V.Verbovskiy
Doctor of physical-matehematical sciences
I.Tulepbergenova
master student,
Suleyman Demirel University
Almaty/Kazakhstan

On commutativity of weakly o-minimal lattice ordered groups

Introduction

Notion of a weakly o-minimal structure has been introduced by D. Macpherson, D. Marker and Ch. Steinhorn in [4] for totally ordered structures. In their paper they investigate properties of such structures, of definable unary functions, topological properties of definable subsets and properties of weakly o-minimal totally ordered groups and fields. They proved that a totally ordered weakly o-minimal groups is Abelian, divisible and dense and that a totally ordered weakly o-minimal field is real closed.

In this paper we consider a generalization of notion of weak o-minimality to the class of partially ordered structures, namely, to the class of lattice ordered groups. We prove that a weakly o-minimal lattice ordered group is Abelian, divisible and dense.

Recall that a semigroup S is said to be *partially ordered* or a *partially ordered semigroup* if it admits a partial order $\leq$ which is compatible with the left and right multiplication:

$$a \leq b \text{ implies } xay \leq xby \text{ for all } x, y \in S^1.$$

A good review on partially ordered groups one can find in [1–3].

Below we recall some basic properties of partially ordered groups.

Proposition 1.1 [1] *Let G be a partially ordered group and suppose that a,b* $\in$ *G. Then a and b have a least upper bound* a $\vee$ *b in G if and only if they have a greatest lower bound a b; this occurs if and only if* a^{-1} *and* b^{-1} *have a least upper bound. Indeed,*

$a \wedge b = a(a \vee b)^{-1}b$ $\qquad$ $a \vee b = a(a \wedge b)^{-1}b$

$a \wedge b=(a^{-1} \vee b^{-1})^{-1}$ $\qquad$ $a \vee b=(a^{-1} \wedge b^{-1})^{-1}$.

Further for every g $\in$ *G,*

$g(a \vee b) = ga \vee gb$ $\qquad$ $(a \vee b)g = ag \vee bg$

$g(a \wedge b) = ga \wedge gb$ $\qquad$ $(a \wedge b)g = ag \wedge bg$.

Corollary 1.2 [1] *The following are equivalent for a partially ordered group G.*

(i) *G is a* $\vee$-semilattice under $\leq$;

(ii) *G* is *a* $\wedge$-semilattice under $\leq$;

(iii) $a \vee 1$ exists for each $a \in G$;

(iv) $a \wedge 1$ exists for each $a \in G$;

(v) $a \vee b$ exists for each $a, b \in G^+$;

(vi) $a \wedge b$ exists for each $a, b \in G^+$;

(vii) for each $a, b \in G^+$ there exists $c \in G^+$ such that $G^+a \cap G^+b = G^+c$.

If *G* satisfies one (thus all) of the conditions in the Corollary we say that *G* is a *lattice ordered group* or *a latticed group*.

Theorem 1.3 [1] *Let G be a lattice ordered group under a partial order* $\leq$ *then G is a distributive lattice under* $\leq$.

Definition: Recall, that two elements *a* and *b* of a lattice ordered group *G* are said to be *orthogonal* if $a \wedge b = 1$.Orthogonal elements are very common in lattice ordered groups and play a crucial role in the theory of such groups. The next proposition lists a number of their basic properties.

Proposition 1.4 [1] *Let G be a lattice ordered group and let a, b, c* $\in$ *G.*

(i) *if* $a \wedge b = 1$ *then* $ab = ba$;

(ii) *if* $a \wedge b = 1$ *and* $c \geq 1$ *then* $a \wedge bc = a \wedge c$;

(iii) $(a \vee 1) \wedge (a^{-1} \boxminus 1) = 1$; $a \boxminus a^{-1} = (a \boxminus 1)(a^{-1} \boxminus 1)$;

(iv) $(a \vee 1)^n = a^n \vee 1$; $(a \wedge 1)^n = a^n \wedge 1$;

(v) $a^n \geq 1$ *implies* $a \geq 1$.

Theorem 1.5 [1] *Let G be an Abelian group. Then G can be lattice ordered (totally ordered) if and only if G is torsion free.*

1. Weakly o-minimal lattice ordered groups

Definition: An interval (a, b) for a<b in a partially ordered set *M* is a set of all element *c* such that $a < c < b$.

Definition: A subset *A* of a partially ordered set *M* is called *convex* if for any $a, b \in A$ if $a < b$, then the interval (a, b) is a subset of *A*.

Definition: A partially ordered structure $(M, \leq, \ldots)$ is called *weakly o-minimal* if any definable subset is a finite union of convex sets.

Lemma 2.1 Let $(G, \leq, \cdot, e)$ be a partially ordered weakly o-minimal group. Let *H* be a definable subgroup of *G*. Then for any $h \in H$ if $h > e$, then $(e, h) \in H$.

Proof: Let $h \in$ H be positive. Assume the contrary that there is $g \in G$ such that $g \in (e, h)$ and $g \notin H$. Since *h* is positive, it is of infinite order, so $h, h^2, h^3, \ldots, h^n, \ldots$ are all different.

By weak o-minimality $H = A_1 \cup \cdots \cup A_r$, where the set A_i are convex. So there is some *k* such that A_k contains at least two elements h^n and h^m with $n < m$. Then $(h^n, h^m) \subseteq A_k$ because A_k is convex.

By the other hand $e < g < h$ and $h^n < h^n \cdot g < h^{n-1} \leq h^m$. Hence, $h^n \cdot g \in A_k$ and $h^n \cdot g \in H$. But since $h^{-n} \in H$, so $g = h^{-n} \cdot h^n \cdot g \in H$, for a contradiction.

Lemma 2.2 Let $(G, \leq, \cdot, e)$ be a partially ordered weakly o-minimal group. Then any two positive elements commute, that is if $g_1 > e, g_2 > e$ then $g_1 g_2 = g_2 g_1$.

Proof: Consider the centralizator *C(g)* of some element $g \in G$. It is definable by $C(x) = (x \cdot g = g \cdot x)$. Clearly, the centralizator *C(g)* is a subgroup.

Case 1) $g_1 < g_2$. Then $g_1 \in (e, g_2) \boxminus C(g)$ by Lemma 2.1. So $g_1 \in C(g_2)$ and $g_1 g_2 = g_2 g_1$.

Case 2) $g_2 < g_2$ is similar.

Case 3) g_1 and g_2 are incomparable. Since $e < g_1$ we obtain that $g_2 < g_1 g_2$. By Case 1) g_2 and $g_1 g_2$ commute, that is $g_2(g_1 g_2) = (g_1 g_2) g_2$.

Then $g_2 g_1 g_2 = g_1 g_2 g_2$ and canceling g_2 from the right we obtain that $g_1 2\ g_1 1 = g_1(12$*from the right we obtain that*'.

Lemma 2.3 Let $(G, \leq, \cdot, e)$ be a partially ordered weakly o-minimal group. Then any two elements comparable with e commute.

Proof: Let g_1, g_2 be comparable with e.

1) $g_1 > e, g_2 > e$ follows from Lemma 2.2.

$g_1 < e, g_2 > e$

2) . Then $g_1^{-1} g_2 = g_2 g_1^{-1}$ because $g_1^{-1} > e$. Then $g_1 2\ g_1 1 = g_1(12$*from the right we obtain that*".

3) $g_1 > e, g_2 < e$ is similar to Case 2

4) $g_1 < e, g_2 < e$ follows from Case 1.

Theorem 2.4 Let $(G, \leq, \cdot, e)$ be a lattice ordered weakly o-minimal group. Then *G* is abelian.

Proof. Let $g_1, g_2 \in$ G. Let $h_1 = e \boxminus g_1, h_2 = e \boxminus g_2$. Then both h_1 and h_2 are positive, so by Lemma 2.2 $h_1 h_2 = h_2 h_1$.

Let

$f_1 = e \boxminus g_1, f_2 = e \boxminus g_2$

$f_i = e \boxminus g_i = e(e \boxminus g_i)^{-1} g_i = h_i^{-1} \cdot g_i$

$g_i = h_i \cdot f_i$
$g_1 g_2 = h_1 f_1 h_2 f_2$

Since h_1, h_2, f_1, f_2 are comparable with e, all of them commute. Then

$$g_1 g_2 = h_1 f_1 h_2 f_2 = h_2 f_2 h_1 f_1 = g_2 g_1$$

From now on we shall use the additive notation of the group operation.

Theorem 2.5 Let $(G, \leq, +, 0)$ be a lattice ordered weakly o-minimal group. Then G is divisible.

Proof. Let $g \in G$. We have to prove that for each positive integer n there is some element $h \in G$ such that $nh = g$, that is the element g is n-divisible. Assume the contrary, that there exists an element g which is not n-divisible for some n.

Case 1. The element g is positive, that is $g > 0$. Consider the subgroup nG. It is definable by the formula $\exists y(ny = x)$. By weak o-minimality it is a finite union of convex sets $K_1, \ldots, K_m$.

Let h be in nG. Consider the following sequence: $a_k = h + kg$. Since the element g is positive, this sequence is strictly increasing. Note that a_k belongs to nG iff k is divisible by n. So, infinitely many elements from the sequence belongs to nG. Since n is finite there is some index j such that for some $0 < s < q$ it holds that both a_{sn} and a_{qn} belong to K_j. Then the whole interval (a_{sn}, a_{qn}) is a subset of K_j, and hence of nG. But a_{sn+1} belongs to this interval, but is not n-divisible, for a contradiction.

Case 2. The element g is negative. Then the element $-g$ is positive and is not n-divisible. So we obtain Case 1.

Case 3. The element g is not comparable with 0 and each element, which is comparable with 0, is n-divisible. By Proposition 1.1 it holds that $g \vee 0 = g - (g \wedge 0) + 0 = g - (g \wedge 0)$. Or equivalently, $g = (g \wedge 0) + (g \vee 0)$. But the last two elements are comparable with 0, so they are divisible by n, as well as their sum, which is equal to g. We obtain a contradiction.

Theorem 2.6 Let $(G, \leq, +, 0)$ be a lattice ordered weakly o-minimal group. Then G is dense.

Proof. Let $g \in G$ be positive. By Theorem 2.5 there is some element $h \in G$ such that $2h = g$. By Proposition 1.4 the element h is positive. Thus, $0 < h$ implies $0 + h < h + h$, that is, $h < 2h$. Thus, there is no minimal positive elements.

Let $a < b$ be two elements of G. Let $2h = b - a$. Since $0 < b - a$, so by Proposition 1.4 it holds that $0 < h$. Then $a < a + h < a + h + h = b$. That is why the order is dense.

References:

[1] T.S. Blyth, Lattices and Ordered Algebraic Structures, Springer, 2005

[2] L. Fuchs, Partially ordered algebraic systems, Pergamon, 1963.

[3] A.I. Kokorin, V.M. Kopytov, Fully ordered groups, Israel Program Sci. Transl., 1974 (Translated from Russian).

[4] Macpherson D., Marker D. and Steinhorn Ch. Weakly o-minimal structures and real closed fields // Transaction of American Mathematical Society. 2000. № 352. P. 5435–5483.

Түйін

Егер кез-келген формулдық көпмүшенің жиынтығының біріккен көпмүшелігінің жиынтығы болса, онда торланып реттелген құрылым босан о-минималды деп аталады. Босан о-минималды торланып реттелген топтардың коммутативтік пен бөлінгіштігі дәлелденді.

Resume

A lattice ordered structure is called weakly o-minimal if any definable subset is a finite union of convex sets. In the paper we prove that any weakly o-minimal lattice ordered group is Abelian, divisible and dense.

Özet

Herhangi tanımlanabilen alt dışbükey kümeler sonlu bir sendika ise bir kafes sıralanmıştır yapısı zayıf o-minimal denir. Yazıda, herhangi bir zayıf halka-minimum kafes sipariş grubu, Abel bölünebilir ve yoğun olduğunu kanıtlaması söz konusu oldu.