ФИЗИКА-МАТЕМАТИКА ҒЫЛЫМДАРЫ

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SOME PROPERTIES OF FINITELY QUILTED ORDERED STRUCTURES

Abstract: It is knownthat any cut in an o-minimal structure has a unique extension up to a complete type over the model. For weakly o-minimal structures a cut can have at most two extensions up to complete types, and the sets of realizations of these types are convex in any elementary extensions. Generalizing weak o-minimality we obtain the following notion of an *n*-quilted structure: a totally ordered structure is said to be n-quilted if any cut has at most *n* extensions up to complete types. Note that we omit here condition that the set of all realizations of a type must be convex. In this article we investigate basic properties of *n*-quilted structures.

Since notion of o-minimality had appeared in [1] a series of generalizations were suggested such as weak o-minimality, quasi-o-minimality, almost o-minimality, quasi weak o-minimality, C-minimality, variants of o-minimality and others.

Key words: Mathematical logic, model theory, o-minimality, ordered structures, a cut, a complete type.

Here we suggest a new notion of a finitely quilted structure. But before we explain why we suggest this notion.

Definition 1 [1] *A totally ordered structure* $(M,<,\dots)$ *is called o-minimal if for any formula* $\varphi(x,\overline{y})$ *and any parameter* $\overline{a}$ *the set* $\varphi(M,\overline{c})$ *is a finite union of intervals and points.*

It is well known that a totally ordered structure is o-minimal iff any cut has a unique extension up to a complete type.

Definition 2[2] *A totally ordered structure* $(M,<,\dots)$ *is called weakly o-minimal if for any formula* $\varphi(x,\overline{y})$ *and any parameter* $\overline{a}$ *the set* $\varphi(M,\overline{a})$ *is a finite union of convex sets.*

Theorem 3[3] $(M,<,\dots)$ *is weakly o-minimal if and only if for any cut* (C,D) *there are at most two complete types extending* (C,D) *and if there are two types then the set of realizations of these two types in any elementary extensions are convex.*

So we see that the number of extensions of a cut characterize weak o-minimality. That is why the following definition is quite natural.

Definition 4 *A totally ordered structure* $(M,<,\dots)$ *is called n-quilted if any cut in* M *has at most* n *extensions to complete types over* M*. A totally ordered structure is called finitely quilted if it is n-quileted for some natural number n.*

Obviously, any weakly o-minimal structure is 2-quilted.

Now we give an example of a 2-quilted structure which is not weakly o-minimal.

Example 5 $M=(\mathbb{R},<,Q)$; $M\vDash Q(a)\Leftrightarrow a\in\mathbb{Q}$.

Theorem 6 *The elementary theory of* $(R,<,Q)$ *admits quantifier elimination.*

Proof: First we write axioms for $(R,<,Q)$:

1) $\forall x \forall y(x < y \rightarrow y \not\leq x)$;

2) $\forall x \forall y \forall z(x < y \wedge y < z \rightarrow x < z)$;

3) $\forall x \forall y(x < y \vee x < y \vee x > y)$;

4) $\forall x \exists y(y < x)$;

5) $\forall x \exists y(x < y)$;

6) $\forall x \forall y \exists z(x < y \rightarrow x < z \wedge z < y)$;

7) $\forall x \forall y \exists z(x < y \rightarrow x < z \wedge z < y \wedge Q(z))$;

8) $\forall x \forall y \exists z(x < y \rightarrow x < z \wedge z < y \wedge \neg Q(z))$.

By Tarski's test it is sufficient to eliminate existential quantifier in a formula of the form $\exists x \vee_{i=1}^{n} \varphi_i(x, \overline{y}_i)$, where φ_i are literals, that is of the form

$$x = y, x \neq y, x < y, x \not\leq y, x > y, x \not\geq y, Q(x), \neg Q(x).$$

Observe that we have the following equivalences:

$$x \neq y \Leftrightarrow (x < y) \vee (x > y);$$
$$xy \Leftrightarrow (x = y) \vee (x > y);$$
$$xy \Leftrightarrow (x < y) \vee (x = y).$$

So if we consider $\exists x(x \neq y \wedge \varphi(x, \overline{y}))$. This is equivalent to

$$\exists x(((x < y) \vee (y < x)) \wedge \varphi(x, \overline{y})).$$

We open parenthesis by distributive low:

$$\exists x[((x < y) \wedge \varphi(x, \overline{y})) \vee ((x > y) \wedge \varphi(x, \overline{y}))]$$

And this is equivalent to

$$\exists x((x < y) \wedge \varphi(x, \overline{y})) \vee \exists x((x > y) \wedge \varphi(x, \overline{y}))$$

Now we can eliminate these quantifiers separately. So we may assume that we do not have φ_i of the form $x \neq y, x \leq y, x \geq y$. Similarly,

$$((x < y) \wedge (x < z)) \Leftrightarrow ((x < y) \wedge (y < z)) \vee ((x < z) \wedge (yz))$$

So we may assume that at most one of φ_i's is of the form $x > y$.

$$\exists x((x < y) \wedge (x < z) \wedge \varphi(x, \overline{y})) \Leftrightarrow$$
$$\exists x[(((x < y) \wedge (y < z)) \vee ((x < z) \wedge (yz))) \wedge \varphi(x, \overline{y})] \Leftrightarrow$$
$$\exists x[((x < y) \wedge (y < z) \wedge \varphi(x, \overline{y})) \vee ((x < z) \wedge (yz) \wedge \varphi(x, \overline{y}))] \Leftrightarrow$$
$$\exists x((x < y) \wedge (y < z) \wedge \varphi(x, \overline{y})) \vee \exists x((x < z) \wedge (yz) \wedge \varphi(x, \overline{y})) \Leftrightarrow$$
$$[\exists x((x < y) \wedge \varphi(x, \overline{y})) \wedge (y < z)] \wedge [\exists x((x < z) \wedge \varphi(x, \overline{y})) \wedge (yz)]$$

If one of φ_i's is $x = z$ formula $\exists x \vee_{i=1}^{n} \varphi_i(x, \overline{y}_i)$ is equivalent to $\vee_{i=1}^{n} \varphi_i(z, \overline{y}_i)$, which is quantifier-free. So we may assume that there is no φ_i of the form $x = y$.

Thus we have the following kinds of formulas:
1.1 $\exists x((y < x < z) \wedge Q(x))$;
1.2 $\exists x((y < x < z) \wedge \neg Q(x))$;
1.3 $\exists x(y < x < z)$;
2.1 $\exists x((x < z) \wedge Q(x))$;
2.2 $\exists x((x < z) \wedge \neg Q(x))$;
2.3 $\exists x(x < z)$;
3.1 $\exists x((y < x) \wedge Q(x))$;
3.2 $\exists x((y < x) \wedge \neg Q(x))$;
3.3 $\exists x(y < x)$;
4.1 $\exists x(Q(x))$;
4.2 $\exists x(\neg Q(x))$.

By axiom (7) the formula 1.1 is equivalent to $y < z$.The formula 1.2 is equivalent to $y < z$ by axiom 8.The formula 1.3 is equivalent to $y < z$ by axiom 7.By axioms 4 and 6,7,8 the formulas 2.1, 2.2, 2.3 are true, that is equivalent to z = z.Similarly 3.1, 3.2, 3.3 are equivalent to y = y by axioms 5, 6, 7, 8.4.1 is equivalent to y = y by axiom 7 and 4.2 is equivalent to y = y by axiom 8.

Consider any cut $\langle C, D\rangle$. Let φ be consistent with $\langle C, D\rangle$. By quantifier elimination φ is $C < x\ x < d, Q(x), \neg Q(x), x = a$. Since formulas $c < x$ and $x < a$ already belongs to $\langle C, D\rangle$ if $c \in C, d \in D$.So the only way to extend the is to add $Q(x)$ or $\neg Q(x)$. Thus we have that each cut has exactly 2 extension, so this structure is 2-quilted.

Example 7 $(\mathbb{R}, <, Z)$ *is 2-quilted. Any bounded cut has a unique extension, the cuts* $+\infty = \langle R, \emptyset\rangle$ *and* $-\infty = \langle \emptyset, R\rangle$*are consisted with* $Z(x); \neg Z(x)$.

Example 2.7 is similar to the example 2.6.

Now we give an example of a 2-quilted structure which is not weakly quasi o-minimal.

Let $M = \mathbb{Q} \cup (\mathbb{Q} + \pi)$ and $\Sigma = \{=, <, P^2\}$. the order $<$ on M is the restriction of the natural order on $\mathbb{R}$ to M. If $a \in \mathbb{Q}$ then $P(M, a) = (a - 1, a + 1) \cap \mathbb{Q}$, otherwise $P(M, a) = (a - 1, a + 1) \cap (\mathbb{Q} + \pi)$.

Also we define function $s(x)$ as $s(a) = supP(M, a)$

$$y = s(x) \Leftrightarrow \forall z(P(z, x) \rightarrow z < y) \wedge \forall t[\forall z(P(z, x) \rightarrow z < t) \rightarrow y \leq t]$$

Theorem 8$Th(M, <, P, s, s^{-1})$ *admits quantifier elimination.*

Proof: Any term has the following form: $s^n(x)$,where $n \in Z$ So we have the following literals:

1. $s^n(x) = s^k(y)$;
2. $s^n(x) \neq s^k(y)$;
3. $s^n(x) < s^k(y)$;
4. $s^n(x)s^k(y)$;
5. $s^n(x) > s^k(y)$;

6. $s^n(x)s^k(y)$;
7. $P(s^n(x), s^k(y))$;
8. $\neg P(s^n(x), s^k(y))$.

Similarly to Theorem 6 we may assume that in $\exists x \vee_{i=1}^{n} \varphi_i(x, \overline{y}_i)$ there is no φ_i of the forms 2, 4, 6.

Note that $s^n(x) = s^k(y)$ is equivalent to $s^{n+m}(x) = s^{k+m}(y)$ for any $m \in Z$ and $P(s^n(x), s^k(y))$ if and only if $P(s^{n+m}(x), s^{k-m}(y))$ for any $m \in Z$. So we may assume that n in each φ_i is the same and $n = 0$. If at least one of $\varphi_i(x, y)$ is one of the form $s^n(x) = s^k(y)$ then

$$\exists x \vee_{i=1}^{n} \varphi_i(s^n(x), \overline{y}_i) \Leftrightarrow \vee_{i=1}^{n} \varphi_i(s^k(y), \overline{y}_i).$$

Thus we may assume that no φ_i of the form $s^n(x) = s^k(y)$. Also we may assume that at most one φ_i of the form 3 and 5. So the general form is

$$\exists x(s^m(y_1) < x < s^k(y_2) \wedge \vee_i P(x, z_i) \wedge \vee_j \neg P_j(x, u_j)).$$

Note that $P(x, z_1) \wedge P(x, z_2)$ is equivalent to $P(x, z_1) \wedge s^{-1}(z_2) < x < s(z_2) \wedge (P(z_2, z_1) \vee P(s(z_2), z_1) \vee P(z_2, s(z_1)))$. So we may assume that at most one φ_i is of the form $P(x, z)$.

$P(x, z) \wedge \neg P(x, y) \Leftrightarrow P(x, z) \wedge [x < s^{-1}(u) \vee x > s(u) \vee (s^{-1}(u) < x < s(u) \wedge \neg P(x, u))]$
$\Leftrightarrow (P(x, z) \wedge x < s^{-1}(u)) \vee (P(x, z) \wedge x < s(u)) \vee (P(x, z) \wedge s^{-1}(u) < x$
$< s(u) \wedge \neg P(x, u)) \Leftrightarrow (P(x, z) \wedge x < s^{-1}(u)) \vee (P(x, z) \wedge x$
$< s(u)) \vee P(x, z) \wedge s^{-1}(u) < x < s(u) \wedge (P(z, u) \vee P(s(z), u) \vee P(z, s(u)))]$
$P(x, z) \wedge s^{-1}(u) < x < s(u) \wedge \neg P(z, u) \wedge \neg P(s(z), u) \wedge \neg P(z, s(u))$

So if we have at least one φ_i of the form $P(x, z)$ we may assume that there is no φ_1 of the form $\neg P(x, u)$. Consider

$$\exists x(s^m(y_1) < x < s^k(y_2) \wedge P(x, z))$$

which is equivalent to $\exists x(s^m(y_1) < x < s^k(y_2) \wedge s^{-1}(z) < x < s(z) \Leftrightarrow s^m(y_1) < s^k(y_2) \wedge s^m(y_1) < s(z) \wedge s^{-1}(z) < s^k(y_2) \wedge s^{-1}(z) < s(z)$.

So it is sufficient to consider

$$\exists x(s^m(y_1) < x < s^k(y_2) \wedge \vee_{j=1}^{2} n \neg P(x, u_j)) \Leftrightarrow$$

$s^m(y_1) < s^k(y_2) \wedge [s^{m+4n}(y_1) < s^k(y_2) \vee (s^{m+4n}(y_1) > s^k(y_2) \wedge \neg \vee_{\tau \in S_{2n}} (s^{-1}(u_{\tau(1)})$
$\leq s^m(y_1) \wedge s^{-1}(u_{\tau(2)}) \leq s(u_{\tau(1)}) \wedge s^{-1}(u_{\tau(3)}) \leq s(u_{\tau(2)}) \wedge \ldots \wedge s^{-1}(u_{\tau(n)})$
$\leq s(u_{\tau(n-1)}) \wedge s(u_{\tau(n)})$
$\geq s^k(y_2) \wedge P(u_{\tau(1)}, u_{\tau(2)}) \wedge P(u_{\tau(2)}, u_{\tau(3)}) \wedge \ldots \wedge P(u_{\tau(n-1)}, u_{\tau(n)}) \wedge s^{-1}(u_{\tau(n+1)})$
$\leq s^m(y_1) \wedge s^{-1}(u_{\tau(n+2)}) \leq s(u_{\tau(n+1)}) \wedge s^{-1}(u_{\tau(n+3)}) \leq s(u_{\tau(n+2)}) \wedge \ldots \wedge s^{-1}(u_{\tau(2n)})$
$\leq s(u_{\tau(2n-1)}) \wedge s(u_{\tau(2n)})$
$\geq s^k(y_2) \wedge P(u_{\tau(n+1)}, u_{\tau(n+2)}) \wedge \ldots \wedge P(u_{\tau(2n-1)}, u_{\tau(2n)}) \wedge \neg P(u_{\tau(1)}, u_{\tau(n+1)})))]$

From quantifier elimination it is easy to prove that the considered structure is 2-quilted.

Obviously, any finitely quilted structure is o-ω-stable. Since any o-stable ordered group is Abelian and divisible [4], we obtain that any finitely quilted ordered group is Abelian and divisible.

Lemma 9 Let G be an ordered finitely quilted groups. Then any definable subgroup is convex.

Proof. Assume the contrary, that a definable subgroup H of the group G is not convex. Since G is divisible the subgroup H has the infinite index in its convex hull. So the cut which is defined by sup$H$ has infinitely many extensions by formulae $H(x) + g$.

Theorem 10 Let G be an ordered finitely quilted groups as well as its any elementary extensions. Then Th(G) is weakly o-minimal.

Proof. Assume that a definable subset A has infinitely many convex components. Let a cut $\langle C, D\rangle$ be consistent with both A and the complement of A. This cut does exists, becouse the set A has infinitely many components. Consider an infinitesimal element b in some elementary extensions, such that for any c, realizing the cut $\langle C, D\rangle$ the element $b + c$ also realizes this cut. Fix som sufficiently saturated elementary extension and consider cuts defined by sup$C$ and inf$D$. It is easy to see that the definable sets $A + nb$are consistent with these cuts for any natural nunmber n. For details one can see [5]. So this cut has infinitely many extensions up to complete types.

Question Is there a 2-quilted ordered group which is not weakly o-minimal?

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НЕКОТОРЫЕ СВОЙСТВА КОНЕЧНО ПРОСТЕГАННЫХ УПОРЯДОЧЕННЫХ СТРУКТУР

Аннотация: Известно, что любое сечение в о-минимальной структуре имеет единственное расширение до полного типа над моделью. Для слабо о-минимальных структур сечение может иметь максимум два расширения до полных типов, причем множества всех реализаций этих типов являются выпуклыми в любых элементарных расширениях. Обобщая слабую о-минимальность, получаем следующее понятие n-стеганых структур: линейно упорядоченная структура называется n-стеганой, если любое сечение имеет не более n расширений до полного типов. Обратите внимание, что мы здесь опускаем условие, что множество всех реализаций типа должно быть выпуклым. В этой статье мы исследуем основные свойства n-стеганых структур.

Ключевые слова: математическая логика, теория моделей, о-минимальность, упорядоченные структуры, сечение, полный тип.