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The solution of the heat equation in domains with moving boundaries by the Integral Error Functions method

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INTRODUCTION

In mathematics , development of new analytical methods of solution of the heat transfer problems is very important for various applications because it enables one to analyze an interrelationship of various input parameters on the dynamics of investigating phenomena, while the use of numerical methods is a problem when the number of parameters is great. And from mathematical point of view, most mathematical models based on Verigin, Stefan and inverse Stefan type boundary value problems. Such problems are among the most complicated, formidable and difficult problems in the theory of nonlinear parabolic equations in mathematical physics, since the corresponding integral equations are singular and require new approaches in solving problems analytically and numerically and also which long with the desired solutions of the equations, moving boundaries have to be found. In some cases, heat potentials can be constructed by which the boundary-value problems can be reduced to integral equations. However, in the case of domains that are degenerate at the initial time, additional difficulties arise due to the singularity of the integral equations, which belong to the class of pseudo-Volterra equations that are unsolvable in the general case by the method of successive approximations. These results are obtained by S.N. Kharin [2].

The method of solving heat transfer problems with moving boundaries and phase transformation is represented by Integral Error Functions and its properties.

The results indicate that Integral Error Functions enable to solve, many practical problems described above in the easier way than classical methods, and could be implemented into the course of teaching mathematical physics, as special methods of solving heat transfer problems with moving boundaries.

1 LITERATURE REVIEW: THE THEORICAL ASPECTS AND METHODS OF SOLVING HEAT EQUATIONS IN THE DOMAINS WITH FIXED AND MOVING BOUNDARIES

1.1 General overview of theorical aspects for heat equations.

In the history there are many methods used for solving Heat Equations. For parabolic partial differential equations (in our work, for heat equations) in the domain with fixed and finite boundaries Method of Separable Variables or Generalized Fourier Series applied. As is well known for domains with fixed infinite or semi-infinite boundaries, it's more suitable to use Laplace and Fourier Transforms. The most remarkable method that applied to solve Heat Equations with fixed and moving boundaries is Heat Potential (for Single and Double layer). And methods of solving Heat Problems with moving boundaries are based on the reduction of Heat Equation to a system of integral equations which cause great difficulties.

The theory on solving Parabolic Equations in domains with fixed boundaries is well developed and discussed in details almost in all "Partial Differential Equations" course books, and as it was said above it is out of scope of this chapter to overview all of them only some examples will be discussed to outline the concept.

1.2 Heat Potentials (Solutions of Heat Equations with fixed boundaries)

Heat Potentials of Single layer and Double layer are one of the powerful methods applied to solve Heat Equations in domains with fixed and moving boundaries. It is well known that Heat Equation can be reduced to the system of Integral Equations by the help of Heat Potentials which are represented in the following form [1]:

$$W_1(x,t)=\int_0^t \frac{a}{\sqrt{\pi(t-\tau)}} e^{\frac{(x-l)^2}{4a^2(t-\tau)}} \mu(\tau)\,dt$$

$$W_2(x,t)=\int_0^t \frac{x-l}{2a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} e^{\frac{(x-l)^2}{4a^2(t-\tau)}} \mu(\tau)\,dt$$

From physical point of view it means that the distribution of the temperature U(x,t) along an infinite bar $-\infty < x < \infty$ caused by an instantaneous heat source of the unit power density placed at the point x=0 at the time t=0. If the heat source located at the x=l act continuously from the time τ=0 to the time τ=t and its power density at τ is μ(t), then the distribution of the temperature is the convolution G*μ; i.e

$$U(x,t) = \int_0^t \frac{e^{-\frac{(x-l)^2}{4a^2(t-\tau)}}}{2a\sqrt{\pi(t-\tau)}} \mu(\tau)\,d\tau \qquad (1)$$

1.3 Solutions of Heat Equations in the domains with moving boundaries

Development of analytical methods of solution of free boundary value problems are of great theoretical and practical interest. From mathematical point of view the well-known analytical method is based on the representation of a solution in the form of heat potentials with following reduction of the given problem to integral equation [16].However if the domain with moving boundary degenerates into a point at the initial time, the integral equations become singular and cannot be solved by Picard's method. Asymptotic properties of such equations have been investigated in [17]. In the next three chapters it will be shown that Integral Error Functions enable to find qualitative and quantitative solution of heat equations with moving boundaries in easier and more precise way than above classical methods. Moreover from practical point of view Integral Error Functions allows us to solve problems concerned with analysis of dynamics of phenomena of heat and mass transfer with phase transformation, hydrodynamic flows and many other problems which will be shown in the last chapter.

1.4 IEF method

There is another new special method which helps to solve Heat Equations in the domains with fixed and moving boundaries. Heat equations are solved by the help of Integral Error Functions (IEF method) and its properties, which were introduced by Hartree in 1935 and reasonably sometimes called Hartree functions. This method can be used to solve first, second and third boundary value problems for Heat Equations with fixed and moving finite, semi-infinite and infinite boundaries. Even though it is not the most powerful side of IEF method for the domains with fixed boundaries and it is hard to say that the introduced method is more advantageous than classical Fourier, Laplace transforms and Heat Potentials except the difficulties, that it is sometimes hard to find inverse transformation and almost every time impossible to evaluate and find quantitative values of bulky integrals , however it is possible to see that in the domains with moving boundaries especially in the domains with degenerating boundary conditions at the initial time, IEF method is much more preferable than classical methods, as from theoretical same from practical points of view.

The integral error functions determined by recurrent formulas

$$i^n erfcx = \int_x^\infty i^{n-1} erfcv dv, \qquad n=1,2,\ldots \qquad i^0 erfcx \equiv erfcx = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-v^2) dv \qquad (2)$$

Where

$$erfx = 1 - erfcx = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-v^2) dv \qquad (3)$$

One can obtain from

$$i^n erfcx = \frac{2}{\sqrt{\pi}} \frac{1}{n!} \int_x^{\infty} (v-x)^n \exp(-v^2) dv \tag{4}$$

Expressions (2) satisfy the differential equation

$$\frac{d^2}{dx^2} i^n erfcx + 2x \frac{d}{dx} i^n erfcx - 2ni^n erfcx = 0 \tag{5}$$

and recurrent formulas

$$2ni^n erfcx = i^{n-2} erfcx - 2xi^{n-1} erfcx \tag{6}$$

Integral Error Functions are very useful for investigation of heat transfer, diffusion and other phenomena which can be described by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{7}$$

in a region $D(t > 0, 0 < x < \alpha(t))$ with free boundary $x = \alpha(t)$, since the functions

$$u_n(\pm x, t) = t^{\frac{n}{2}} i^n erfc \frac{\pm x}{2a\sqrt{t}}$$

suffice the equation (7) as well as their linear combination or even series

$$u(x,t) = \sum_{n=0}^{\infty} [A_n u_n(x,t) + B_n u_n(-x,t)]$$

For any constants A_n, B_n. We can choose these constants to satisfy the boundary conditions at

$x = 0$ and $x = \alpha(t)$, if given boundary functions can be expanded into Taylor series with powers

t or $\sqrt{t}$.

1.5 Properties of Integral Error Function

It is possible to derive new properties of Integral Error Functions.

1. If n is an integer, then

$$i^n erfc(-x)+(-1)^n i^n erfcx=\frac{1}{2^{n-1}n!i^n}H_n(ix)=\frac{1}{2^{n-1}n!}e^{-x^2}\frac{d^n}{dx^n}e^{x^2} \text{with} \qquad i=\sqrt{-1}$$

and Hermite polynomials $H_n(x)$ in the right side. Indeed, using formula (4) one can write

$$i^n erfc(-x)+(-1)^n i^n erfcx=\frac{2}{\sqrt{\pi}}\frac{1}{n!}\int_{-x}^{\infty}(v+x)^n\exp(-v^2)dv+$$

$$\frac{(-1)^n 2}{n!\sqrt{\pi}}\int_{x}^{\infty}(v-x)^n\exp(-v^2)dv=\frac{2}{n!\sqrt{\pi}}\int_{-\infty}^{\infty}(v+x)^n\exp(-v^2)dv=\frac{1}{2^{n-1}n!i^n}H_n(ix) \quad (8)$$

2. Using formula for Hermite polynomials one can derive

$$i^n erfc(-x)+(-1)^n i^n erfcx=\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{x^{n-2m}}{2^{2m-1}m!(n-2m)!} \qquad (9)$$

If $n=2k$, then

$$i^{2k} erfc\, x+i^{2k} erfc\, (-x)=\sum_{m=0}^{k}\frac{x^{2(k-m)}}{2^{2m-1}m!(2k-2m)!}$$

In particular

$$erfc\, x+erfc\, (-x)=2,$$

$$i^2 erfc\, x+i^2 erfc\, (-x)=\frac{1}{2}+x^2 ,$$

$$i^4 erfc\, x+i^4 erfc\, (-x)=\frac{1}{8}+\frac{1}{4}x^2+\frac{1}{12}x^4 .$$

If $n=2k+1$, then

$$i^{2k+1} erfc(-x)-i^{2k+1} erfc\, x=\sum_{m=0}^{k}\frac{x^{2(k-m)+1}}{2^{2m-1}m!(2k-2m+1)!} \qquad (10)$$

In particular

$$ierfc(-x)-ierfc\, x=2x,$$

$$i^3 erfc(-x)-i^3 erfc\, x=\frac{1}{2}x+\frac{1}{3}x^3 ,$$

$$i^5 erfc\,(-x) - i^5 erfc\, x = \frac{1}{2^3 \cdot 2!}x + \frac{1}{2 \cdot 2! \cdot 3!}x^3 + \frac{2}{5!}x^5 .$$

The proof of the formula

$$i^n erfc(-x) - (-1)^n i^n erfcx = \frac{1}{2^{n-1} n!} e^{-x^2} \frac{d^n}{dx^n}(e^{x^2} erfx) \tag{11}$$

where

$$erfx = 1 - erfcx = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-v^2) dv$$

can be obtained by mathematical induction method using recurrent formula (1.5).

3. Differentiating the right side of formula (11), we obtain

$$i^n erfc(-x) - (-1)^n i^n erfcx = P_n(x) erfx - Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2), \tag{12}$$

where polynomials $P_n(x)$ and $Q_n(x)$ are defined by formulas

$$P_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2m}}{2^{2m-1} m!(n-2m)!}, \qquad Q_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^{n-k} H_{n-k-1}(x)}{2^{n-k}(n-k)!} P_k(x)$$

4. From (11), (12) we can obtain the explicit expressions for Integral Error Functions of an integer index

$$i^n erfcx = \frac{(-1)^n}{2}[P_n(x) erfcx + Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2)] \tag{13}$$

$$i^n erfc(-x) = \frac{1}{2}[P_n(x) erfc(-x) - Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2)] \tag{14}$$

5. Using L'Hopital rule and representation (1.1), it is not difficult to show that

$$\lim_{x \to \infty} \frac{i^n erfc(-x)}{x^n} = \frac{2}{n!} \tag{15}$$

1.6 Corollaries for IEF method

Following corollaries will be helpful to so solve Heat equations.

1. Using property 2 one can derive following formula

$$u(x,t)=\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^{m}\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^{m}\beta_{2n+1,m}\right\}$$

Where u(x,t) is a solution of Heat Equation in polynomial form and

$$\beta(n,m):=\frac{1}{2^{n+m-1}\cdot m!\cdot(n-2m)!}$$

2. Expression u(x,t) can be expanded in the following form

$$u(x,t)=A_0\beta_{0,0}+$$

$$+A_2\left(x^2\beta_{2,0}+t\beta_{2,1}\right)+$$

$$+A_4\left(x^4\beta_{4,0}+x^2t\beta_{4,1}+t\beta_{4,2}\right)+\cdots+$$

$$+A_{2k}\left(x^{2k}\beta_{2k,0}+x^{2k-2}t\beta_{2k,1}+\cdots+x^2t^{k-1}\beta_{2k,k-1}+t^k\beta_{2k,k}\right)+$$

$$+A_1x\beta_{1,0}+$$

$$+A_3\left(x^3\beta_{3,0}+xt\beta_{3,1}\right)+$$

$$+A_5\left(x^5\beta_{5,0}+x^3t\beta_{5,1}+xt^2\beta_{5,2}\right)+\cdots+$$

$$+A_{2k+1}\left(x^{2k+1}\beta_{2k+1,0}+x^{2k-1}t\beta_{2k+1,1}+\cdots+x^3t^{k-1}\beta_{2k+1,k-1}+xt^k\beta_{2k+1,k}\right)$$

3. Following expression will be frequently used in solutions of Heat Equations

$$\frac{du}{dx} = 2A_2x\beta_{2,0} +$$

$$+A_4\left(4x^3\beta_{4,0} + 2xt\beta_{4,1}\right) +$$

$$+A_6\left(6x^5\beta_{6,0} + 4x^3t\beta_{6,1} + 2xt^2\beta_{6,2}\right) + \cdots +$$

$$+A_{2k}\left(2kx^{2k-1}\beta_{2k,0} + (2k-1)x^{2k-3}t\beta_{2k,1} + \cdots + 2xt^k\beta_{2k,k}\right) +$$

$$+A_1\beta_{1,0} +$$

$$+A_3\left(3x^2\beta_{3,0} + t\beta_{3,1}\right) +$$

$$+A_5\left(5x^4\beta_{5,0} + 3x^2t\beta_{5,1} + t^2\beta_{5,2}\right) + \cdots +$$

$$+A_{2k+1}\left((2k+1)x^{2k}\beta_{2k+1,0} + (2k-1)x^{2k-2}t\beta_{2k+1,1} + \cdots + 3x^2t^{k-1}\beta_{2k+1,k-1} + t^k\beta_{2k+1,k}\right)$$

4. Expression $u(x,t) = \sum_{n=0}^{k}\left(\sqrt{t}\right)^n\left[A_n i^n erfc\frac{x}{2a\sqrt{t}} + B_n i^n erfc\frac{-x}{2a\sqrt{t}}\right]$ can be expanded

$$u(x,t) = \left(\sqrt{t}\right)^0\left[A_0 i^0 erfc\frac{x}{2a\sqrt{t}} + B_0 i^0 erfc\frac{-x}{2a\sqrt{t}}\right] +$$

$$+\left(\sqrt{t}\right)^1\left[A_1 i^1 erfc\frac{x}{2a\sqrt{t}} + B_1 i^1 erfc\frac{-x}{2a\sqrt{t}}\right] +$$

$$+\left(\sqrt{t}\right)^2\left[A_2 i^2 erfc\frac{x}{2a\sqrt{t}} + B_2 i^2 erfc\frac{-x}{2a\sqrt{t}}\right] + \cdots +$$

$$+\left(\sqrt{t}\right)^n\left[A_n i^n erfc\frac{x}{2a\sqrt{t}} + B_n i^n erfc\frac{-x}{2a\sqrt{t}}\right]$$

5. Partial derivative of u(x,t) can be written as

$$\frac{\partial u}{\partial x} = \frac{1}{2a\sqrt{t}}\left[-A_0\exp\left(\frac{x^2}{4a^2t}\right) + B_0 exp\left(\frac{-x^2}{4a^2t}\right)\right] +$$

$$+\frac{1}{2a}\left[-A_1 i^0 erfc\frac{x}{2a\sqrt{t}} + B_1 i^0 erfc\frac{-x}{2a\sqrt{t}}\right] +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^1 \left[-A_2 i^1 erfc\frac{x}{2a\sqrt{t}} + B_2 i^1 erfc\frac{-x}{2a\sqrt{t}}\right] + \cdots +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^{n-1} \left[-A_n i^{n-1} erfc\frac{x}{2a\sqrt{t}} + B_n i^{n-1} erfc\frac{-x}{2a\sqrt{t}}\right]$$

1.7 Solution of Heat Equations in the domains with moving, given boundaries by IEF method

Integral Error Functions method enable investigate heat transfer, diffusion and other phenomena which can be described by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

In a region $D(t > 0, 0 < x < \alpha(t))$ with free boundary $x = \alpha(t)$, and solution is considered in the following form

$$u(x,t) = \sum_{n=0}^{\infty} [A_n u_n(x,t) + B_n u_n(-x,t)]$$

For any constants A_n, B_n. Which have to be found from boundary conditions at $x = 0$ and $x = \alpha(t)$, if given boundary functions can be expanded into Taylor series with powers t or $\sqrt{t}$.It is possible to see in the following examples (where all types of boundary value problems considered) and paragraphs, that analytic solutions of Heat Equations are found.

Analytic solution of Heat Equation with the first type boundary conditions in the $\beta\sqrt{t} < x < \alpha\sqrt{t}$ domain by IEF method

Analytical solution of Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad \beta\sqrt{t} < x < \alpha\sqrt{t}\ , \quad t > 0 \tag{16}$$

Subject to

I.C: $\qquad u(x,0) = 0, \qquad (17)$

B.C: $\qquad u(0,t) = \varphi(t), \qquad (18)$

$$u(l,t) = \phi(t), \tag{19}$$

$$u(0,0) = 0, \tag{20}$$

If functions $\varphi(t), \phi(t)$ are definite functions given in the form $\varphi(t) = \sum_{n=0}^{k} \mu_n t^{\frac{n}{2}}, \phi(t) = \sum_{n=0}^{m} \nu_n t^{\frac{n}{2}}$ Then solution can be represented in the form

$$u(x,t) = \sum_{n=0}^{\gamma} \left(\sqrt{t}\right)^n \left[A_n i^n erfc\frac{x}{2a\sqrt{t}} + B_n i^n erfc\frac{-x}{2a\sqrt{t}}\right]$$

Substituting expression into the boundary conditions

for $x = \beta\sqrt{t}$

$$u\left(\beta\sqrt{t}, t\right) = \sum_{n=0}^{\gamma} \left(\sqrt{t}\right)^n \left[A_n i^n erfc\frac{\beta}{2a} + B_n i^n erfc\frac{-\beta}{2a}\right]$$

where $\gamma = sup\{m, n\}$

for $x = \alpha\sqrt{t}$

Finally coefficients $A_0, A_1, A_2, \dots, A_\gamma$ and $B_0, B_1, B_2, \dots, B_\gamma$ are determined from system of linear equations

$$A_0 i^0 erfc\frac{\beta}{2a} + B_0 i^0 erfc\frac{-\beta}{2a} = \mu_0$$

$$A_0 i^0 erfc\frac{\alpha}{2a} + B_0 i^0 erfc\frac{-\alpha}{2a} = \nu_0$$

$$A_1 i^1 erfc\frac{\beta}{2a} + B_1 i^1 erfc\frac{-\beta}{2a} = \mu_1$$

$$A_1 i^1 erfc\frac{\alpha}{2a} + B_1 i^1 erfc\frac{-\alpha}{2a} = \nu_1$$

$$A_2 i^2 erfc\frac{\beta}{2a} + B_2 i^2 erfc\frac{-\beta}{2a} = \mu_2$$

$$A_2 i^2 erfc\frac{\alpha}{2a} + B_2 i^2 erfc\frac{-\alpha}{2a} = \nu_2$$

... ...

$$A_\gamma i^\gamma erfc\frac{\beta}{2a} + B_\gamma i^\gamma erfc\frac{-\beta}{2a} = \mu_\gamma$$

$$A_\gamma i^\gamma erfc\frac{\alpha}{2a} + B_\gamma i^\gamma erfc\frac{-\alpha}{2a} = \nu_\gamma$$

where $i^\gamma erfc\frac{\beta}{2a}, i^\gamma erfc\frac{-\beta}{2a}, i^\gamma erfc\frac{\alpha}{2a}, i^\gamma erfc\frac{-\alpha}{2a}$, γ=0,1,2... are identified from tables.

Remark: One of key points in solving Heat Equations in the domains with moving boundaries of the first type is to correctly identify value of γ, which takes maximum value between m and k in the boundary conditions.

Approximate solution of Heat Equation with the first type boundary conditions in the $0 < x < \alpha(t)$ domain by IEF method

Solution of the Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \alpha(t), \quad t > 0 \tag{21}$$

Subject to

B.C: $\qquad u(0,t) = \varphi(t), \qquad t > 0 \qquad$ (22)

$$u(\alpha(t),t)=\phi(t),t>0 \qquad (23)$$

can be represented in the following form

$$U(x,t)=\sum_{n=0}^{\infty}\left\{A_n\cdot t^n\left[i^{2n}erfc\frac{x}{2a\sqrt{t}}+i^{2n}erfc(\frac{-x}{2a\sqrt{t}})\right]+B_n\cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc\frac{x}{2a\sqrt{t}}-i^{2n+1}erfc(\frac{-x}{2a\sqrt{t}})\right]\right\} \qquad (24)$$

or using property 2 of IEF, solution can be represented in the polynomial form

$$U(x,t)=\sum_{n=0}^{k}\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\} \qquad (25)$$

In this case it is more convenient to use expression (24) for solution of (24)-(25).

From (19) for x=0, we have

$$\varphi(t)=\sum_{n=0}^{\infty}A_n t^n i^{2n}erfc0 \qquad (26)$$

And coefficient A_n of expression (24) can be found from

$$A_n=\frac{\varphi^n(0)}{i^{2n}\,n!\,erfc0} \qquad (27)$$

From (23) for x=$\alpha(t)$, we have

$$\psi(t)=\sum_{n=0}^{\infty}\left\{A_n\cdot t^n\left[i^{2n}erfc\frac{\alpha(t)}{2a\sqrt{t}}+i^{2n}erfc(\frac{-\alpha(t)}{2a\sqrt{t}})\right]+B_n\cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc\frac{\alpha(t)}{2a\sqrt{t}}-i^{2n+1}erfc(\frac{-\alpha(t)}{2a\sqrt{t}})\right]\right\}$$

$$B_n = -\frac{\left[\psi(\tau^2)\right]_{\tau=0}^{(2n+1)}}{(2n+1)!\cdot\binom{2n+1}{0}\cdot\left[i^{2n+1}\operatorname{erfc}\beta(\tau) - i^{2n+1}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}} +$$

$$+\frac{\sum_{k=0}^{n}\{A_k\cdot(2k)!\cdot\binom{2n+1}{2n+1-2k}\cdot\left[i^{2k}\operatorname{erfc}\beta(\tau) + i^{2k}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)}}{(2n+1)!\cdot\binom{2n+1}{0}\cdot\left[i^{2n+1}\operatorname{erfc}\beta(\tau) - i^{2n+1}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}} +$$

$$+\frac{\sum_{k=0}^{n-1}B_k\cdot(2k+1)!\cdot\binom{2n+1}{2n-2k}\cdot\left[i^{2k+1}\operatorname{erfc}\beta(\tau) - i^{2k+1}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}}{(2n+1)!\cdot\binom{2n+1}{0}\cdot\left[i^{2n+1}\operatorname{erfc}\beta(\tau) - i^{2n+1}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}} \qquad (28)$$

Thus required A_n and B_n coefficients are obtained and can be calculated from formulas (27) and (28) respectively.

2.1.1 Analytic solution of Heat Equation with the second type boundary conditions in the $\beta\sqrt{t} < x < \alpha\sqrt{t}$ domain by IEF method

Analytic solution of Heat Equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad \beta\sqrt{t} < x < \alpha\sqrt{t}\ , \quad t > 0 \qquad (29)$$

subject to

I.C: $$u(x,0) = 0, \qquad (30)$$

B.C: $$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \varphi(t), \qquad (31)$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=\alpha(t)} = \phi(t), \qquad (32)$$

$$u(0,0) = 0, \qquad (33)$$

where $\varphi(t) = \sum_{n=0}^{k} \mu_n t^{\frac{n}{2}}, \phi(t) = \sum_{n=0}^{k} \nu_n t^{\frac{n}{2}}$ analytical functions, can be represented in the form

$$u(x,t) = \sum_{n=0}^{k} \left(\sqrt{t}\right)^n \left[A_n i^n erfc\frac{x}{2a\sqrt{t}} + B_n i^n erfc\frac{-x}{2a\sqrt{t}}\right] \tag{34}$$

Substituting expression (20) into the boundary conditions (31) and (32) and applying UC method (undetermined coefficients method)

for $x = \beta\sqrt{t}$ we have:

$$\left.\frac{\partial u}{\partial x}\right|_{x=\beta\sqrt{t}} = \frac{1}{2a\sqrt{t}}\left[-A_0 \exp\left(\frac{\beta^2}{4a^2}\right) + B_0 i^0 exp(-\frac{\beta^2}{4a^2})\right] +$$

$$+\frac{1}{2a}\left[-A_1 i^0 erfc\frac{\beta}{2a} + B_1 i^0 erfc\frac{-\beta}{2a}\right] +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^1 \left[-A_2 i^1 erfc\frac{\beta}{2a} + B_2 i^1 erfc\frac{-\beta}{2a}\right] + \cdots +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^k \left[-A_{k+1} i^k erfc\frac{\beta}{2a} + B_{k+1} i^k erfc\frac{-\beta}{2a}\right] =$$

$$= \sum_{n=0}^{k} \mu_n t^{\frac{n}{2}}$$

and for $x = \alpha\sqrt{t}$ we have

$$\left.\frac{\partial u}{\partial x}\right|_{x=\alpha\sqrt{t}} = \frac{1}{2a\sqrt{t}}\left[-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right) + B_0 i^0 exp(-\frac{\alpha^2}{4a^2})\right] +$$

$$+\frac{1}{2a}\left[-A_1 i^0 erfc\frac{\alpha}{2a} + B_1 i^0 erfc\frac{-\alpha}{2a}\right] +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^1\left[-A_2 i^1 erfc\frac{\alpha}{2a}+B_2 i^1 erfc\frac{-\alpha}{2a}\right]+\cdots+$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^k\left[-A_{k+1} i^k erfc\frac{\alpha}{2a}+B_{k+1} i^k erfc\frac{-\alpha}{2a}\right]=$$

$$=\sum_{n=0}^{k}\nu_n t^{\frac{n}{2}}$$

Thus following system obtained where coefficient $A_0, A_1, A_2, \ldots, A_y$ and $B_0, B_1, B_2, \ldots, B_y$ can be determined

$$-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right)+B_0 exp\left(-\frac{\alpha^2}{4a^2}\right)=0$$

$$-A_0 \exp\left(\frac{\beta^2}{4a^2}\right)+B_0 exp\left(-\frac{\beta^2}{4a^2}\right)=0$$

$$-A_1 i^0 erfc\frac{\beta}{2a}+B_1 i^0 erfc\frac{-\beta}{2a}=\mu_1$$

$$-A_1 i^0 erfc\frac{\alpha}{2a}+B_1 i^0 erfc\frac{-\alpha}{2a}=\nu_1$$

$$-A_2 i^1 erfc\frac{\beta}{2a}+B_2 i^1 erfc\frac{-\beta}{2a}=\mu_2$$

$$-A_2 i^1 erfc\frac{\alpha}{2a}+B_2 i^1 erfc\frac{-\alpha}{2a}=\nu_2$$

... ...

$$-A_k i^{k-1} erfc\frac{\beta}{2a}+B_k i^{k-1} erfc\frac{-\beta}{2a}=\mu_k$$

$$-A_k i^{k-1} erfc\frac{\alpha}{2a}+B_k i^{k-1} erfc\frac{-\alpha}{2a}=\nu_k$$

If the functions $\varphi(t), \phi(t)$ are given in the form $\varphi(t)=\sum_{n=0}^{k}\mu_n t^{\frac{n}{2}}, \phi(t)=\sum_{n=0}^{m}\nu_n t^{\frac{n}{2}}$

then for $m \geq k$,

solution will be considered in the form

$$u(x,t) = \sum_{n=0}^{\gamma}\left(\sqrt{t}\right)^n\left[A_n i^n erfc\frac{x}{2a\sqrt{t}} + B_n i^n erfc\frac{-x}{2a\sqrt{t}}\right] \tag{35}$$

where $\gamma = m + 1$

for $x = \beta\sqrt{t}$ we have

$$\left.\frac{\partial u}{\partial x}\right|_{x=\beta\sqrt{t}} = \frac{1}{2a\sqrt{t}}\left[-A_0 \exp\left(\frac{\beta^2}{4a^2}\right) + B_0 i^0 exp\left(-\frac{\beta^2}{4a^2}\right)\right] +$$

$$+\frac{1}{2a}\left[-A_1 i^0 erfc\frac{\beta}{2a} + B_1 i^0 erfc\frac{-\beta}{2a}\right] +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^1\left[-A_2 i^1 erfc\frac{\beta}{2a} + B_2 i^1 erfc\frac{-\beta}{2a}\right] + \cdots +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^k\left[-A_{k+1} i^k erfc\frac{\beta}{2a} + B_{k+1} i^k erfc\frac{-\beta}{2a}\right] =$$

$$= \sum_{n=0}^{k} \mu_n t^{\frac{n}{2}}$$

For $x = \alpha\sqrt{t}$

$$\left.\frac{\partial u}{\partial x}\right|_{x=\alpha\sqrt{t}} = \frac{1}{2a\sqrt{t}}\left[-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right) + B_0 i^0 exp\left(-\frac{\alpha^2}{4a^2}\right)\right] +$$

$$+\frac{1}{2a}\left[-A_1 i^0 erfc\frac{\alpha}{2a} + B_1 i^0 erfc\frac{-\alpha}{2a}\right] +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^1\left[-A_2 i^1 erfc\frac{\alpha}{2a} + B_2 i^1 erfc\frac{-\alpha}{2a}\right] + \cdots +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^{m+1}\left[-A_{m+1} i^m erfc\frac{\alpha}{2a} + B_{m+1} i^m erfc\frac{-\alpha}{2a}\right] =$$

$$= \sum_{n=0}^{m+1} v_n t^{\frac{n}{2}}$$

Finally coefficients $A_0, A_1, A_2, \ldots, A_\gamma$ and $B_0, B_1, B_2, \ldots, B_\gamma$ are determined from system of linear equations

$$-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right) + B_0 exp\left(-\frac{\alpha^2}{4a^2}\right) = 0$$

$$-A_0 \exp\left(\frac{\beta^2}{4a^2}\right) + B_0 exp\left(-\frac{\beta^2}{4a^2}\right) = 0$$

$$-A_1 i^0 erfc\frac{\beta}{2a} + B_1 i^0 erfc\frac{-\beta}{2a} = \mu_1$$

$$-A_1 i^0 erfc\frac{\alpha}{2a} + B_1 i^0 erfc\frac{-\alpha}{2a} = v_1$$

$$-A_2 i^1 erfc\frac{\beta}{2a} + B_2 i^1 erfc\frac{-\beta}{2a} = \mu_2$$

$$-A_2 i^1 erfc\frac{\alpha}{2a} + B_2 i^1 erfc\frac{-\alpha}{2a} = v_2$$

... ...

... ...

$$-A_{m+1} i^m erfc\frac{\beta}{2a} + B_{m+1} i^m erfc\left(-\frac{\beta}{2a}\right) = 0$$

$$-A_{m+1} i^m erfc\frac{\alpha}{2a} + B_{m+1} i^m erfc\frac{-\alpha}{2a} = v_m$$

For $m < k$,

solution is considered in the form

$$u(x,t) = \sum_{n=0}^{\gamma} \left(\sqrt{t}\right)^n \left[A_n i^n erfc\frac{x}{2a\sqrt{t}} + B_n i^n erfc\frac{-x}{2a\sqrt{t}}\right]$$

where $\gamma = k + 1$

for $x = \beta\sqrt{t}$

$$\left.\frac{\partial u}{\partial x}\right|_{x=\beta\sqrt{t}} = \frac{1}{2a\sqrt{t}}\left[-A_0 \exp\left(\frac{\beta^2}{4a^2}\right) + B_0 i^0 exp\left(-\frac{\beta^2}{4a^2}\right)\right] +$$

$$+\frac{1}{2a}\left[-A_1 i^0 erfc\frac{\beta}{2a} + B_1 i^0 erfc\frac{-\beta}{2a}\right] +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^1\left[-A_2 i^1 erfc\frac{\beta}{2a} + B_2 i^1 erfc\frac{-\beta}{2a}\right] + \cdots +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^k\left[-A_{k+1} i^k erfc\frac{\beta}{2a} + B_{k+1} i^k erfc\frac{-\beta}{2a}\right] =$$

$$= \sum_{n=0}^{k+1} \mu_n t^{\frac{n}{2}}$$

and for $x = \alpha\sqrt{t}$ we have

$$\left.\frac{\partial u}{\partial x}\right|_{x=\alpha\sqrt{t}} = \frac{1}{2a\sqrt{t}}\left[-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right) + B_0 i^0 exp\left(-\frac{\alpha^2}{4a^2}\right)\right] +$$

$$+\frac{1}{2a}\left[-A_1 i^0 erfc\frac{\alpha}{2a} + B_1 i^0 erfc\frac{-\alpha}{2a}\right] +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^1\left[-A_2 i^1 erfc\frac{\alpha}{2a} + B_2 i^1 erfc\frac{-\alpha}{2a}\right] + \cdots +$$

$$+\left(\frac{\sqrt{t}}{2a}\right)^k\left[-A_{k+1} i^k erfc\frac{\alpha}{2a} + B_{k+1} i^k erfc\frac{-\alpha}{2a}\right] =$$

$$= \sum_{n=0}^{k+1} \nu_n t^{\frac{n}{2}}$$

Thus following system obtained where coefficient $A_0, A_1, A_2, \ldots, A_?$ and $B_0, B_1, B_2, \ldots, B_?$ can be determined

$$-A_0 \exp\left(\frac{\alpha^2}{4a^2}\right) + B_0 exp\left(-\frac{\alpha^2}{4a^2}\right) = 0$$

$$-A_0 \exp\left(\frac{\beta^2}{4a^2}\right) + B_0 exp\left(-\frac{\beta^2}{4a^2}\right) = 0$$

$$-A_1 i^0 erfc\frac{\beta}{2a} + B_1 i^0 erfc\frac{-\beta}{2a} = \mu_1$$

$$-A_1 i^0 erfc\frac{\alpha}{2a} + B_1 i^0 erfc\frac{-\alpha}{2a} = \nu_1$$

$$-A_2 i^1 erfc\frac{\beta}{2a} + B_2 i^1 erfc\frac{-\beta}{2a} = \mu_2$$

$$-A_2 i^1 erfc\frac{\alpha}{2a} + B_2 i^1 erfc\frac{-\alpha}{2a} = \nu_2$$

... ...

$$-A_{k+1} i^k erfc\frac{\beta}{2a} + B_{k+1} i^k erfc\frac{-\beta}{2a} = \mu_{k+1}$$

$$-A_{k+1} i^k erfc\frac{\alpha}{2a} + B_{k+1} i^k erfc\frac{-\alpha}{2a} = \nu_{k+1}$$

where $i^\gamma erfc\frac{\beta}{2a}, i^\gamma erfc\frac{-\beta}{2a}, i^\gamma erfc\frac{\alpha}{2a}, i^\gamma erfc\frac{-\alpha}{2a}$, γ=0,1,2… are treated as constants which can be determined from erfc tables.

Approximate solution of Heat Equation in the domain with moving boundary of the second type, obtained by IEF method

For the Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad \alpha(t) < x < \beta(t), \quad t > 0$$

Subject to

I.C: $u(x, o) = 0, \quad \alpha(t) < x < \beta(t),$

B.C: $\left.\frac{\partial u}{\partial x}\right|_{x=\alpha(t)} = \varphi(t), \quad t > 0,$

$\left.\frac{\partial u}{\partial x}\right|_{x=\beta(t)} = \phi(t), \quad t > 0.$

Solution:

Solution of the Heat Equation is represented in the form

$$u(x,t) = \sum_{n=0}^{k} (t^{\frac{n}{2}}) \left\{ A_n i^n erfc\left(\frac{x}{2a\sqrt{t}}\right) + B_n i^n erfc\left(-\frac{x}{2a\sqrt{t}}\right) \right\}$$

or

$$u(x,t) = \sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1.m} \right\}$$

Remark In both cases to find approximate solution it is necessary to give certain values for time *t*. For *k* values of *t* we obtain *k* coefficients from *2k* linear equations.

If we use IEF for solution then for boundary conditions at $x = \alpha(t)$and $x = \beta(t)$ we have system of *2k* linear equations.

$$\sum_{n=0}^{k} (t_k)^{\frac{n-1}{2}} \left\{ -A_n i^{n-1} erfc\left(\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right) + B_n i^{n-1} erfc\left(-\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right) \right\} = \varphi(t_k),$$

$$\sum_{n=0}^{k} (t_k)^{\frac{n-1}{2}} \left\{ -A_n i^{n-1} erfc\left(\frac{\beta(t_k)}{2a\sqrt{t_k}}\right) + B_n i^{n-1} erfc\left(-\frac{\beta(t_k)}{2a\sqrt{t_k}}\right) \right\} = \phi(t_k)$$

In the same manner solution in the polynomial form of IEF allows us to find even and odd coefficients A_{2n}, A_{2n+1} from system of linear equations

$$\sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} (2n-2m)(\alpha(t_k))^{2n-2m-1} t_k^m \alpha_{2n,m} + \right.$$
$$\left. + A_{2n+1} \sum_{m=0}^{n} (2n-2m+1)(\alpha(t_k))^{2n-2m} t_k^m \alpha_{2n+1,m} \right\} = \varphi(t_k)$$

$$\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(2n-2m)(\beta(t_k))^{2n-2m-1}t_k^m\beta_{2n,m}+\right.$$
$$\left.+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)(\beta(t_k))^{2n-2m}t_k^m\beta_{2n+1,m}\right\}=\phi(t_k)$$

2 SOLUTION OF INVERSE STEFAN PROBLEM

2.1 SOLUTION OF INVERSE STEFAN PROBLEM BY IEF METHOD

Problem statement:

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}\,, \qquad 0<x<\propto(t), \quad t>0 \tag{36}$$

I.C: $u(x,0)=0, \quad 0<x<\propto(t)$ (37)

B.C: $(\rho u+\theta\frac{\partial u}{\partial x})|_{x=0}=P(t)\,, \qquad t>0$ (38)

$$u(\propto(t),t)=\psi(t), \quad t>0 \tag{39}$$

Stefan condition:

$$(\lambda u-\propto(t)\frac{\partial u}{\partial x})|_{x=\propto(t)}=\propto^2(t)\,\frac{d\propto(t)}{dt} \tag{40}$$

$$u(0,0)=0 \tag{41}$$

Solution can be represented in the following form:

$$u(x,t)=\sum_{n=0}^{k}\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\} \tag{42}$$

Where A_{2k}, A_{2k+1} and $f \times n\ P(t)$ has to be determined.

We represent $P(t)$ as

$$P(t) = p_1 t \quad + p_2 t^2 + p_3 t^3 + \ \dots\dots + p_{k+1} t^{k+1} \tag{43}$$

where $p_1, p_2, p_3, \dots, p_{k+1}$ has to be determined.

SOLUTION 1:

To find appropriate number of coefficients A_{2n}, A_{2n+1} in (42) and coefficients $p_1, p_2, p_3, \dots, p_k$ of unknown $f \times n\ P(t)$, it is necessary to substitute same number of values for time t, in boundary conditions (38), (39) and (40). Thus from (38) and (39) we obtain system of linear equations which allows to determine A_{2n} and A_{2n+1} coefficients, and in the same manner coefficients $p_1, p_2, p_3, \dots, p_{k+1}$ can be obtained from (37).

From (38) for $x = \propto (t)$

$$\sum_{n=0}^{k} \{A_{2n} \sum_{m=0}^{n} (\propto (t_n)^{2n-2m} t_n^m \beta_{2n,m}\ A_{2n+1} \sum_{m=0}^{n} (\propto (t_n)^{2n-2m+1} t_n^m \beta_{2n+1,m}\} =$$

$$= \psi(t_k) \tag{44}$$

From (39) for $x = \propto (t)$

$$\lambda \sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} (\propto (t_n)^{2n-2m} t_n^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} (\propto (t_n)^{2n-2m+1} t_n^m \beta_{2n+1,m} \right\} -$$

$$-\propto$$

$$(t_k) \sum_{n=0}^{k} \{A_{2n} \sum_{m=0}^{n} (2n-2m)(\propto (t_n)^{2n-2m-1} t_n^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} (2n -$$

$$-2m+1)\cdot(\propto(t_n)^{2n-2m}t_n^m\beta_{2n+1,m}\}=\propto^2(t_k)\frac{d\propto}{dt}|_{t=t_k}$$

(45)

can be determined.

As it was told above A_{2n} and A_{2n+1} from (44) and (45).

Finally from (37)

$$\rho\left(A_0\beta_{0,0}+A_2t_k\beta_{2,1}+A_4t_k^2\beta_{4,2}+\ \dots\dots+A_{2k}t_k^{2k}\beta_{2k,k}\right)+$$

$$+\theta\left(A_1\beta_{1,0}+A_3t_k\beta_{3,1}+A_5t_k^2\beta_{5,2}+\ \dots\dots+A_{2k+1}t_k^{2k}\beta_{2k+1,k}\right)=$$

$$=p_1t_k^{\frac{1}{2}}+p_2t_k+p_3t_k^{\frac{3}{2}}+\ \dots+p_kt_k^{\frac{k}{2}}$$

It is possible to determine $p_1, p_2, p_3, \dots, p_{k+1}$.

We determine deviation of solution by Maximum principle.

SOLUTION 2:

Approximate solution of Inverse Stefan problem by IEF method.

We represent function $P(t)$ in (37) as following

$$P(t)=\{y_1(t), y_2(t), y_3(t), \dots\dots, y_k(t)\}$$

where

$$y_1(t)=h_1+m_1t\,,\ \ h_1=0$$

$$y_2(t)=h_2+m_2t\,,$$

$$y_3(t)=h_3+m_3t\,,$$

.

.

.

$$y_k(t) = h_k + m_k t\,,$$

and h_k, m_k has to be found and $P(t)$ is as shown in the figure.

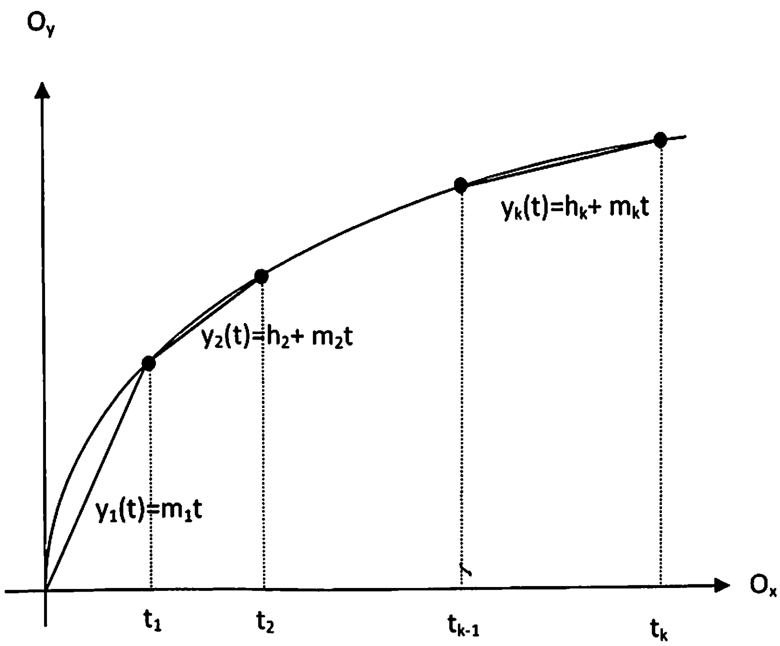

Figure 2

It is clear that $y_1(t_1) = y_2(t_1)$, then $m_1 t_1 = h_2 + m_2 t_1$ implies $h_2 = (m_1 - m_2)t_1$ and $y_2(t) = (m_1 - m_2)t_1 + m_2 t$.

In the same manner $y_2(t_2) = y_3(t_2)$, then $(m_1 - m_2)t_1 + m_2 t_2 = h_3 + m_3 t_2$ implies $h_3 = (m_1 - m_2)t_1 + (m_2 - m_3)t_2$ and $y_3(t) = (m_1 - m_2)t_1 + (m_2 - m_3)t_2 + m_3 t$, etc.

$$y_k(t) = (m_1 - m_2)t_1 + (m_2 - m_3)t_2 + \ \dots\ + m_k t\,,$$

Thus all $y_n(t)$ linear functions are expressed in terms of unknown $m_1, m_2, \dots \dots m_{k+1}$ coefficients. After $m_1, m_2, \dots \dots m_{k+1}$ found $y_k(t)$ $f \times n\ \rho$ can be easily determined. Using programs like Mathcad it is possible to construct analytical form of function $P(t)$.

To find $P(t)$ and $u(x,t)$ subject to above boundary conditions we substitute (41) into (38) and (39) in terms to find A_{2k} and A_{2k+1} coefficients.

In the same manner from (37) for $x = 0$

$$\rho\left(A_0\beta_{0,0} + A_2 t_k \beta_{2,1} + A_4 t_k^2 \beta_{4,2} + \dots \dots + A_{2k} t_k^{2k} \beta_{2k,k}\right) +$$

$$+\theta\left(A_1\beta_{1,0} + A_3 t_k \beta_{3,1} + A_5 t_k^2 \beta_{5,2} + \dots \dots + A_{2k+1} t_k^{2k} \beta_{2k+1,k}\right) = y_k(t_k) \tag{46}$$

Thus from (44), (45) and (46) we obtain A_{2k} , A_{2k+1} and $\{m_1, m_2, \dots \dots m_{k+1}\}$ coefficient respectively and apply Maximum principle to calculate deviation of solution.

CONCLUSION

The results indicate that Integral Error Functions enable to solve, many practical problems described above in the easier way than classical methods, and could be implemented into the course of teaching mathematical physics, as special methods of solving heat transfer problems with moving boundaries.

So, we solved problem of inverse Stefan problem by using IEF.

For first problem we have two solutions:

1. By taking unknown heat flow P(t) in form $P(t) = p_1 t + p_2 t^2 + p_3 t^3 + \dots\dots + p_{k+1} t^{k+1}$ and determining $p_1, p_2, p_3, \dots, p_{k+1}$ coefficients. We found the solution of this problem.

2. By representing the function P(t) in form $P(t) = \{y_1(t), y_2(t), y_3(t), \dots\dots, y_k(t)\}$ and by taking $y_n(t)$ linear functions then we found A_{2k}, A_{2k+1} in the solution.

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2. By representing the function P(t) in form $P(t) = \{y_1(t), y_2(t), y_3(t), \dots\dots, y_k(t)\}$ and by taking $y_n(t)$ linear functions then we found A_{2k}, A_{2k+1} in the solution.

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