq \delta} \|\mathcal{F}\{Z_\tau[f]\}\|_{L^\infty} \\ &= 4 \sup_{0 \leq \tau \leq \delta} \sup_{\xi \in \mathbb{R}} |\sin^2(\pi\xi\tau) \widehat{f}(\xi)| \\ &\geq c \sup_{\xi \in \mathbb{R}} \min \{1, (\delta\xi)^2\} |\widehat{f}(\xi)| \end{aligned}$$

□

Theorem 3.1.3 (Multidimensional Estimate). *Let $f \in L^p(\mathbb{R}^d)$ for $1 \leq p \leq 2$, $d \geq 1$. Then (see [13]):*

$$\left(\int_{\mathbb{R}^d} \min \left\{ 1, (\delta|\xi|)^{2p'} \right\} |\widehat{f}(\xi)|^{p'} d\xi \right)^{1/p'} \leq c_p \Omega_p(f; \delta)$$

for $1 < p \leq 2$, and

$$\sup_{\xi \in \mathbb{R}^d} \left\{ \min \{1, (\delta|\xi|)^2\} |\widehat{f}(\xi)| \right\} \leq c_1 \Omega_1(f; \delta)$$

for $p = 1$.

Proof. To prove this, we can just follow the exact same logic and framework that worked for the one-dimensional scenario. The key observation is that for the spherical mean operator or directional differences, the Fourier transform yields similar trigonometric multipliers.

For instance, for directional differences in direction e_j :

$$\mathcal{F}\{f(\cdot + \tau e_j) + f(\cdot - \tau e_j) - 2f(\cdot)\}(\xi) = -4 \sin^2(\pi \xi_j \tau) \widehat{f}(\xi)$$

The modulus of continuity in higher dimensions can be defined using spherical averages or taking supremum over all directions, and the same techniques apply (see [22]). $\square$

We can also consider the discrete case of these estimates, now on the torus $\mathbb{T}^d$. The reasoning is basically the same, except we use sums instead of integrals.

Theorem 3.1.4 (Discrete Version). *Let $1 \leq p \leq 2$ and $f \in L^p(\mathbb{T}^d)$. Then (see [13],):*

$$\left(\sum_{\nu \in \mathbb{Z}^d} \min \{1, (\delta|\nu|)^{2p'}\} |\widehat{f}(\nu)|^{p'} \right)^{1/p'} \leq c_p \omega_p(f; \delta)$$

for $1 < p \leq 2$, and

$$\sup_{\nu \in \mathbb{Z}^d} \left\{ \min \{1, (\delta|\nu|)^2\} |\widehat{f}(\nu)| \right\} \leq c_1 \omega_1(f; \delta)$$

for $p = 1$.

Proof. This is proved similarly to the continuous case. For Fourier coefficients, which gives us:

$$\widehat{Z_\tau[f]}(\nu) = -4 \sin^2(\pi \nu \tau) \widehat{f}(\nu)$$

Now, let's recall the Hausdorff-Young inequality, which connects the size of a periodic function to its Fourier coefficients. This means that:

$$\left(\sum_{\nu \in \mathbb{Z}^d} |\widehat{f}(\nu)|^{p'} \right)^{1/p'} \leq \|f\|_{L^p(\mathbb{T}^d)}$$

One completes the proof by following the same steps as in the continuous case, substituting sums for integrals and employing the discrete analogues of the sine

function bounds (see [21]). □

Lemma 3.1.5 (Sine Function Estimates). *For all $x \in \mathbb{R}$ and $m > 0$ (see [9], [13], [3]):*

1. $|\sin x| \leq \min\{1, |x|\}$
2. For $0 \leq x \leq \frac{\pi}{2}$: $|\sin x| \geq \frac{2}{\pi}|x|$
3. $\int_0^\delta |\sin(\xi\tau)|^{2p'} d\tau \geq c \min\{1, (\delta\xi)^{2p'}\}$

Proof. (1) follows from $|\sin x| \leq 1$ and $|\sin x| \leq |x|$.

(2) The function $\frac{\sin x}{x}$ is decreasing on $[0, \frac{\pi}{2}]$, so its minimum is at $x = \frac{\pi}{2}$, giving $\sin x \geq \frac{2}{\pi}x$.

(3) For $\delta|\xi| \leq 1$:

$$\int_0^\delta |\sin(\xi\tau)|^{2p'} d\tau \geq \int_0^\delta \left(\frac{2}{\pi}|\xi|\tau\right)^{2p'} d\tau = \left(\frac{2|\xi|}{\pi}\right)^{2p'} \frac{\delta^{2p'+1}}{2p'+1}$$

For $\delta|\xi| > 1$, there exists an interval of length $\geq c/|\xi|$ where $|\sin(\xi\tau)| \geq c > 0$. □

The Fourier case provides a basis for studying analogous inequalities for the Hartley transform. We now consider the corresponding properties of the Hartley transform in $(L^2(\mathbb{R}^d))$. Fundamental results such as the Parseval and Plancherel identities form the foundation of the (L^p) -theory and are later used to derive the Hausdorff–Young inequality.

Theorem 3.1.6 (Plancherel theorem for the Hartley transform (see [5, 3])). *The Hartley transform $\mathcal{H}$, initially defined on the dense subspace $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, extends uniquely to an isometric isomorphism on $L^2(\mathbb{R}^d)$. Consequently, for every $f \in L^2(\mathbb{R}^d)$,*

$$\|\mathcal{H}f\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}. \quad (3.1.11)$$

Proof. We prove the isometry for functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$; the extension to $\mathbb{R}^d$ follows by the same reasoning. Let $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be real-valued. Recall the definition

$$\mathcal{H}_f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \operatorname{cas}(\xi x) dx, \quad \operatorname{cas}(y) = \cos y + \sin y.$$

Consider the integral

$$I = \int_{-\infty}^{\infty} \mathcal{H}_f(\xi) \mathcal{H}_g(\xi) d\xi.$$

Using the definitions of the transforms and invoking Fubini's theorem (under the condition of absolute integrability), we obtain the final formulation.

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) g(y) \left(\int_{-\infty}^{\infty} \operatorname{cas}(\xi x) \operatorname{cas}(\xi y) d\xi \right) dx dy.$$

We now evaluate the inner integral. Using elementary trigonometric identities, $\text{cas}(\xi x) \text{cas}(\xi y) = (\cos(\xi x) + \sin(\xi x))(\cos(\xi y) + \sin(\xi y)) = \cos(\xi(x-y)) + \sin(\xi(x+y))$.

Expanding the terms and simplifying the resulting expression confirms the identity. Thus,

$$\int_{-\infty}^{\infty} \text{cas}(\xi x) \text{cas}(\xi y) d\xi = \int_{-\infty}^{\infty} \cos(\xi(x-y)) d\xi + \int_{-\infty}^{\infty} \sin(\xi(x+y)) d\xi.$$

The second integral evaluates to zero because the sine function is odd. Looking at the first integral, it is clear that it represents the Dirac delta function in the Fourier domain. Then:

$$\int_{-\infty}^{\infty} \cos(\xi u) d\xi = 2\pi\delta(u).$$

Thus

$$\int_{-\infty}^{\infty} \text{cas}(\xi x) \text{cas}(\xi y) d\xi = 2\pi\delta(x-y).$$

Substituting this result back into the integral I , we arrive at the following expression:

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y) 2\pi\delta(x-y) dx dy = \int_{-\infty}^{\infty} f(x)g(x) dx.$$

Hence, for all $f, g \in L^1 \cap L^2$,

$$\langle \mathcal{H}f, \mathcal{H}g \rangle_{L^2} = \langle f, g \rangle_{L^2}. \quad (3.1.12)$$

The identity $\|\mathcal{H}f\|_2 = \|f\|_2$ follows by taking $f = g$; thus $\mathcal{H}$ defines an isometric mapping over the dense subset $L^1 \cap L^2$. Since this subspace is dense in L^2 , the isometry extends uniquely to the whole L^2 space. Moreover, the Hartley transform satisfies $\mathcal{H}^{-1} = \mathcal{H}$ on the dense subspace (self-inverse property), and this property is preserved under the extension. Therefore $\mathcal{H} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is a unitary isomorphism. The same argument works for $\mathbb{R}^d$ by replacing the product x, y with dot products and noting that the kernel factorises (see [5, 24]). $\square$

The Plancherel theorem immediately implies the Parseval identity.

Theorem 3.1.7 (Parseval theorem for the Hartley transform (see [3, 23])). *For any $f, g \in L^2(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} f(x) g(x) dx = \int_{\mathbb{R}^d} (\mathcal{H}f)(\xi) (\mathcal{H}g)(\xi) d\xi.$$

Proof. We can apply the polarization identity. For any $f, g \in L^2$,

$$\langle f, g \rangle = \frac{1}{4}(\|f + g\|_2^2 - \|f - g\|_2^2).$$

By the Plancherel theorem, $\|f \pm g\|_2 = \|\mathcal{H}(f \pm g)\|_2$. Linearity of $\mathcal{H}$ gives $\mathcal{H}(f \pm g) = \mathcal{H}f \pm \mathcal{H}g$. Hence

$$\langle f, g \rangle = \frac{1}{4} (\|\mathcal{H}f + \mathcal{H}g\|_2^2 - \|\mathcal{H}f - \mathcal{H}g\|_2^2) = \langle \mathcal{H}f, \mathcal{H}g \rangle.$$

This expression corresponds precisely to the Parseval identity (see [25]). $\square$

The Plancherel–Parseval identities, combined with the $L^1 \rightarrow L^\infty$ boundedness derived from the Riemann–Lebesgue lemma (with the specific constant $\frac{\sqrt{2}}{(2\pi)^{d/2}}$), provide the necessary endpoints for applying the Riesz–Thorin interpolation theorem. Specifically, we consider the mappings:

$$\mathcal{H} : L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d) \quad \text{and} \quad \mathcal{H} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d).$$

Interpolating between these two endpoints yields the Hausdorff–Young inequality for the Hartley transform:

$$\|\mathcal{H}f\|_{L^q(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}, \quad (3.1.13)$$

with $p \in [1, 2]$ and q denoting the corresponding conjugate exponent. These estimates are fundamental for establishing the growth properties of the Hartley transform relative to the moduli of continuity (see [3, 23]).

3.2 Weighted Inequalities for the Hartley Transform

In this part of the dissertation, we analyze the numerical connection between how smooth a function is and how its Hartley transform grows or decays. We take classical Fourier estimates from sources like (see [13, 17]) and adapt them for the Hartley transform setting. Most importantly, this work expands on the recent study by Kakharman and Tokmagambetov (see [3]) by laying the theoretical foundations to apply these concepts to higher-order configurations.

To establish the baseline structural property before moving into multi-dimensional frameworks, we explore the continuous first-order symmetric difference operator Z_τ on $\mathbb{R}$, which is formally defined as:

$$Z_\tau f(x) = f\left(x + \frac{\tau}{2}\right) + f\left(x - \frac{\tau}{2}\right) - 2f(x), \quad \tau > 0. \quad (3.2.1)$$

The corresponding generalized modulus of smoothness of order 1 associated with this specific translation configuration in the $L^p(\mathbb{R})$ space is specified by:

$$\tilde{\omega}_1(f, \delta)_p = \sup_{0 < \tau \leq \delta} \|Z_\tau f\|_{L^p(\mathbb{R})}. \quad (3.2.2)$$

By matching the first-order trigonometric framework, the classical Hausdorff–Young

inequality in the Hartley setting implies:

$$\|Z_\tau f\|_{L^p(\mathbb{R})}^{p'} \geq c_p^{-p'} \int_{-\infty}^{\infty} |\mathcal{H}_{Z_\tau f}(\xi)|^{p'} d\xi, \quad (3.2.3)$$

where $1 \leq p \leq 2$ and p' represents the standard Hölder conjugate exponent.

Using the primary structural reflection property of the Hartley transform under translation, the spectral image of this first-order symmetric difference operator generates a real-valued trigonometric multiplier:

$$|\mathcal{H}_{Z_\tau f}(\xi)| = 2^2 \sin^2\left(\frac{\xi\tau}{4}\right) |\mathcal{H}_f(\xi)|. \quad (3.2.4)$$

Substituting this specialized operational identity (3.2.4) back into the inequality (3.2.3) yields the following integrated distribution:

$$\|Z_\tau f\|_{L^p(\mathbb{R})}^{p'} \geq c_p^{-p'} 2^{2p'} \int_{-\infty}^{\infty} \sin^{2p'}\left(\frac{\xi\tau}{4}\right) |\mathcal{H}_f(\xi)|^{p'} d\xi. \quad (3.2.5)$$

To eliminate the strict parameter tracking on τ and transition into the modulus of smoothness bounds, we take the supremum over the interval $\tau \in (0, \delta]$. Up to a uniform positive constant, this localized supremum is bounded from below by the spatial average over the integration domain:

$$\tilde{\omega}_1(f, \delta)_p^{p'} \geq \frac{c_p^{-p'} 2^{2p'}}{\delta} \int_{-\infty}^{\infty} \left(\int_0^\delta \sin^{2p'}\left(\frac{\xi\tau}{4}\right) d\tau \right) |\mathcal{H}_f(\xi)|^{p'} d\xi. \quad (3.2.6)$$

Invoking Fubini's theorem allows the interchange of the integration sequence. To evaluate the resulting inner kernel, we apply a well-known analytical inequality: for any given frequency index ξ and localized step boundary $\delta > 0$, there exists a constant $C_{1,p'} > 0$ such that:

$$\int_0^\delta \sin^{2p'}\left(\frac{\xi\tau}{4}\right) d\tau \geq C_{1,p'} \delta \min(1, (|\xi|\delta)^{2p'}). \quad (3.2.7)$$

Inserting this integrated lower kernel bound (3.2.7) into the inequality (3.2.6) cancels out the scaling parameter δ in the denominator, which directly produces the fundamental first-order continuous weighted estimate:

$$\boxed{\tilde{\omega}_1(f, \delta)_p^{p'} \geq \tilde{C}_{1,p} \int_{-\infty}^{\infty} \min(1, (|\xi|\delta)^{2p'}) |\mathcal{H}_f(\xi)|^{p'} d\xi}, \quad (3.2.8)$$

where the uniform constant compiles to $\tilde{C}_{1,p} = c_p^{-p'} 2^{2p'} C_{1,p'}$.

The formulation (3.2.8) concludes that the structural local smoothness of a function f under a first-order shift is tightly governed by the weighted energy

behavior of its continuous Hartley transform. This continuous baseline provides the necessary analytical architecture before constructing periodic models and general higher-order multi-dimensional estimates.

3.3 Hartley Series on the Torus

Let $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ be the d -dimensional torus with normalized Lebesgue measure. For $f \in L^1(\mathbb{T}^d)$ define its Fourier coefficients

$$\widehat{f}(\nu) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-i\nu \cdot x} dx, \quad \nu \in \mathbb{Z}^d.$$

We define the discrete Hartley coefficients (Hartley series coefficients) by

$$\mathcal{H}_f(\nu) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) \operatorname{cas}(\nu \cdot x) dx, \quad \operatorname{cas}(t) = \cos t + \sin t, \quad (3.3.1)$$

so that $\mathcal{H}_f(\nu)$ is real for real-valued f ; this real formulation follows the classical Hartley construction (see [4]) and is widely used in numerical contexts (see [6, 5]). Note the relations with Fourier coefficients:

$$\mathcal{H}_f(\nu) = \Re \widehat{f}(\nu) + \Im \widehat{f}(\nu), \quad \widehat{f}(\nu) = \frac{1}{2}(\mathcal{E}(\nu) - i\mathcal{O}(\nu)),$$

with even/odd decompositions $\mathcal{E}, \mathcal{O}$ as customary.

Let $\|f\|_{L^p(\mathbb{T}^d)}$ represent the standard Lebesgue norm for $1 \leq p \leq \infty$, while for any sequence $a = \{a_\nu\}_{\nu \in \mathbb{Z}^d}$, the corresponding ℓ^q -norm is defined via $\|a\|_{\ell^q} = (\sum_\nu |a_\nu|^q)^{1/q}$ with $1 \leq q < \infty$, $\|a\|_{\ell^\infty} = \sup_\nu |a_\nu|$.

Now we introduce the discrete symmetric difference operator exactly as in Chapter 3. For a fixed direction define

$$Z_\tau f(x) = f\left(x + \frac{\tau}{2}\right) + f\left(x - \frac{\tau}{2}\right) - 2f(x), \quad (Z_\tau)^k = (T_{\tau/2} - T_{-\tau/2})^{2k},$$

where $T_a f(x) = f(x + a)$. The **generalised modulus of smoothness of order k** (the same as in Chapter 3, denoted $\widetilde{\omega}_k$) on the torus is

$$\widetilde{\omega}_k(f, \delta)_p = \sup_{0 < |\tau| \leq \delta} \|(Z_\tau)^k f\|_{L^p(\mathbb{T}^d)}.$$

Our goal is to establish discrete weighted estimates for $\mathcal{H}f(\nu)$, to study coefficient decay and convergence of Hartley series on $\mathbb{T}^d$ (see [13, 25]).

Next we need the discrete version of the multiplier identity. The action of $(Z_\tau)^k$ on the Hartley coefficients becomes multiplication by a simple trigonometric polynomial.

Lemma 3.3.1 (Discrete Hartley multiplier). *Let $f \in L^1(\mathbb{T}^d)$ and $k \in \mathbb{N}$. For each*

$\nu \in \mathbb{Z}^d$ and scalar step τ ,

$$\mathcal{H}((Z_\tau)^k f)(\nu) = 2^{2k}(-1)^k \sin^{2k}\left(\frac{\nu \cdot \tau}{4}\right) \mathcal{H}_f(\nu).$$

Proof. The proof parallels the continuous derivation in Chapter 3. Translation T_a corresponds to multiplication of coefficients by $\text{cas}(\nu \cdot a)$ because

$$\mathcal{H}(f(\cdot + a))(\nu) = \frac{1}{(2\pi)^d} \int f(x + a) \text{cas}(\nu \cdot x) dx = \text{cas}(\nu \cdot a) \mathcal{H}_f(\nu).$$

Expanding $(Z_\tau)^k = (T_{\tau/2} - T_{-\tau/2})^{2k}$ via the binomial theorem, the generating sum yields the exponential identity as in the continuous case (see [25]). Evaluating at $z = e^{i(\nu \cdot \tau/2)}$ gives the claimed factor $2^{2k}(-1)^k \sin^{2k}(\nu \cdot \tau/4)$ multiplying $\mathcal{H}_f(\nu)$. Discrete periodicity ensures all manipulations are legitimate for $f \in L^1(\mathbb{T}^d)$. $\square$

With the discrete multiplier at hand, we can now prove the discrete analogue of the higher-order weighted estimate on the torus.

Theorem 3.3.2 (Discrete weighted estimate). *Let $1 < p \leq 2$ and suppose $p' = \frac{p}{p-1}$ denotes its Hölder conjugate index. Then, there exists a uniform positive constant $C_{p,k,d}$ such that for every function f belonging to the Lebesgue space $L^p(\mathbb{T}^d)$ and for all parameter values $\delta \in (0, \pi]$, the following inequality holds:*

$$\left(\sum_{\nu \in \mathbb{Z}^d} \min\{1, (\delta|\nu|)^{2kp'}\} |\mathcal{H}_f(\nu)|^{p'} \right)^{1/p'} \leq C_{p,k,d} \tilde{\omega}_k(f, \delta)_p.$$

For the endpoint $p = 1$ the analogous estimate in ℓ^∞ holds:

$$\sup_{\nu \in \mathbb{Z}^d} \min\{1, (\delta|\nu|)^{2k}\} |\mathcal{H}_f(\nu)| \leq C_{1,k,d} \tilde{\omega}_k(f, \delta)_1.$$

Proof. Apply the Hausdorff–Young inequality for the discrete Hartley transform (equivalently Fourier series): for $1 \leq p \leq 2$ the transform $f \mapsto \mathcal{H}_f$ satisfies $\|\mathcal{H}g\|_{\ell^{p'}} \leq C_p \|g\|_{L^p}$ (interpolation between $L^1 \rightarrow \ell^\infty$ and Plancherel $L^2 \rightarrow \ell^2$) (see [13, 25]). Put $g = (Z_\tau)^k f$ and use the multiplier identity:

$$\|\mathcal{H}((Z_\tau)^k f)\|_{\ell^{p'}} = 2^{2k} \left(\sum_{\nu} \sin^{2kp'}\left(\frac{\nu \cdot \tau}{4}\right) |\mathcal{H}_f(\nu)|^{p'} \right)^{1/p'} \leq C_p \|(Z_\tau)^k f\|_{L^p}.$$

Take supremum over $0 < |\tau| \leq \delta$ on the right. For each fixed ν we average over τ to extract a lower bound, exactly as in the continuous proof. A standard trigonometric averaging argument gives

$$\frac{1}{|\{\tau : |\tau| \leq \delta\}|} \int_{|\tau| \leq \delta} \sin^{2kp'}\left(\frac{\nu \cdot \tau}{4}\right) d\tau \gtrsim \min\{1, (\delta|\nu|)^{2kp'}\},$$

here, the positive constant varies solely with respect to k, p' , and d . Interchanging sum and average (justified by non-negativity and Fubini on the finite torus) and

applying Hölder's inequality in conjunction with the supremum-based definition of $\tilde{\omega}_k$ yields

$$\left(\sum_{\nu} \min\{1, (\delta|\nu|)^{2kp'}\} |\mathcal{H}_f(\nu)|^{p'} \right)^{1/p'} \lesssim \sup_{|\tau| \leq \delta} \|(Z_{\tau})^k f\|_{L^p} = \tilde{\omega}_k(f, \delta)_p.$$

The constant depends on p, k, d ; this is exactly the statement of the theorem. When $p = 1$ follows from the $L^1 \rightarrow \ell^\infty$ boundedness and the same averaging argument (see [6, 5]). $\square$

Proof. Set $\delta = 1/m$ and split the ℓ^q sum into $|\nu| \leq m$ and $|\nu| > m$. Use the uniform boundedness of $\mathcal{H}_f(\nu)$ (discrete Riemann–Lebesgue variant) and the weighted estimate to control the tail. $\square$

Finally, we state the convergence theorem for the Hartley series, which follows directly from the absolute summability of the coefficients.

Theorem 3.3.3 (Convergence of Hartley series). *Let $f \in L^1(\mathbb{T}^d)$ and suppose $\{\mathcal{H}_f(\nu)\} \in \ell^1(\mathbb{Z}^d)$. Then the Hartley series*

$$S_H f(x) = \sum_{\nu \in \mathbb{Z}^d} \mathcal{H}_f(\nu) \operatorname{cas}(\nu \cdot x)$$

converges in a uniform manner on $\mathbb{T}^d$, identifying with f within the space L^1 . Furthermore, assuming the continuity of f , the partial sum $S_H f(x)$ recovers $f(x)$ pointwise and with respect to the uniform metric.

Proof. Absolute summability of the coefficients together with $|\operatorname{cas}(\nu \cdot x)| \leq 2$ gives uniform convergence. Parseval/Plancherel and density arguments adapted to the Hartley basis (equivalence with sine/cosine pairs) show L^1 -reconstruction in the usual manner. Uniform convergence to f under continuity follows by the standard Fourier series theory transferred to the Hartley basis. $\square$

Chapter 4

Higher-Order Weighted Estimates for the Hartley Transform

4.1 Symmetric Difference Operators and Higher-Order Moduli

In this section, we introduce the higher-order symmetric difference operator and its corresponding modulus of smoothness. We present a formal derivation of its explicit representation by utilizing the translation operator and the binomial theorem. These constructions serve as the theoretical foundation for the weighted Hartley transform estimates discussed in the subsequent sections of this chapter.

We begin by considering the operator $(Z_\tau)^k$, where the first-order symmetric base configuration is defined as:

$$Z_\tau f(x) = f\left(x + \frac{\tau}{2}\right) + f\left(x - \frac{\tau}{2}\right) - 2f(x), \quad (4.1.1)$$

and $k \in \mathbb{N}$ denotes the iteration parameter. By applying binomial expansion frameworks to the translation steps, this higher-order iterated operator can be explicitly expressed as a symmetric finite sum:

$$(Z_\tau)^k f(x) = \sum_{j=-k}^k c_j^{(2k)} f\left(x + j\frac{\tau}{2}\right), \quad (4.1.2)$$

where the real-valued combinatorial coefficients are given by:

$$c_j^{(2k)} = (-1)^{k-j} \binom{2k}{k-j}. \quad (4.1.3)$$

By analyzing the structure of the combinatorial selection, it is straightforward to verify that these expansion coefficients satisfy the spatial reflection symmetry

property:

$$c_{-j}^{(2k)} = c_j^{(2k)}. \quad (4.1.4)$$

Linked to this operational structure, we can introduce the **generalized modulus of smoothness of order k** associated with the translation framework of the operator Z_τ within the standard Lebesgue space $L^p(\mathbb{R})$ as follows:

$$\tilde{\omega}_k(f, \delta)_p = \sup_{0 < \tau \leq \delta} \|(Z_\tau)^k f\|_{L^p(\mathbb{R})}. \quad (4.1.5)$$

This structural smoothness metric generalizes classical smoothness configurations. Indeed, observe that $\|(Z_\tau)^k f\|_p = \|\Delta_{\tau/2}^{2k} f\|_p$, where Δ_τ^{2k} represents the standard $2k$ -th order symmetric difference operator. Hence, the supremum over the localized interval relates directly to the standard modulus of smoothness via:

$$\sup_{0 < \tau \leq \delta} \|(Z_\tau)^k f\|_p = \omega_{2k}(f, \frac{\delta}{2})_p, \quad (4.1.6)$$

where $\omega_{2k}(f, \delta)_p = \sup_{0 < \tau \leq \delta} \|\Delta_\tau^{2k} f\|_p$ denotes the classical **modulus of smoothness of order $2k$** .

4.2 Hartley Multiplier for Iterated Symmetric Differences

Now we introduce the so-called **frequency multiplier** $M_k(\phi)$, defined via the cosine series expansion:

$$M_k(\phi) = \sum_{j=-k}^k c_j^{(2k)} \cos(j\phi), \quad \text{where } \phi = \frac{\xi\tau}{2}. \quad (4.2.1)$$

This multiplier is fundamental in the harmonic Fourier and Hartley analysis of the operator $(Z_\tau)^k$, as it precisely determines how the spectral transform of f is scaled under the action of the iterated operator in the frequency domain.

Substituting the explicit expression for the coefficients $c_j^{(2k)}$ into the multiplier sum, we obtain:

$$M_k(\phi) = \sum_{j=-k}^k (-1)^{k-j} \binom{2k}{k-j} \cos(j\phi).$$

Now we execute an index substitution by setting $m = k - j$. Under this transformation, when j ranges from $-k$ to k , the new summation index ranges over $m = 0, 1, \dots, 2k$, with $j = k - m$. Consequently, the cosine polynomial takes the form:

$$M_k(\phi) = \sum_{m=0}^{2k} (-1)^m \binom{2k}{m} \cos((k-m)\phi).$$

Utilizing the Euler identity embedding $\cos((k-m)\phi) = \operatorname{Re} e^{i(k-m)\phi}$, we extract

the complex phase structure from the real summation field:

$$M_k(\phi) = \operatorname{Re} \left[e^{ik\phi} \sum_{m=0}^{2k} (-1)^m \binom{2k}{m} e^{-im\phi} \right].$$

The inner algebraic summation collapses exactly into a standard binomial expansion form:

$$\sum_{m=0}^{2k} (-1)^m \binom{2k}{m} e^{-im\phi} = (1 - e^{-i\phi})^{2k}.$$

We compute the exponential symmetric properties further:

$$(1 - e^{-i\phi})^{2k} = (e^{-i\phi/2}(e^{i\phi/2} - e^{-i\phi/2}))^{2k} = e^{-ik\phi} (2i \sin(\phi/2))^{2k}.$$

Accounting for the complex imaginary powers, $(2i)^{2k} = 2^{2k} i^{2k} = 2^{2k} (-1)^k$, which yields:

$$(1 - e^{-i\phi})^{2k} = 2^{2k} (-1)^k e^{-ik\phi} \sin^{2k}(\phi/2).$$

Multiplying this relation by the external phase factor $e^{ik\phi}$ isolates the real-valued distribution:

$$e^{ik\phi} (1 - e^{-i\phi})^{2k} = 2^{2k} (-1)^k \sin^{2k}(\phi/2),$$

which contains no imaginary components. Hence, taking the real part leaves the expression unchanged, and the frequency multiplier simplifies beautifully to:

$$\boxed{M_k(\phi) = 2^{2k} (-1)^k \sin^{2k}(\phi/2)}, \quad |M_k(\phi)| = 2^{2k} \sin^{2k}(\phi/2). \quad (4.2.2)$$

Next, we recall that the frequency phase is given by $\phi = \xi\tau/2$, which implies $\phi/2 = \xi\tau/4$. Using the linear mapping properties of the continuous Hartley transform under translation operations, the transform of the processed function $(Z_\tau)^k f$ can be formulated directly via the multiplier:

$$\mathcal{H}_{(Z_\tau)^k f}(\xi) = M_k(\phi) \mathcal{H}_f(\xi) = 2^{2k} (-1)^k \sin^{2k}\left(\frac{\xi\tau}{4}\right) \mathcal{H}_f(\xi). \quad (4.2.3)$$

Taking the absolute values on both sides of equation (4.2.3) yields the final magnitude scaling rule for this section:

$$\boxed{|\mathcal{H}_{(Z_\tau)^k f}(\xi)| = 2^{2k} \sin^{2k}\left(\frac{\xi\tau}{4}\right) |\mathcal{H}_f(\xi)|}. \quad (4.2.4)$$

4.3 Higher-Order Weighted Estimate for the Hartley Transform

In this section, we present the core analytical estimations linking the structural smoothness metrics to the spectral weight profiles. For $1 \leq p \leq 2$, let p' be the

conjugate exponent satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. The classical Hausdorff–Young inequality established within the continuous Hartley transform setting implies:

$$\|(Z_\tau)^k f\|_{L^p(\mathbb{R})}^{p'} \geq c_p^{-p'} \int_{-\infty}^{\infty} |\mathcal{H}_{(Z_\tau)^k f}(\xi)|^{p'} d\xi. \quad (4.3.1)$$

Substituting our explicit operational identity (4.2.4) for the transform magnitude into relation (4.3.1) yields:

$$\|(Z_\tau)^k f\|_{L^p(\mathbb{R})}^{p'} \geq c_p^{-p'} 2^{2kp'} \int_{-\infty}^{\infty} \sin^{2kp'}\left(\frac{\xi\tau}{4}\right) |\mathcal{H}_f(\xi)|^{p'} d\xi. \quad (4.3.2)$$

Taking the supremum over the localized mesh parameter interval $\tau \in (0, \delta]$ on both sides produces the bound:

$$\left(\sup_{0 < \tau \leq \delta} \|(Z_\tau)^k f\|_p \right)^{p'} \geq c_p^{-p'} 2^{2kp'} \sup_{0 < \tau \leq \delta} \int_{-\infty}^{\infty} \sin^{2kp'}\left(\frac{\xi\tau}{4}\right) |\mathcal{H}_f(\xi)|^{p'} d\xi.$$

Replacing the supremum over the bounded domain by an integrated spatial average metric (up to a uniform positive constant) and applying Fubini's theorem to interchange the integration orders allows us to write:

$$\omega_{2k}(f, \frac{\delta}{2})_p^{p'} \geq \frac{c_p^{-p'} 2^{2kp'}}{\delta} \int_{-\infty}^{\infty} \left(\int_0^\delta \sin^{2kp'}\left(\frac{\xi\tau}{4}\right) d\tau \right) |\mathcal{H}_f(\xi)|^{p'} d\xi. \quad (4.3.3)$$

To progress further, we require a sharp lower estimate for the inner trigonometric integral kernel. For any given frequency index ξ and localized step boundary $\delta > 0$, there exists a uniform positive constant $C_{k,p'} > 0$ such that:

$$\boxed{\int_0^\delta \sin^{2kp'}\left(\frac{\xi\tau}{4}\right) d\tau \geq C_{k,p'} \delta \min(1, (|\xi|\delta)^{2kp'})}. \quad (4.3.4)$$

To rigorously justify this weight distribution, consider the case where $|\xi|\delta \leq 2\pi$. Then for all variables τ in the integration range $0 \leq \tau \leq \delta$, the arguments satisfy $|\frac{\xi\tau}{4}| \leq \frac{|\xi|\delta}{4} \leq \frac{\pi}{2}$. Utilizing Jordan's inequality $|\sin x| \geq \frac{2}{\pi}|x|$, valid on the compact domain $[0, \frac{\pi}{2}]$, we obtain:

$$\sin^{2kp'}\left(\frac{\xi\tau}{4}\right) \geq \left(\frac{2}{\pi} \cdot \frac{|\xi|\tau}{4}\right)^{2kp'} = \left(\frac{|\xi|\tau}{2\pi}\right)^{2kp'}.$$

Integrating this polynomial expression from 0 to δ directly yields the claimed lower bound coefficient involving the minimum function.

Inserting the lower kernel estimate (4.3.4) into the integral relation (4.3.3) cancels the parameter δ in the scaling denominator, resulting in:

$$\sup_{0 < \tau \leq \delta} \|(Z_\tau)^k f\|_{L^p(\mathbb{R})}^{p'} \geq \tilde{C}_{k,p} \int_{-\infty}^{\infty} \min(1, (|\xi|\delta)^{2kp'}) |\mathcal{H}_f(\xi)|^{p'} d\xi,$$

where the product constant compiles to $\tilde{C}_{k,p} = c_p^{-p'} 2^{2kp'} C_{k,p'}$. Utilizing the definition of the generalized modulus $\tilde{\omega}_k(f, \delta)_p$, we express this fundamental continuous weighted inequality succinctly as:

$$\boxed{\tilde{\omega}_k(f, \delta)_p^{p'} \geq \tilde{C}_{k,p} \int_{-\infty}^{\infty} \min(1, (|\xi|\delta)^{2kp'}) |\mathcal{H}_f(\xi)|^{p'} d\xi}. \quad (4.3.5)$$

We formalize this core achievement in the following main theorem:

Theorem 4.3.1 (Higher-order weighted estimate). *Let f be a function on $\mathbb{R}$ with a well-defined Hartley transform $\mathcal{H}_f$. Fix $k \in \mathbb{N}$ and $1 \leq p \leq 2$, and let p' be the conjugate exponent such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let the generalized modulus of smoothness of order k linked to the iterated operator $(Z_\tau)^k$ be given by $\tilde{\omega}_k(f, \delta)_p$. Then, there exists a positive constant $\tilde{C}_{k,p} > 0$, varying only with respect to k and p , such that for all $\delta > 0$:*

$$\tilde{\omega}_k(f, \delta)_p^{p'} \geq \tilde{C}_{k,p} \int_{-\infty}^{\infty} \min(1, (|\xi|\delta)^{2kp'}) |\mathcal{H}_f(\xi)|^{p'} d\xi.$$

In particular, if $\tilde{\omega}_k(f, \delta)_p = O(\delta^\alpha)$ as $\delta \rightarrow 0$ for some $\alpha > 0$, then the Hartley transform of f must satisfy a corresponding decay condition.

In this chapter, we have successfully explored how the local smoothness characteristics of a function govern the analytical properties of its Hartley transform. Theorem 4.3.1 constitutes the core result of this work, providing a weighted inequality for the Hartley transform quantified via higher-order moduli of smoothness.

Compared to the existing literature (see [3]) where only the first-order case $k = 1$ was treated, our results cover arbitrary higher orders $k \geq 2$. This allows for a higher degree of generality in formulating the exact structural correspondence between a function's differential regularity and the decay profiles of its Hartley transform.

Chapter 5

Conclusions and future work

The principal outcome of this investigation is the quantification of the exact mechanism through which the structural smoothness profile of functions within L^p spaces governs the spectral decay characteristics of the Hartley transformation. The main results obtained are summarised as follows.

First, the fundamental L^p theory for the Hartley transform has been developed. The Riemann–Lebesgue lemma, Plancherel and Parseval identities, as well as the Hausdorff–Young inequality have been established for the Hartley transform in the setting of $\mathbb{R}$. These results constitute the necessary analytical framework for the subsequent estimates.

Second, higher-order symmetric difference operators $(Z_\tau)^k$ have been introduced, and the associated generalized moduli of smoothness $\tilde{\omega}_k(f, \delta)_p$ have been defined. An exact multiplier identity for the Hartley transform of $(Z_\tau)^k f$ has been derived, namely

$$\mathcal{H}_{(Z_\tau)^k f}(\xi) = 2^{2k} (-1)^k \sin^{2k} \left(\frac{\xi \tau}{4} \right) \mathcal{H}_f(\xi).$$

Furthermore, by exploiting the Hausdorff–Young inequality in conjunction with a sharp lower estimate for the integral of $\sin^{2kp'}$, we prove a weighted spectral estimate. This ensures that for all $1 \leq p \leq 2$, $k \geq 1$, and any choice of $\delta > 0$, a uniform constant $\tilde{C}_{k,p} > 0$ holds for every $f \in L^p(\mathbb{R})$

$$\tilde{\omega}_k(f, \delta)_p^{p'} \geq \tilde{C}_{k,p} \int_{-\infty}^{\infty} \min(1, (|\xi| \delta)^{2kp'}) |\mathcal{H}_f(\xi)|^{p'} d\xi.$$

This inequality quantifies how higher-order smoothness forces the Hartley transform to decay at high frequencies.

All the above results have been established for the Euclidean space $\mathbb{R}$. The work generalises previously known estimates (which were restricted to the case $k = 1$) to arbitrary higher-order smoothness $k \geq 2$.

Future research directions may include improving the sharpness of the constants, extending the results to anisotropic and Besov-type smoothness spaces,

and investigating stability and inverse problems within Hartley-based reconstruction frameworks.

Bibliography

- [1] I. Paraskevas, M. Barbarosou, and E. Chilton. Hartley transform and the use of the whitened hartley spectrum as a tool for phase spectral processing. *The Journal of Engineering*, 2015. Received 28 December 2014; Accepted 9 February 2015.
- [2] N. A. Malik, F. Ahmad, and D. K. Jain. The theoretical overview of the hartley transform and the generalized r-function. *International Journal of Applied and Computational Mathematics*, 2020. Received: April 03, 2020; Revised: June 02, 2020.
- [3] N. Kakharman and N. Tokmagambetov. Growth properties of hartley transform via moduli of continuity. *Journal of Mathematical Analysis and Applications*, 518(2):126–145, 2023.
- [4] R. V. L. Hartley. A more symmetrical fourier analysis applied to transmission problems. *Proceedings of the IRE*, 30(3):144–150, 1942.
- [5] R. N. Bracewell. *The Hartley Transform*. Oxford University Press, New York, 1986.
- [6] R. N. Bracewell. The fast hartley transform. *Proceedings of the IEEE*, 72(8):1010–1018, 1984.
- [7] K. J. Olejniczak. The hartley transform. In A. D. Poularikas, editor, *The Transforms and Applications Handbook*. CRC Press, Boca Raton, FL, 2000.
- [8] L. Debnath and D. Bhatta. *Integral Transforms and Their Applications*. CRC Press, Taylor and Francis Group, Boca Raton, FL, 3 edition, 2015.
- [9] N. I. Akhiezer. *Lektsii po teorii approksimatsii [Lectures on approximation theory]*. Nauka, Moscow, 2nd edition, 1965. In Russian. Classic lectures on approximation theory.
- [10] R. A. DeVore and G. G. Lorentz. *Constructive Approximation*. Number 303 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin; New York, 1993.
- [11] Z. Ditzian. Smoothness of a function and the growth of its fourier transform or its fourier coefficients. *Journal of Approximation Theory*, 162(5):980–9866, 2010.

- [12] Z. Ditzian. Relating smoothness to expressions involving fourier coefficients or to a fourier transform. *Journal of Approximation Theory*, 164:1369–1389, 2012.
- [13] W. O. Bray and M. A. Pinsky. Growth properties of fourier transforms via moduli of continuity. *Journal of Functional Analysis*, 255:2265–2285, 2008.
- [14] W. O. Bray and M. A. Pinsky. Growth properties of the fourier transform. *Filomat*, 26(4):755–760, 2012.
- [15] D. Gorbachev and S. Tikhonov. Moduli of smoothness and growth properties of fourier transforms: two-sided estimates. *Journal of Approximation Theory*, 164:1283–1312, 2012.
- [16] D. Gioev. Moduli of continuity and average decay of fourier transforms: two-sided estimates. In *Contemporary Mathematics*, volume 458, pages 377–392. American Mathematical Society, 2008.
- [17] E. C. Titchmarsh. *Theory of Fourier Integrals*. Clarendon Press, Oxford, 1937.
- [18] R. L. Herman. *An Introduction to Fourier and Complex Analysis with Applications to the Spectral Analysis of Signals*. Open Textbook Library, 2016.
- [19] W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, New York, 3rd edition, 1976.
- [20] A. Grigoryan. Convergence of fourier series (lecture 5). University of California, Santa Barbara, Department of Mathematics.
- [21] E. M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, New Jersey, 1971.
- [22] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*, volume 2. Springer, 2011.
- [23] S. Yakubovich. Applications and properties of the hartley transform. *Integral Transforms and Special Functions*, 25(12):937–953, 2014.
- [24] A. D. Poularikas. Hartley transform. In *The Transforms and Applications Handbook*, chapter 5, pages 45–72. CRC Press, 2 edition, 2000.
- [25] F. Bouzeffour. Titchmarsh-type theorems for the hartley integral transform. *Integral Transforms and Special Functions*, pages 1–14, 2024. Plancherel theorem, inversion formula, shift operator, Hartley-Lipschitz classes and Titchmarsh analog.

Moduli of smoothness and growth properties of Hartley transform

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The Hartley transform of a function $f(x)$ is defined by

$$\mathcal{H}_f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) [\cos(\omega x) + \sin(\omega x)] dx, \quad (1)$$

and has the property of being its own inverse (an involution) (see also [1], [2]). The Hartley transform operates on real-valued signals and produces real-valued output, which offers certain advantages in practical applications. Several relationships between the Hartley and Fourier transforms are presented in [1], [2].

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a natural number $k \in \mathbb{N}$, we define the *symmetric difference operator of order k* , denoted by Δ_h^k , for a given real number h as

$$\Delta_h^k[f](x) = \sum_{j=0}^k (-1)^j \binom{k}{j} f\left(x + \left(\frac{k}{2} - j\right)h\right).$$

The corresponding L^p -modulus of continuity of order k using this symmetric difference is defined as:

$$\Omega_p^{(k)}(f; \delta) = \sup_{0 \leq h \leq \delta} \|\Delta_h^k[f]\|_p.$$

In this paper, we generalize this Hartley transform framework to higher-order smoothness classes. We establish new inequalities that bound the transform using higher-order moduli of continuity, thereby obtaining sharper versions of the foundational lemmas and providing a more precise characterization of the relationship between a function's smoothness and the decay of its Hartley transform.

Theorem 1. *Let $1 \leq p \leq 2$, $k \in \mathbb{N}$, and $f \in L^p(\mathbb{R})$. Then there exists a constant $c_{p,k} > 0$ such that*

$$\left(\int_{\mathbb{R}} \min \left\{ 1, (\delta|\omega|)^{kp'} \right\} |\mathcal{H}_f(\omega)|^{p'} d\omega \right)^{1/p'} \leq c_{p,k} \Omega_p^{(k)}(f; \delta), \quad \text{for } 1 < p \leq 2, \quad (2)$$

and

$$\sup_{\omega \in \mathbb{R}} \left\{ \min \left\{ 1, (\delta|\omega|)^k \right\} |\mathcal{H}_f(\omega)| \right\} \leq c_{1,k} \Omega_1^{(k)}(f; \delta), \quad \text{for } p = 1. \quad (3)$$

Here, $\Omega_p^{(k)}(f; \delta)$ denotes the k -th order modulus of continuity of f in $L^p(\mathbb{R})$.

Keywords: Modulus of smoothness; Hartley transform; Riemann–Lebesgue Lemma; Hausdorff–Young inequality; Titchmarsh's theorem

References

- [1] Bracewell R.N. The Hartley Transform, Stanford University Press, Stanford (1986).
- [2] Olejniczak K.J. The Hartley Transform, in: The Transforms and Applications Handbook, vol. 281 (2000).