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IEF METHOD
IN HEAT EQUATIONS

Supplementary handbook for Partial Differential Equations course

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It can be used during the instruction Partial Differential Equations course and can be useful for engineers and students of technical specialties, such as 5B060100 — “Mathematics”, 5B010900 — “Mathematics”.

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Introduction

Development of new analytical methods of solution of the heat transfer problems is very important for various applications because it enables one to analyze an interrelationship of various input parameters on the dynamics of investigating phenomena, while the use of numerical methods is a problem when the number of parameters is great. Therefore the research investigation in this direction is very topical. The method of Integral Error Functions elaborated in this study can be applied in many fields of science and industry for mathematical modeling of heat conduction with phase transformation. In particular, the phenomena occurring at interaction of electrical arc with electrode can be described in dynamic using the presented method for very short arc duration, even in nanosecond diapason, when experimental investigation is very difficult. It is possible now to describe different mechanisms of electric erosion including melting, evaporation and thermo-elastic ejecting. It should be noted that approximate solutions of problems considered in the study are estimated using maximum principle for the heat equation with an error less than 1%.

The book consists of three chapters: in the first chapter properties of the integral error function and corollaries necessary to solve heat equations are represented. In the second chapter theoretical aspects and methods for solving heat problems in domains with given and moving boundaries are represented. In the third chapter different methods of solving nonlinear Stefan problem are given.

CONTENTS

1 IEF METHOD

1.1 Introduction to IEF method

This chapter is devoted to introduce special methods by the help of which Heat Equations in the domains with fixed and moving boundaries are solved. Heat equations are solved by the help of Integral Error Functions (IEF method) and its properties, which were introduced by Hartree in 1935 and reasonably sometimes called Hartree functions. As it will be shown in further paragraphs, method can be used to solve first, second and third boundary value problems for Heat Equations with fixed and moving finite, semi-infinite and infinite boundaries. The integral error functions are determined by recurrent formulas

$$i^n erfcx = \int_x^\infty i^{n-1} erfcv\,dv, \qquad n=1,2,\ldots \qquad i^0 erfcx \equiv erfcx = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-v^2)\,dv \tag{1}$$

Where $\qquad erfx = 1 - erfcx = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x \exp(-v^2)\,dv$

One can obtain from (1)

$$i^n erfcx = \frac{2}{\sqrt{\pi}} \frac{1}{n!} \int_x^\infty (v-x)^n \exp(-v^2)\,dv \tag{2}$$

Expressions (1) satisfy the differential equation

$$\frac{d^2}{dx^2} i^n erfcx + 2x \frac{d}{dx} i^n erfcx - 2n i^n erfcx = 0 \tag{3}$$

and recurrent formulas

$$2n i^n erfcx = i^{n-2} erfcx - 2x i^{n-1} erfcx \tag{4}$$

Integral Error Functions are very useful for investigation of heat transfer, diffusion and other phenomena which can be described by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{5}$$

in a region $D(t > 0, 0 < x < \alpha(t))$ with free boundary $x = \alpha(t)$, since the functions

$$u_n(\pm x, t) = t^{\frac{n}{2}} i^n erfc \frac{\pm x}{2a\sqrt{t}} \tag{6}$$

suffice the equation (5) as well as their linear combination or even series

$$u(x,t) = \sum_{n=0}^{\infty} [A_n u_n(x,t) + B_n u_n(-x,t)] \tag{7}$$

For any constants A_n, B_n.We can choose these constants to satisfy the boundary conditions at $x = 0$ and $x = \alpha(t)$, if given boundary functions can be expanded into Taylor series with powers t or $\sqrt{t}$.

1.2 Properties of Integral Error Functions

It is possible to derive new properties of Integral Error Functions.

1. If n is an integer, then

$$i^n erfc(-x)+(-1)^n i^n erfcx=\frac{1}{2^{n-1}n!i^n}H_n(ix)=\frac{1}{2^{n-1}n!}e^{-x^2}\frac{d^n}{dx^n}e^{x^2}$$ with $i=\sqrt{-1}$ and Hermite polynomials $H_n(x)$ in the right side. Indeed, using formula (6) one can write

$$i^n erfc(-x)+(-1)^n i^n erfcx=\frac{2}{\sqrt{\pi}}\frac{1}{n!}\int_{-x}^{\infty}(v+x)^n\exp(-v^2)dv+$$

$$\frac{(-1)^n 2}{n!\sqrt{\pi}}\int_{x}^{\infty}(v-x)^n\exp(-v^2)dv=\frac{2}{n!\sqrt{\pi}}\int_{-\infty}^{\infty}(v+x)^n\exp(-v^2)dv=\frac{1}{2^{n-1}n!i^n}H_n(ix) \quad (8)$$

2. Using formula for Hermite polynomials one can derive

$$i^n erfc(-x)+(-1)^n i^n erfcx=\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{x^{n-2m}}{2^{2m-1}m!(n-2m)!} \quad (9)$$

If $n=2k$, then

$$i^{2k}erfc\ x+i^{2k}erfc\ (-x)=\sum_{m=0}^{k}\frac{x^{2(k-m)}}{2^{2m-1}m!(2k-2m)!}$$

In particular

$$erfc\ x+erfc\ (-x)=2,$$

$$i^2erfc\ x+i^2erfc\ (-x)=\frac{1}{2}+x^2 ,$$

$$i^4erfc\ x+i^4erfc\ (-x)=\frac{1}{8}+\frac{1}{4}x^2+\frac{1}{12}x^4 .$$

If $n=2k+1$, then

$$i^{2k+1}erfc(-x)-i^{2k+1}erfc\ x=\sum_{m=0}^{k}\frac{x^{2(k-m)+1}}{2^{2m-1}m!(2k-2m+1)!} \quad (10)$$

In particular

$$ierfc(-x)-ierfc\ x=2x ,$$

$$i^3erfc(-x)-i^3erfc\ x=\frac{1}{2}x+\frac{1}{3}x^3 ,$$

$$i^5erfc\ (-x)-i^5erfc\ x=\frac{1}{2^3\cdot 2!}x+\frac{1}{2\cdot 2!\cdot 3!}x^3+\frac{2}{5!}x^5 .$$

The proof of the formula

$$i^n erfc(-x)-(-1)^n i^n erfcx=\frac{1}{2^{n-1}n!}e^{-x^2}\frac{d^n}{dx^n}(e^{x^2}erfx) \quad (11)$$

where

$$erfx=1-erfcx=\frac{2}{\sqrt{\pi}}\int_0^x\exp(-v^2)dv$$

can be obtained by mathematical induction method using recurrent formula (3).

3. Differentiating the right side of formula (11), we obtain

$$i^n erfc(-x) - (-1)^n i^n erfcx = P_n(x) erfx - Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2), \tag{12}$$

where polynomials $P_n(x)$ and $Q_n(x)$ are defined by formulas

$$P_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2m}}{2^{2m-1} m!(n-2m)!}, \qquad Q_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^{n-k} H_{n-k-1}(x)}{2^{n-k}(n-k)!} P_k(x)$$

4. From (11), (12) we can obtain the explicit expressions for Integral Error Functions of an integer index

$$i^n erfcx = \frac{(-1)^n}{2}\left[P_n(x) erfcx + Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2)\right] \tag{13}$$

$$i^n erfc(-x) = \frac{1}{2}\left[P_n(x) erfc(-x) - Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2)\right] \tag{14}$$

5. Using L'Hopital rule and representation (1.1), it is not difficult to show that

$$\lim_{x \to \infty} \frac{i^n erfc(-x)}{x^n} = \frac{2}{n!} \tag{15}$$

1.3 Corollaries for IEF method

Following corollaries will be helpful to so solve Heat equations.

1. Using property 2 one can derive following formula

$$u(x,t) = \sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right\}$$

Where u(x,t) is a solution of Heat Equation in polynomial form and

$$\beta(n,m) := \frac{1}{2^{n+m-1} \cdot m! \cdot (n-2m)!}$$

2. Expression u(x,t) can be expanded in the following form

$$u(x,t) = A_0 \beta_{0,0} +$$

$$+A_2\left(x^2 \beta_{2,0} + t\beta_{2,1}\right) +$$

$$+A_4\left(x^4 \beta_{4,0} + x^2 t\beta_{4,1} + t\beta_{4,2}\right) + \cdots +$$

$$+A_{2k}\left(x^{2k} \beta_{2k,0} + x^{2k-2} t\beta_{2k,1} + \cdots + x^2 t^{k-1} \beta_{2k,k-1} + t^k \beta_{2k,k}\right) +$$

$$+A_1 x \beta_{1,0} +$$

$$+A_3\left(x^3 \beta_{3,0} + xt\beta_{3,1}\right) +$$

$$+A_5\left(x^5 \beta_{5,0} + x^3 t\beta_{5,1} + xt^2 \beta_{5,2}\right) + \cdots +$$

$$+A_{2k+1}\left(x^{2k+1}\beta_{2k+1,0}+x^{2k-1}t\beta_{2k+1,1}+\cdots+x^3t^{k-1}\beta_{2k+1,k-1}\right.$$
$$\left.+xt^k\beta_{2k+1,k}\right)$$

3. Following expression will be frequently used in solutions of Heat Equations

$$\frac{du}{dx}=2A_2x\beta_{2,0}+$$
$$+A_4\left(4x^3\beta_{4,0}+2xt\beta_{4,1}\right)+$$
$$+A_6\left(6x^5\beta_{6,0}+4x^3t\beta_{6,1}+2xt^2\beta_{6,2}\right)+\cdots+$$
$$+A_{2k}\left(2kx^{2k-1}\beta_{2k,0}+(2k-1)x^{2k-3}t\beta_{2k,1}+\cdots+2xt^k\beta_{2k,k}\right)+$$
$$+A_1\beta_{1,0}+$$
$$+A_3\left(3x^2\beta_{3,0}+t\beta_{3,1}\right)+$$
$$+A_5\left(5x^4\beta_{5,0}+3x^2t\beta_{5,1}+t^2\beta_{5,2}\right)+\cdots+$$
$$+A_{2k+1}\left((2k+1)x^{2k}\beta_{2k+1,0}+(2k-1)x^{2k-2}t\beta_{2k+1,1}+\cdots\right.$$
$$\left.+3x^2t^{k-1}\beta_{2k+1,k-1}+t^k\beta_{2k+1,k}\right)$$

4. Expression $u(x,t)=\sum_{n=0}^{k}\left(\sqrt{t}\right)^n\left[A_ni^nerfc\frac{x}{2a\sqrt{t}}+B_ni^nerfc\frac{-x}{2a\sqrt{t}}\right]$ can be expanded

$$u(x,t)=\left(\sqrt{t}\right)^0\left[A_0i^0erfc\frac{x}{2a\sqrt{t}}+B_0i^0erfc\frac{-x}{2a\sqrt{t}}\right]+$$
$$+\left(\sqrt{t}\right)^1\left[A_1i^1erfc\frac{x}{2a\sqrt{t}}+B_1i^1erfc\frac{-x}{2a\sqrt{t}}\right]+$$
$$+\left(\sqrt{t}\right)^2\left[A_2i^2erfc\frac{x}{2a\sqrt{t}}+B_2i^2erfc\frac{-x}{2a\sqrt{t}}\right]+\cdots+$$
$$+\left(\sqrt{t}\right)^n\left[A_ni^nerfc\frac{x}{2a\sqrt{t}}+B_ni^nerfc\frac{-x}{2a\sqrt{t}}\right]$$

5. Partial derivative of u(x,t) can be written as

$$\frac{\partial u}{\partial x}=\frac{1}{2a\sqrt{t}}\left[-A_0\exp\left(\frac{x^2}{4a^2t}\right)+B_0exp\left(\frac{-x^2}{4a^2t}\right)\right]+$$
$$+\frac{1}{2a}\left[-A_1i^0erfc\frac{x}{2a\sqrt{t}}+B_1i^0erfc\frac{-x}{2a\sqrt{t}}\right]+$$
$$+\left(\frac{\sqrt{t}}{2a}\right)^1\left[-A_2i^1erfc\frac{x}{2a\sqrt{t}}+B_2i^1erfc\frac{-x}{2a\sqrt{t}}\right]+\cdots+$$
$$+\left(\frac{\sqrt{t}}{2a}\right)^{n-1}\left[-A_ni^{n-1}erfc\frac{x}{2a\sqrt{t}}+B_ni^{n-1}erfc\frac{-x}{2a\sqrt{t}}\right]$$

6. Theorem: Maximum Principle [1, p. 194]

If the function u(x,t) defined and continuous on the domains $0 \leq x \leq l,\ \ 0 \leq t \leq T,$ and suffices the Heat Equation

$$a^2 u_{xx} = u_t$$

in the points of domains $0 < x < l,\ \ 0 < t \leq T$, then the function u(x,t) achieves its maximum and minimum points at the initial time or in the boundary values x=0 or x=*l*

This theorem is widely used in applications and enables to see the deviation between exact solution and searched solution. When we apply the solution for the boundary conditions it is possible to obtain functions whose graphs are easy to build by the help of any mathematical program like Mathcad, and so is for the functions at boundaries. Finally it is possible to compare these graphs and determine the deviation of the function (that is obtained by application of searched solution to the boundary conditions) from the boundary value function. Hence according to the Maximum Principle Theorem it is possible to say that the error of searched solution doesn't exceed the error between the boundary value function and function obtained with application of searched solution to the boundary conditions. We can observe in the following examples that deviation of solutions doesn't exceed than 1%.

2 APPROXIMATE SOLUTIONS OF HEAT EQUATIONS WITH GIVEN BOUNDARIES

It is possible to observe in further paragraphs and examples that IEF method enables to solve boundary value problems for Heat equations in domains with fixed boundaries approximately.

2.1 Approximate solution of the first boundary value problem

The solution of the problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < l \ , \quad t > 0 \tag{16}$$

Subject to

I.C: $u(x,0) = 0,$ (17)

B.C: $u(0,t) = \varphi(t),$ (18)

$u(l,t) = \phi(t),$ (19)

$u(0,0) = 0,$ (20)

where $\varphi(t), \phi(t)$ are analytical functions , can be represented in the form

$$u(x,t) = \sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right\}$$

or

$$u(x,t) = A_0 \beta_{0,0} +$$

$$+A_2 \left(x^2 \beta_{2,0} + t\beta_{2,1} \right) +$$

$$+A_4 \left(x^4 \beta_{4,0} + x^2 t \beta_{4,1} + t\beta_{4,2} \right) + \cdots +$$

$$+A_{2k} \left(x^{2k} \beta_{2k,0} + x^{2k-2} t \beta_{2k,1} + \cdots + x^2 t^{k-1} \beta_{2k,k-1} + t^k \beta_{2k,k} \right) +$$

$$+A_1 x \beta_{1,0} +$$

$$+A_3 \left(x^3 \beta_{3,0} + xt\beta_{3,1} \right) +$$

$$+A_5\left(x^5\beta_{5,0}+x^3t\beta_{5,1}+xt^2\beta_{5,2}\right)+\cdots+$$

$$+A_{2k+1}\left(x^{2k+1}\beta_{2k+1,0}+x^{2k-1}t\beta_{2k+1,1}+\cdots+x^3t^{k-1}\beta_{2k+1,k-1}+xt^k\beta_{2k+1,k}\right)$$

This function satisfies the heat equation (69), as it was mentioned above and coefficients A_{2n}, A_{2n+1} has to be found. To satisfy the initial and boundary conditions (71) and (72) we expand the functions $\varphi(t), \phi(t)$ in Maclaurin's series:

$$\varphi(t)=\sum_{n=0}^{\infty}\frac{\varphi^n(0)}{n!}\cdot t^n, \tag{74}$$

and

$$\phi(t)=\sum_{k=0}^{\infty}\frac{\phi^n(0)}{n!}\cdot t^n \tag{21}$$

for t=0, $A_0=0$

and for x=0, even coefficients A_{2k} can be found from

$$A_0\beta_{0,0}+A_2t\beta_{2,1}+A_4t^2\beta_{4,2}+\cdots+A_{2k}t^{2k}\beta_{2k,k}=\sum_{k=0}^{\infty}\frac{\varphi^{2k}(0)}{(2k)!}t^{2k}$$

substituting even coefficients into the boundary condition for x=l and combining like terms, odd coefficients A_{2k+1} can be found from

$$A_0\beta_{0,0}+$$

$$+A_2\left(l^2\beta_{2,0}+t\beta_{2,1}\right)+$$

$$+A_2\left(l^4\beta_{4,0}+l^2t\beta_{4,1}+t\beta_{4,2}\right)+\cdots+$$

$$+A_{2k}\left(l^{2k}\beta_{2k,0}+l^{2k-2}t\beta_{2k,1}+\cdots+l^2t^{k-1}\beta_{2k,k-1}+t^k\beta_{2k,k}\right)+$$

$$+A_1l\beta_{1,0}+$$

$$+A_3\left(l^3\beta_{3,0}+lt\beta_{3,1}\right)+$$

$$+A_5\left(l^5\beta_{5,0}+l^3t\beta_{5,1}+lt^2\beta_{5,2}\right)+\cdots+$$

$$+A_{2k+1}\Big(l^{2k+1}\beta_{2k+1,0}+l^{2k-1}t\beta_{2k+1,1}+\cdots+l^3t^{k-1}\beta_{2k+1,k-1}$$
$$+lt^k\beta_{2k+1,k}\Big)=\sum_{k=0}^{\infty}\frac{\phi^{2k+1}(0)}{(2k+1)!}t^{2k+1}$$

2.2 Approximate solution of the first boundary value problem by IEF method

Approximate solution of the same problem

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\qquad 0<x<l\ ,\quad t>0 \tag{22}$$

Subject to

I.C: $$u(x,0)=0, \tag{23}$$

B.C: $$u(0,t)=\varphi(t), \tag{24}$$

$$u(l,t)=\phi(t), \tag{25}$$

where $\varphi(t),\phi(t)$ are analytical functions, can be represented in the form

$$u(x,t)=\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\right\}$$

or

$$u(x,t)=\sum_{n=0}^{k}\left(2a\sqrt{t}\right)^n\left[A_n i^n erfc\frac{x}{2a\sqrt{t}}+B_n i^n erfc\frac{-x}{2a\sqrt{t}}\right]$$

In both cases it is necessary to substitute above expressions into the boundary conditions for certain values of t with appropriate step. For polynomial form of IEF we perform similar procedure as in the analytic solution but only for certain values of t.

As for IEF solution A_n and B_n coefficients can be obtained in the following way *for*

$$x=0 \qquad \varphi(t)=\sum_{k=0}^{n}(A_k+B_k)t^{k/2}i^k erfc\,0$$

$$x=l \qquad 0=\sum_{k=0}^{n}(A_k t^{k/2}i^k erfc\frac{l}{2a\sqrt{t}}+B_k t^{k/2}i^k erfc\frac{-l}{2a\sqrt{t}})$$

we make substitution

$$A_k+B_k=C_k \qquad A_k=C_k-B_k$$

Then

$$\sum_{k=0}^{n}C_k t^{k/2}i^k erfc\,0=\varphi(t) \qquad (*)$$

$$\sum_{k=0}^{n}B_k(t^{k/2}i^k erfc\frac{-l}{2a\sqrt{t}}-t^{k/2}i^k erfc\frac{l}{2a\sqrt{t}})=\phi(t)-\sum_{k=0}^{n}C_k t^{k/2}i^k erfc\frac{\sqrt{t}}{2} \qquad (**)$$

Where (*), (**) are systems of linear equations and $i^n erfc\frac{l}{2a\sqrt{t}}, i^n erfc\frac{-l}{2a\sqrt{t}}$ are calculated from tables. Recall that instead of t numerical values are substituted.

2.3 Approximate solution of the second type boundary value problem

For the same problem but different boundary conditions

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}, \qquad 0<x<l\ , \quad t>0 \qquad (26)$$

Subject to

$$\text{I.C:} \qquad u(x,0)=0, \qquad t>0 \qquad (27)$$

$$\text{B.C:} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0}=\varphi(t), \qquad 0<x<l \qquad (28)$$

$$\left.\frac{du}{dx}\right|_{x=l}=\phi(t), \qquad 0<x<l \qquad (29)$$

$$u(0,0)=0 \qquad (30)$$

where $\varphi(t), \phi(t)$ are analytical functions , and solution represented in the form $u(x,t)=\sum_{n=0}^{k}\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\}$ (31) $u(x,t)=\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\right\}$ (32)

Where A_{2n}, A_{2n+1} have to be found. To satisfy the initial and boundary conditions (27) - (29) we expand the functions $\varphi(t), \phi(t)$ in Maclaurin's series:

$$\varphi(t)=\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\cdot t^n \qquad (33)$$

$$\phi(t)=\sum_{n=0}^{\infty}\frac{\phi^{(n)}(0)}{n!}\cdot t^n \qquad (34)$$

Using undetermined coefficients method (equating coefficients in like terms) for x=0 even coefficients A_{2k} can be obtained from

$$A_0\beta_{0,0} + A_2 t\beta_{2,1} + A_4 t^2\beta_{4,2} + \cdots + A_{2k} t^{2k}\beta_{2k,k} = \sum_{k=0}^{\infty} \frac{\varphi^{2k}(0)}{(2k)!} t^{2k}$$

In the same manner, using corollaries (3) and substituting even coefficients into the boundary condition for x=*l*, odd coefficients A_{2k+1} can be determined from

$$\left.\frac{du}{dx}\right|_{x=l} \equiv 2A_2 l\beta_{2,0} +$$

$$+A_4\left(4l^3\beta_{4,0} + 2lt\beta_{4,1}\right) +$$

$$+A_6\left(6l^5\beta_{6,0} + 4l^3 t\beta_{6,1} + 2lt^2\beta_{6,2}\right) + \cdots +$$

$$+A_{2k}\left(2kl^{2k-1}\beta_{2k,0} + (2k-1)l^{2k-3}t\beta_{2k,1} + \cdots + 2lt^k\beta_{2k,k}\right) +$$

$$+A_1\beta_{1,0} +$$

$$+A_3\left(3l^2\beta_{3,0} + t\beta_{3,1}\right) +$$

$$+A_5\left(5l^4\beta_{5,0} + 3l^2 t\beta_{5,1} + t^2\beta_{5,2}\right) + \cdots +$$

$$+A_{2k+1}\Big((2k+1)l^{2k}\beta_{2k+1,0} + (2k-1)l^{2k-2}t\beta_{2k+1,1} + \cdots$$

$$+ 3l^2 t^{k-1}\beta_{2k+1,k-1} + t^k\beta_{2k+1,k}\Big) = \sum_{n=0}^{2k+1} \frac{\phi^{(2k+1)}(0)}{(2k+1)!} t^{2k+1}$$

2.4 Approximate solution of the second type boundary value problem by IEF method

To find approximate solution of Heat Equation with fixed boundary conditions of the second type we use IEF or polynomial form of IEF. Boundary conditions provide coefficients only for certain values of time *t*.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < l \ , \quad t > 0 \tag{35}$$

I.C: $u(x,0) = 0,$ t>0 (36)

B.C: $\left.\frac{\partial u}{\partial x}\right|_{x=0} = \varphi(t),$ $0 < x < l$ (37)

$\left.\frac{du}{dx}\right|_{x=l} = \phi(t),$ $0 < x < l$ (38)

$u(0,0) = 0$ (39)

where $\varphi(t), \phi(t)$ are analytical functions , solution can be represented in the following form

$$u(x,t) = \sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right\}$$

or

$$u(x,t) = \sum_{n=0}^{k} \left(\sqrt{t}\right)^n \left[A_n i^n erfc \frac{x}{2a\sqrt{t}} + B_n i^n erfc \frac{-x}{2a\sqrt{t}} \right]$$

If we use polynomial form then even coefficients A_{2n} determined from boundary condition for x=0, and odd coefficients are determined from boundary condition for x=*l*.

2.5 Approximate solution of Heat Equation with fixed boundary conditions of the third type by IEF method

For the solution of Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < l \ , \quad t > 0 \tag{40}$$

subject to initial and boundary conditions

$$u(x,0) = 0, \qquad 0 < x < l \tag{41}$$

$$\left. \left(\alpha \frac{du}{dx} + \beta u \right) \right|_{x=0} = \varphi(t), \qquad t > 0 \tag{42}$$

$$\left. \left(\mu \frac{du}{dx} + \nu u \right) \right|_{x=l} = \phi(t), \qquad t > 0 \tag{43}$$

$$u(0,0) = 0, \tag{44}$$

where $\varphi(x)$ and $\phi(t)$ are analytical functions, solution can be represented in the polynomial form of IEF

$$u(x,t) = \sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right\}$$

which satisfy initial conditions (27) with $A_0 = 0$. As in previous cases to determine even and odd coefficients A_{2n}, A_{2n+1} it is necessary to expand into Maclaurin's series right parts in boundary conditions

for x=0

$$\alpha\left(A_0\beta_{0,0}+A_2t\beta_{2,1}+A_4t^2\beta_{4,2}+\cdots+A_{2k}t^{2k}\beta_{2k,k}\right)+\beta\{A_1\beta_{1,0}+ +A_3t\beta_{3,1}+ +A_5t^2\beta_{5,2}+\cdots+A_{2k+1}t^k\beta_{2k+1,k}\}=\sum_{k=0}^{\infty}\frac{\varphi^{2k+1}(0)}{(2k+1)!}t^{2k+1}$$

and for x=*l*

$$\mu\{A_0\beta_{0,0}+$$

$$+A_2\left(l^2\beta_{2,0}+t\beta_{2,1}\right)+$$

$$+A_2\left(l^4\beta_{4,0}+l^2t\beta_{4,1}+t\beta_{4,2}\right)+\cdots+$$

$$+A_{2k}\left(l^{2k}\beta_{2k,0}+l^{2k-2}t\beta_{2k,1}+\cdots+l^2t^{k-1}\beta_{2k,k-1}+t^k\beta_{2k,k}\right)+$$

$$+A_1l\beta_{1,0}+$$

$$+A_3\left(l^3\beta_{3,0}+lt\beta_{3,1}\right)+$$

$$+A_5\left(l^5\beta_{5,0}+l^3t\beta_{5,1}+lt^2\beta_{5,2}\right)+\cdots+$$

$$+A_{2k+1}\left(l^{2k+1}\beta_{2k+1,0}+l^{2k-1}t\beta_{2k+1,1}+\cdots+l^3t^{k-1}\beta_{2k+1,k-1}+lt^k\beta_{2k+1,k}\right)$$

$$\}+$$

$$+\nu\{2A_2l\beta_{2,0}+$$

$$+A_4\left(4l^3\beta_{4,0}+2lt\beta_{4,1}\right)+$$

$$+A_6\left(6l^5\beta_{6,0}+4l^3t\beta_{6,1}+2lt^2\beta_{6,2}\right)+\cdots+$$

$$+A_{2k}\left(2kl^{2k-1}\beta_{2k,0}+(2k-1)l^{2k-3}t\beta_{2k,1}+\cdots+2lt^k\beta_{2k,k}\right)+$$

$$+A_1\beta_{1,0}+$$

$$+A_3\left(3l^2\beta_{3,0}+t\beta_{3,1}\right)+$$

$$+A_5\left(5l^4\beta_{5,0}+3l^2t\beta_{5,1}+t^2\beta_{5,2}\right)+\cdots+$$

$$+A_{2k+1}\Big((2k+1)l^{2k}\beta_{2k+1,0}+(2k-1)l^{2k-2}t\beta_{2k+1,1}+\cdots$$
$$+3l^2t^{k-1}\beta_{2k+1,k-1}+t^k\beta_{2k+1,k}\Big)\}=\sum_{n=0}^{2k+1}\frac{\phi^{(2k+1)}(0)}{(2k+1)!}t^{2k+1}$$

then combine like terms in left parts, and equate coefficients using UC method (undetermined coefficients method). Finally after obtaining recurrent formulas and solving systems of linear equations even and odd coefficients can be determined.

2.6 Approximate solution of Heat Equation with fixed boundary conditions of the third type by IEF method

As in previous cases we follow same procedure. We consider solution in the polynomial form of IEF. To determine coefficients we substitute expression (3) from chapter 1.3 into the boundary conditions but only for certain values of time *t*. Boundary conditions provide system of linear equations and necessary coefficients can be obtained or calculated by the help of Mathcad.

2.7 Solution of Heat Equations in the domains with moving, given boundaries by IEF method

As it was told before Integral Error Functions enable investigate heat transfer, diffusion and other phenomena which can be described by the equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$$

In a region $D(t>0, 0<x<\alpha(t))$ with free boundary $x=\alpha(t)$, and solution is considered in the following form

$$u(x,t)=\sum_{n=0}^{\infty}[A_n u_n(x,t)+B_n u_n(-x,t)]$$

For any constants A_n, B_n. Which have to be found from boundary conditions at $x=0$ and $x=\alpha(t)$, if given boundary functions can be expanded into Taylor series with powers t or $\sqrt{t}$.It is possible to see in the following examples (where all types of boundary value problems considered) and paragraphs, that analytic solutions of Heat Equations are found.

2.8 Approximate solution of Heat Equation with the first type boundary conditions in the $0<x<\alpha(t)$ domain by IEF method

Solution of the Heat Equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\qquad 0<x<\alpha(t)\ ,\quad \mathrm{t}>0 \tag{45}$$

Subject to

B.C: $$u(0,t)=\varphi(t), \qquad t>0 \tag{46}$$

$$u(\alpha(t),t)=\phi(t), t>0 \tag{47}$$

can be represented in the following form

$$U(x,t)=\sum_{n=0}^{\infty}\left\{A_n\cdot t^n\left[i^{2n}\mathrm{erfc}\frac{x}{2a\sqrt{t}}+i^{2n}\mathrm{erfc}(\frac{-x}{2a\sqrt{t}})\right]+B_n\cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}\mathrm{erfc}\frac{x}{2a\sqrt{t}}-i^{2n+1}\mathrm{erfc}(\frac{-x}{2a\sqrt{t}})\right]\right\} \tag{48}$$

or using property 2 of IEF, solution can be represented in the polynomial form

$$u(x,t)=\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\right\} \tag{49}$$

In this case it is more convenient to use expression (48) for solution of (46)-(47).

where

$$\alpha(t)=\alpha_1 t^{\frac{1}{2}}+\alpha_2 t+\alpha_3 t^{\frac{3}{2}}+\ldots+\alpha_n t^{\frac{n}{2}}+\ldots \tag{50}$$

for $\tau=\sqrt{t}$ (51)

we have

$$\alpha(\tau^2)=\alpha_1\tau+\alpha_2\tau^2+\alpha_3\tau^3+\ldots+\alpha_n\tau^n+\ldots \tag{52}$$

From (2) for x=0, we have

$$\varphi(t)=\sum_{n=0}^{\infty}A_n t^n i^{2n}erfc0 \tag{53}$$

And coefficient A_n of expression (129) can be found from

$$A_n=\frac{\varphi^n(0)}{i^{2n}\,n!\,erfc0} \tag{54}$$

From (128) for x=$\alpha(t)$, we have

$$\psi(t)=\sum_{n=0}^{\infty}\left\{A_n\cdot t^n\left[i^{2n}\mathrm{erfc}\frac{\alpha(t)}{2a\sqrt{t}}+i^{2n}\mathrm{erfc}(\frac{-\alpha(t)}{2a\sqrt{t}})\right]+B_n\cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}\mathrm{erfc}\frac{\alpha(t)}{2a\sqrt{t}}-i^{2n+1}\mathrm{erfc}(\frac{-\alpha(t)}{2a\sqrt{t}})\right]\right\} \tag{55}$$

Let

$$\frac{\alpha(t)}{2a\sqrt{t}}=\frac{\alpha_1}{2a}+\frac{\alpha_2}{2a}t^{\frac{1}{2}}+,\frac{\alpha_3}{2a}t+\frac{\alpha_3}{2a}t^{\frac{3}{2}}+\frac{\alpha_4}{2a}t^2+\ldots+\frac{\alpha_n}{2a}t^{\frac{n}{2}}+\ldots \tag{56}$$

for $\tau=\sqrt{t}$

$$\beta(\tau)=\frac{\alpha_1}{2a}+\frac{\alpha_2}{2a}\tau+,\frac{\alpha_3}{2a}\tau^2+\frac{\alpha_3}{2a}\tau^3+\frac{\alpha_4}{2a}\tau^4+...+\frac{\alpha_n}{2a}\tau^n+..., \tag{57}$$

Let's say

$\beta_0=\frac{\alpha_1}{2a},$

$\beta_1=\frac{\alpha_2}{2a},$

$\beta_3=\frac{\alpha_2}{2a},$

……………..

$\beta_n=\frac{\alpha_{n-1}}{2a}$

then

$$\beta(\tau)=\beta_0+\beta_1\tau+\beta_2\tau^2+\beta_3\tau^3+\beta_4\tau^4+...+\beta_n\tau^n+..., \tag{58}$$

Finally

$$\psi(\tau^2)=\sum_{n=0}^{\infty}\left\{A_n\cdot\tau^{2n}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc\left(-\beta(\tau)\right)\right]+B_n\cdot\tau^{2n+1}\left[i^{2n+1}erfc\beta(\tau)-i^{2n+1}erfc\left(-\beta(\tau)\right)\right]\right\} \tag{59}$$

Taking n times derivatives of expression (59) and making $\tau=0,$ it's possible to determine B_n coefficients.

Corollaries: [2, p. 34-35]

1. $\frac{d^n}{dx^n}f(x)=\frac{U_1}{1!}\cdot F'(y)+\frac{U_2}{2!}\cdot F''(y)+\frac{U_3}{3!}\cdot F'''(y)+...+\frac{U_n}{n!}\cdot F^{(n)}(y)$

$$U_k=\frac{d^n}{dx^n}y^k-\frac{k}{1!}\cdot y\cdot\frac{d^n}{dx^n}y^{k-1}+\frac{k(k-1)}{2!}\cdot y^2\cdot\frac{d^n}{dx^n}y^{k-2}-...+(-1)^{k-1}k\cdot y^{k-1}\cdot\frac{d^n}{dx^n}y$$

2. $(u\cdot v)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}\cdot u^{(n-k)}\cdot v^{(k)}$

Firstly let's expand (59)

$$\psi(\tau^2)=\sum_{n=0}^{\infty}\left\{A_n\cdot\tau^{2n}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc\left(-\beta(\tau)\right)\right]+B_n\cdot\tau^{2n+1}\left[i^{2n+1}erfc\beta(\tau)-i^{2n+1}erfc\left(-\beta(\tau)\right)\right]\right\}=$$
$$=A_0\cdot\left[erfc\beta(\tau)+erfc\left(-\beta(\tau)\right)\right]+B_0\cdot\tau\left[ierfc\beta(\tau)-ierfc\left(-\beta(\tau)\right)\right]+$$
$$+A_1\cdot\tau^2\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]+B_1\cdot\tau^3\left[i^3erfc\beta(\tau)-i^3erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_2\cdot\tau^4\left[i^4erfc\beta(\tau)+i^4erfc\left(-\beta(\tau)\right)\right]+B_2\cdot\tau^5\left[i^5erfc\beta(\tau)-i^5erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_3\cdot\tau^6\left[i^6erfc\beta(\tau)+i^6erfc\left(-\beta(\tau)\right)\right]+B_3\cdot\tau^7\left[i^7erfc\beta(\tau)-i^7erfc\left(-\beta(\tau)\right)\right]+$$
$$+\ldots$$
$$.$$
$$.$$
$$.$$
$$+A_n\cdot\tau^{2n}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc\left(-\beta(\tau)\right)\right]+B_n\cdot\tau^{2n+1}\left[i^{2n+1}erfc\beta(\tau)-i^{2n+1}erfc\left(-\beta(\tau)\right)\right]$$

Recall that A_n coefficients are already determined from (54). Using formulas in corollaries now it is possible to determine B_n coefficients. To determine B_n coefficients it is necessary to take 2n+1 times derivatives of (140) and make $\tau=0$. It is required to determine B_0 from following expression

$$\left[\psi(\tau^2)\right]_{\tau=0}^{(1)}=\left\{A_0\cdot\left[erfc\beta(\tau)+erfc\left(-\beta(\tau)\right)\right]+B_0\cdot\tau\left[ierfc\beta(\tau)-ierfc\left(-\beta(\tau)\right)\right]\right\}_{\tau=0}^{(1)}$$

B_1 from

$$\left[\psi(\tau^2)\right]_{\tau=0}^{(3)}=\{A_0\cdot\left[erfc\beta(\tau)+erfc\left(-\beta(\tau)\right)\right]+B_0\cdot\tau\left[ierfc\beta(\tau)-ierfc\left(-\beta(\tau)\right)\right]+$$
$$+A_1\cdot\tau^2\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]+B_1\cdot\tau^3\left[i^3erfc\beta(\tau)-i^3erfc\left(-\beta(\tau)\right)\right]\}_{\tau=0}^{(3)}$$

B_2 from

$$\left[\psi(\tau^2)\right]_{\tau=0}^{(5)}=\{A_0\cdot\left[erfc\beta(\tau)+erfc\left(-\beta(\tau)\right)\right]+B_0\cdot\tau\left[ierfc\beta(\tau)-ierfc\left(-\beta(\tau)\right)\right]+$$
$$+A_1\cdot\tau^2\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]+B_1\cdot\tau^3\left[i^3erfc\beta(\tau)-i^3erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_2\cdot\tau^4\left[i^4erfc\beta(\tau)+i^4erfc\left(-\beta(\tau)\right)\right]+B_2\cdot\tau^5\left[i^5erfc\beta(\tau)-i^5erfc\left(-\beta(\tau)\right)\right]\}_{(\tau=0)}^{(5)}$$

B_3 from

$$\left[\psi(\tau^2)\right]_{\tau=0}^{(7)}=\{A_0\cdot\left[erfc\beta(\tau)+erfc\left(-\beta(\tau)\right)\right]+B_0\cdot\tau\left[ierfc\beta(\tau)-ierfc\left(-\beta(\tau)\right)\right]+$$
$$+A_1\cdot\tau^2\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]+B_1\cdot\tau^3\left[i^3erfc\beta(\tau)-i^3erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_2\cdot\tau^4\left[i^4erfc\beta(\tau)+i^4erfc\left(-\beta(\tau)\right)\right]+B_2\cdot\tau^5\left[i^5erfc\beta(\tau)-i^5erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_3\cdot\tau^6\left[i^6erfc\beta(\tau)+i^6erfc\left(-\beta(\tau)\right)\right]+B_3\cdot\tau^7\left[i^7erfc\beta(\tau)-i^7erfc\left(-\beta(\tau)\right)\right]\}_{\tau=0}^{(7)}$$

and so on. Finally B_n from the following expression

$$\left[\psi(\tau^2)\right]_{\tau=0}^{(2n+1)} = \{A_0 \cdot \left[\mathrm{erfc}\beta(\tau) + \mathrm{erfc}(-\beta(\tau))\right] + B_0 \cdot \tau\left[\mathrm{ierfc}\beta(\tau) - \mathrm{ierfc}(-\beta(\tau))\right] +$$
$$+ A_1 \cdot \tau^2\left[i^2\mathrm{erfc}\beta(\tau) + i^2\mathrm{erfc}(-\beta(\tau))\right] + B_1 \cdot \tau^3\left[i^3\mathrm{erfc}\beta(\tau) - i^3\mathrm{erfc}(-\beta(\tau))\right] +$$
$$+ A_2 \cdot \tau^4\left[i^4\mathrm{erfc}\beta(\tau) + i^4\mathrm{erfc}(-\beta(\tau))\right] + B_2 \cdot \tau^5\left[i^5\mathrm{erfc}\beta(\tau) - i^5\mathrm{erfc}(-\beta(\tau))\right] +$$
$$+ A_3 \cdot \tau^6\left[i^6\mathrm{erfc}\beta(\tau) + i^6\mathrm{erfc}(-\beta(\tau))\right] + B_3 \cdot \tau^7\left[i^7\mathrm{erfc}\beta(\tau) - i^7\mathrm{erfc}(-\beta(\tau))\right] +$$
$$+ \ldots$$
.
.
.
$$+ A_n \cdot \tau^{2n}\left[i^{2n}\mathrm{erfc}\beta(\tau) + i^{2n}\mathrm{erfc}(-\beta(\tau))\right] + B_n \cdot \tau^{2n+1}\left[i^{2n+1}\mathrm{erfc}\beta(\tau) - i^{2n+1}\mathrm{erfc}(-\beta(\tau))\right]\}_{\tau=0}^{(2n+1)}$$

or

$$\left[\psi(\tau^2)\right]_{\tau=0}^{(2n+1)} = \sum_{k=0}^{n} \begin{aligned}&\{A_k \cdot \tau^{2k}\left[i^{2k}\mathrm{erfc}\beta(\tau) + i^{2k}\mathrm{erfc}(-\beta(\tau))\right] + \\ &+ B_k \cdot \tau^{2k+1}\left[i^{2k+1}\mathrm{erfc}\beta(\tau) - i^{2k+1}\mathrm{erfc}(-\beta(\tau))\right]\}_{\tau=0}^{(2n+1)}\end{aligned} \qquad (61)$$

For expression $\left[\tau^m \cdot i^m\mathrm{erfc}(-\beta(\tau))\right]^{(n)}$ it is necessary to apply formula (2) in corollaries where $\tau^m = u$ and $i^m erfc\beta(\tau) = v$.

$$\left[\tau^m \cdot i^m\mathrm{erfc}\beta(\tau)\right]^{(n)} = \left(\tau^m\right)^{(n)} \cdot i^m\mathrm{erfc}\beta(\tau) + \binom{n}{1} \cdot \left(\tau^m\right)^{(n-1)} \cdot \left(i^m\mathrm{erfc}\beta(\tau)\right)^{(1)} +$$
$$+ \binom{n}{2} \cdot \left(\tau^m\right)^{(n-2)} \cdot \left(i^m\mathrm{erfc}\beta(\tau)\right)^{(2)} + \ldots + \tau^m \cdot \left(i^m\mathrm{erfc}\beta(\tau)\right)^{(n)}.$$

(62)

For $\tau = 0$,

1. if $m > n$ then $\left[\tau^m \cdot i^m erfc\beta(\tau)\right]_{\tau=0}^{(n)} = 0$
2. if $m \le n$ then $\left[\tau^m \cdot i^m erfc\beta(\tau)\right]_{\tau=0}^{(n)} = m! \cdot \binom{n}{n-m} \cdot \left[i^m erfc\beta(\tau)\right]_{\tau=0}^{(n-m)}$ (63)

In the same manner for expression $\left[\tau^m \cdot i^m erfc(-\beta(\tau))\right]^{(n)}$

1. if $m > n$ then $\left[\tau^m \cdot i^m\mathrm{erfc}(-\beta(\tau))\right]_{\tau=0}^{(n)} = 0$ (64)
2. if $m \le n$ then $\left[\tau^m \cdot i^m\mathrm{erfc}(-\beta(\tau))\right]_{\tau=0}^{(n)} = m! \cdot \binom{n}{n-m} \cdot \left[i^m\mathrm{erfc}(-\beta(\tau))\right]_{\tau=0}^{(n-m)}$

(65)

Finally

$$\left\{\tau^m \cdot \left[i^m\mathrm{erfc}\beta(\tau) \pm i^m\mathrm{erfc}(-\beta(\tau))\right]\right\}_{\tau=0}^{(n)} = m! \cdot \binom{n}{n-m} \cdot \left[i^m\mathrm{erfc}\beta(\tau) \pm i^m\mathrm{erfc}(-\beta(\tau))\right]_{\tau=0}^{(n-m)}$$

(66)

yields

$$\sum_{k=0}^{n}\{A_k \cdot \tau^{2k}\left[i^{2k}\operatorname{erfc}\beta(\tau)+i^{2k}\operatorname{erfc}(-\beta(\tau))\right]+B_k \cdot \tau^{2k+1}\left[i^{2k+1}\operatorname{erfc}\beta(\tau)-i^{2k+1}\operatorname{erfc}(-\beta(\tau))\right]\}_{\tau=0}^{(2n+1)}=$$

$$=\sum_{k=0}^{n}\{A_k \cdot (2k)!\cdot\binom{2n+1}{2n+1-2k}\cdot\left[i^{2k}\operatorname{erfc}\beta(\tau)+i^{2k}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)}+$$

$$+B_k \cdot (2k+1)!\cdot\binom{2n+1}{2n-2k}\cdot\left[i^{2k+1}\operatorname{erfc}\beta(\tau)-i^{2k+1}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}\}$$

(67)

then

$$\left[\psi(\tau^2)\right]_{\tau=0}^{(2n+1)}=\sum_{k=0}^{n}\{A_k \cdot (2k)!\cdot\binom{2n+1}{2n+1-2k}\cdot\left[i^{2k}\operatorname{erfc}\beta(\tau)+i^{2k}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)}+$$

$$+B_k \cdot (2k+1)!\cdot\binom{2n+1}{2n-2k}\cdot\left[i^{2k+1}\operatorname{erfc}\beta(\tau)-i^{2k+1}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}\}$$

(68)

Implies

$$\sum_{k=0}^{n}\{A_k \cdot (2k)!\cdot\binom{2n+1}{2n+1-2k}\cdot\left[i^{2k}\operatorname{erfc}\beta(\tau)+i^{2k}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)}+$$

$$+B_k \cdot (2k+1)!\cdot\binom{2n+1}{2n-2k}\cdot\left[i^{2k+1}\operatorname{erfc}\beta(\tau)-i^{2k+1}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}\}=$$

$$=\sum_{k=0}^{n}\left\{A_k \cdot (2k)!\cdot\binom{2n+1}{2n+1-2k}\cdot\left[i^{2k}\operatorname{erfc}\beta(\tau)+i^{2k}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)}\right\}+$$

$$+\sum_{k=0}^{n-1}\left\{B_k \cdot (2k+1)!\cdot\binom{2n+1}{2n-2k}\cdot\left[i^{2k+1}\operatorname{erfc}\beta(\tau)-i^{2k+1}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}\right\}+$$

$$+B_n \cdot (2n+1)!\cdot\binom{2n+1}{0}\cdot\left[i^{2n+1}\operatorname{erfc}\beta(\tau)-i^{2n+1}\operatorname{erfc}(-\beta(\tau))\right]_{\tau=0}$$

(69)

Yields

$$B_n = -\frac{\left[\psi(\tau^2)\right]_{\tau=0}^{(2n+1)}}{(2n+1)!\cdot\begin{pmatrix}2n+1\\0\end{pmatrix}\cdot\left[i^{2n+1}\mathrm{erfc}\beta(\tau) - i^{2n+1}\mathrm{erfc}\left(-\beta(\tau)\right)\right]_{\tau=0}} +$$

$$+\frac{\sum_{k=0}^{n}\{A_k\cdot(2k)!\cdot\begin{pmatrix}2n+1\\2n+1-2k\end{pmatrix}\cdot\left[i^{2k}\mathrm{erfc}\beta(\tau) + i^{2k}\mathrm{erfc}\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n+1-2k)}}{(2n+1)!\cdot\begin{pmatrix}2n+1\\0\end{pmatrix}\cdot\left[i^{2n+1}\mathrm{erfc}\beta(\tau) - i^{2n+1}\mathrm{erfc}\left(-\beta(\tau)\right)\right]_{\tau=0}} +$$

$$+\frac{\sum_{k=0}^{n-1}B_k\cdot(2k+1)!\cdot\begin{pmatrix}2n+1\\2n-2k\end{pmatrix}\cdot\left[i^{2k+1}\mathrm{erfc}\beta(\tau) - i^{2k+1}\mathrm{erfc}\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}}{(2n+1)!\cdot\begin{pmatrix}2n+1\\0\end{pmatrix}\cdot\left[i^{2n+1}\mathrm{erfc}\beta(\tau) - i^{2n+1}\mathrm{erfc}\left(-\beta(\tau)\right)\right]_{\tau=0}} \qquad (70)$$

Thus required A_n and B_n coefficients are obtained and can be calculated from formulas (54) and (70) respectively.

Example 1: Analytic solution of one Heat Equation with the first type boundary conditions in the $0 < x < t$ domain

Solution of the Heat Equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < t\ , \quad t > 0 \qquad (71)$$

Subject to

I.C: $u(x,0)=0, \qquad t>0$ (72)

B.C: $u(0,t)=\phi(t),t>0$ (73)

$$u(t,t)=\phi(t),t>0 \qquad (74)$$

can be represented in the following form

$$U(x,t)=\sum_{n=0}^{\infty}\left\{\begin{array}{l}A_n\cdot t^n\left[i^{2n}erfc\frac{x}{2a\sqrt{t}}+i^{2n}erfc(\frac{-x}{2a\sqrt{t}})\right]+\\ +B_n\cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc\frac{x}{2a\sqrt{t}}-i^{2n+1}erfc(\frac{-x}{2a\sqrt{t}})\right]\end{array}\right\} \qquad (75)$$

or using property 2 of IEF solution can represented in the polynomial form

$$U(x,t) = \sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\right\}$$

or

$$u(x,t) = A_0\beta_{0,0} + \qquad (76)$$

$$+A_2\left(x^2\beta_{2,0}+t\beta_{2,1}\right)+$$

$$+A_4\left(x^4\beta_{4,0}+x^2t\beta_{4,1}+t\beta_{4,2}\right)+\cdots+$$

$$+A_{2k}\left(x^{2k}\beta_{2k,0}+x^{2k-2}t\beta_{2k,1}+\cdots+x^2t^{k-1}\beta_{2k,k-1}+t^k\beta_{2k,k}\right)+$$

$$+A_1x\beta_{1,0}+$$

$$+A_3\left(x^3\beta_{3,0}+xt\beta_{3,1}\right)+$$

$$+A_5\left(x^5\beta_{5,0}+x^3t\beta_{5,1}+xt^2\beta_{5,2}\right)+\cdots+$$

$$+A_{2k+1}\left(x^{2k+1}\beta_{2k+1,0}+x^{2k-1}t\beta_{2k+1,1}+\cdots+x^3t^{k-1}\beta_{2k+1,k-1}+xt^k\beta_{2k+1,k}\right)$$

where

$$\beta(n,m):=\frac{1}{2^{n+m-1}\cdot m!\cdot(n-2m)!}$$

Let's expand $\varphi(t)$ into Taylor'series then we have $\varphi(t)=\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\cdot t^n$

Then for x=0 from (73)

$$\sum_{k=0}^{\infty}A_{2k}\beta_{2k,k}\cdot t^n=\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}\cdot t^k$$

$$\Rightarrow A_{2k}=\frac{\varphi^{(k)}(0)}{k!\cdot\beta_{2k,k}}$$

In the same manner let's expand $\phi(t)$ into Taylor's series then we have

$$\phi(t)=\sum_{n=0}^{\infty}\frac{\phi^{(n)}(0)}{n!}\cdot t^n$$

Then for x=t from (74) we have

$$u(t,t)=A_0\beta_{0,0}+$$

$$+A_2\left(t^2\beta_{2,0}+t\beta_{2,1}\right)+$$

$$+A_4\left(t^4\beta_{4,0}+t^3\beta_{4,1}+t\beta_{4,2}\right)+$$

$$+A_6\left(t^6\beta_{6,0}+t^5\beta_{6,1}+t^4\beta_{6,2}+t^3\beta_{6,3}\right)+$$

$$+A_8(t^8\beta_{8,0}+t^7\beta_{8,1}+t^6\beta_{8,2}+t^5\beta_{8,3}+t^4\beta_{8,4})+\cdots+$$

$$+A_{2k}\left(t^{2k}\beta_{2k,0}+t^{2k-1}\beta_{2k,1}+\cdots+t^{k+1}\beta_{2k,k-1}+t^k\beta_{2k,k}\right)+$$

$$+A_1t\beta_{1,0}+$$

$$+A_3\left(t^3\beta_{3,0}+t^2\beta_{3,1}\right)+$$

$$+A_5\left(t^5\beta_{5,0}+t^4\beta_{5,1}+t^3\beta_{5,2}\right)+$$

$$+A_7\left(t^7\beta_{7,0}+t^6\beta_{7,1}+t^5\beta_{7,2}+t^4\beta_{7,3}\right)+$$

$$+A_9\left(t^9\beta_{9,0}+t^8\beta_{9,1}+t^7\beta_{9,2}+t^6\beta_{9,3}+t^5\beta_{9,4}\right)+\cdots+$$

$$+A_{2k+1}\left(t^{2k+1}\beta_{2k+1,0}+t^{2k}\beta_{2k+1,1}+\cdots+t^{k+1}\beta_{2k+1,k-1}+t^k\beta_{2k+1,k}\right)$$

From above expression it’s possible to see that

$$A_0\cdot\beta_{0,0}=\phi_0$$

$$A_1\cdot\beta_{1,0}+A_2\cdot\beta_{2,1}=\phi_1$$

$$A_2\cdot\beta_{2,0}+A_3\cdot\beta_{3,1}+A_4\cdot\beta_{4,2}=\phi_2$$

$$A_3\cdot\beta_{3,0}+A_4\cdot\beta_{4,1}+A_5\cdot\beta_{5,2}+A_6\cdot\beta_{6,3}=\phi_3$$

$$A_4\cdot\beta_{4,0}+A_5\cdot\beta_{5,1}+A_6\cdot\beta_{6,2}+A_7\cdot\beta_{7,3}+A_8\cdot\beta_{8,4}=\phi_4$$

……………………………………………………………………………

$$A_k\cdot\beta_{k,0}+A_{k+1}\cdot\beta_{k+1,1}+A_{k+2}\cdot\beta_{k+2,2}+\ldots+A_{2k-1}\cdot\beta_{2k-1,k-1}+A_{2k}\cdot\beta_{2k,k}=\phi_k$$

where $\phi_k=\frac{\varphi^{(k)}(0)}{k!}$

and

$$t^k \sum_{n=k}^{2k} A_n \cdot \beta_{n,n-k} = \frac{\phi^{(k)}(0)}{k!} \cdot t^k$$

Finally following recurrent formula obtained

$$A_{2k-1} \cdot \beta_{2k-1,k-1} = \frac{\phi^{(k)}(0)}{k!} - A_{2k} \cdot \beta_{2k,k} - \sum_{n=k}^{2k-2} A_n \cdot \beta_{n,n-k}$$

2.9 Approximate solution of Heat Equation with moving boundaries of the first type by IEF method

Results of studies and experimental data (links) show that it is possible to solve Heat Equations with moving boundaries approximately by the help of IEF method. Solutions obtained by IEF method are easier and give quite good precision, which can be used in applications.

Solution is represented in the form of Integral Error Functions or in polynomial form which is derived from its properties.

For the Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \alpha(t), \quad t > 0$$

Subject to

I.C: $u(x, o) = 0, \qquad 0 < x < \alpha(t),$

B.C: $u(0,t) = \varphi(t), \qquad t > 0,$

$u(\alpha(t), t) = \phi(t), \qquad t > 0.$

Solution:

Solution of the Heat Equation is represented in the form

$$u(x,t) = \sum_{n=0}^{k} (t^{\frac{n}{2}}) \left\{ A_n i^n erfc\left(\frac{x}{2a\sqrt{t}}\right) + B_n i^n erfc\left(-\frac{x}{2a\sqrt{t}}\right) \right\}$$

or

$$u(x,t) = \sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right\}$$

Remark In both cases to find approximate solution it is necessary to give certain values for time *t*. For *k* values of *t* we obtain *k* coefficients from *2k* linear equations.

If we use IEF for solution then for boundary conditions at x=0 and $x = \alpha(t)$ we have system of *2k* linear equations.

$$\sum_{n=0}^{k} (t_k^{\frac{n}{2}}) \{ A_n i^n erfc0 + B_n i^n erfc0 \} = \varphi(t_k),$$

$$\sum_{n=0}^{k} (t_k^{\frac{n}{2}}) \left\{ A_n i^n erfc\left(\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right) + B_n i^n erfc\left(-\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right) \right\} = \phi(t_k)$$

In the same manner solution in the polynomial form of IEF allows us to find even and odd coefficients A_{2n}, A_{2n+1} from system of linear equations

$$\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n} x^{2n-2m}t_k^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t_k^m\beta_{2n+1,m}\right\} = \varphi(t_k),$$

$$\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n} x^{2n-2m}t_k^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t_k^m\beta_{2n+1,m}\right\} = \phi(t_k)$$

Test problem: Approximate solution of one Heat Equation in the domain with moving boundary of the first type, degenerating at the initial time by polynomial form of IEF

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < t, \quad 0 < t < 1$$

$$u|_{t=0} = 0$$

$$u|_{x=0} = e^t$$

$$u|_{x=t} = 1$$

Making substitution for

$$u(x,t) = v(x,t) + 1$$

We obtain

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \quad 0 < x < t, \quad 0 < t < 1$$

$$v|_{t=0} = 0$$

$$v|_{x=0} = e^t - 1$$

$$v|_{x=t} = 0$$

Using corollaries we consider solution in the following form

$$V(x,t) = \sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n} x^{2n-2m}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t^m\beta_{2n+1,m}\right\}$$

And it is required to determine even and odd coefficients A_{2n}, A_{2n+1}. The number of coefficients determined with respect to the number of the values given for t. If there're 6 values given for t, 6 odd and 6 even coefficients will be found for the solution above. As more values are given for t, as more precise will be solution.

If above expression substituted into the boundary conditions where values of t are: t1=0, t2=0,2, t3=0,4, t4=0,6, t5=0,8 and t6=1. Even and odd coefficients can be obtained from systems of linear equations

for x=0,
$e^t - 1 = \sum_{n=0}^{k}\{A_{2n}\sum_{m=0}^{n} x^{2n-2m}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t^m\beta_{2n+1,m}\}$

yield

$$\sum_{n=0}^{k} A_{2n}\sum_{m=0}^{n} x^{2n-2m}t^m\beta_{2n,m} = e^t - 1 - \sum_{n=0}^{k} A_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t^m\beta_{2n+1,m}$$

Where even coefficients expressed in terms of odd coefficients and substituted into the linear system of equations for $x = t$, where values of t are same t1=0, t2=0,2, t3=0,4, t4=0,6, t5=0,8 and t6=1.

for $x = t$,
$0 = \sum_{n=0}^{k}\{A_{2n}\sum_{m=0}^{n} x^{2n-2m}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t^m\beta_{2n+1,m}\}$

To solve above systems and build graphs of functions programs like Mathcad can be helpful. After even and odd coefficients A_{2n}, A_{2n+1} already found it is necessary to substitute the expression into the boundary conditions and compare with functions at boundary conditions. When the procedure performed it is possible to see almost one to one satisfaction in the figures 2, 3, 4,5,6,7 when we compare functions at boundaries for x=0 and x=t respectively with the solutions obtained by Integral Error Functions.

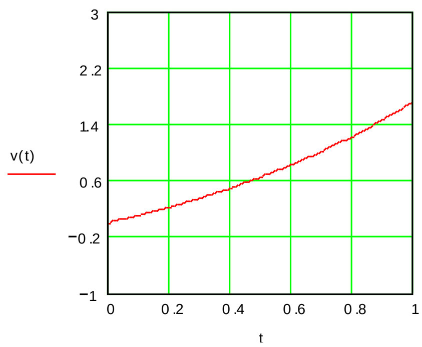


Figure 1 – Graph of function at x=0

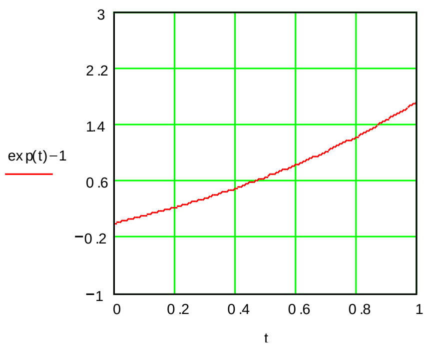
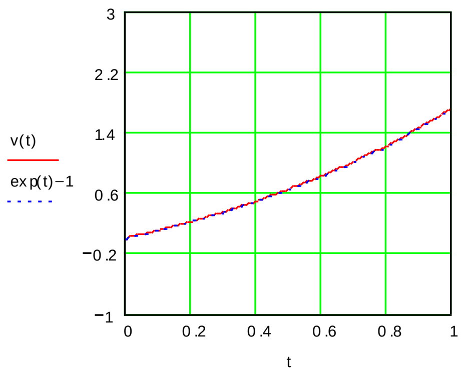


Figure 2 –Graph of function at x=0

In the figure 1 we see function obtained by IEF at x=0, and in the figure 3 given function at boundary x=0.



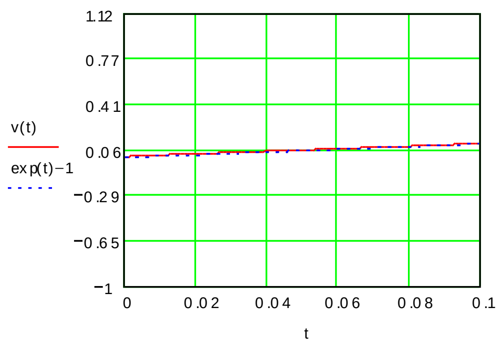


Figure 3 – Graphs of boundary function and V(x,t) at x=0

Figure 4 – Graphs of boundary function and V(x,t) at x=0

To compare above graphs we put them on each other and observe one to one satisfaction and coincidence in the figure 4 and 5.

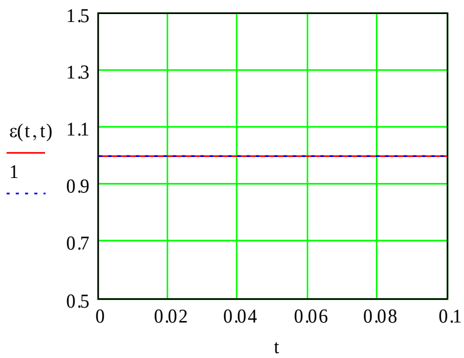


Figure 5 - Graphs of constant 1

and V(x,t) at x=t

It is possible to see same precision at the boundary $x = t$.

Conclusion

In the same manner it is possible to obtain same results if solution is

$$u(x,t) = \sum_{n=0}^{\infty} (\sqrt{t})^n \left[A_n i^n erfc \frac{x}{2a\sqrt{t}} + B_n i^n erfc \frac{-x}{2a\sqrt{t}} \right]$$

Comparison of functions obtained after substitution into the boundary conditions with given boundary functions give same precision which possible to observe in the below pictures.

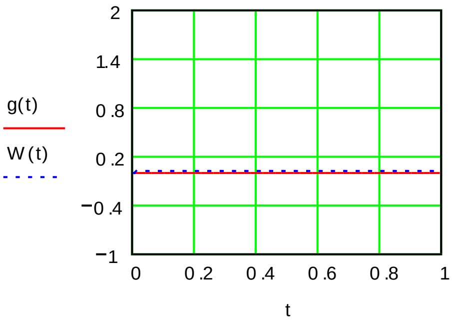


Figure 6 – Graphs of functions at x=0

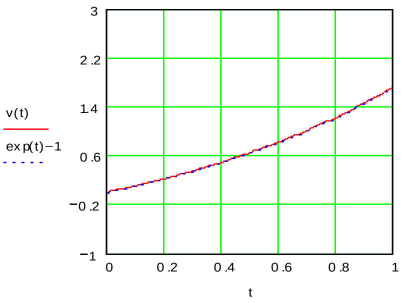


Figure 7 – Graphs of functions at x=t

Finally according to maximum principle, in the above figures it is possible to see that for same values of t: t=0, t2=0,2, t3=0,4, t4=0,6, t5=0,8, t6=1 solution of Heat Equation with moving boundary, degenerating at the initial time, obtained by IEF method is much more precise and easier than traditional methods.

2.10 Approximate solution of Heat Equation with the second type boundary conditions in the $0 < x < \alpha(t)$ domain by IEF method

Solution of the Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \alpha(t)\ , \quad t > 0 \tag{77}$$

Subject to

I.C: $$u(x,0) = 0, \tag{78}$$

B.C: $$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \varphi(t), \tag{79}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=\alpha(t)} = \phi(t), \tag{80}$$

$$u(0,0) = 0, \tag{81}$$

can be represented in the following form

$$U(x,t) = \sum_{n=0}^{\infty}\left\{A_n \cdot t^n \left[i^{2n} erfc\frac{x}{2a\sqrt{t}} + i^{2n} erfc(\frac{-x}{2a\sqrt{t}})\right] + B_n \cdot t^{\frac{2n+1}{2}}\left[i^{2n+1} erfc\frac{x}{2a\sqrt{t}} - i^{2n+1} erfc(\frac{-x}{2a\sqrt{t}})\right]\right\} \tag{82}$$

or using property 2 of IEF, solution can be represented in the polynomial form

$$U(x,t) = \sum_{n=0}^{k}\{A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m}\} \tag{83}$$

In this case it is more convenient to use expression (82) for solution of (78)- (80).

where

$$\alpha(t)=\alpha_1 t^{\frac{1}{2}}+\alpha_2 t+\alpha_3 t^{\frac{3}{2}}+\ldots+\alpha_n t^{\frac{n}{2}}+\ldots \tag{84}$$

for $\tau=\sqrt{t}$ (85)

we have

$$\alpha(\tau^2)=\alpha_1\tau+\alpha_2\tau^2+\alpha_3\tau^3+\ldots+\alpha_n\tau^n+\ldots \tag{86}$$

From (79) for x=0, we have

$$\varphi(t)=-2\sum_{n=0}^{\infty}B_n t^n i^{2n} erfc0 \tag{87}$$

And coefficients B_n of expression (82) can be found from

$$B_n=-\frac{\varphi^n(0)}{2\cdot n!\cdot i^{2n} erfc0} \tag{88}$$

From (80) for x=$\alpha(t)$, we have

$$\frac{\partial U}{\partial x}=\sum_{n=0}^{\infty}\left\{\begin{array}{l} A_n\cdot t^{n-1/2}\left[-i^{2n-1}erfc\frac{x}{2a\sqrt{t}}+i^{2n-1}erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right]- \\ -B_n\cdot t^n\left[i^{2n}erfc\frac{x}{2a\sqrt{t}}+i^{2n}erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right] \end{array}\right\} \tag{89}$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=\alpha(t)}=\sum_{n=0}^{\infty}\left\{\begin{array}{l} A_n\cdot t^{n-1/2}\left[-i^{2n-1}erfc\frac{x}{2a\sqrt{t}}+i^{2n-1}erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right]- \\ -B_n\cdot t^n\left[i^{2n}erfc\frac{x}{2a\sqrt{t}}+i^{2n}erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right] \end{array}\right\}$$

Let

$$\frac{\alpha(t)}{2a\sqrt{t}}=\frac{\alpha_1}{2a}+\frac{\alpha_2}{2a}t^{\frac{1}{2}}+\frac{\alpha_3}{2a}t+\frac{\alpha_3}{2a}t^{\frac{3}{2}}+\frac{\alpha_4}{2a}t^2+\ldots+\frac{\alpha_n}{2a}t^{\frac{n}{2}}+\ldots \tag{90}$$

for $\tau=\sqrt{t}$

$$\beta(\tau)=\frac{\alpha_1}{2a}+\frac{\alpha_2}{2a}\tau+\frac{\alpha_3}{2a}\tau^2+\frac{\alpha_3}{2a}\tau^3+\frac{\alpha_4}{2a}\tau^4+\ldots+\frac{\alpha_n}{2a}\tau^n+\ldots, \tag{91}$$

Let’s say

$$\beta_0=\frac{\alpha_1}{2a},$$

$$\beta_1=\frac{\alpha_2}{2a},$$

$$\beta_3 = \frac{\alpha_2}{2a},$$

……………..

$$\beta_n = \frac{\alpha_{n-1}}{2a}$$

then

$$\beta(\tau) = \beta_0 + \beta_1\tau + \beta_2\tau^2 + \beta_3\tau^3 + \beta_4\tau^4 + \ldots + \beta_n\tau^n + \ldots, \tag{92}$$

thus

$$\varphi(\tau^2) = \sum_{n=0}^{\infty}\left\{A_n \cdot \tau^{2n-1}\left[-i^{2n-1}erfc\beta(\tau) + i^{2n-1}erfc(-\beta(\tau))\right] - B_n \cdot \tau^{2n}\left[i^{2n}erfc\beta(\tau) + i^{2n}erfc(-\beta(\tau))\right]\right\} \tag{93}$$

Taking 2n times derivatives of expression (93) and making $\tau = 0,$ it's possible to determine B_n coefficients.

Corollaries: [2, c. 34-35]

1. $$\frac{d^n}{dx^n} f(x) = \frac{U_1}{1!} \cdot F'(y) + \frac{U_2}{2!} \cdot F''(y) + \frac{U_3}{3!} \cdot F'''(y) + \ldots + \frac{U_n}{n!} \cdot F^{(n)}(y)$$

$$U_k = \frac{d^n}{dx^n} y^k - \frac{k}{1!} \cdot y \cdot \frac{d^n}{dx^n} y^{k-1} + \frac{k(k-1)}{2!} \cdot y^2 \cdot \frac{d^n}{dx^n} y^{k-2} - \ldots + (-1)^{k-1} k \cdot y^{k-1} \cdot \frac{d^n}{dx^n} y$$

2. $$\frac{d^n}{dx^n} f(x) = \sum \frac{n!}{i!\, j!\, h! \ldots k!} \cdot \frac{d^m F}{dy^m}\left(\frac{y'}{1!}\right)^i \cdot \left(\frac{y''}{2!}\right)^j \cdot \left(\frac{y'''}{3!}\right)^h \ldots \left(\frac{y^{(l)}}{l!}\right)^k$$

3. $$(u \cdot v)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \cdot u^{(n-k)} \cdot v^{(k)}$$

Firstly let's expand (93)

$$\varphi(\tau^2)=\sum_{n=0}^{\infty}\left\{A_n\cdot\tau^{2n-1}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]-B_n\cdot\tau^{2n}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]\right\}=$$

$$=A_0\cdot\frac{1}{\tau}\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc(-\beta(\tau))\right]-B_0\cdot\left[i^0erfc\beta(\tau)+i^0erfc(-\beta(\tau))\right]+$$
$$+A_1\cdot\tau\cdot\left[-ierfc\beta(\tau)+ierfc(-\beta(\tau))\right]-B_1\cdot\tau^2\cdot\left[i^2erfc\beta(\tau)+i^2erfc(-\beta(\tau))\right]+$$
$$+A_2\cdot\tau^3\left[-i^3erfc\beta(\tau)+i^3erfc(-\beta(\tau))\right]-B_2\cdot\tau^4\left[i^4erfc\beta(\tau)+i^4erfc(-\beta(\tau))\right]+$$
$$+A_3\cdot\tau^5\left[-i^5erfc\beta(\tau)+i^5erfc(-\beta(\tau))\right]-B_3\cdot\tau^6\left[i^6erfc\beta(\tau)+i^6erfc(-\beta(\tau))\right]+$$
$$+\ldots$$
$$.$$
$$.$$
$$.$$
$$+A_n\cdot\tau^{2n-1}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]-B_n\cdot\tau^{2n}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]+$$
$$+\ldots$$
$$\vdots$$
$$+$$

Multiplying both sides by τ following expression obtained

$$\tau\cdot\varphi(\tau^2)=\sum_{n=0}^{\infty}\left\{A_n\cdot\tau^{2n}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]-B_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]\right\}=$$

$$=A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc(-\beta(\tau))\right]-B_0\cdot\tau\left[i^0erfc\beta(\tau)+i^0erfc(-\beta(\tau))\right]+$$
$$+A_1\cdot\tau^2\cdot\left[-ierfc\beta(\tau)+ierfc(-\beta(\tau))\right]-B_1\cdot\tau^3\cdot\left[i^2erfc\beta(\tau)+i^2erfc(-\beta(\tau))\right]+$$
$$+A_2\cdot\tau^4\left[-i^3erfc\beta(\tau)+i^3erfc(-\beta(\tau))\right]-B_2\cdot\tau^5\left[i^4erfc\beta(\tau)+i^4erfc(-\beta(\tau))\right]+$$
$$+A_3\cdot\tau^6\left[-i^5erfc\beta(\tau)+i^5erfc(-\beta(\tau))\right]-B_3\cdot\tau^7\left[i^6erfc\beta(\tau)+i^6erfc(-\beta(\tau))\right]+$$
$$+\ldots$$
$$.$$
$$.$$
$$.$$
$$+A_n\cdot\tau^{2n}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]-B_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]+$$
$$+\ldots$$
$$\vdots$$
$$+$$

Recall that B_n coefficients are already determined from (88). Using formulas in corollaries now it is possible to determine A_n coefficients. To determine A_n coefficients it is necessary to take 2n+1 times derivatives of (93) and make $\tau=0$. It is required to determine B_0 from following expression

$$\left[\tau\cdot\varphi(\tau^2)\right]_{\tau=0}^{(1)}=\left\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc(-\beta(\tau))\right]-B_0\cdot\tau\cdot\left[i^0erfc\beta(\tau)+i^0erfc(-\beta(\tau))\right]\right\}_{\tau=0}^{(1)}$$

A_1 from

$$\left[\tau\cdot\psi(\tau^2)\right]_{\tau=0}^{(3)}=\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc(-\beta(\tau))\right]-B_0\tau\left[i^0erfc\beta(\tau)+i^0erfc(-\beta(\tau))\right]+$$
$$+A_1\cdot\tau^2\cdot\left[-ierfc\beta(\tau)+ierfc(-\beta(\tau))\right]-B_1\cdot\tau^3\cdot\left[i^2erfc\beta(\tau)+i^2erfc(-\beta(\tau))\right]\}_{\tau=0}^{(3)}$$

A_2 from

$$\left[\tau\cdot\psi(\tau^2)\right]_{\tau=0}^{(5)}=\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc\left(-\beta(\tau)\right)\right]-B_0\cdot\tau\cdot\left[i^0 erfc\beta(\tau)+i^0 erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_1\cdot\tau^2\cdot\left[-ierfc\beta(\tau)+ierfc\left(-\beta(\tau)\right)\right]-B_1\cdot\tau^3\cdot\left[i^2 erfc\beta(\tau)+i^2 erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_2\cdot\tau^4\left[-i^3 erfc\beta(\tau)+i^3 erfc\left(-\beta(\tau)\right)\right]-B_2\cdot\tau^5\left[i^4 erfc\beta(\tau)+i^4 erfc\left(-\beta(\tau)\right)\right]\}_{(\tau=0)}^{(5)}$$

A_3 from

$$\left[\tau\cdot\psi(\tau^2)\right]_{\tau=0}^{(7)}=\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc\left(-\beta(\tau)\right)\right]-B_0\tau\left[i^0 erfc\beta(\tau)+i^0 erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_1\cdot\tau^2\left[-ierfc\beta(\tau)+ierfc\left(-\beta(\tau)\right)\right]-B_1\cdot\tau^3\cdot\left[i^2 erfc\beta(\tau)+i^2 erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_2\cdot\tau^4\left[-i^3 erfc\beta(\tau)+i^3 erfc\left(-\beta(\tau)\right)\right]-B_2\cdot\tau^5\left[i^4 erfc\beta(\tau)+i^4 erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_3\cdot\tau^6\left[-i^5 erfc\beta(\tau)+i^6 erfc\left(-\beta(\tau)\right)\right]-B_3\cdot\tau^7\left[i^6 erfc\beta(\tau)+i^7 erfc\left(-\beta(\tau)\right)\right]\}_{\tau=0}^{(7)}$$

and so on. Finally A_n from the following expression

$$\left[\tau\cdot\psi(\tau^2)\right]_{\tau=0}^{(2n+1)}=\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc\left(-\beta(\tau)\right)\right]-B_0\tau\cdot\left[i^0 erfc\beta(\tau)+i^0 erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_1\cdot\tau^2\cdot\left[-ierfc\beta(\tau)+ierfc\left(-\beta(\tau)\right)\right]-B_1\cdot\tau^3\cdot\left[i^2 erfc\beta(\tau)+i^2 erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_2\cdot\tau^4\left[-i^3 erfc\beta(\tau)+i^3 erfc\left(-\beta(\tau)\right)\right]-B_2\cdot\tau^5\left[i^4 erfc\beta(\tau)+i^4 erfc\left(-\beta(\tau)\right)\right]+$$
$$+A_3\cdot\tau^6\left[-i^5 erfc\beta(\tau)+i^5 erfc\left(-\beta(\tau)\right)\right]-B_3\cdot\tau^7\left[i^6 erfc\beta(\tau)+i^6 erfc\left(-\beta(\tau)\right)\right]+$$
$$+\ldots$$
$$.$$
$$.$$
$$.$$
$$+A_n\cdot\tau^{2n}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc\left(-\beta(\tau)\right)\right]-B_n\cdot\tau^{2n+1}\left[i^{2n} erfc\beta(\tau)+i^{2n} erfc\left(-\beta(\tau)\right)\right]\}_{\tau=0}^{(2n+1)}$$

or

$$\left[\tau\cdot\psi(\tau^2)\right]_{\tau=0}^{(2n+1)}=\sum_{k=0}^{n}\left\{A_k\cdot\tau^{2k}\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc\left(-\beta(\tau)\right)\right]-B_k\cdot\tau^{2k+1}\left[i^{2k} erfc\beta(\tau)+i^{2k} erfc\left(-\beta(\tau)\right)\right]\right\}_{\tau=0}^{(2k+1)} \tag{94}$$

For expression $\left[\tau^m\cdot i^m erfc(-\beta(\tau))\right]^{(n)}$ it is necessary to apply formula (2) in corollaries where $\tau^m=u$ and $i^m erfc\beta(\tau)=v.$

$$\left[\tau^m\cdot i^m erfc\beta(\tau)\right]^{(n)}=\left(\tau^m\right)^{(n)}\cdot i^m erfc\beta(\tau)+\binom{n}{1}\cdot\left(\tau^m\right)^{(n-1)}\cdot\left(i^m erfc\beta(\tau)\right)^{(1)}+$$
$$+\binom{n}{2}\cdot\left(\tau^m\right)^{(n-2)}\cdot\left(i^m erfc\beta(\tau)\right)^{(2)}+\ldots+\tau^m\cdot\left(i^m erfc\beta(\tau)\right)^{(n)}. \tag{95}$$

For $\tau=0$,

1. if $m>n$ then $\left[\tau^m\cdot i^m erfc\beta(\tau)\right]_{\tau=0}^{(n)}=0$
2. if $m\le n$ then $\left[\tau^m\cdot i^m erfc\beta(\tau)\right]_{\tau=0}^{(n)}=m!\cdot\binom{n}{n-m}\cdot\left[i^m erfc\beta(\tau)\right]_{\tau=0}^{(n-m)}$

(96)

In the same manner for expression $\left[\tau^m\cdot i^m erfc\left(-\beta(\tau)\right)\right]^{(n)}$

1. if $m > n$ then $\left[\tau^{m} \cdot i^{m} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(n)} = 0$

(97)

2. if $m \le n$ then $\left[\tau^{m} \cdot i^{m} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(n)} = m! \cdot \begin{pmatrix} n \\ n-m \end{pmatrix} \cdot \left[i^{m} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(n-m)}$ (98)

Thus

$$\left\{\tau^{m} \cdot \left[i^{m} erfc\beta(\tau) \pm i^{m} erfc\left(-\beta(\tau)\right)\right]\right\}_{\tau=0}^{(n)} = m! \cdot \begin{pmatrix} n \\ n-m \end{pmatrix} \cdot \left[i^{m} erfc\beta(\tau) \pm i^{m} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(n-m)}$$

(99)

Applying (185) for (180)

$$\begin{aligned} &\sum_{k=0}^{n} \{ A_{k} \cdot \tau^{2k} \left[-i^{2k-1} erfc\beta(\tau) + i^{2k-1} erfc\left(-\beta(\tau)\right)\right] - \\ &- B_{k} \cdot \tau^{2k+1} \left[i^{2k} erfc\beta(\tau) + i^{2k} erfc\left(-\beta(\tau)\right)\right] \}_{\tau=0}^{(2n+1)} = \\ &= \sum_{k=0}^{n} \{ A_{k} \cdot (2k)! \cdot \begin{pmatrix} 2n+1 \\ 2n+1-2k \end{pmatrix} \cdot \left[-i^{2k-1} erfc\beta(\tau) + i^{2k-1} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n+1-2k)} - \\ &- B_{k} \cdot (2k+1)! \cdot \begin{pmatrix} 2n+1 \\ 2n-2k \end{pmatrix} \cdot \left[i^{2k} erfc\beta(\tau) + i^{2k} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)} \} \end{aligned} \quad (100)$$

then

$$\begin{aligned} &\left[\tau \cdot \psi(\tau^{2})\right]_{\tau=0}^{(2n)} = \sum_{k=0}^{n} \{ A_{k} \cdot (2k)! \cdot \begin{pmatrix} 2n+1 \\ 2n+1-2k \end{pmatrix} \cdot \left[-i^{2k-1} erfc\beta(\tau) + i^{2k-1} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n+1-2k)} - \\ &- B_{k} \cdot (2k+1)! \cdot \begin{pmatrix} 2n+1 \\ 2n-2k \end{pmatrix} \cdot \left[i^{2k} erfc\beta(\tau) + i^{2k} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)} \} \end{aligned}$$

(101)

Implies

$$\sum_{k=0}^{n}\{A_k\cdot(2k)!\cdot\binom{2n+1}{2n+1-2k}\cdot\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)}-$$

$$-B_k\cdot(2k+1)!\cdot\binom{2n+1}{2n-2k}\cdot\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}\}=$$

$$=A_n\cdot(2n)!\cdot\binom{2n+1}{1}\cdot\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(1)}+$$

$$+\sum_{k=0}^{n-1}\left\{A_k\cdot(2k)!\cdot\binom{2n+1}{2n+1-2k}\cdot\left[i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)}\right\}-$$

$$-\sum_{k=0}^{n}\left\{B_k\cdot(2k+1)!\cdot\binom{2n+1}{2n-2k}\cdot\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}\right\}$$

(102)

Where $B_n=-\frac{\varphi^n(0)}{2\cdot n!\cdot i^{2n}erfc0}$. Yields

$$A_n=\frac{\left[\tau\cdot\psi(\tau^2)\right]_{\tau=0}^{(2n+1)}}{(2n)!\cdot\binom{2n+1}{1}\cdot\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(1)}}-$$

$$-\frac{\sum_{k=0}^{n-1}\left\{A_k\cdot(2k)!\cdot\binom{2n+1}{2n+1-2k}\cdot\left[i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)}\right\}}{(2n)!\cdot\binom{2n+1}{1}\cdot\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(1)}}+$$

$$+\frac{\sum_{k=0}^{n}\left\{B_k\cdot(2k+1)!\cdot\binom{2n+1}{2n-2k}\cdot\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}\right\}}{(2n)!\cdot\binom{2n+1}{1}\cdot\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(1)}}$$

2.11 Approximate solution of Heat Equation in the domain with moving boundary of the second type, obtained by IEF method

For the Heat Equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\qquad \alpha(t)<x<\beta(t),\qquad t>0$$

Subject to

I.C: $u(x,o)=0,\qquad \alpha(t)<x<\beta(t),$

B.C: $\left.\frac{\partial u}{\partial x}\right|_{x=\alpha(t)}=\varphi(t),\qquad t>0,$

$\left.\frac{\partial u}{\partial x}\right|_{x=\beta(t)}=\phi(t),\qquad t>0.$

Solution:

Solution of the Heat Equation is represented in the form

$$u(x,t) = \sum_{n=0}^{k} (t^{\frac{n}{2}}) \left\{ A_n i^n erfc\left(\frac{x}{2a\sqrt{t}}\right) + B_n i^n erfc\left(-\frac{x}{2a\sqrt{t}}\right) \right\}$$

or

$$u(x,t) = \sum_{n=0}^{k} \{ A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \}$$

Remark In both cases to find approximate solution it is necessary to give certain values for time *t*. For *k* values of *t* we obtain *k* coefficients from *2k* linear equations.

If we use IEF for solution then for boundary conditions at $x = \alpha(t)$and $x = \beta(t)$ we have system of *2k* linear equations.

$$\sum_{n=0}^{k} (t_k)^{\frac{n-1}{2}} \left\{ -A_n i^{n-1} erfc\left(\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right) + B_n i^{n-1} erfc\left(-\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right) \right\} = \varphi(t_k),$$

$$\sum_{n=0}^{k} (t_k)^{\frac{n-1}{2}} \left\{ -A_n i^{n-1} erfc\left(\frac{\beta(t_k)}{2a\sqrt{t_k}}\right) + B_n i^{n-1} erfc\left(-\frac{\beta(t_k)}{2a\sqrt{t_k}}\right) \right\} = \phi(t_k)$$

In the same manner solution in the polynomial form of IEF allows us to find even and odd coefficients A_{2n}, A_{2n+1} from system of linear equations

$$\sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} (2n-2m)(\alpha(t_k))^{2n-2m-1} t_k^m \alpha_{2n,m} + \right.$$
$$\left. + A_{2n+1} \sum_{m=0}^{n} (2n-2m+1)(\alpha(t_k))^{2n-2m} t_k^m \alpha_{2n+1,m} \right\} = \varphi(t_k)$$

$$\sum_{n=0}^{k} \left\{ A_{2n} \sum_{m=0}^{n} (2n-2m)(\beta(t_k))^{2n-2m-1} t_k^m \beta_{2n,m} + \right.$$
$$\left. + A_{2n+1} \sum_{m=0}^{n} (2n-2m+1)(\beta(t_k))^{2n-2m} t_k^m \beta_{2n+1,m} \right\} = \phi(t_k)$$

2.12 Approximate solutions of Heat Equation in the domain with moving boundaries of the mixed type by IEF method

In Heat Equations with moving boundary subject to the second type boundary conditions it is important to determine value of γ in

$$u(x,t) = \sum_{n=0}^{\gamma} (\sqrt{t})^n \left[A_n i^n erfc\frac{x}{2a\sqrt{t}} + B_n i^n erfc\frac{-x}{2a\sqrt{t}} \right]$$

which belongs to degrees of polynomials into which functions located on the right parts of boundary conditions expanded.

2.13 Approximate solution of Heat Equation with the mixed type boundary conditions in the $0<x<\alpha(t)$ domain by IEF method

Solution of the Heat Equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}, \qquad 0<x<\alpha(t)\ , \quad t>0 \tag{103}$$

Subject to

I.C: $$u(x,0)=0, \tag{104}$$

B.C: $$u(0,t)=\varphi(t), \tag{105}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=\alpha(t)}=\phi(t), \tag{106}$$

$$u(0,0)=0, \tag{107}$$

can be represented in the following form

$$U(x,t)=\sum_{n=0}^{\infty}\left\{\begin{matrix}A_n\cdot t^n\left[i^{2n}erfc\dfrac{x}{2a\sqrt{t}}+i^{2n}erfc(\dfrac{-x}{2a\sqrt{t}})\right]+ \\ +B_n\cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc\dfrac{x}{2a\sqrt{t}}-i^{2n+1}erfc(\dfrac{-x}{2a\sqrt{t}})\right]\end{matrix}\right\} \tag{108}$$

or using property 2 of IEF, solution can be represented in the polynomial form

$$U(x,t)=\sum_{n=0}^{k}\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\} \tag{109}$$

In this case it is more convenient to use expression (108) for solution of (104)- (106)

where

$$\alpha(t)=\alpha_1 t^{\frac{1}{2}}+\alpha_2 t+\alpha_3 t^{\frac{3}{2}}+...+\alpha_n t^{\frac{n}{2}}+... \tag{110}$$

for $\tau=\sqrt{t}$

(111)

we have

$$\alpha(\tau^2)=\alpha_1\tau+\alpha_2\tau^2+\alpha_3\tau^3+...+\alpha_n\tau^n+... \tag{112}$$

From (105) for x=0, we have

$$\varphi(t) = \sum_{n=0}^{\infty} A_n t^n i^{2n} erfc0 \tag{113}$$

And coefficient A_n of expression (109) can be found from

$$A_n = \frac{\varphi^n(0)}{i^{2n}\, n!\, erfc0} \tag{114}$$

From (106) for x=$\alpha(t)$, we have

$$\frac{\partial U}{\partial x} = \sum_{n=0}^{\infty} \left\{ \begin{array}{l} A_n \cdot t^{n-1/2}\left[-i^{2n-1} erfc \frac{x}{2a\sqrt{t}} + i^{2n-1} erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right] - \\ - B_n \cdot t^n \left[i^{2n} erfc \frac{x}{2a\sqrt{t}} + i^{2n} erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right] \end{array} \right\} \tag{115}$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=\alpha(t)} = \sum_{n=0}^{\infty} \left\{ \begin{array}{l} A_n \cdot t^{n-1/2}\left[-i^{2n-1} erfc \frac{x}{2a\sqrt{t}} + i^{2n-1} erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right] - \\ - B_n \cdot t^n \left[i^{2n} erfc \frac{x}{2a\sqrt{t}} + i^{2n} erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right] \end{array} \right\}$$

Let

$$\frac{\alpha(t)}{2a\sqrt{t}} = \frac{\alpha_1}{2a} + \frac{\alpha_2}{2a} t^{\frac{1}{2}} + \frac{\alpha_3}{2a} t + \frac{\alpha_3}{2a} t^{\frac{3}{2}} + \frac{\alpha_4}{2a} t^2 + \ldots + \frac{\alpha_n}{2a} t^{\frac{n}{2}} + \ldots \tag{116}$$

for $\tau = \sqrt{t}$

$$\beta(\tau) = \frac{\alpha_1}{2a} + \frac{\alpha_2}{2a}\tau + \frac{\alpha_3}{2a}\tau^2 + \frac{\alpha_3}{2a}\tau^3 + \frac{\alpha_4}{2a}\tau^4 + \ldots + \frac{\alpha_n}{2a}\tau^n + \ldots, \tag{117}$$

Let's say

$$\beta_0 = \frac{\alpha_1}{2a},$$

$$\beta_1 = \frac{\alpha_2}{2a},$$

$$\beta_3 = \frac{\alpha_2}{2a},$$

……………..

$$\beta_n = \frac{\alpha_{n-1}}{2a}$$

then

$$\beta(\tau) = \beta_0 + \beta_1\tau + \beta_2\tau^2 + \beta_3\tau^3 + \beta_4\tau^4 + \ldots + \beta_n\tau^n + \ldots, \tag{118}$$

thus

$$\varphi(\tau^2) = \sum_{n=0}^{\infty} \begin{Bmatrix} A_n \cdot \tau^{2n-1}\left[-i^{2n-1} erfc\beta(\tau) + i^{2n-1} erfc\left(-\beta(\tau)\right)\right] - \\ - B_n \cdot \tau^{2n}\left[i^{2n} erfc\beta(\tau) + i^{2n} erfc\left(-\beta(\tau)\right)\right] \end{Bmatrix} \tag{119}$$

Taking 2n times derivatives of expression (119) and making $\tau = 0$, it's possible to determine B_n coefficients.

Corollaries: [2, c. 34-35]

4. $$\frac{d^n}{dx^n} f(x) = \frac{U_1}{1!} \cdot F'(y) + \frac{U_2}{2!} \cdot F''(y) + \frac{U_3}{3!} \cdot F'''(y) + \ldots + \frac{U_n}{n!} \cdot F^{(n)}(y)$$

$$U_k = \frac{d^n}{dx^n} y^k - \frac{k}{1!} \cdot y \cdot \frac{d^n}{dx^n} y^{k-1} + \frac{k(k-1)}{2!} \cdot y^2 \cdot \frac{d^n}{dx^n} y^{k-2} - \ldots + (-1)^{k-1} k \cdot y^{k-1} \cdot \frac{d^n}{dx^n} y$$

5. $$\frac{d^n}{dx^n} f(x) = \sum \frac{n!}{i!\, j!\, h! \ldots k!} \cdot \frac{d^m F}{dy^m} \left(\frac{y'}{1!}\right)^i \cdot \left(\frac{y''}{2!}\right)^j \cdot \left(\frac{y'''}{3!}\right)^h \ldots \left(\frac{y^{(l)}}{l!}\right)^k$$

6. $$\left(u \cdot v\right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \cdot u^{(n-k)} \cdot v^{(k)}$$

Firstly let's expand (119)

$$\varphi(\tau^2) = \sum_{n=0}^{\infty} \left\{A_n \cdot \tau^{2n-1}\left[-i^{2n-1} erfc\beta(\tau) + i^{2n-1} erfc\left(-\beta(\tau)\right)\right] - B_n \cdot \tau^{2n}\left[i^{2n} erfc\beta(\tau) + i^{2n} erfc\left(-\beta(\tau)\right)\right]\right\} =$$

$$= A_0 \cdot \frac{1}{\tau} \cdot \left[-i^{-1} erfc\beta(\tau) + i^{-1} erfc\left(-\beta(\tau)\right)\right] - B_0 \cdot \left[i^0 erfc\beta(\tau) + i^0 erfc\left(-\beta(\tau)\right)\right] +$$
$$+ A_1 \cdot \tau \cdot \left[-ierfc\beta(\tau) + ierfc\left(-\beta(\tau)\right)\right] - B_1 \cdot \tau^2 \cdot \left[i^2 erfc\beta(\tau) + i^2 erfc\left(-\beta(\tau)\right)\right] +$$
$$+ A_2 \cdot \tau^3\left[-i^3 erfc\beta(\tau) + i^3 erfc\left(-\beta(\tau)\right)\right] - B_2 \cdot \tau^4\left[i^4 erfc\beta(\tau) + i^4 erfc\left(-\beta(\tau)\right)\right] +$$
$$+ A_3 \cdot \tau^5\left[-i^5 erfc\beta(\tau) + i^5 erfc\left(-\beta(\tau)\right)\right] - B_3 \cdot \tau^6\left[i^6 erfc\beta(\tau) + i^6 erfc\left(-\beta(\tau)\right)\right] +$$
$$+ \ldots$$
$$.$$
$$.$$
$$.$$
$$+ A_n \cdot \tau^{2n-1}\left[-i^{2n-1} erfc\beta(\tau) + i^{2n-1} erfc\left(-\beta(\tau)\right)\right] - B_n \cdot \tau^{2n}\left[i^{2n} erfc\beta(\tau) + i^{2n} erfc\left(-\beta(\tau)\right)\right] +$$
$$+ \ldots$$
$$\vdots$$
$$+$$

Multiplying both sides by τ following expression obtained

$$\tau\cdot\varphi(\tau^2)=\sum_{n=0}^{\infty}\left\{\begin{array}{l}A_n\cdot\tau^{2n}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]-\\-B_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]\end{array}\right\}=$$
$$=A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc(-\beta(\tau))\right]-B_0\cdot\tau\left[i^0erfc\beta(\tau)+i^0erfc(-\beta(\tau))\right]+$$
$$+A_1\cdot\tau^2\cdot\left[-ierfc\beta(\tau)+ierfc(-\beta(\tau))\right]-B_1\cdot\tau^3\cdot\left[i^2erfc\beta(\tau)+i^2erfc(-\beta(\tau))\right]+$$
$$+A_2\cdot\tau^4\left[-i^3erfc\beta(\tau)+i^3erfc(-\beta(\tau))\right]-B_2\cdot\tau^5\left[i^4erfc\beta(\tau)+i^4erfc(-\beta(\tau))\right]+$$
$$+A_3\cdot\tau^6\left[-i^5erfc\beta(\tau)+i^5erfc(-\beta(\tau))\right]-B_3\cdot\tau^7\left[i^6erfc\beta(\tau)+i^6erfc(-\beta(\tau))\right]+$$
$$+\dots$$
.

.

.
$$+A_n\cdot\tau^{2n}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]-B_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]+$$
$$+\dots$$
$\vdots$

$+$

Recall that A_n coefficients are already determined from (114). Using formulas in corollaries now it is possible to determine B_n coefficients. To determine B_n coefficients it is necessary to take 2n+1 times derivatives of (119) and make $\tau=0$. It is required to determine B_0 from following expression
$$\left[\tau\cdot\varphi(\tau^2)\right]_{\tau=0}^{(1)}=\left\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc(-\beta(\tau))\right]-B_0\cdot\tau\cdot\left[i^0erfc\beta(\tau)+i^0erfc(-\beta(\tau))\right]\right\}_{\tau=0}^{(1)}$$

B_1 from
$$\left[\tau\cdot\psi(\tau^2)\right]_{\tau=0}^{(3)}=\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc(-\beta(\tau))\right]-B_0\tau\left[i^0erfc\beta(\tau)+i^0erfc(-\beta(\tau))\right]+$$
$$+A_1\cdot\tau^2\cdot\left[-ierfc\beta(\tau)+ierfc(-\beta(\tau))\right]-B_1\cdot\tau^3\cdot\left[i^2erfc\beta(\tau)+i^2erfc(-\beta(\tau))\right]\}_{\tau=0}^{(3)}$$

B_2 from
$$\left[\tau\cdot\psi(\tau^2)\right]_{\tau=0}^{(5)}=\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc(-\beta(\tau))\right]-B_0\cdot\tau\cdot\left[i^0erfc\beta(\tau)+i^0erfc(-\beta(\tau))\right]+$$
$$+A_1\cdot\tau^2\cdot\left[-ierfc\beta(\tau)+ierfc(-\beta(\tau))\right]-B_1\cdot\tau^3\cdot\left[i^2erfc\beta(\tau)+i^2erfc(-\beta(\tau))\right]+$$
$$+A_2\cdot\tau^4\left[-i^3erfc\beta(\tau)+i^3erfc(-\beta(\tau))\right]-B_2\cdot\tau^5\left[i^4erfc\beta(\tau)+i^4erfc(-\beta(\tau))\right]\}_{(\tau=0)}^{(5)}$$

B_3 from
$$\left[\tau\cdot\psi(\tau^2)\right]_{\tau=0}^{(7)}=\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc(-\beta(\tau))\right]-B_0\tau\left[i^0erfc\beta(\tau)+i^0erfc(-\beta(\tau))\right]+$$
$$+A_1\cdot\tau^2\left[-ierfc\beta(\tau)+ierfc(-\beta(\tau))\right]-B_1\cdot\tau^3\cdot\left[i^2erfc\beta(\tau)+i^2erfc(-\beta(\tau))\right]+$$
$$+A_2\cdot\tau^4\left[-i^3erfc\beta(\tau)+i^3erfc(-\beta(\tau))\right]-B_2\cdot\tau^5\left[i^4erfc\beta(\tau)+i^4erfc(-\beta(\tau))\right]+$$
$$+A_3\cdot\tau^6\left[-i^5erfc\beta(\tau)+i^6erfc(-\beta(\tau))\right]-B_3\cdot\tau^7\left[i^6erfc\beta(\tau)+i^7erfc(-\beta(\tau))\right]\}_{\tau=0}^{(7)}$$

and so on. Finally B_n from the following expression

$$\left[\tau\cdot\psi(\tau^2)\right]_{\tau=0}^{(2n+1)}=\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc(-\beta(\tau))\right]-B_0\tau\cdot\left[i^0erfc\beta(\tau)+i^0erfc(-\beta(\tau))\right]+$$
$$+A_1\cdot\tau^2\cdot\left[-ierfc\beta(\tau)+ierfc(-\beta(\tau))\right]-B_1\cdot\tau^3\cdot\left[i^2erfc\beta(\tau)+i^2erfc(-\beta(\tau))\right]+$$
$$+A_2\cdot\tau^4\left[-i^3erfc\beta(\tau)+i^3erfc(-\beta(\tau))\right]-B_2\cdot\tau^5\left[i^4erfc\beta(\tau)+i^4erfc(-\beta(\tau))\right]+$$
$$+A_3\cdot\tau^6\left[-i^5erfc\beta(\tau)+i^5erfc(-\beta(\tau))\right]-B_3\cdot\tau^7\left[i^6erfc\beta(\tau)+i^6erfc(-\beta(\tau))\right]+$$
$$+\dots$$
.

.

.
$$+A_n\cdot\tau^{2n}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]-B_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]\}_{\tau=0}^{(2n+1)}$$

or

$$\left[\tau\cdot\psi(\tau^2)\right]_{\tau=0}^{(2n+1)}=\sum_{k=0}^{n}\left\{\begin{matrix}A_k\cdot\tau^{2k}\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc(-\beta(\tau))\right]- \\ -B_k\cdot\tau^{2k+1}\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]\end{matrix}\right\}_{\tau=0}^{(2k+1)} \tag{120}$$

For expression $\left[\tau^m\cdot i^m erfc(-\beta(\tau))\right]^{(n)}$ it is necessary to apply formula (2) in corollaries where $\tau^m=u$ and $i^m erfc\beta(\tau)=v$.

$$\left[\tau^m\cdot i^m erfc\beta(\tau)\right]^{(n)}=\left(\tau^m\right)^{(n)}\cdot i^m erfc\beta(\tau)+\binom{n}{1}\cdot\left(\tau^m\right)^{(n-1)}\cdot\left(i^m erfc\beta(\tau)\right)^{(1)}+$$
$$+\binom{n}{2}\cdot\left(\tau^m\right)^{(n-2)}\cdot\left(i^m erfc\beta(\tau)\right)^{(2)}+\dots+\tau^m\cdot\left(i^m erfc\beta(\tau)\right)^{(n)}.$$

(121)

For $\tau=0$,

1. if $m>n$ then $\left[\tau^m\cdot i^m erfc\beta(\tau)\right]_{\tau=0}^{(n)}=0$
2. if $m\le n$ then $\left[\tau^m\cdot i^m erfc\beta(\tau)\right]_{\tau=0}^{(n)}=m!\cdot\binom{n}{n-m}\cdot\left[i^m erfc\beta(\tau)\right]_{\tau=0}^{(n-m)}$ (122)

In the same manner for expression $\left[\tau^m\cdot i^m erfc(-\beta(\tau))\right]^{(n)}$

1. if $m>n$ then $\left[\tau^m\cdot i^m erfc(-\beta(\tau))\right]_{\tau=0}^{(n)}=0$ (123)
2. if $m\le n$ then $\left[\tau^m\cdot i^m erfc(-\beta(\tau))\right]_{\tau=0}^{(n)}=m!\cdot\binom{n}{n-m}\cdot\left[i^m erfc(-\beta(\tau))\right]_{\tau=0}^{(n-m)}$ (124)

Thus

$$\left\{\tau^m\cdot\left[i^m erfc\beta(\tau)\pm i^m erfc(-\beta(\tau))\right]\right\}_{\tau=0}^{(n)}=m!\cdot\binom{n}{n-m}\cdot\left[i^m erfc\beta(\tau)\pm i^m erfc(-\beta(\tau))\right]_{\tau=0}^{(n-m)}$$

(125)

Applying (234) for (229)

$$\sum_{k=0}^{n}\{ A_k \cdot \tau^{2k}\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc(-\beta(\tau))\right]- \\ - B_k \cdot \tau^{2k+1}\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]\}_{\tau=0}^{(2n+1)} =$$

$$= \sum_{k=0}^{n}\{ A_k \cdot (2k)! \cdot \binom{2n+1}{2n+1-2k} \cdot \left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)} - \tag{126}$$

$$- B_k \cdot (2k+1)! \cdot \binom{2n+1}{2n-2k} \cdot \left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)} \}$$

then

$$\left[\tau \cdot \psi(\tau^2)\right]_{\tau=0}^{(2n)} = \sum_{k=0}^{n}\{ A_k \cdot (2k)! \cdot \binom{2n+1}{2n+1-2k} \cdot \left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)} -$$

$$- B_k \cdot (2k+1)! \cdot \binom{2n+1}{2n-2k} \cdot \left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)} \} \tag{127}$$

Implies

$$\sum_{k=0}^{n}\{ A_k \cdot (2k)! \cdot \binom{2n+1}{2n+1-2k} \cdot \left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)} -$$

$$- B_k \cdot (2k+1)! \cdot \binom{2n+1}{2n-2k} \cdot \left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)} \} =$$

$$= \sum_{k=0}^{n}\left\{ A_k \cdot (2k)! \cdot \binom{2n+1}{2n+1-2k} \cdot \left[i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)} \right\} -$$

$$- \sum_{k=0}^{n-1}\left\{ B_k \cdot (2k+1)! \cdot \binom{2n+1}{2n-2k} \cdot \left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)} \right\} -$$

$$- B_n \cdot (2n+1)! \cdot \binom{2n+1}{0} \cdot \left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]_{\tau=0}^{(0)} \tag{128}$$

Yields

$$B_n = -\frac{\left[\tau \cdot \psi(\tau^2)\right]_{\tau=0}^{(2n+1)}}{(2n+1)! \cdot \binom{2n+1}{0} \cdot \left[i^{2n} erfc\beta(\tau) + i^{2n} erfc(-\beta(\tau))\right]_{\tau=0}^{(0)}} +$$

$$+\frac{\sum_{k=0}^{n}\left\{A_k \cdot (2k)! \cdot \binom{2n+1}{2n+1-2k} \cdot \left[i^{2k-1} erfc\beta(\tau) + i^{2k-1} erfc(-\beta(\tau))\right]_{\tau=0}^{(2n+1-2k)}\right\}}{(2n+1)! \cdot \binom{2n+1}{0} \cdot \left[i^{2n} erfc\beta(\tau) + i^{2n} erfc(-\beta(\tau))\right]_{\tau=0}^{(0)}} -$$

$$-\frac{-\sum_{k=0}^{n-1}\left\{B_k \cdot (2k+1)! \cdot \binom{2n+1}{2n-2k} \cdot \left[i^{2k} erfc\beta(\tau) + i^{2k} erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}\right\}}{(2n+1)! \cdot \binom{2n+1}{0} \cdot \left[i^{2n} erfc\beta(\tau) + i^{2n} erfc(-\beta(\tau))\right]_{\tau=0}^{(0)}}$$

2.14 Approximate solution of Heat Equation in the domain with moving boundary of the mixed type, obtained by IEF

For the Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad \alpha(t) < x < \beta(t), \quad t > 0$$

Subject to

I.C: $\quad u(x, o) = 0, \qquad \alpha(t) < x < \beta(t),$

B.C: $\quad u(\alpha(t), t) = \varphi(t), \qquad t > 0,$

$$\left.\frac{\partial u}{\partial x}\right|_{x=\beta(t)} = \phi(t), \qquad t > 0.$$

Solution:

Solution of the Heat Equation is represented in the form

$$u(x,t) = \sum_{n=0}^{k} (t^{\frac{n}{2}}) \left\{A_n i^n erfc\left(\frac{x}{2a\sqrt{t}}\right) + B_n i^n erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right\}$$

or

$$u(x,t) = \sum_{n=0}^{k} \left\{A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m}\right\}$$

Remark In both cases to find approximate solution it is necessary to give certain values for time *t*. For *k* values of *t* we obtain *k* coefficients from *2k* linear equations.

If we use IEF for solution then for boundary conditions at $x = \alpha(t)$and $x = \beta(t)$ we have system of *2k* linear equations.

$$\sum_{n=0}^{k} (t_k)^{\frac{n}{2}} \left\{A_n i^n erfc\left(\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right) + B_n i^n erfc\left(-\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right)\right\} = \varphi(t_k),$$

$$\sum_{n=0}^{k}(t_k)^{\frac{n}{2}-1}\left\{-A_n i^n erfc\left(\frac{\beta(t_k)}{2a\sqrt{t_k}}\right)+B_n i^n erfc\left(-\frac{\beta(t_k)}{2a\sqrt{t_k}}\right)\right\}=\phi(t_k)$$

In the same manner solution in the polynomial form of IEF allows us to find even and odd coefficients A_{2n}, A_{2n+1} from system of linear equations

$$\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(\alpha(t_k))^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(\alpha(t_k))^{2n-2m+1}t^m\beta_{2n+1,m}\right\}$$
$$=\varphi(t_k)$$

$$\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(2n-2m)(\beta(t_k))^{2n-2m-1}t_k^m\beta_{2n,m}+\right.$$
$$\left.+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)(\beta(t_k))^{2n-2m}t_k^m\beta_{2n+1,m}\right\}=\phi(t_k)$$

2.15 Approximate solutions of Heat Equation in the domain with moving boundaries of the third type by IEF method

To solve Heat Equation of the third type boundary condition analytically, as in previous case it is important to determine value of γ in the expression

$$u(x,t)=\sum_{n=0}^{\gamma}(t^{\frac{n}{2}})\left\{A_n i^n erfc\left(\frac{x}{2a\sqrt{t}}\right)+B_n i^n erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right\}$$

As for the approximate solution it is necessary to give values for time t and solve systems of linear equations. Solution can be represented both in the polynomial form of IEF and in the form of IEF.

2.16 Approximate solution of Heat Equation with the third type boundary conditions in the $0<x<\alpha(t)$ domain by IEF method

Solution of the Heat Equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}, \qquad 0<x<\alpha(t)\ , \quad t>0 \tag{129}$$

Subject to

$$u(x,0)=0, \tag{130}$$

B.C:
$$\left.\left(\rho u+\theta\frac{\partial u}{\partial x}\right)\right|_{x=0}=\varphi(t), \tag{131}$$

$$\left.\left(\varepsilon u+\xi\frac{\partial u}{\partial x}\right)\right|_{x=\alpha(t)}=\phi(t), \tag{132}$$

$$u(0,0)=0, \tag{133}$$

can be represented in the following form

$$U(x,t)=\sum_{n=0}^{\infty}\left\{\begin{array}{l} A_n\cdot t^n\left[i^{2n}erfc\frac{x}{2a\sqrt{t}}+i^{2n}erfc(\frac{-x}{2a\sqrt{t}})\right]+ \\ +B_n\cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc\frac{x}{2a\sqrt{t}}-i^{2n+1}erfc(\frac{-x}{2a\sqrt{t}})\right]\end{array}\right\} \tag{134}$$

or using property 2 of IEF, solution can be represented in the polynomial form

$$U(x,t)=\sum_{n=0}^{k}\left\{\begin{array}{c} A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+ \\ +A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\end{array}\right\} \tag{135}$$

In this case it is more convenient to use expression (134) for solution of (130)- (132)

$$\frac{\partial U}{\partial x}=\sum_{n=0}^{\infty}\left\{\begin{array}{l} A_n\cdot t^{n-1/2}\left[-i^{2n-1}erfc\frac{x}{2a\sqrt{t}}+i^{2n-1}erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right]- \\ \_B_n\cdot t^n\left[i^{2n}erfc\frac{x}{2a\sqrt{t}}+i^{2n}erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right]\end{array}\right\} \tag{136}$$

$$\begin{aligned}\Rightarrow\left.\frac{\partial U}{\partial x}\right|_{x=0}&=\sum_{n=0}^{\infty}\left\{\begin{array}{l} A_n\cdot t^{n-1/2}\left[-i^{2n-1}erfc0+i^{2n-1}erfc0\right]- \\ \_B_n\cdot t^n\left[i^{2n}erfc0+i^{2n}erfc0\right]\end{array}\right\}= \\ &=-\sum_{n=0}^{\infty}2B_n\cdot t^n\cdot i^{2n}erfc0\end{aligned} \tag{137}$$

$$U(0,t)=\sum_{n=0}^{\infty}\left\{\begin{array}{l} A_n\cdot t^n\left[i^{2n}erfc0+i^{2n}erfc0\right]+ \\ +B_n\cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc0-i^{2n+1}erfc0\right]\end{array}\right\}=\sum_{n=0}^{\infty}2A_n\cdot t^n\, i^{2n}erfc0 \tag{138}$$

From (131)

$$\sum_{n=0}^{\infty}\left\{t^n\cdot i^{2n}erfc0\cdot\left[\rho\cdot A_n-\theta\cdot B_n\right]\right\}=\frac{\varphi(t)}{2}, \tag{139}$$

$$\Rightarrow\rho A_n+b\cdot B_n=\frac{\varphi^{(n)}(0)}{n!\cdot i^{2n}erfc0}, \tag{140}$$

$$\Rightarrow A_n=\frac{1}{\rho}\cdot\left[\frac{\varphi_n}{i^{2n}erfc0}-b\cdot B_n\right], \tag{141}$$

where

$$\varphi_n = \frac{\varphi^{(n)}(0)}{n!} \tag{142}$$

From (132) for x=$\alpha(t)$, we have

$$\varepsilon\sum_{n=0}^{\infty}\left\{A_n \cdot t^{n-1/2}\left[-i^{2n-1}erfc\frac{\alpha(t)}{2a\sqrt{t}} + i^{2n-1}erfc\left(-\frac{\alpha(t)}{2a\sqrt{t}}\right)\right] - B_n \cdot t^n\left[i^{2n}erfc\frac{\alpha(t)}{2a\sqrt{t}} + i^{2n}erfc\left(-\frac{\alpha(t)}{2a\sqrt{t}}\right)\right]\right\} +$$
$$+\xi\sum_{n=0}^{\infty}\left\{A_n \cdot t^n\left[i^{2n}erfc\frac{\alpha(t)}{2a\sqrt{t}} + i^{2n}erfc\left(-\frac{\alpha(t)}{2a\sqrt{t}}\right)\right] + B_n \cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc\frac{\alpha(t)}{2a\sqrt{t}} - i^{2n+1}erfc\left(-\frac{\alpha(t)}{2a\sqrt{t}}\right)\right]\right\} =$$
$$= \phi(t) \tag{143}$$

Let

$$\alpha(t) = \alpha_1 t^{\frac{1}{2}} + \alpha_2 t + \alpha_3 t^{\frac{3}{2}} + \ldots + \alpha_n t^{\frac{n}{2}} + \ldots \tag{144}$$

$$\frac{\alpha(t)}{2a\sqrt{t}} = \frac{\alpha_1}{2a} + \frac{\alpha_2}{2a}t^{\frac{1}{2}} +, \frac{\alpha_3}{2a}t + \frac{\alpha_3}{2a}t^{\frac{3}{2}} + \frac{\alpha_4}{2a}t^2 + \ldots + \frac{\alpha_n}{2a}t^{\frac{n}{2}} + \ldots \tag{145}$$

for $\tau = \sqrt{t}$

$$\beta(\tau) = \frac{\alpha_1}{2a} + \frac{\alpha_2}{2a}\tau +, \frac{\alpha_3}{2a}\tau^2 + \frac{\alpha_3}{2a}\tau^3 + \frac{\alpha_4}{2a}\tau^4 + \ldots + \frac{\alpha_n}{2a}\tau^n + \ldots, \tag{146}$$

Let’s say

$\beta_0 = \frac{\alpha_1}{2a}$,

$\beta_1 = \frac{\alpha_2}{2a}$,

$\beta_3 = \frac{\alpha_2}{2a}$,

……………..

$\beta_n = \frac{\alpha_{n-1}}{2a}$

then

$$\beta(\tau) = \beta_0 + \beta_1\tau + \beta_2\tau^2 + \beta_3\tau^3 + \beta_4\tau^4 + \ldots + \beta_n\tau^n + \ldots, \tag{147}$$

Finally

$$\varepsilon\sum_{n=0}^{\infty}\left\{A_n\cdot\tau^{2n-1}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]-B_n\cdot\tau^{2n}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]\right\}+$$
$$+\xi\sum_{n=0}^{\infty}\left\{A_n\cdot\tau^{2n}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]+B_n\cdot\tau^{2n+1}\left[i^{2n+1}erfc\beta(\tau)-i^{2n+1}erfc(-\beta(\tau))\right]\right\}= \quad (148)$$
$$=\phi(\tau^2)$$

Taking 2n+1 times derivatives of expression (148) and making $\tau=0$, it's possible to determine B_n coefficients.

Corollaries: [2, c. 34-35]

1. $$\frac{d^n}{dx^n}f(x)=\frac{U_1}{1!}\cdot F'(y)+\frac{U_2}{2!}\cdot F''(y)+\frac{U_3}{3!}\cdot F'''(y)+...+\frac{U_n}{n!}\cdot F^{(n)}(y)$$

$$U_k=\frac{d^n}{dx^n}y^k-\frac{k}{1!}\cdot y\cdot\frac{d^n}{dx^n}y^{k-1}+\frac{k(k-1)}{2!}\cdot y^2\cdot\frac{d^n}{dx^n}y^{k-2}-...+(-1)^{k-1}k\cdot y^{k-1}\cdot\frac{d^n}{dx^n}y$$

2. $$\frac{d^n}{dx^n}f(x)=\sum\frac{n!}{i!\,j!\,h!...k!}\cdot\frac{d^mF}{dy^m}\left(\frac{y'}{1!}\right)^i\cdot\left(\frac{y''}{2!}\right)^j\cdot\left(\frac{y'''}{3!}\right)^h...\left(\frac{y^{(l)}}{l!}\right)^k$$

3. $$(u\cdot v)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}\cdot u^{(n-k)}\cdot v^{(k)}$$

Firstly let's expand (148)

$$\varphi(\tau^2)=\varepsilon\sum_{n=0}^{\infty}\begin{Bmatrix}A_n\cdot\tau^{2n-1}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]-\\-B_n\cdot\tau^{2n}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]\end{Bmatrix}+$$
$$+\xi\sum_{n=0}^{\infty}\left\{A_n\cdot\tau^{2n}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]+B_n\cdot\tau^{2n+1}\left[i^{2n+1}erfc\beta(\tau)-i^{2n+1}erfc(-\beta(\tau))\right]\right\}=$$
$$=\varepsilon\left\{A_0\frac{1}{\tau}\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc(-\beta(\tau))\right]-B_0\cdot\left[i^0erfc\beta(\tau)-i^0erfc(-\beta(\tau))\right]\right\}+$$
$$+\xi\left\{A_0\cdot\left[i^0erfc\beta(\tau)+i^0erfc(-\beta(\tau))\right]+B_0\cdot\tau\left[ierfc\beta(\tau)-ierfc(-\beta(\tau))\right]\right\}$$
$$+\varepsilon\left\{A_1\cdot\tau^1\left[-ierfc\beta(\tau)+ierfc(-\beta(\tau))\right]-B_1\cdot\tau^2\left[i^2erfc\beta(\tau)+i^2erfc(-\beta(\tau))\right]\right\}+$$
$$+\xi\left\{A_1\cdot\tau^2\left[i^2erfc\beta(\tau)+i^2erfc(-\beta(\tau))\right]+B_1\cdot\tau^3\left[i^3erfc\beta(\tau)-i^3erfc(-\beta(\tau))\right]\right\}+$$
$$+...$$
.

.

.
$$+\varepsilon\left\{A_n\cdot\tau^{2n-1}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]-B_n\cdot\tau^{2n}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]\right\}+$$
$$+\xi\left\{A_n\cdot\tau^{2n}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc(-\beta(\tau))\right]+B_n\cdot\tau^{2n+1}\left[i^{2n+1}erfc\beta(\tau)-i^{2n+1}erfc(-\beta(\tau))\right]\right\}$$

Multiplying both sides of equation by τ yields

$$\tau\cdot\varphi(\tau^2)=\varepsilon\sum_{n=0}^{\infty}\left\{A_n\cdot\tau^{2n}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc\left(-\beta(\tau)\right)\right]-B_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc\left(-\beta(\tau)\right)\right]\right\}+$$
$$+\xi\sum_{n=0}^{\infty}\left\{A_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc\left(-\beta(\tau)\right)\right]+B_n\cdot\tau^{2n+2}\left[i^{2n+1}erfc\beta(\tau)-i^{2n+1}erfc\left(-\beta(\tau)\right)\right]\right\}=$$
$$=\varepsilon\left\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc\left(-\beta(\tau)\right)\right]-B_0\tau\cdot\left[i^{0}erfc\beta(\tau)-i^{0}erfc\left(-\beta(\tau)\right)\right]\right\}+$$
$$+\xi\left\{A_0\tau\cdot\left[i^{0}erfc\beta(\tau)+i^{0}erfc\left(-\beta(\tau)\right)\right]+B_0\cdot\tau^2\left[ierfc\beta(\tau)-ierfc\left(-\beta(\tau)\right)\right]\right\}$$
$$+\varepsilon\left\{A_1\cdot\tau^2\left[-ierfc\beta(\tau)+ierfc\left(-\beta(\tau)\right)\right]-B_1\cdot\tau^3\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]\right\}+$$
$$+\xi\left\{A_1\cdot\tau^3\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]+B_1\cdot\tau^4\left[i^3erfc\beta(\tau)-i^3erfc\left(-\beta(\tau)\right)\right]\right\}+$$
$$+\ldots$$
$$.$$
$$.$$
$$.$$
$$+\varepsilon\left\{A_n\cdot\tau^{2n}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc\left(-\beta(\tau)\right)\right]-B_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc\left(-\beta(\tau)\right)\right]\right\}+$$
$$+\xi\left\{A_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc\left(-\beta(\tau)\right)\right]+B_n\cdot\tau^{2n+2}\left[i^{2n+1}erfc\beta(\tau)-i^{2n+1}erfc\left(-\beta(\tau)\right)\right]\right\}\tag{149}$$

Using formulas in corollaries and constructing systems of linear equations from (141) and (149) it is possible to determine A_n and B_n coefficients by taking 2n+1 times derivatives of (148) and make $\tau=0$.
It is required to take first derivative of expression (264) to obtain:
expression in terms of unknown A_0 and B_0

$$\left[\tau\cdot\varphi(\tau^2)\right]^{(1)}_{\tau=0}=\left\{\begin{array}{l}\varepsilon\left\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc\left(-\beta(\tau)\right)\right]-B_0\tau\cdot\left[i^{0}erfc\beta(\tau)-i^{0}erfc\left(-\beta(\tau)\right)\right]\right\}+\\+\xi\cdot A_0\tau\cdot\left[i^{0}erfc\beta(\tau)+i^{0}erfc\left(-\beta(\tau)\right)\right]\end{array}\right\}^{(1)}_{\tau=0}\tag{150}$$

take third derivative of expression (150) to obtain expression in terms of unknown A_1 and B_1

$$\left[\tau\cdot\varphi(\tau^2)\right]^{(3)}_{\tau=0}=\left\{\begin{array}{l}\varepsilon\left\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc\left(-\beta(\tau)\right)\right]-B_0\tau\cdot\left[i^{0}erfc\beta(\tau)-i^{0}erfc\left(-\beta(\tau)\right)\right]\right\}+\\+\xi\left\{A_0\tau\cdot\left[i^{0}erfc\beta(\tau)+i^{0}erfc\left(-\beta(\tau)\right)\right]+B_0\cdot\tau^2\left[ierfc\beta(\tau)-ierfc\left(-\beta(\tau)\right)\right]\right\}\\+\varepsilon\left\{A_1\cdot\tau^2\left[-ierfc\beta(\tau)+ierfc\left(-\beta(\tau)\right)\right]-B_1\cdot\tau^3\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]\right\}+\\+\xi\left\{A_1\cdot\tau^3\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]\right\}\end{array}\right\}^{(3)}_{\tau=0}\tag{151}$$

take fifth derivative of expression (151) to obtain expression in terms of unknown A_2 and B_2

$$\left[\tau\cdot\varphi(\tau^2)\right]^{(5)}_{\tau=0}=\left\{\begin{array}{l}\varepsilon\left\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc\left(-\beta(\tau)\right)\right]-B_0\tau\cdot\left[i^{0}erfc\beta(\tau)-i^{0}erfc\left(-\beta(\tau)\right)\right]\right\}+\\+\xi\left\{A_0\tau\cdot\left[i^{0}erfc\beta(\tau)+i^{0}erfc\left(-\beta(\tau)\right)\right]+B_0\cdot\tau^2\left[ierfc\beta(\tau)-ierfc\left(-\beta(\tau)\right)\right]\right\}\\+\varepsilon\left\{A_1\cdot\tau^2\left[-ierfc\beta(\tau)+ierfc\left(-\beta(\tau)\right)\right]-B_1\cdot\tau^3\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]\right\}+\\+\xi\left\{A_1\cdot\tau^3\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]-B_1\cdot\tau^4\left[i^3erfc\beta(\tau)+i^3erfc\left(-\beta(\tau)\right)\right]\right\}+\\+\varepsilon\left\{A_2\cdot\tau^4\left[-i^3erfc\beta(\tau)+i^3erfc\left(-\beta(\tau)\right)\right]-B_2\cdot\tau^5\left[-i^4erfc\beta(\tau)+i^4erfc\left(-\beta(\tau)\right)\right]\right\}+\\+\xi\left\{A_2\cdot\tau^5\left[i^4erfc\beta(\tau)+i^4erfc\left(-\beta(\tau)\right)\right]\right\}\end{array}\right\}^{(5)}_{\tau=0}\tag{152}$$

take seventh derivative of expression (152) to obtain expression in terms of unknown A_3 and B_3

$$
\left[\tau\cdot\varphi(\tau^2)\right]_{\tau=0}^{(7)}=\left\{\begin{array}{l}
\varepsilon\left\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc\left(-\beta(\tau)\right)\right]-B_0\tau\cdot\left[i^0erfc\beta(\tau)-i^0erfc\left(-\beta(\tau)\right)\right]\right\}+\\
+\xi\left\{A_0\tau\cdot\left[i^0erfc\beta(\tau)+i^0erfc\left(-\beta(\tau)\right)\right]+B_0\cdot\tau^2\left[ierfc\beta(\tau)-ierfc\left(-\beta(\tau)\right)\right]\right\}\\
+\varepsilon\left\{A_1\cdot\tau^2\left[-ierfc\beta(\tau)+ierfc\left(-\beta(\tau)\right)\right]-B_1\cdot\tau^3\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]\right\}+\\
+\xi\left\{A_1\cdot\tau^3\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]-B_1\cdot\tau^4\left[i^3erfc\beta(\tau)+i^3erfc\left(-\beta(\tau)\right)\right]\right\}+\\
+\varepsilon\left\{A_2\cdot\tau^4\left[-i^3erfc\beta(\tau)+i^3erfc\left(-\beta(\tau)\right)\right]-B_2\cdot\tau^5\left[-i^4erfc\beta(\tau)+i^4erfc\left(-\beta(\tau)\right)\right]\right\}+\\
+\xi\left\{A_2\cdot\tau^5\left[i^4erfc\beta(\tau)+i^4erfc\left(-\beta(\tau)\right)\right]-B_2\cdot\tau^6\left[-i^5erfc\beta(\tau)+i^5erfc\left(-\beta(\tau)\right)\right]\right\}+\\
+\varepsilon\left\{A_3\cdot\tau^6\left[-i^5erfc\beta(\tau)+i^5erfc\left(-\beta(\tau)\right)\right]-B_3\cdot\tau^7\left[-i^5erfc\beta(\tau)+i^5erfc\left(-\beta(\tau)\right)\right]\right\}+\\
+\xi\left\{A_3\cdot\tau^7\left[i^6erfc\beta(\tau)+i^6erfc\left(-\beta(\tau)\right)\right]\right\}
\end{array}\right\}_{\tau=0}^{(7)} \tag{153}
$$

and so on. Finally take 2n+1 derivative of expression (153) to obtain expression in terms of unknown A_n and B_n

$$
\left(\tau\cdot\varphi(\tau^2)\right)^{(2n+1)}=\left\{\begin{array}{l}
\varepsilon\left\{A_0\cdot\left[-i^{-1}erfc\beta(\tau)+i^{-1}erfc\left(-\beta(\tau)\right)\right]-B_0\tau\cdot\left[i^0erfc\beta(\tau)-i^0erfc\left(-\beta(\tau)\right)\right]\right\}+\\
+\xi\left\{A_0\tau\cdot\left[i^0erfc\beta(\tau)+i^0erfc\left(-\beta(\tau)\right)\right]+B_0\cdot\tau^2\left[ierfc\beta(\tau)-ierfc\left(-\beta(\tau)\right)\right]\right\}\\
+\varepsilon\left\{A_1\cdot\tau^2\left[-ierfc\beta(\tau)+ierfc\left(-\beta(\tau)\right)\right]-B_1\cdot\tau^3\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]\right\}+\\
+\xi\left\{A_1\cdot\tau^3\left[i^2erfc\beta(\tau)+i^2erfc\left(-\beta(\tau)\right)\right]+B_1\cdot\tau^4\left[i^3erfc\beta(\tau)-i^3erfc\left(-\beta(\tau)\right)\right]\right\}+\\
+\ldots\\
.\\
.\\
.\\
+\varepsilon\left\{A_n\cdot\tau^{2n}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc\left(-\beta(\tau)\right)\right]-B_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc\left(-\beta(\tau)\right)\right]\right\}+\\
+\xi\left\{A_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc\left(-\beta(\tau)\right)\right]\right\}
\end{array}\right\}_{\tau=0}^{(2n+1)}
$$

or

$$
\left[\tau\cdot\varphi(\tau^2)\right]_{\tau=0}^{(2n+1)}=\varepsilon\sum_{n=0}^{\infty}\left\{A_n\cdot\tau^{2n}\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc\left(-\beta(\tau)\right)\right]-B_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc\left(-\beta(\tau)\right)\right]\right\}_{\tau=0}^{(2n+1)}+
$$
$$
+\xi\sum_{n=0}^{\infty}\left\{A_n\cdot\tau^{2n+1}\left[i^{2n}erfc\beta(\tau)+i^{2n}erfc\left(-\beta(\tau)\right)\right]\right\}_{\tau=0}^{(2n+1)} \tag{154}
$$

For expression $\left[\tau^m\cdot i^m erfc\left(-\beta(\tau)\right)\right]^{(n)}$ it is necessary to apply formula (2) in corollaries where $\tau^m=u$ and $i^m erfc\beta(\tau)=v.$

$$
\left[\tau^m\cdot i^m erfc\beta(\tau)\right]^{(n)}=\left(\tau^m\right)^{(n)}\cdot i^m erfc\beta(\tau)+\binom{n}{1}\cdot\left(\tau^m\right)^{(n-1)}\cdot\left(i^m erfc\beta(\tau)\right)^{(1)}+
$$
$$
+\binom{n}{2}\cdot\left(\tau^m\right)^{(n-2)}\cdot\left(i^m erfc\beta(\tau)\right)^{(2)}+\ldots+\tau^m\cdot\left(i^m erfc\beta(\tau)\right)^{(n)}. \tag{155}
$$

For $\tau=0,$

1. if $m>n$ then $\left[\tau^m\cdot i^m erfc\beta(\tau)\right]_{\tau=0}^{(n)}=0$
2. if $m\le n$ then $\left[\tau^m\cdot i^m erfc\beta(\tau)\right]_{\tau=0}^{(n)}=m!\cdot\binom{n}{n-m}\cdot\left[i^m erfc\beta(\tau)\right]_{\tau=0}^{(n-m)}$

(156)

In the same manner for expression $\left[\tau^m \cdot i^m erfc\left(-\beta(\tau)\right)\right]^{(n)}$

1. if $m > n$ then $\left[\tau^m \cdot i^m erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(n)} = 0$ (157)
2. if $m \le n$ then $\left[\tau^m \cdot i^m erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(n)} = m! \cdot \begin{pmatrix} n \\ n-m \end{pmatrix} \cdot \left[i^m erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(n-m)}$ (158)

Finally

$$\left\{\tau^m \cdot \left[i^m erfc\beta(\tau) \pm i^m erfc\left(-\beta(\tau)\right)\right]\right\}_{\tau=0}^{(n)} = m! \cdot \begin{pmatrix} n \\ n-m \end{pmatrix} \cdot \left[i^m erfc\beta(\tau) \pm i^m erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(n-m)} \quad (159)$$

Applying (273) for (268)

$$\varepsilon \sum_{n=0}^{\infty} \left\{A_n \cdot \tau^{2n}\left[-i^{2n-1} erfc\beta(\tau) + i^{2n-1} erfc\left(-\beta(\tau)\right)\right] - B_n \cdot \tau^{2n+1}\left[i^{2n} erfc\beta(\tau) + i^{2n} erfc\left(-\beta(\tau)\right)\right]\right\}_{\tau=0}^{(2n+1)} +$$
$$+ \xi \sum_{n=0}^{\infty} \left\{A_n \cdot \tau^{2n+1}\left[i^{2n} erfc\beta(\tau) + i^{2n} erfc\left(-\beta(\tau)\right)\right]\right\}_{\tau=0}^{(2n+1)} =$$
$$= \varepsilon \sum_{k=0}^{n} A_k \cdot (2k)!\left[-i^{2k-1} erfc\beta(\tau) + i^{2k-1} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k+1)} - B_k \cdot (2k+1)!\left[i^{2k} erfc\beta(\tau) + i^{2k} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)} +$$
$$+ \xi \sum_{k=0}^{n} A_k \cdot (2k+1)! \cdot \left[i^{2k} erfc\beta(\tau) + i^{2k} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}$$
(160)

then

$$\left[\tau \cdot \varphi(\tau^2)\right]_{\tau=0}^{(2n+1)} = \varepsilon \sum_{k=0}^{n} \left\{\begin{matrix} A_k \cdot (2k)!\left[-i^{2k-1} erfc\beta(\tau) + i^{2k-1} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k+1)} - \\ - B_k \cdot (2k+1)!\left[i^{2k} erfc\beta(\tau) + i^{2k} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)} \end{matrix}\right\} +$$
$$+ \xi \sum_{k=0}^{n} A_k \cdot (2k+1)! \cdot \left[i^{2k} erfc\beta(\tau) + i^{2k} erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)} \quad (161)$$

From (141) and (161) it is possible to construct systems of equations where coefficients A_n and B_n can be determined. If from (141) A_n expressed in terms of B_n recurrent formula for B_n can be obtained in the following way

$$A_n = \frac{1}{\rho} \cdot \left[\frac{\varphi_n}{i^{2n} erfc0} - b \cdot B_n\right],$$

Implies

$$
\begin{aligned}
&\left[\tau\cdot\varphi(\tau^2)\right]_{\tau=0}^{(2n+1)}=\varepsilon\sum_{k=0}^{n}\left\{\begin{array}{l}\dfrac{1}{\rho}\cdot\left[\dfrac{\varphi_k}{i^{2k}erfc0}-b\cdot B_k\right]\cdot(2k)!\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k+1)}-\\ -B_k\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}\end{array}\right\}+\\
&+\xi\sum_{k=0}^{n}\frac{1}{\rho}\cdot\left[\frac{\varphi_k}{i^{2k}erfc0}-b\cdot B_k\right]\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}=\\
&=\sum_{k=0}^{n}\left\{\begin{array}{l}\dfrac{\varphi_k}{\rho\cdot i^{2k}erfc0}\cdot\left\{\varepsilon\cdot(2k)!\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k+1)}+\xi\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}\right\}-\\ --B_k\cdot\left\{\begin{array}{l}\dfrac{\varepsilon b}{\rho}\cdot(2k)!\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k+1)}+\varepsilon\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}+\\ +\dfrac{\xi b}{\rho}\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}\end{array}\right\}\end{array}\right\}
\end{aligned}
\tag{162}
$$

Yields

$$
\begin{aligned}
&\sum_{k=0}^{n}\left\{\begin{array}{l}\dfrac{\varphi_k}{\rho\cdot i^{2k}erfc0}\cdot\left\{\varepsilon\cdot(2k)!\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k+1)}+\xi\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}\right\}-\\ -B_k\cdot\left\{\begin{array}{l}\dfrac{\varepsilon b}{\rho}\cdot(2k)!\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k+1)}+\varepsilon\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}+\\ +\dfrac{\xi b}{\rho}\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}\end{array}\right\}\end{array}\right\}=\\
&=\sum_{k=0}^{n}\frac{\varphi_k}{\rho\cdot i^{2k}erfc0}\cdot\left\{\varepsilon\cdot(2k)!\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k+1)}+\xi\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}\right\}-\\
&-\sum_{k=0}^{n-1}B_k\cdot\left\{\begin{array}{l}\dfrac{\varepsilon b}{\rho}\cdot(2k)!\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k+1)}+\varepsilon\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}+\\ +\dfrac{\xi b}{\rho}\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(2n-2k)}\end{array}\right\}-\\
&-B_n\cdot\left\{\begin{array}{l}\dfrac{\varepsilon b}{\rho}\cdot(2n)!\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc\left(-\beta(\tau)\right)\right]_{\tau=0}^{(1)}+\varepsilon\cdot(2n+1)!\left[i^{2n}erfc\beta(0)+i^{2n}erfc\left(-\beta(0)\right)\right]+\\ +\dfrac{\xi b}{\rho}\cdot(2n+1)!\left[i^{2n}erfc\beta(0)+i^{2n}erfc\left(-\beta(0)\right)\right]\end{array}\right\}
\end{aligned}
\tag{163}
$$

Yields

$$
\begin{aligned}
&\left[\tau\cdot\varphi(\tau^2)\right]_{\tau=0}^{(2n+1)}=\\
&=\sum_{k=0}^{n}\frac{\varphi_k}{\rho\cdot i^{2k}erfc0}\cdot\left\{\varepsilon\cdot(2k)!\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc\left(-\beta(\tau)\right)\right]^{(2n-2k+1)}+\xi\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]^{(2n-2k)}\right\}-\\
&-\sum_{k=0}^{n-1}B_k\cdot\left\{\begin{array}{l}\dfrac{\varepsilon b}{\rho}\cdot(2k)!\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc\left(-\beta(\tau)\right)\right]^{(2n-2k+1)}+\varepsilon\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]^{(2n-2k)}+\\ +\dfrac{\xi b}{\rho}\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc\left(-\beta(\tau)\right)\right]^{(2n-2k)}\end{array}\right\}-\\
&-B_n\cdot\left\{\begin{array}{l}\dfrac{\varepsilon b}{\rho}\cdot(2n)!\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc\left(-\beta(\tau)\right)\right]^{(1)}+\varepsilon\cdot(2n+1)!\left[i^{2n}erfc\beta(0)+i^{2n}erfc\left(-\beta(0)\right)\right]+\\ +\dfrac{\xi b}{\rho}\cdot(2n+1)!\left[i^{2n}erfc\beta(0)+i^{2n}erfc\left(-\beta(0)\right)\right]\end{array}\right\}
\end{aligned}
$$

Finally

$$B_n=\frac{\sum_{k=0}^{n}\frac{\varphi_k}{\rho\cdot i^{2k}erfc0}\cdot\left\{\varepsilon\cdot(2k)!\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k+1)}+\xi\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}\right\}}{\left\{\begin{array}{l}\frac{\varepsilon b}{\rho}\cdot(2n)!\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(1)}+\varepsilon\cdot(2n+1)!\left[i^{2n}erfc\beta(0)+i^{2n}erfc(-\beta(0))\right]+\\+\frac{\xi b}{\rho}\cdot(2n+1)!\left[i^{2n}erfc\beta(0)+i^{2n}erfc(-\beta(0))\right]\end{array}\right\}}-$$

$$-\frac{\sum_{k=0}^{n-1}B_k\cdot\left\{\begin{array}{l}\frac{\varepsilon b}{\rho}\cdot(2k)!\left[-i^{2k-1}erfc\beta(\tau)+i^{2k-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k+1)}+\varepsilon\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}+\\+\frac{\xi b}{\rho}\cdot(2k+1)!\left[i^{2k}erfc\beta(\tau)+i^{2k}erfc(-\beta(\tau))\right]_{\tau=0}^{(2n-2k)}\end{array}\right\}}{\left\{\begin{array}{l}\frac{\varepsilon b}{\rho}\cdot(2n)!\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(1)}+\varepsilon\cdot(2n+1)!\left[i^{2n}erfc\beta(0)+i^{2n}erfc(-\beta(0))\right]+\\+\frac{\xi b}{\rho}\cdot(2n+1)!\left[i^{2n}erfc\beta(0)+i^{2n}erfc(-\beta(0))\right]\end{array}\right\}}-$$

$$-\frac{\left[\tau\cdot\varphi(\tau^2)\right]_{\tau=0}^{(2n+1)}}{\left\{\begin{array}{l}\frac{\varepsilon b}{\rho}\cdot(2n)!\left[-i^{2n-1}erfc\beta(\tau)+i^{2n-1}erfc(-\beta(\tau))\right]_{\tau=0}^{(1)}+\varepsilon\cdot(2n+1)!\left[i^{2n}erfc\beta(0)+i^{2n}erfc(-\beta(0))\right]+\\+\frac{\xi b}{\rho}\cdot(2n+1)!\left[i^{2n}erfc\beta(0)+i^{2n}erfc(-\beta(0))\right]\end{array}\right\}}$$

(164)

Thus recurrent formula for B_n is derived.

2.17 Approximate solution of Heat Equation in the domain with moving boundaries of the third type by IEF method

For the Heat Equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\qquad \beta(t)<x<\alpha(t),\qquad t>0$$

subject to

$$I.C:\ u(x,o)=0,\qquad \alpha(t)<x<\beta(t),$$

$$B.C:\left(\rho u+\theta\frac{\partial u}{\partial x}\right)\Big|_{x=\beta\sqrt{t}}=\varphi(t),\qquad t>0,$$

$$\left(\varepsilon u+\xi\frac{\partial u}{\partial x}\right)\Big|_{x=\alpha\sqrt{t}}=\phi(t),\qquad t>0.$$

Solution of the Heat Equation can be represented in the following form

$$u(x,t)=\sum_{n=0}^{k}(t^{\frac{n}{2}})\left\{A_n i^n erfc\left(\frac{x}{2a\sqrt{t}}\right)+B_n i^n erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right\}$$

or

$$u(x,t)=\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\right\}$$

Remark In both cases to find approximate solution it is necessary to give certain values for time *t*. For *k* values of *t* we obtain *k* coefficients from *2k* linear equations.

If we use IEF for the solution, then for boundary conditions at $x=\beta(t)$and $x=\alpha(t)$ we have system of *2k* linear equations.

$$\rho\sum_{n=0}^{k}(t_k)^{\frac{n}{2}}\left\{A_n i^n erfc\left(\frac{\beta(t_k)}{2a\sqrt{t_k}}\right)+B_n i^n erfc\left(-\frac{\beta(t_k)}{2a\sqrt{t_k}}\right)\right\}$$
$$+\theta\sum_{n=0}^{k}(t_k)^{\frac{n}{2}-1}\left\{-A_n i^n erfc\left(\frac{\beta(t_k)}{2a\sqrt{t_k}}\right)+B_n i^n erfc\left(-\frac{\beta(t_k)}{2a\sqrt{t_k}}\right)\right\}=\varphi(t_k),$$

$$\varepsilon\sum_{n=0}^{k}(t_k)^{\frac{n}{2}}\left\{A_n i^n erfc\left(\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right)+B_n i^n erfc\left(-\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right)\right\}$$
$$+\xi\sum_{n=0}^{k}(t_k)^{\frac{n}{2}-1}\left\{-A_n i^n erfc\left(\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right)+B_n i^n erfc\left(-\frac{\alpha(t_k)}{2a\sqrt{t_k}}\right)\right\}=\phi(t_k)$$

In the same manner solution in the polynomial form of IEF allows us to find even and odd coefficients A_{2n}, A_{2n+1} from system of linear equations

$$\rho\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(\beta(t_k))^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(\beta(t_k))^{2n-2m+1}t^m\beta_{2n+1,m}\right\}$$
$$+\theta\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(2n-2m)(\beta(t_k))^{2n-2m-1}t_k^m\beta_{2n,m}+\right.$$
$$\left.+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)(\beta(t_k))^{2n-2m}t_k^m\beta_{2n+1,m}\right\}=\varphi(t_k)$$

$$\varepsilon\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(\alpha(t_k))^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(\alpha(t_k))^{2n-2m+1}t^m\beta_{2n+1,m}\right\}$$
$$+\xi\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(2n-2m)(\alpha(t_k)))^{2n-2m-1}t_k^m\beta_{2n,m}+\right.$$
$$\left.+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)(\alpha(t_k)))^{2n-2m}t_k^m\beta_{2n+1,m}\right\}=\phi(t_k)$$

Example 3: Approximate solution of one Heat Transfer Problem, in the domain with moving boundary of the third type, degenerating at the initial time, obtained in polynomial form of IEF

Approximate solution of the Heat Equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < t, \quad t > 0$$

Subject to

$$I.C\colon\ u(x,o) = 0, \qquad 0 < x < t,$$

$$B.C\colon \left(u + 2\frac{\partial u}{\partial x}\right)\Big|_{x=0} = e^t, \qquad t > 0,$$

$$\left(3u + 4\frac{\partial u}{\partial x}\right)\Big|_{x=\mathrm{t}} = 1, \quad t > 0.$$

represented in the following form

$$u(x,t) = \sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n} x^{2n-2m}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t^m\beta_{2n+1,m}\right\}$$

where even and odd coefficients A_{2n}, A_{2n+1} have to be determined.

Substituting expression (1) from chapter 1.3 into the boundary conditions for certain values of t_k: $t_1 = 0, t_2 = 0.2, t_3 = 0.4, t_4 = 0{,}6,\ t_5 = 0{,}8$ we obtain

For x=0

$$\left(A_0\beta_{0,0} + A_2t_k\beta_{2,1} + A_4{t_k}^2\beta_{4,2} + \cdots + A_{2k}{t_k}^{2k}\beta_{2k,k}\right) + 2\{\mathrm{A}_1\beta_{1,0} + +\mathrm{A}_3t_k\beta_{3,1} + +\mathrm{A}_5{t_k}^2\beta_{5,2} + \cdots + \mathrm{A}_{2k+1}{t_k}^{\mathrm{k}}\beta_{2\mathrm{k}+1,\mathrm{k}}\} =$$

$$= e^{t_k}$$

For $x = t$

$$3\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(t_k)^{2n-2m}{t_k}^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n}(t_k)^{2n-2m+1}{t_k}^m\beta_{2n+1,m}\right\} + 4\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(2n-2m)(t_k)^{2n-2m-1}t_k^m\beta_{2n,m} + + A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)(t_k)^{2n-2m}t_k^m\beta_{2n+1,m}\right\} = 1$$

Using programs like Mathcad it is possible to calculate coefficients from above systems of linear equations.

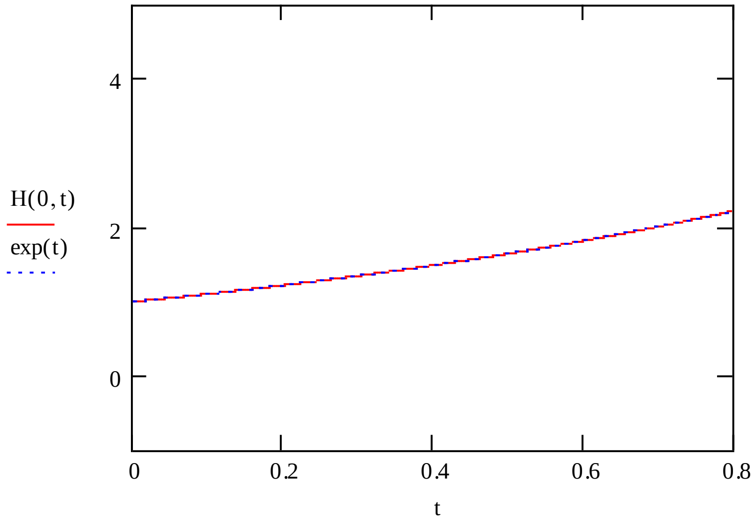


Figure 8 - Graphs of functions exp(t) and (1) from chapter 1.3 for x=0

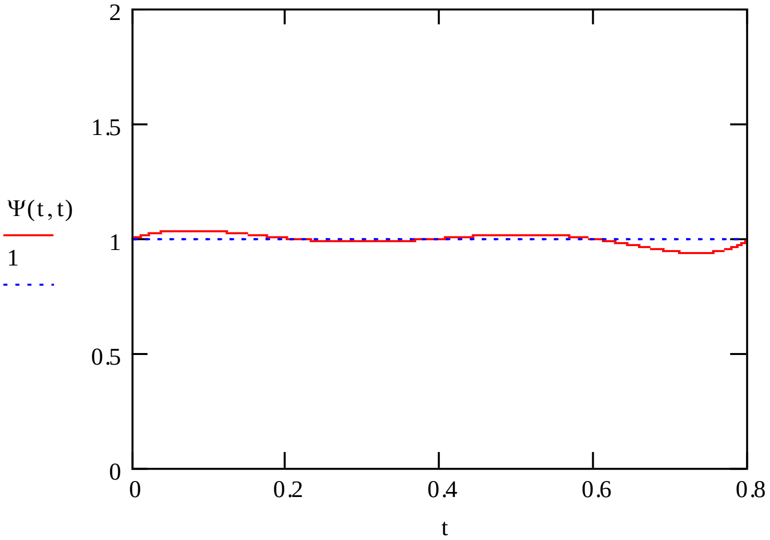


Figure 9 - Graphs of constant 1 and function (1) from chapter 1.3 for x=t

Where

$$\mathrm{H}(0,\mathrm{t}) = \left(A_0\beta_{0,0} + A_2 t_k \beta_{2,1} + A_4 {t_k}^2 \beta_{4,2} + \cdots + A_{2k} {t_k}^{2k} \beta_{2k,k}\right) + 2\{\mathrm{A}_1\beta_{1,0} +$$

$$+\mathrm{A}_3 t_k \beta_{3,1} + \mathrm{A}_5 {t_k}^2 \beta_{5,2} + \cdots + \mathrm{A}_{2k+1} {t_k}^{\mathrm{k}} \beta_{2k+1,k}\}$$

and $\exp(t) = e^{t_k}$

$$\Psi(t,t) = 3\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(t_k)^{2n-2m}{t_k}^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n}(t_k)^{2n-2m+1}{t_k}^m\beta_{2n+1,m}\right\}$$

$$+ 4\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(2n-2m)(t_k)^{2n-2m-1}t_k^m\beta_{2n,m} +\right.$$

$$\left. + A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)(t_k)^{2n-2m}t_k^m\beta_{2n+1,m}\right\}$$

It is possible to observe in above figures, that deviation among given boundary functions and functions obtained by substitution solution into the boundary conditions, is less than 10%. Maximum Principle allows us to say that error of the solution doesn't exceed than 10% for 5 points of time *t*. As more precise solution demanded as more values for time *t* should be taken.

3 APPROXIMATE SOLUTIONS OF HEAT EQUATIONS WITH UNKNOWN BOUNDARIES OBTAINED BY IEF METHOD

3.1 Approximate solution of Stefan problem by IEF method

It is required to solve equation

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < \beta(t) \qquad (165)$$

For $x = 0,$

$$\left[U - b\frac{\partial U}{\partial x}\right]_{x=0} = \frac{b^2}{\lambda} \cdot P(t), \qquad t > 0 \qquad (166)$$

For $x = \beta(t), \quad U(\beta(t), t) = \beta(t) \cdot \theta_m, \qquad t > 0 \qquad (167)$

Stefan condition

$$\left[U - \beta(t)\frac{\partial U}{\partial x}\right]_{x=\beta(t)} = \frac{L\gamma}{\lambda} \cdot \beta^2(t) \cdot \frac{d\beta(t)}{dt}, \qquad t > 0 \qquad (168)$$

where $\beta(t) = \beta_1 \cdot t^{\frac{1}{2}} + \beta_2 \cdot t + \beta_3 \cdot t^{\frac{3}{2}} + ... + \beta_n \cdot t^{\frac{n}{2}} + ...$ and P(t) is an analytic function

Solution is considered to be found by the help of IEF method

$$U(x,t) = \sum_{n=0}^{\infty}\left\{A_n \cdot t^n \left[i^{2n}erfc\frac{x}{2a\sqrt{t}} + i^{2n}erfc(\frac{-x}{2a\sqrt{t}})\right] + B_n \cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc\frac{x}{2a\sqrt{t}} - i^{2n+1}erfc(\frac{-x}{2a\sqrt{t}})\right]\right\}$$

where coefficients A_n and B_n has to be found.

Using properties of IEF it is possible to find following expression

$$\frac{\partial U}{\partial x} = \sum_{n=0}^{\infty}\left\{\begin{array}{l} A_n \cdot t^{n-1/2}\left[-i^{2n-1}erfc\frac{x}{2a\sqrt{t}} + i^{2n-1}erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right] - \\ -B_n \cdot t^n\left[i^{2n}erfc\frac{x}{2a\sqrt{t}} + i^{2n}erfc\left(-\frac{x}{2a\sqrt{t}}\right)\right]\end{array}\right\} \qquad (169)$$

$$\Rightarrow \left.\frac{\partial U}{\partial x}\right|_{x=0} = \sum_{n=0}^{\infty}\left\{A_n \cdot t^{n-1/2}\left[-i^{2n-1}erfc0 + i^{2n-1}erfc0\right] - B_n \cdot t^n\left[i^{2n}erfc0 + i^{2n}erfc0\right]\right\} =$$
$$= -\sum_{n=0}^{\infty} 2B_n \cdot t^n \cdot i^{2n}erfc0 \qquad (170)$$

$$U(0,t) = \sum_{n=0}^{\infty}\left\{\begin{array}{l} A_n \cdot t^n\left[i^{2n}erfc0 + i^{2n}erfc0\right] + \\ + B_n \cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc0 - i^{2n+1}erfc0\right]\end{array}\right\} = \sum_{n=0}^{\infty} 2A_n \cdot t^n \cdot i^{2n}erfc0 \qquad (171)$$

From (166)

$$\sum_{n=0}^{\infty}\left\{t^n \cdot i^{2n}erfc0 \cdot \left[A_n + b\cdot B_n\right]\right\} = \rho \cdot P(t), \rho = \frac{b^2}{2\lambda} \tag{172}$$

$$\Rightarrow A_n + b\cdot B_n = \frac{\rho \cdot P^{(n)}(0)}{n! i^{2n}erfc0}, \tag{173}$$

$$\Rightarrow A_n = \frac{\rho \cdot P_n}{i^{2n}erfc0} - b\cdot B_n, \tag{174}$$

$$\text{where } P_n = \frac{P^n(0)}{n!} \tag{175}$$

From (167) and for $\sqrt{t} = \tau$

$$U(\beta(t),t) = \sum_{n=0}^{\infty}\left\{A_n \cdot t^n\left[i^{2n}erfc\frac{\beta(t)}{2a\sqrt{t}} + i^{2n}erfc(\frac{-\beta(t)}{2a\sqrt{t}})\right] + B_n \cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc\frac{\beta(t)}{2a\sqrt{t}} - i^{2n+1}erfc(\frac{-\beta(t)}{2a\sqrt{t}})\right]\right\} =$$
$$= \sum_{n=0}^{\infty}\left\{A_n \cdot \tau^{2n}\left[i^{2n}erfc\frac{\beta(\tau)}{2a} + i^{2n}erfc(\frac{-\beta(\tau)}{2a})\right] + B_n \cdot \tau^{2n+1}\left[i^{2n+1}erfc\frac{\beta(\tau)}{2a} - i^{2n+1}erfc(\frac{-\beta(\tau)}{2a})\right]\right\} = \beta(\tau)\cdot\Theta_m \tag{176}$$

Substitute $\frac{\rho \cdot P_n}{i^{2n}erfc0} - b\cdot B_n$ instead of A_n to express B_n coefficients in terms of unknown $\beta_0, \beta_1, \beta_2, \beta_3, ..., \beta_n$. It is necessary to take 2n+1 derivative of expression (177)

for τ=0

$$\begin{aligned} U(\beta(t),t) &= \sum_{n=0}^{\infty}\left\{\begin{matrix}\left[\frac{\rho \cdot P_n}{i^{2n}erfc0} - b\cdot B_n\right]\cdot t^n\left[i^{2n}erfc\frac{\beta(t)}{2a\sqrt{t}} + i^{2n}erfc(\frac{-\beta(t)}{2a\sqrt{t}})\right] + \\ + B_n \cdot t^{\frac{2n+1}{2}}\left[i^{2n+1}erfc\frac{\beta(t)}{2a\sqrt{t}} - i^{2n+1}erfc(\frac{-\beta(t)}{2a\sqrt{t}})\right]\end{matrix}\right\} = \\ &= \sum_{n=0}^{\infty}\left\{\begin{matrix}\left[\frac{\rho \cdot P_n}{i^{2n}erfc0} - b\cdot B_n\right]\cdot \tau^{2n}\left[i^{2n}erfc\frac{\beta(\tau)}{2a} + i^{2n}erfc(\frac{-\beta(\tau)}{2a})\right] + \\ + B_n \cdot \tau^{2n+1}\left[i^{2n+1}erfc\frac{\beta(\tau)}{2a} - i^{2n+1}erfc(\frac{-\beta(\tau)}{2a})\right]\end{matrix}\right\} = \beta(\tau)\cdot\Theta_m \end{aligned} \tag{178}$$

Let's expand (179)

$$U(\beta(t),t)=\sum_{n=0}^{\infty}\left\{\begin{array}{l}\left[\dfrac{\rho\cdot P_n}{i^{2n}erfc0}-b\cdot B_n\right]\cdot\tau^{2n}\left[i^{2n}erfc\dfrac{\beta(\tau)}{2a}+i^{2n}erfc(\dfrac{-\beta(\tau)}{2a})\right]+\\ +B_n\cdot\tau^{2n+1}\left[i^{2n+1}erfc\dfrac{\beta(\tau)}{2a}-i^{2n+1}erfc(\dfrac{-\beta(\tau)}{2a})\right]\end{array}\right\}=$$

$$=\left[\frac{\rho\cdot P_0}{i^0erfc0}-b\cdot B_0\right]\cdot\left[erfc\frac{\beta(\tau)}{2a}+erfc\left(-\frac{\beta(\tau)}{2a}\right)\right]+B_0\cdot\tau\left[ierfc\frac{\beta(\tau)}{2a}-ierfc\left(-\frac{\beta(\tau)}{2a}\right)\right]+$$

$$+\left[\frac{\rho\cdot P_1}{i^2erfc0}-b\cdot B_1\right]\cdot\tau^2\left[i^2erfc\frac{\beta(\tau)}{2a}+i^2erfc\left(-\frac{\beta(\tau)}{2a}\right)\right]+B_1\cdot\tau^3\left[i^3erfc\frac{\beta(\tau)}{2a}-i^3erfc\left(-\frac{\beta(\tau)}{2a}\right)\right]+$$

$$+\left[\frac{\rho\cdot P_2}{i^4erfc0}-b\cdot B_2\right]\cdot\tau^4\left[i^4erfc\frac{\beta(\tau)}{2a}+i^4erfc\left(-\frac{\beta(\tau)}{2a}\right)\right]+B_2\cdot\tau^5\left[i^5erfc\beta(\tau)-i^5erfc\left(-\frac{\beta(\tau)}{2a}\right)\right]+$$

$$+\left[\frac{\rho\cdot P_3}{i^6erfc0}-b\cdot B_3\right]\cdot\tau^6\left[i^6erfc\frac{\beta(\tau)}{2a}+i^6erfc\left(-\frac{\beta(\tau)}{2a}\right)\right]+B_3\cdot\tau^7\left[i^7erfc\beta(\tau)-i^7erfc\left(-\frac{\beta(\tau)}{2a}\right)\right]+$$

$$+\ldots$$

.

.

.

$$+\left[\frac{\rho\cdot P_n}{i^{2n}erfc0}-b\cdot B_n\right]\cdot\tau^{2n}\left[i^{2n}erfc\frac{\beta(\tau)}{2a}+i^{2n}erfc\left(-\frac{\beta(\tau)}{2a}\right)\right]+B_n\cdot\tau^{2n+1}\left[i^{2n+1}erfc\frac{\beta(\tau)}{2a}-i^{2n+1}erfc\left(-\frac{\beta(\tau)}{2a}\right)\right]$$

for $\sqrt{t}=\tau$

$$\beta(\tau)=\beta_0+\beta_1\tau+\beta_2\tau^2+\beta_3\tau^3+\beta_4\tau^4+\ldots+\beta_n\tau^n+\ldots$$

Let

$$\alpha(\tau)=\alpha_0+\alpha_1\tau+\alpha_2\tau^2+\alpha_3\tau^3+\ldots+\alpha_n\tau^n+\ldots$$

where

$$\alpha_0=\frac{\beta_0}{2a},\ \alpha_1=\frac{\beta_1}{2a},\ \alpha_2=\frac{\beta_2}{2a},\ldots,\alpha_n=\frac{\beta_n}{2a},\ldots$$

and

$$\frac{d\beta(t)}{dt}=\frac{1}{\tau}\cdot\frac{d\alpha(\tau)}{d\tau}$$

Then from (180)

$$\tau\cdot\sum_{n=0}^{\infty}\left\{\begin{matrix}\left[\left[\frac{\rho\cdot P_n}{i^{2n}erfc0}-b\cdot B_n\right]\tau^{2n}\left[\underbrace{i^{2n}erfc\,\alpha(\tau)+i^{2n}erfc(-\alpha(\tau))}_{M_n}\right]+B_n\tau^{2n+1}\left[\underbrace{i^{2n+1}erfc\,\alpha(\tau)-i^{2n+1}erfc(-\alpha(\tau))}_{L_n}\right]\right]-\\ -\alpha(\tau)\cdot\left[\begin{matrix}\left[\frac{\rho\cdot P_n}{i^{2n}erfc0}-b\cdot B_n\right]\tau^{2n-1}\left[\underbrace{-i^{2n-1}erfc\,\alpha(\tau)+i^{2n-1}erfc(-\alpha(\tau))}_{P_n}\right]-\\ -B_n\tau^{2n}\left[\underbrace{i^{2n}erfc\,\alpha(\tau)+i^{2n}erfc(-\alpha(\tau))}_{Q_n}\right]\end{matrix}\right]\end{matrix}\right\}=$$

$$=\mu\cdot\alpha^2(\tau)\cdot\frac{d\alpha(\tau)}{d\tau}$$

where $\mu=\frac{L\gamma}{2\lambda}$

Multiplying terms of the sum by τ

$$\sum_{k=0}^{n}\left\{\left[\left[\frac{\rho\cdot P_k}{i^{2k}erfc0}-b\cdot B_k\right]\tau^{2k+1}M_k+B_k\tau^{2k+2}L_k\right]-\alpha(\tau)\cdot\left[\left[\frac{\rho\cdot P_k}{i^{2k}erfc0}-b\cdot B_k\right]\tau^{2k}P_k-B_k\tau^{2k+1}Q_k\right]\right\}=$$

$$=\mu\cdot\alpha^2(\tau)\cdot\frac{d\alpha(\tau)}{d\tau} \tag{181}$$

Second property of IEF implies

$$M_k=\sum_{m=0}^{k}\frac{(\alpha(\tau))^{2(k-m)}}{2^{2m-1}m!(2k-2m)!},L_k=\sum_{m=0}^{k}\frac{(\alpha(\tau))^{2(k-m)+1}}{2^{2m-1}m!(2k-2m+1)!},$$
$$P_k=\sum_{m=0}^{k}\frac{(\alpha(\tau))^{2(k-m)-1}}{2^{2m-1}m!(2k-2m-1)!},Q_k=\sum_{m=0}^{k}\frac{(\alpha(\tau))^{2(k-m)}}{2^{2m-1}m!(2k-2m)!}$$

Taking 2n+1 derivative of expression (292) and using corollaries for $\tau=0$

$$\left\{\sum_{k=0}^{n}\left\{\left[\left[\frac{\rho\cdot P_k}{i^{2k}erfc0}-b\cdot B_k\right]\tau^{2k+1}M_k+B_k\tau^{2k+2}L_k\right]-\alpha(\tau)\cdot\left[\left[\frac{\rho\cdot P_k}{i^{2k}erfc0}-b\cdot B_k\right]\tau^{2k}P_k-B_k\tau^{2k+1}Q_k\right]\right\}\right\}^{(2n+1)}=$$

$$=\left\{\mu\cdot\alpha^2(\tau)\cdot\frac{d\alpha(\tau)}{d\tau}\right\}^{(2n+1)}$$

yields

$$\sum_{k=0}^{n}\left\{\left[\frac{\rho\cdot P_n}{i^{2n}erfc0}-b\cdot B_n\right](M_k)^{(2n-2k)}+B_k(L_k)^{(2n-2k-1)}-\left[\frac{\rho\cdot P_n}{i^{2n}erfc0}-b\cdot B_n\right][\alpha(\tau)P_k]^{(2n-2k+1)}+B_k[\alpha(\tau)Q_k]^{(2n-2k)}\right\}=$$

$$=\mu\cdot\sum_{k=0}^{2n+1}\binom{2n+1}{k}\cdot\left[\alpha^2(\tau)\right]^{(2n-k+1)}\cdot\left[\frac{d\alpha(\tau)}{d\tau}\right]^k$$

thus

$$\sum_{k=0}^{n}\left\{\begin{array}{l}\left[\dfrac{\rho\cdot P_n}{i^{2n}erfc0}-b\cdot B_n\right](M_k)^{(2n-2k)}+B_k(L_k)^{(2n-2k-1)}-\left[\dfrac{\rho\cdot P_n}{i^{2n}erfc0}-b\cdot B_n\right][\alpha(\tau)P_k]^{(2n-2k+1)}+\\+B_k[\alpha(\tau)Q_k]^{(2n-2k)}\end{array}\right\}=$$

$$=\mu\cdot\sum_{k=0}^{2n+1}\binom{2n+1}{k}\cdot\left[\alpha^2(\tau)\right]^{(2n-k+1)}\cdot\left[\frac{d\alpha(\tau)}{d\tau}\right]^{k}$$

Finally $\alpha_0,\alpha_1,\alpha_2,\alpha_3,...,\alpha_n$ can be determined from the following system of nonlinear equations

$$\sum_{k=0}^{n}\left\{\begin{array}{l}\left[\dfrac{\rho\cdot P_n}{i^{2n}erfc0}-b\cdot B_n\right](M_k)^{(2n-2k)}+B_k(L_k)^{(2n-2k-1)}-\\-\left[\dfrac{\rho\cdot P_n}{i^{2n}erfc0}-b\cdot B_n\right]\displaystyle\sum_{p=0}^{2n-2k+1}\binom{2n-2k+1}{p}\cdot[\alpha(\tau)]^{(2n-2k-p+1)}[P_k]^{(p)}+\\+B_k\displaystyle\sum_{p=0}^{2n-2k}\binom{2n-2k}{p}[\alpha(\tau)]^{(2n-2k)}[Q_k]^{(p)}\end{array}\right\}_{\tau=0}=$$

$$=\mu\cdot\sum_{k=0}^{2n+1}\binom{2n+1}{k}\cdot\left[\alpha^2(\tau)\right]_{\tau=0}^{(2n-k+1)}\cdot\left[\frac{d\alpha(\tau)}{d\tau}\right]_{\tau=0}^{(k)}$$

3.2 Approximate solution of the Stefan problem by IEF method

Approximate solution of Heat Equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\qquad 0<x<\alpha(t)\ ,\quad \mathrm{t}>0 \tag{182}$$

Subject to

I.C: $u(x,0)=0,\qquad 0<x<\alpha(t)$ (183)

B.C: $\left.\left(\rho u+\theta\dfrac{\partial u}{\partial x}\right)\right|_{x=0}=P(t),\qquad \mathrm{t}>0$ (184)

$u(\alpha(t),t)=\phi(t),\qquad \mathrm{t}>0$ (185)

Stephan condition:

$$\left.\left(\lambda u-\alpha(t)\frac{\partial u}{\partial x}\right)\right|_{x=\alpha(t)}=\alpha^2(t)\frac{d\alpha(t)}{dt}, \tag{186}$$

$$u(0,0)=0, \tag{187}$$

can be represented in the polynomial form of IEF

$$u(x,t)=\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\right\} \tag{188}$$

where unknown moving boundary $\alpha(t)$ and even and odd coefficients A_{2n}, A_{2n+1} have to be determined. Making assumptions concerned with the function $\alpha(t)$, problem can be solved in following two ways.

3.3 Approximate solution of the Stefan problem by IEF method with $\boldsymbol{\alpha(t) == \alpha_1 t^{\frac{1}{2}} + +\alpha_2 t + \alpha_3 t^{\frac{3}{2}} + +\alpha_4 t^2 + \alpha_5 t^{\frac{5}{2}} + \cdots + \alpha_n t^{\frac{n}{2}}}$ unknown moving boundary

We consider solution in the form of (188). Unknown moving boundary $\alpha(t)$ and even and odd coefficients A_{2n}, A_{2n+1} have to be determined. Making assumptions concerned with the function $\alpha(t)$, problem can be solved in the following ways.

Remark: To find appropriate number of coefficients in expressions (183) - (186) and coefficients of unknown moving boundary, it is necessary to give same number of values for time *t*. Thus we obtain two systems of linear equations from the first two boundary conditions and one system of nonlinear equations from Stefan condition.

Solution: Let $\alpha(t)$ be a function in the following form

$$\alpha(t) = \alpha_1 t^{\frac{1}{2}} + +\alpha_2 t + \alpha_3 t^{\frac{3}{2}} + \alpha_4 t^2 + \alpha_5 t^{\frac{5}{2}} + \cdots + \alpha_n t^{\frac{n}{2}}$$

where coefficients $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ have to be determined.

Substituting expression (188) into the boundary conditions (183) and (184) for k values of time *t*, we obtain 2 systems of k linear equations for each boundary condition, and 1 system of k nonlinear equations for Stefan condition.

for x=0

$$\rho\left(A_0\beta_{0,0} + A_2 t_k \beta_{2,1} + A_4 {t_k}^2 \beta_{4,2} + \cdots + A_{2k} {t_k}^{2k} \beta_{2k,k}\right) + \theta\{\mathrm{A}_1\beta_{1,0} + \mathrm{A}_3 t_k \beta_{3,1} + \mathrm{A}_5 {t_k}^2 \beta_{5,2} + \cdots + \mathrm{A}_{2k+1} {t_k}^{k} \beta_{2k+1,k}\} = P(t_k)$$

where even coefficients A_{2k} expressed in terms of odd coefficients A_{2k+1}.

for $x = \alpha(t)$

$$\sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}(\alpha(t_k))^{2n-2m}{t_k}^{m}\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n}(\alpha(t_k))^{2n-2m+1}{t_k}^{m}\beta_{2n+1,m}\right\} = \phi(t_k)$$

where odd coefficients A_{2k+1} expressed in terms of coefficients $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$.

Finally for Stefan condition (186) we have system of nonlinear equations

$$\lambda \sum_{n=0}^{k} \{A_{2n} \sum_{m=0}^{n} (y_k(t_k))^{2n-2m} t_k{}^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} (y_k(t_k))^{2n-2m+1} t_k{}^m \beta_{2n+1,m}\} -$$
$$\alpha(t) \sum_{n=0}^{k} \left\{A_{2n} \sum_{m=0}^{n} (2n-2m) \left(y_k(t_k)\right)^{2n-2m-1} t_k^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} (2n-2m+\right.$$
$$\left. +1) \left(y_k(t_k)\right)^{2n-2m} t_k^m \beta_{2n+1,m}\right\} = \alpha^2(t) \left\{\frac{\alpha_1}{t_k^{\frac{1}{2}}} + \alpha_2 + \alpha_3 t_k^{\frac{1}{2}} + \alpha_4 t_k + \alpha_5 t_k^{\frac{3}{2}} + \cdots + \alpha_n t_k^{\frac{n}{2}-1}\right\}$$

where $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ have to be determined.

Above two systems of linear equations obtained from boundary conditions can be solved parametrically, and system of nonlinear equations obtained from Stefan condition can be solved by Mathcad computer program.

It is possible to determine deviation of solution using Maximum principle.

3.4 Approximate solution of the Stefan problem by IEF method in the linearly approximated $\alpha(t)$ boundary

It is possible to approximate function $\alpha(t)$ linearly in the following way:

$\alpha(t) = \{y_1(t), y_2(t), y_3(t), \dots, y_n(t)\}$

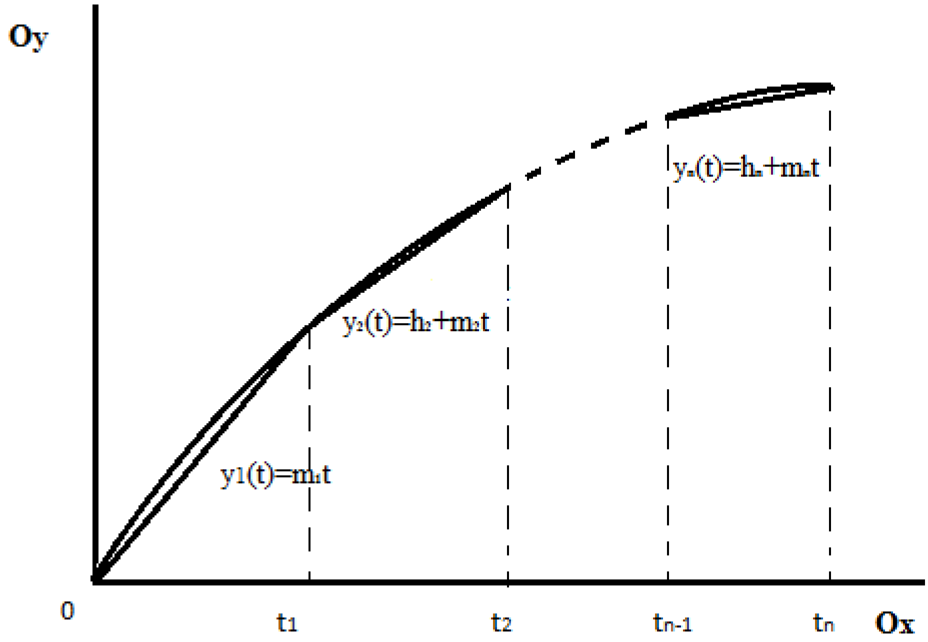
where $y_1(t), y_2(t), y_3(t), \dots, y_n(t)$ are linear functions

$$y_1(t) = h_1 + m_1 t, \quad h_1 = 0$$
$$y_2(t) = h_2 + m_2 t,$$
$$y_3(t) = h_3 + m_3 t,$$
$$\dots \dots \dots \dots \dots$$
$$y_n(t) = h_n + m_n t.$$

h_n, μ_n have to be found and $\alpha(t)$ is as shown in the figure. It's clear that $y_1(t_1) = y_2(t_1)$, then $m_1 t_1 = h_2 + m_2 t_1$, implies $h_2 = (m_1 - m_2)t_1$, and $y_2(t) = (m_1 - m_2)t_1 + m_2 t$. In the similar way we have $y_2(t_2) = y_3(t_3)$, then $(m_1 - m_2)t_1 + m_2 t_2 = h_3 + m_3 t_2$ implies $h_3 = (m_1 - m_2)t_1 + m_2 t_2 - m_3 t_2$, and $y_3(t) = (m_1 - m_2)t_1 + (m_2 - m_3)t_2 + m_3 t$, etc.

$y_n(t) = (m_1 - m_2)t_1 + (m_2 - m_3)t_2 + +(m_3 - m_4)t_3 + \cdots + m_n t$

Thus all $y_n(t)$ linear functions are expressed in terms of unknown $m_1, m_2 \dots m_n$ coefficients.



After coefficients $m_1, m_2 \dots m_n$ found $y_n(t)$ functions can be easily determined. Substituting values of time *t* into the functions $y_n(t)$ from their domains it is possible to build a graph of function $\alpha(t)$, and using Mathcad it is possible to construct analytical form.

To find $\alpha(t)$ and u(x,t) subject to above boundary conditions and special Stephan condition we substitute expression (299) into the boundary conditions and obtain systems of linear equations

for x=0

$$\rho\left(A_0\beta_{0,0} + A_2 t_k \beta_{2,1} + A_4 {t_k}^2 \beta_{4,2} + \dots + A_{2k} {t_k}^{2k} \beta_{2k,k}\right) + \theta\{A_1\beta_{1,0} + A_3 t_k \beta_{3,1} + A_5 {t_k}^2 \beta_{5,2} + \dots + A_{2k+1} {t_k}^{k} \beta_{2k+1,k}\} = P(t_k) \quad (189)$$

where even coefficients A_{2k} are expressed in terms of odd coefficients A_{2k+1} and substituted into the next boundary condition

for $x = \alpha(t)$

$$\varepsilon \sum_{n=0}^{k} \{A_{2n} \sum_{m=0}^{n} (y_k(t_k))^{2n-2m} {t_k}^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} (y_k(t_k))^{2n-2m+1} {t_k}^m \beta_{2n+1,m}\} + \xi \sum_{n=0}^{k} \left\{ \begin{array}{c} A_{2n} \sum_{m=0}^{n} (2n-2m)\left(y_k(t_k)\right)^{2n-2m-1} t_k^m \beta_{2n,m} + \\ + A_{2n+1} \sum_{m=0}^{n} (2n-2m+1)(y_k(t_k))^{2n-2m} t_k^m \beta_{2n+1,m} \end{array} \right\} = \phi(t_k) \quad (190)$$

where odd coefficients A_{2k+1} expressed in terms of $m_1, m_2 \dots m_n$ and substituted into the system of nonlinear equations obtained from Stephan condition.

$\lambda\sum_{n=0}^{k}\{A_{2n}\sum_{m=0}^{n}(y_k(t_k))^{2n-2m}{t_k}^m\beta_{2n,m} +$
$A_{2n+1}\sum_{m=0}^{n}(y_k(t_k))^{2n-2m+1}{t_k}^m\beta_{2n+1,m}\} -$
$\eta\sum_{n=0}^{k}\{A_{2n}\sum_{m=0}^{n}(2n-2m)(y_k(t_k))^{2n-2m-1}t_k^m\beta_{2n,m} + + A_{2n+1}\sum_{m=0}^{n}(2n-2m+$
$1)(y_k(t_k))^{2n-2m}t_k^m\beta_{2n+1,m}\} = m_k$

(191)

System of nonlinear equations (189) with unknown $m_1, m_2 \ldots m_n$ and parametric solutions of systems (190), (191) can be obtained and solved in Mathcad.

Example 4: Approximate solution of Stefan problem by IEF method in the linearly approximated $\boldsymbol{\alpha(t)}$ boundary

Solve the Heat Equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \alpha(t)\ , \quad 0 < \mathrm{t} < 1 \tag{192}$$

Subject to

I.C: $$u(x,0) = 0, \tag{193}$$

B.C: $$-\left.\frac{\partial u}{\partial x}\right|_{x=0} = e^t, \qquad 0 < t < 1, \tag{194}$$

$$u_{x=\alpha(t)} = 0, \qquad 0 < t < 1, \tag{195}$$

Stephan condition:

$$-\left.\frac{\partial u}{\partial x}\right|_{x=\alpha(t)} = \frac{\partial \alpha(t)}{\partial t}, \qquad 0 < x < \alpha(t) \tag{196}$$

$$u(0,0) = 0, \tag{197}$$

Exact solution:

$$u(x,t) = e^{t-x} - 1 \ \text{ and } \ \alpha(t) = t$$

Solution:

We shall solve the problem in the second way and approximate solution can be represented in the polynomial form of IEF. We subdivide time interval (0,1) into two intervals$(0;0{,}4]$, $(0{,}4;0{,}8]$ and take two points for time *t*, $t_1 = 0.4$ and $t_1 = 0.8$

$$u(x,t) = \sum_{n=0}^{k}\left\{A_{2n}\sum_{m=0}^{n}x^{2n-2m}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n}x^{2n-2m+1}t^m\beta_{2n+1,m}\right\}$$

and

$\alpha(t) = \{y_1(t), y_2(t)\}$, where $y_1(t) = m_1 t$ defined in the domain $t \in (0;0{,}4]$ and

$y_2(t) = (m_1 - m_2)t_1 + m_2 t$ defined in the domain $t \in (0{,}4;0{,}8]$. Since two points are taken two even A_0, A_2 and two odd A_1, A_3 coefficients have to be determined.

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = A_1\beta_{1,0} + A_3\beta_{3,1}t_k = e^{t_k}, where\ k = 1,2$$

Odd coefficients A_1, A_3 can be obtained from above system. Making calculations we have $A_1 = 0.758$, $A_3 = 14.674$

System of equations obtained from second boundary condition, can be solved parametrically in terms of m_1, m_2 (even coefficients A_0, A_2 can be expressed in terms of m_1, m_2).

$$A_0\beta_{0,0} + A_2\left(y_k{}^2(t_k)\beta_{2,0} + t_k\beta_{2,1}\right) + A_1 y_k(t_k)\beta_{1,0} + A_3\left(y_k{}^3(t_k)\beta_{3,0} + y_k(t_k)t_k\beta_{3,1}\right) = 0$$

or

$$A_0\beta_{0,0} + A_2\left(y_k{}^2(t_k)\beta_{2,0} + t_k\beta_{2,1}\right) + 0.758 y_k(t_k)\beta_{1,0} + 14.674\left(y_k{}^3(t_k)\beta_{3,0} + y_k(t_k)t_k\beta_{3,1}\right) = 0$$

Using Mathcad it is possible to obtain A_0, A_2 expressed in terms of m_1, m_2

$$A0 := -6.4000000000000000000 \cdot 10^{-8} \cdot \frac{\left(3044855 \cdot m1^3 - 115190 \cdot m1^2 \cdot m2 + 616308 \cdot m1^4 \cdot m2 + 1232616 \cdot m1^3 \cdot m2^2 + 616308 \cdot m1^2 \cdot m2^3 - 11843750 \cdot m1 + 34771875 \cdot m2 - 4700815 \cdot m1 \cdot m2^2 + 1540770 \cdot m2^3\right)}{\left(5. + 4. \cdot m1 \cdot m2 + 2. \cdot m2^2\right)}$$

$$A2 := 3.2000000000000000000 \cdot 10^{-6} \cdot \frac{\left(4585625 \cdot m1 + 13908750 \cdot m2 + 1848924 \cdot m1^2 \cdot m2 + 1848924 \cdot m1 \cdot m2^2 + 616308 \cdot m2^3\right)}{\left(5. + 4. \cdot m1 \cdot m2 + 2. \cdot m2^2\right)}$$

Finally to find values of m_1, m_2 it is necessary to solve below system of nonlinear equations obtained from Stefan condition.

$$\left.\frac{\partial u}{\partial x}\right|_{x=y_k(t_k)} = 2A_2 y_k(t_k)\beta_{2,0} + A_4\left(4\left(y_k(t_k)\right)^3\beta_{4,0} + 2\left(y_k(t_k)\right)t_k\beta_{4,1}\right) + A_1\beta_{1,0} + A_3\left(3\left(y_k(t_k)\right)^2\beta_{3,0} + t_k\beta_{3,1}\right) = -m_k$$

Thus $m_1 = 0.723$, $m_2 = 0.926$ implies $A_0 = -0.758$ and $A_2 = -14.674$

Conclusion:

Even though only two values were taken for time *t*, it is possible to see that values of m_1, m_2 are close to 1, as it was expected from exact solution.

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