Süleyman Demirel University
Engineering Faculty

Department of Natural Sciences, Mathematics and Informatics

H.N.Aliyev

Calculus and analytic geometry

Solved problems and exercises. Part 1.

Almaty – 2005

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ББК 22.151
А 36

Рекомендовано к печати Ученым Советом Университета имени Сулеймана Демиреля.

Aliyev H.N..
A 36 Calculus and analytic geometry. Solved problems and exercises. Part 1. – Алматы, 2005 - 244 с.

ISBN № 9965-9605-1-8

ББК 22.151

А $\frac{1602050000}{00\ (05)\text{-}05}$

ISBN № 9965-9605-1-8

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Chapter 7. Inverse trigonometric and hyperbolic functions

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To the students

There are a few lucky students who seem to learn even the hardest math almost effortlessly. The rest of us can only envy them and try to pick their brains. You are like the majority of us who cannot learn math without working hard at it. Do not fool yourself into thinking that you can get by without working at it. You will only get yourself into more trouble than you climb out of by mid-semester.

Tip 1: Do the homework exercises. Many professors do not require you hand in the homework. The homework is for your benefit, not the professor's. You cannot learn to play the piano without endlessly practicing scales. You cannot make the football team without endlessly running windsprints. You cannot learn to paint without endlessly painting still lives. Math is no different. The exercises will train your mind and sharpen your intuition. So do the work. It will pay off in the end.

Tip 2: Math books are meant to be read slowly. You cannot speed- read it and expect to get any benefit out of it at all. When you encounter a new concept in a math book, do not expect to understand it on the first reading, no matter how carefully you read it. You should go over each difficult paragraph several times. If you are still uncomfortable with it, read ahead a page or so, then come back to the difficult passage. And remember that math books are meant to be read *with paper and pencil in hand.*

Tip 3: Always use a pencil to do math (and exams). Don't ever try to do math in ink. You will make mistakes. Everybody does. So be equipped to clean them up. If you like mechanical pencils, great. If you prefer the old wooden kind, then sharpen several of them before you start each homework. Make sure you have a clean, usable eraser as well.

Although neatness might not get you extra points, it does help keep you from confusion. Keep your work organized. Skip a line (or even two) between each row of written calculations. You will be surprised at how much easier it will be for you to follow your own work when it's not so densely packed onto the page. Paper is cheap. Do not be afraid to use lots of it.

Tip 4: Your greatest assets are in the class with you. Your classmates are in the same boat as you. Organize a study group. Try to coax at least one of the top students in the class into your group. I recommend that the group size be three to five. Try to meet at least once per week. You will be working together on homework and comparing your lecture notes.

Tip5: You will be tested as an individual. Despite the helpfulness of your group activities, in the end your grade will be based upon your individual performance at solving problems. Following your group get-togethers, be sure to go solo on a few exercises.

Tip 6: Try to see more than just procedures. Again I urge you, learn the concepts, and the procedures will seem obvious. And try to have some fun with it. Humanity invented math largely because it is fascinating.

As for this book, it was written with the aim to help students to understand how to solve problems. The chapters are divided into sections. Each section contains necessary theoretical background and solved problems. This book also contains a lot of exercises, which would be useful to strength your understanding of sections. Answers are right after exercises.

In the end, any suggestions from readers would be greatly appreciated.

Chapter 1.
Coordinates, lines, functions, graphs.

1.1. Rectangular coordinate systems.

A rectangular coordinate system (also called a Cartesian coordinate system) is a pair of perpendicular coordinate lines, called coordinate axes, which are placed so that they intersect at their origin.
The horizontal axis is usually called x-axis and the vertical axis is called y-axes.(Fig. 1.1).
A plane in which a rectangular coordinate system has been introduced is called a coordinate plane.
Every point P in a coordinate plane can be associated with a unique ordered pair of real numbers (a, b) .

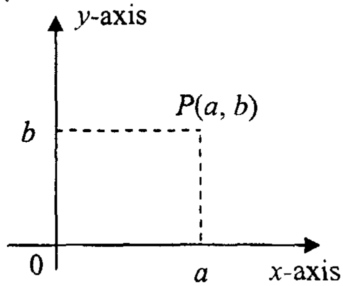


Fig.1.1.

The number a is called the x- coordinate or abscissa of P and the number b is called the y-coordinate or ordinate of P.
In a rectangular coordinate system the coordinate axes divide the plane into four regions called quadrants. These are numbered counter-clockwise with roman numerals as shown in Fig.1.2. It is easy to determine the quadrant in which a given point lies from the signs of its coordinates: a point with two positive coordinates (+, +) lies in quadrant I, a point with a negative x – coordinate and positive y – coordinate (–,+) lies in quadrant II, and so on. Points with a zero x – coordinate lie on the – y axis and points with a zero y – coordinate lie on the x – axis.

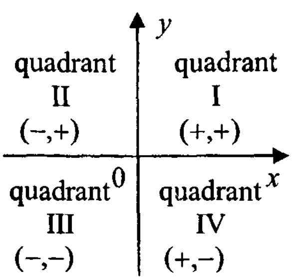


Fig. 1.2

1.2. Increments.

Increments are net changes. When a particle moves from (x_1, y_1) to (x_2, y_2) the increments in its coordinates are $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ (Fig.1.3.).

Example:

From A(2,5) to B(7,2) the increments are $\Delta x = 7 - 2 = 5$ and $\Delta y = 2 - 5 = -3$.

Example:

A particle starts at A(-5,6) and its coordinates change by increments $\Delta x = 7$ and $\Delta y = -4$. Find its new position.

Solution:

From $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ using $x_1 = -5$ and $y_1=6$ we get

$x_2 - (-5) = 7$ and

$y_2 - 6 = -4$. So particle's new position is $x_2 = 2$ and $y_2 = 2$.

1.3. Slopes of nonvertical lines.

If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points on a nonvertical line (Fig. 1.3), then slope m of the line is defined by:

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

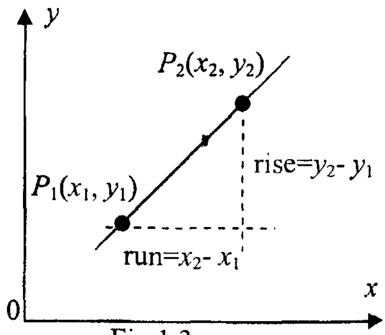


Fig.1.3

Example:

Find the slope of the line through the points (2,9) and (4,3).

Solution:

$$m = \frac{3-9}{4-2} = \frac{-6}{2} = -3$$

Example:

Find the slope of the line through the points (6,2) and (9,8).

Solution:

$$m=\frac{8-2}{9-6}=\frac{6}{3}=2$$

m is sometimes called the rate of change of y with respect to x along the line.

1.4. Lines that are parallel or perpendicular.

Let L_1 and L_2 be nonvertical lines with slopes m_1 and m_2 respectively.

a) The lines are parallel if and only if $m_1 = m_2$.

b) The lines are perpendicular if and only if $m_1 \cdot m_2 = -1$.

Example:

Use slopes to show that the points A (1, 3), B (3, 7) and C (7, 5) are vertices of right triangle.

Solution:

We shall show that the line through A and B is perpendicular to the line through B and C. The slopes of these lines are

m_1= (7-3)/ (3-1) =2 and m_2= (5-7)/ (7-3) =-1/2

Since $m_1 \cdot m_2$=-1, the line through A and B is perpendicular to the line through B and C; thus, ABC is a right triangle.

1.5. Equations for lines.

a) Horizontal and vertical lines.

The vertical line through $(a,0)$ and the horizontal line through $(0,b)$ are represented, respectively, by the equations

$$x=a \text{ and } y=b$$

Example:

Find an equation for a) the vertical line and b) the horizontal line through (-5, 7).

Solution:

a) The graph of $x=-5$ is the vertical line and

b) The graph of $y=7$ is the horizontal line through (-5, 7).

b) Point- slope equation for lines.

The line passing through $P_1(x_1, y_1)$ and having slope m is given by the equation:

$$(1) \qquad y - y_1 = m(x - x_1)$$

This is called the point- slope equation for the line.

Example:

Find the point slope equation of the line through (4,-3) with slope 5.

Solution:

Substituting the values $x_1 = 4$, $y_1 = -3$ and m=5 in (1) yields the point- slope equation:

$$y + 3 = 5(x - 4) \text{ or } y = 5x - 23.$$

Example:

Write an equation for the line that passes through (-2, 1) and (2, 2).

Solution:

First of all we calculate the slope and then use equation (1).

$$m = (2-1)/(2-(-2)) = 1/4$$

The (x_1, y_1) in equation (1) can be either (-2, 1) or (2, 2).

Let (x_1, y_1) be (-2, 1).

Then

$$y - 1 = \frac{1}{4}(x - (-2))$$

$$y = \frac{x}{4} + \frac{3}{2}$$

Check for yourself that with $(x_1, y_1) = (2,2)$ the equation is the same.

c) Lines determined by slope and *y*-intercept.

The line with y intercept b and slope m is given by the equation

$$(2) \qquad y = mx + b$$

This is called the slope-intercept form of the line.

Example:

Find the slope intercept form of the equation of the line that satisfies the stated conditions:

a) slope is –9; crosses the y-axis at (0, -4);

b) slope is 1; passes through the origin;

c) passes through (-5, 1); perpendicular to y=3x+4.

Solution:

a) From the given conditions we have $m = -9$; $b = -4$, so equation (2) yields $y = -9x - 4$.

b) From the given conditions $m = 1$ and the line passes through $(0,0)$, so b=0.

Thus, it follows from (2) that y=x.

c) The given line $y = 3x + 4$ has a slope m=3, so the line to be determined will have slope (–1)/3. Substituting this slope and given point in the point slope form (2):

$$y - 1 = -\frac{1}{3}(x + 5)$$

and simplifying yields

$$y = -\frac{x}{3} - \frac{2}{3}.$$

Example:

Find equations for the lines through P (-2, 2) that are

a) parallel and

b) perpendicular to the line $2x+y=4$.

Solution:

a) We shall first write equation $2x + y = 4$ in the form

$$y = -2x + 4.$$

This line has slope $m = -2$. Parallel line will have same slope. So using $m = -2$ and $P(-2,2)$ from point-slope form we will have

$$y - 2 = -2(x + 2) \text{ or } y = -2x - 2.$$

b) Two lines are perpendicular if $m_1 \cdot m_2 = -1$. So if $m_1 = -2$ then m_2=1/2. Again using point-slope form we will get

$$y - 2 = \frac{1}{2}(x + 2) \text{ or } y = \frac{x}{2} + 3.$$

Exercises.

In exercises 1-2, a particle moves from A to B. Find the net changes Δx and Δy. in the particles coordinates.

1. A (-5, 11), B (3, 6)
2. A ($\sqrt{6}$, 2.7), B $(0, -1.3)$
3. A particle starts at A (-2, 3) and its coordinates change by increments Δx=5, Δy=-6. Find its new position.

4. The coordinates of a particular change by $\Delta x=3$, $\Delta y=-5$ as it moves from A (x, y) to B (-6, 4). Find x and y.
5. Find the slope of the line through
a) (-1, 2) and (3, 4)
b) (5, 3) and (7, 1)
c) $(4, \sqrt{2})$ and $(-3, \sqrt{2})$
d) (-2,-6) and (-2, 12)
6. Let points A (8, 1), B (2, 10), C (-4, 6), D (2, -3) be given. Determine whether the line through AB is perpendicular or parallel to the line through CD.

In exercises 7-8 find an equation for **a)** the vertical line and **b)** the horizontal line through the given point.

7. (-2, 3/2) 8. $(-3\pi, 5)$

In exercises 9-10, write an equation for the line through P with slope m.

9. P (-2, 3), $m=2$ 10. P $(\pi, 0)$, $m=-3$

In exercises 11-12, write an equation for the line through the two points.

11. (2, 4), (1, -7) 12. (-3, 6), (-2, 1)

In exercises 13-14 write an equation for the line with slope m and y-intercept b.

13. $m=-2/3$, $b=\sqrt{3}$
14. $m=2$, $b=3.5$
15. Find equations for the lines through P (1, 2) that are **a)** parallel and **b)** perpendicular to the line $x+2y=3$.
16. Find equation for the line which is parallel to $y=4x-2$ and its y-intercept is 7.
17. Find equation for the line which is perpendicular to $y=5x+9$ and has y-intercept 6.
18. For what value of k the line $3x + k \cdot y = 4$ will

a) have slope 2
b) have y-intercept 5
c) pass through the point $(-2,4)$
d) be parallel to the line $2x-5y=1$
e) be perpendicular to the line $4x+3y=2$?

Answers.

1. Δx=8, Δy= -5; **2.** Δx=-$\sqrt{6}$, Δy=-4; **3.** (3, -3); **4.** (-9, 9); **5.** a) 1/2; b)-1; c) 0; d) not defined; **7.** a) x=-2 b) y=3/2; **8.** a) x=-3π; b) $y = 5$;
9. $y = 2x + 7$; **10.** y=-3x+3π; **11.** y=11x-18; **12.** y=-5x-9;
13. y=-2x/3+$\sqrt{3}$; **14.** y=2x+3.5; **15.** a) y=-x/2+5/2; b) y=2x;
16. y=4x+7; **17.** y=-x/5+6; **18.** a)-3/2; b) 4/5; c) 5/2; d) –15/2; e) –4.

1.6. The properties of inequalities.

1. If $a<b$ and $c<d$, then $a+c< b+d$
2. If $a<b$ and c is any number, then $a+c< b+c$
3. If $a<b$ and c is any number, then $a-c< b-c$
4. If $a<b$ and c is a positive number, then $a \cdot c < b \cdot c$
5. If $a<b$ and c is a negative number, then $a \cdot c > b \cdot c$
6. If $a<b$ and $c<d$ and a, b, c, and d are positive, then $a \cdot c < b \cdot d$
7. If $a<b$ and c is a positive number, then $a/c< b/c$
8. If $a<b$ and c is a negative number, then $a/c> b/c$
9. If a and b are positive numbers and $a< b$, then $1/a> 1/b$.

All 1-9 properties hold for $\leq$ as well as for $<$.

1.7. The absolute value function.

The absolute value or magnitude of a real number a is denoted by $|a|$ and is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Example:

$|5|=5; \quad |-6/7|=6/7; \quad |0|=0$

Remark:

Inequality $|x| \leq a$, ($a>0$) is equivalent to the inequality

$$-a \leq x \leq a$$

1.8. Properties of absolute value.

If a and b are real numbers, then

1. $|-a|=|a|$
2. $|a \cdot b|=|a| \cdot |b|$
3. $|a/b|=|a|/|b|$; $b \neq 0$
4. $|a+b| \leq |a|+|b|$
5. $|a-b| \geq |a|-|b|$

To solve an equation or inequality that contains absolute values, we write equivalent equation or inequality and then solve as usual.

Example:

Solve: $|x-3|< 4$

Solution:

This inequality can be rewritten as

$-4 < x-3 < 4$ or, on adding 3 throughout, $-1 < x < 7$.

This can be written in interval notation as (-1, 7).

Example:

Solve: $|5(x-1)|< 3$

Solution:

$|5(x-1)|< 3$ given.

$5 \cdot |(x-1)|< 3$ absolute value of product

$|(x-1)|< 3/5$ dividing by 5

$-3/5<(x-1) < 3/5$ $(x-1)$ has absolute value less than 3/5

$-3/5+1<x < 3/5+1$ adding 1 to an inequality

$2/5<x < 8/5$ arithmetic.

In short, x is in open interval (2/5, 8/5).

Example:

Solve: $\dfrac{1}{|2x-3|} > 5$

Solution:

First of all, we see that $x=3/2$ is not a solution because this value of x results in a division by zero. Let's keep it in mind.

$\dfrac{1}{|2x-3|} > 5$ given

$|2x-3| < 1/5$ taking reciprocals

$\|2(x-3/2)\|< 1/5$	factor out the coefficients of x
$\|2\|\cdot\|x-3/2\|< 1/5$	absolute value of product
$\|x-3/2\|< 1/10$	dividing by 2
$-1/10<x-3/2< 1/10$	$(x-3/2)$ has abs. value less than 1/10
$7/5<x< 8/5$	we added 3/2 throughout.

If, as noted above, we eliminate the value $x=3/2$ to avoid the division by zero, we see that the solution consists of all x that satisfy

$$7/5< x < 3/2 \quad \text{or} \quad 3/2< x < 8/5$$

The solution set consists of all x in the set $(7/5, 3/2)\cup(3/2, 8/5)$.

Example:

Solve the equation $|x-3|= 4$

Solution:

Depending on whether $(x-3)$ is positive or negative, the equation $|x-3|= 4$ can be written as:

$$x-3=4 \quad \text{or} \quad x-3=-4$$

We obtain that equation has two roots $x=7$ and $x=-1$.

Example:

Solve the equation $|x-1|+|x-2|= 1$

Solution:

Let us consider these cases:

1) $x< 1$; 2) $1\le x\le 2$; 3) $x> 2$

1) In this case $|x-1|=-(x-1)$ and $|x-2|=-(x-2)$.
So given equation is equivalent to

$$-(x-1)-(x-2)=1 \quad \text{or} \quad -2x+2=0 \quad \text{and} \quad x=1.$$

Since in this case $x< 1$, and $x=1$ can not be a root of equation.

2) In this case $|x-1|=(x-1)$ and $|x-2|=-(x-2)$.
Given equation is equivalent to

$$x-1-(x-2)=1$$

It leads us to the identity, so any x from $[1, 2]$ satisfies the given equation.

3) In this case $|x-1|=(x-1)$ and $|x-2|=(x-2)$.
Given equation is equivalent to

$$x-1+x-2=1$$
$$2x=4 \quad \text{and} \quad x=2.$$

It contradicts to the condition that $x>2$.

So as a result, the root of the equation is any x from $[1, 2]$.

Exercises.

In exercises 1-11 solve the inequalities.

1. $\|x-3\|< 4$	2. $\|x+2\|\le 5$
3. $\|x-4\|> 7$	4. $x^2\le 25$
5. $x^2\ge 16$	6. $\|x+6\|<3$
7. $\|2x-3\|\le 6$	8. $\|x+2\|\ge 1$
9. $\|5-2x\|\ge 4$	10. $\frac{1}{\|x-1\|}\le 2$
11. $\frac{3}{\|2x-1\|}\ge 4$	

In exercises 12-19 solve the equations.

12. $\|x-5\|= 4$	13. $\|2x+5\|= 1$
14. $\|x+1\|-\|x-4\|= 6$	15. $\|s/2-2\|= 3$
16. $\|6x-2\|=7$	17. $\|6x-7\|=\|3+2x\|$
18. $\|9x\|-11=x$	19. $\left\|\frac{x+5}{2-x}\right\|=6$

Answers.

1. $-1< x < 7$; **2.** $-7\le x\le 3$; **3.** $x< -3$ and $x> 11$; **4.** $-5\le x \le 5$; **5.** $x\le -4$ and $x\ge 4$; **6.** $(-9,-3)$; **7.** $\left[-\frac{3}{2},\frac{9}{2}\right]$; **8.** $(-\infty,-3]\cup[-1,+\infty)$; **9.** $(-\infty,\frac{1}{2}]\cup[\frac{9}{2},+\infty)$ **10.** $\left(-\infty,\frac{1}{2}\right]\cup\left[\frac{3}{2},+\infty\right)$; **11.** $[\frac{1}{8},\frac{1}{2})\cup(\frac{1}{2},\frac{7}{8})$; **12.** $x_1=1$; $x_2=9$; **13.** $x_1=-2$; $x_2=-3$; **14.** $x=4.5$; **15.** $s_1=10$; $s_2=-2$; **16.** $-\frac{5}{6};\frac{3}{2}$; **17.** $\frac{1}{2};\frac{5}{2}$; **18.** $-\frac{11}{10};\frac{11}{8}$; **19.** $1;\frac{17}{5}$.

1.9. Distance formula for points in the plane.

The distance between two points (x_1, y_1) and (x_2, y_2) in a coordinate plane is given by:

$$(1) \qquad d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Example:

Find the distance between the points (-2, 3) and (1, 7)

Solution:

If we let (x_1, y_1) be (-2, 3) and let (x_2, y_2) be (1, 7) then (1) yields

$$d=\sqrt{(1-(-2))^2+(7-3)^2}=\sqrt{3^2+4^2}=5.$$

Remark: When using (1) it does not matter which point is labeled (x_1, y_1) and which is labeled (x_2, y_2). Thus, in example above, if we had let (x_1, y_1) be (1, 7) and (x_2, y_2) be (-2, 3) we would obtained

$$d=\sqrt{(-2-1)^2+(3-7)^2}=\sqrt{(-3)^2+(-4)^2}=5$$

which is the same result we obtained with the opposite labeling.

1.10. Division of line segment into the given ratio.

The coordinates of the point $M(x, y)$ which divides the line segment M_1M_2 with $M_1(x_1,y_1)$ and $M_2(x_2,y_2)$ into the ratio $\frac{M_1M}{MM_2}=l$ (where $l\neq -1$) are defined by

$$(1) \qquad x=\frac{x_1+lx_2}{1+l}, \qquad y=\frac{y_1+ly_2}{1+l}$$

Example:

Find the coordinates of the point $C(x,y)$ which divides the line segment AB between points $A(-4,-2)$ and $B(2,-8)$ into the ratio $\frac{AC}{CB}=\frac{3}{2}$

Solution: $l=\frac{3}{2}$ and using (1) yields

$$x=\frac{-4+\frac{3}{2}\cdot 2}{1+\frac{3}{2}}=-\frac{2}{5};\quad y=\frac{-2+\frac{3}{2}\cdot(-8)}{1+\frac{3}{2}}=-\frac{28}{5}$$

Hence $C(x,y)=\left(-\frac{2}{5};-\frac{28}{5}\right)$.

1.11. The midpoint formula.

The midpoint of the line segment joining two points (x_1,y_1) and (x_2,y_2) in a coordinate plane is

(2) $$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

Example:

Find the midpoint of the line segment joining $(3,-5)$ and $(4,7)$.

Solution:

From $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$, the midpoint is

$$\left(\frac{3+4}{2},\frac{-5+7}{2}\right)=(3.5,1)$$

1.12. Area of the triangle.

Area of the triangle ABC with vertices $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ is defined by

(3) $$S=\frac{1}{2}\left|(x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)\right|$$

Example:

Find the area of the triangle with vertices $A(-3,-1)$, $B(2,5)$, $C(-1,4)$.

Solution: From (3)

$$S = \frac{1}{2}\left|(2-(-3))(4-(-1)) - (-\text{[illegible]} - (-3))(5-(-1))\right| =$$

$$= \frac{1}{2}|5\cdot 5 - 2\cdot 6| = \frac{1}{2}\cdot 13 = \frac{13}{2}\text{ (unit square)}$$

Example:

Find the length of the medians (A median of a triangle is a line that passes through a vertex and the midpoint of the opposite edge) of a triangle PQR with vertices $P(-3,2)$, $Q(5,4)$, $R(7,-2)$.

Solution: Let L, M, N be midpoints of the sides $PQ, QR,$ and PR, respectively. Coordinates of the point L can be found using (2). In this case $x_1 = -3$, $y_1 = 2$, $x_2 = 5$, $y_2 = 4$. If we will denote coordinates of the point L by (x_L, y_L) we would get

$$x_L = \frac{-3+5}{2} = 1, \qquad y_L = \frac{2+4}{2} = 3, \qquad L(1,3)$$

Similarly

$$x_M = \frac{5+7}{2} = 6, \qquad y_M = \frac{4+(-2)}{2} = 1, \qquad M(6,1)$$

$$x_N = \frac{-3+7}{2} = 2, \qquad y_N = \frac{2+(-2)}{2} = 0, \qquad N(2,0)$$

Now let us find the length of median LR. Using distance formula with $x_1 = 1$, $y_1 = 3$, $x_2 = 7$, $y_2 = -2$ we obtain

$$LR = \sqrt{(7-1)^2 + (-2-3)^2} = \sqrt{61}$$

Similarly

$$MP = \sqrt{(6-(-3)^2 + (1-2)^2} = \sqrt{82}$$

$$NQ = \sqrt{(5-2)^2 + (4-0)^2} = 5.$$

Exercises.

In exercises 1-4 find the distance between A and B.

1. A (2, 5), B (-1, 1)
2. A (7, 1), B (1, 9)
3. A (2, 0), B (-3, 6)
4. A (-2, -6), B (-7, -4)

5. The point C divides the line segment AB with $A(2,5)$ and $B(4,8)$ into the ratio $2 \div 3$. Find the coordinates of the point C.
6. The point $C(2,3)$ divides AB into the ratio $1 \div 2$. Find the coordinates of the point B if the coordinates of the point A are $x = 1,\ y = 2$.
7. Find area of the triangle ABC with vertices $A(0,-2)$, $B(4,5)$, and $C(6,-4)$.
8. Let $A(2,1)$, $B(-2,-2)$, and $C(-8,6)$ be vertices of the triangle ABC. Find the height of the triangle through the vertex B.
9. Prove that the triangle with vertices (5, -2), (6, 5), (2, 2) is isosceles.
10. Prove that for all values of t the point (t, $2t$-6) is equidistant from (0, 4) and (8, 0).
11. Prove that points (0, -2), (-4, 8) and (3, 1) lie on a circle with center (-2, 3).

Answers.

1. 5; **2.** 10; **3.** $\sqrt{61}$; **4.** $\sqrt{29}$; **5.** $C(2\frac{4}{5}, 6\frac{1}{5})$; **6.** $B(4,5)$;

7. 25 unit sq. ; **8.** $2\sqrt{5}$.

1.13. Functions.

Definition: Let X and Y be sets. A function from X to Y is a rule (or method) for assigning one (and only one) element in Y to each element in X.

Definition: (Domain and range). Let X and Y be sets. Let f be function from X to Y. The set X is called the domain of the function. The set of all outputs of then function is called the range of the function.

Example:

Find the domain and range of $f(x) = 3 + \sqrt{x-1}$.

Solution:

For $3 + \sqrt{x-1}$ to be meaningful, the square root of $(x-1)$

must make sense; thus, the domain consists of all numbers x such that $x-1\geq 0$ or $x\geq 1$. That is, the domain is the interval $[1,+\infty)$.
As x varies from 1 to larger numbers, $f(x)$ increases from $f(1)=3+\sqrt{1-1}=3$ to arbitrary large values.
Thus, the range of f is $[3,+\infty)$.

Example:

Find the domain and range of the function $f(x)=\dfrac{x+1}{x-1}$.

Solution:

The domain of f consists of all x, except $x=1$. In interval notation the domain is $(-\infty,1)\cup(1,+\infty)$.
To find range of the function, let us introduce a dependent variable

$$y=\frac{x+1}{x-1}$$

Solving this equation for x in terms of y yields

$$(x-1)y=x+1 \Rightarrow \qquad x=\frac{y+1}{y-1}$$

It is now evident from the right side of this equation that $y=1$ is not in the range. So the range of the function f is $(-\infty,1)\cup(1,+\infty)$.

1.14. Operations on functions.

Definition: Given functions f and g, formulas for the sum $f+g$, difference $f-g$, product $f\cdot g$ and quotient f/g are defined by

$$(f+g)(x)=f(x)+g(x)$$
$$(f-g)(x)=f(x)-g(x)$$
$$(f\cdot g)(x)=f(x)\cdot g(x)$$
$$(f/g)(x)=f(x)/g(x)$$

Example: Let $f(x)=2+\sqrt{x-3}$ and $g(x)=x-2$.
Find $(f+g)(x),(f-g)(x),(f\cdot g)(x),(f/g)(x)$

Solution:

$$(f+g)(x)=f(x)+g(x)=(2+\sqrt{x-3})+(x-2)=x+\sqrt{x-3}$$

$$(f-g)(x)=f(x)-g(x)=(2+\sqrt{x-3})-(x-2)=$$

$$=\sqrt{x-3}-x+4$$

$$(f\cdot g)(x)=f(x)\cdot g(x)=(2+\sqrt{x-3})\cdot(x-2)=$$

$$=x\sqrt{x-3}-2\sqrt{x-3}+2x-4$$

$$(f/g)(x)=f(x)/g(x)=\frac{2+\sqrt{x-3}}{x-2}$$

Definition: Given functions f and g ,the composition of f with g , denoted by $f\circ g$, is the function defined by

$$(f\circ g)=f(g(x))$$

($f\circ g$ is read as " f circle g " or as " f composed with g ").

Example:

Let $f(x)=x^2$ and $g(x)=x+3$. Find a) $f(g(5))$ and $g(f(5))$.

Solution:

$$f(g(x))=f(x+3)=(x+3)^2$$

$$g(f(x))=g(x^2)=x^2+3,$$

So

$$f(g(5))=f(8)=64 \text{ and } g(f(5))=g(25)=28$$

Exercises.

1. Given that $f(x)=3x^2+2$. Find

a) $f(-2)$; b) $f(4)$; c) $f(0)$; d) $f(-\sqrt{3})$; e) $f(a+1)$

2. Given that $f(x)=\begin{cases}\frac{1}{x} & ,x>3\\ 2x & ,x\le 3\end{cases}$. Find

a) $f(-4)$; b) $f(4)$; c) $f(0)$; d) $f(3)$; e) $f(t^2+5)$

In exercises 3-8 find domain of the function

3. $f(x)=\frac{1}{x-3}$ 4. $h(x)=\sqrt{\frac{x-1}{x+2}}$

5. $f(x)=\frac{x}{|x|+1}$ 6. $F(x)=3\sqrt{x}-\sqrt{x^2-4}$

7. $G(x)=\sqrt{x^2-2x+5}$ 8. $g(x)=\sin\sqrt{x}$

In exercises 9-16 find the domain and the range of the given function.

9. $f(x)=\sqrt{3-x}$ 10. $g(x)=\sqrt{4-x^2}$

11. $h(x)=3+\sqrt{x}$ 12. $F(x)=x^2+3$

13. $H(x)=3\sin x$ 14. $g(x)=2+\cos x$

15. $f(x)=\frac{1}{x+1}$ 16. $g(x)=\frac{1}{1-x^2}$

17. Given that $f(-1)=4$, $f(2)=5$, $g(-1)=-3$, and $g(2)=-1$. Find

a) $(f-g)(-1)$ b) $(f\cdot g)(-1)$

c) $(f/g)(2)$ d) $(f\circ g)(2)$

In exercises 18-19 find formulas for

a) $(f+g)(x)$ b) $(f-g)(x)$ c) $(f\cdot g)(x)$

d) $(f/g)(x)$ e) $(f\circ g)(x)$ f) $(g\circ f)(x)$

18. $f(x)=\sqrt{x+1}$, $g(x)=x-2$

19. $f(x)=\sqrt{1-x^2}$, $g(x)=\sin 3x$

20. Let $f(x)=\frac{4}{x^2+5}$ and $g(x)=\sqrt{x}$. Find $(f\circ g)(x)$ and $(g\circ f)(x)$.

In exercises 21-22 find $\frac{f(x+h)-f(x)}{h}$ and simplify as much as possible.

21. $f(x)=3x^2-5$ 22. $f(x)=\frac{1}{x}$

Answers.

1. a) 14; b) 50; c) 2; d) 11; e) $3a^2+6a+5$; **2.** a) -8; b) 1/4; c) 0; d) 6; e) $\frac{1}{t^2+5}$; **3.** $(-\infty,3)\cup(3,+\infty)$; **4.** $(-\infty,-2)\cup[1,+\infty)$;

5. $(-\infty,+\infty)$; **6.** $[2,+\infty)$; **7.** $(-\infty,+\infty)$; **8.** $[0,+\infty)$;

9. domain $(-\infty,3]$, range $[0,+\infty)$; **10.** domain $[-2,2]$, range $[0,2]$; **11.** domain $[0,+\infty)$, range $[3,+\infty)$; **12.** domain $(-\infty,+\infty)$, range $[3,+\infty)$; **13.** domain $(-\infty,+\infty)$, range $[-3,3]$; **14.** domain $(-\infty,+\infty)$, range $[1,3]$; **15.** domain $(-\infty,-1)\cup(-1,+\infty)$, range $(-\infty,0)\cup(0,+\infty)$; **16.** domain all real x except $x=\pm1$; range $(-\infty,0)\cup[1,+\infty)$; **17.** a) 7; b) -12; c) -5; d) 4; **18.** a) $\sqrt{x+1}+x-2$; b) $\sqrt{x+1}-x+2$; c) $(x-2)\sqrt{x+1}$; d) $\frac{\sqrt{x+1}}{x-2}$; e) $\sqrt{x-1}$; f) $\sqrt{x+1}-2$; **19.** a) $\sqrt{1-x^2}+\sin 3x$; b) $\sqrt{1-x^2}-\sin 3x$; c) $\sqrt{1-x^2}\cdot\sin 3x$; d) $\sqrt{1-x^2}/\sin 3x$; e) $|\cos 3x|$; f) $\sin(3\sqrt{1-x^2})$; **20.** $(f\circ g)(x)=\frac{4}{x+5},\quad x\ge0$; $(g\circ f)(x)=\frac{2}{\sqrt{x^2+5}}$;

21. $6x+3h,\quad h\neq0$; **22.** $-\frac{1}{x(x+h)},\ h\neq0$.

1.15. Symmetry of odd and even functions.

Definition:

A function f such that $f(-x)=f(x)$ is called an **even function.**

Example: $f(x)=\frac{x^4}{(1-x^2)}$ is an even function, since

$$f(-x)=\frac{(-x)^4}{(1-(-x)^2)}=\frac{x^4}{(1-x^2)}=f(x)$$

The graph of even function is symmetric with respect to the y-axis.

Definition:

A function f such that $f(-x)=-f(x)$ is called an **odd function.**

Example: The function $f(x)=x^3$ is odd function, since

$$f(-x)=(-x)^3=-f(x)$$

The graph of odd function is symmetric with respect to the origin.

Most functions are neither even nor odd. For instance x^3+x^6 is neither even nor odd, since $(-x)^3+(-x)^6=-x^3+x^6$, which is neither x^3+x^6 nor $-(x^3+x^6)$.

1.16. Graphs and graphing.

The points (x,y) in the plane whose coordinates are the input-output pairs of function $y=f(x)$ make up the function's graph.

Example: Sketch the graph of $f(x)=x+4$.

Solution:

By definition the graph of f is input-output pairs of $f(x)=x+4$; this is a line with slope 1 and y-intercept 4. (Fig.1.4)

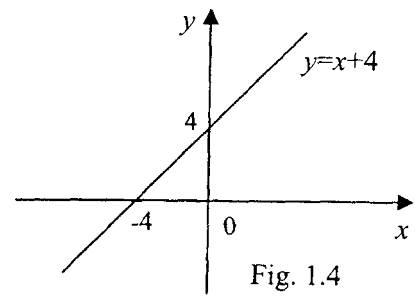


Fig. 1.4

Example:

Sketch the graph of $f(x)=|x|$

Solution:

$$y=f(x)=\begin{cases}x & ,x\ge0\\ -x & ,x<0\end{cases}$$

The graph coincides with the line $y=x$ for $x\ge0$ and with the line $y=-x$ for $x<0$. (Fig.1.5)

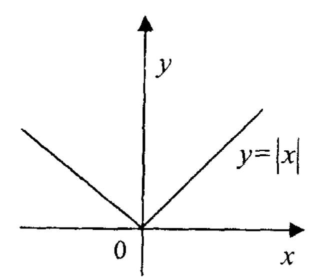


Fig. 1.5

Example:

Sketch $y = f(x) = \begin{cases} 1 & , x \le 2 \\ x+2 & , x > 2 \end{cases}$

Solution:

For $x \le 2$, we have $y = 1$, for $x > 2$ we have $y = x + 2$. The graph of $y = 1$ is a horizontal line, and the graph of $y = x + 2$ is a straight line. (Fig.1.6).In that figure we used the heavy dot and open circle above $x = 2$ to emphasize that the value $f(2) = 1$ lies on the horizontal line and not on the inclined line.

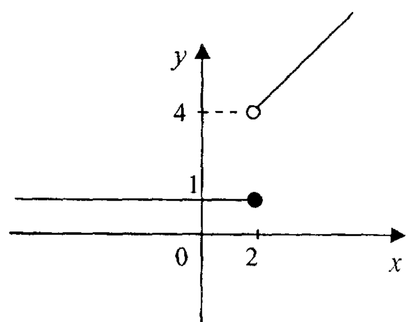


Fig. 1.6

1.17. Graphing functions by translation.

1) If a positive constant is added to $f(x)$, ($y = f(x) + C$), the geometric effect is to translate the graph of the function f parallel to the vertical axis in the positive direction. (Fig.1.7).

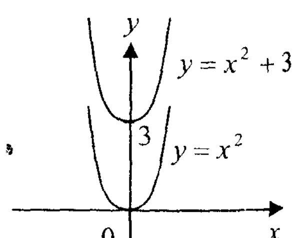


Fig.1.7

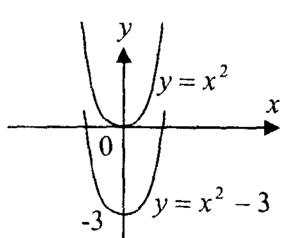


Fig.1.8

2) If a positive constant is subtracted from $f(x)$, ($y = f(x) - C$), translate the graph of the graph of the function f parallel to the vertical axis in the negative direction. (Fig.1.8).

) Similarly, if a positive constant is dded to the independent variable of a unction $f(x)$, ($y = f(x + C)$), the eometric effect is to translate the graph f the function parallel to the horizontal xis in the negative direction.(Fig.1.9).

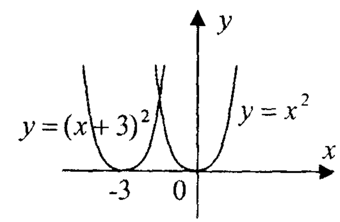


Fig.1.9

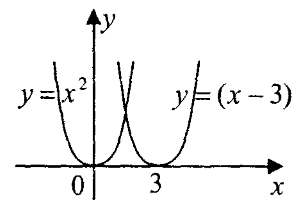


Fig.1.10

4) If a positive constant is subtracted from the independent variable of a function $f(x)$, ($y = f(x - C)$), translate the graph of the graph of the function f parallel to the horizontal axis in the positive direction. (Fig.1.10).

Exercises.

In exercises 1-10 label each function as even, odd, or neither.

1. $f(x) = x^2 + 2$
2. $f(x) = \sqrt{1 - x^2}$
3. $f(x) = x + x^3$
4. $f(x) = 3 + x$
5. $f(x) = (x + 2)^2$
6. $f(x) = \sqrt[5]{x}$
7. $f(x) = x + x^3 + 5x^4$
8. $f(x) = 7x^4 - 5x^2$
9. $f(x) = \sqrt[3]{x^2 + 1}$
10. $f(x) = x^2 + \frac{1}{x}$

In exercises 11-23 sketch the graph of the function.

11. $f(x) = 2x + 1$
12. $f(x) = x, \ 1 \le x \le 2$
13. $f(x) = x^2 - 2, \ -1 \le x \le 1$
14. $h(x) = x^2 - 5$
15. $h(x) = (x - 5)^2$
16. $F(x) = \sqrt{x + 1}$
17. $F(x) = \sqrt{3 - x}$
18. $f(x) = \frac{x^2 - 4}{x + 2}$

19. $f(x)=\dfrac{x^3-x^2}{x-1}$

20. $f(x)=\dfrac{x}{|x|}$

21. $f(x)=\begin{cases} x+2, x\le 3 \\ x+4, x>3 \end{cases}$

22. $h(x)=\begin{cases} 1, & 0<x\le 1 \\ 3, & 1<x\le 2 \\ -1, & 2<x\le 3 \\ 0 & ,elsewhere \end{cases}$

23. $f(x)=\begin{cases} x^2 & ,x>1 \\ 2 & ,x\le 1 \end{cases}$

24. Use the graph of $y=|x|$ to graph the following functions

a) $y=|x-4|$ b) $y=|x|+4$

c) $y=|x-4|+4$ d) $y=|x+6|-2$

Answers.

1. even; 2. even; 3. odd; 4. neither; 5. neither ; 6. odd ; 7.
8. even; 9. even; 10. neither.

Chapter 2.
The limit of a function.

2.1. Definition of limit.

A limit is the value a function tends to take as the limit variable approaches a specified value. For example, the limit of the function $f(x)$ as x approaches the value of zero would be written as:

$$\lim_{x \to 0} f(x)$$

The value will come as close to zero as possible without actually becoming zero. The limit variable (x in example above) could approach some value other than zero. Most limits will have their limit variable approaching zero but that is not necessary and any other value will work just as well.

As an example, consider a function such as $f(x) = 3x^2+2$, and substitute $x+\Delta x$ everywhere the variable x appears in the function to get the function $f(x+\Delta x)$.

$$f(x)=3x^2+2$$

$$f(x+\Delta x)=3(x+\Delta x)^2+2=3(x^2+2x\cdot\Delta x+(\Delta x)^2)+2=3x^2+6x\cdot\Delta x+3(\Delta x)^2+2$$

Then the limit of the function $f(x+\Delta x)$ as Δx approaches zero is written as:

$$\lim_{\Delta x \to 0}(f(x+\Delta x)) = \lim_{\Delta x \to 0}(3x^2+6x\cdot\Delta x+3\cdot(\Delta x)^2+2) =$$

$$=3x^2+6x\cdot 0+3\cdot(0)^2+2=3x^2+2$$

Obviously, in this example, the value of the limit is a function of x and a value can be computed if a value of x is specified. For $x=2$, the limit is neither 14, etc. You can find limit of a function as the limit variable approaches something other than zero. It could approach any other value within its range such as:

$$\lim_{\Delta x \to 14}(f(x+\Delta x)) = 3x^2+6x\cdot 14+3\cdot 14^2+2=3x^2+84x+590$$

This method of evaluating a limit does not work for all functions. For example if:

$$f(x)=\frac{x^2-16}{x-4}$$

then the limit of this function as x approaches the value of 4 cannot be computed by simply substituting for x. This function does not have a value for x=4 since we get a division of zero by zero by substituting 4 for x. This is defined as an **indeterminent result** since there is no real result defined for this type of division.

There are two possibilities here:

1. **Factor the equation.** In example above, the equation can be factored into:

$$f(x) = \frac{(x+4)(x-4)}{x-4} = x+4$$

Now we can substitute the value 4 for x to get the limit:

$$\lim_{x\to 4}(f(x)) = \lim_{x\to 4}(x+4) = 8.$$

This works most of the times but not always.

2. **Approach the value of the limit** from both directions and calculate the value of the function. If you take values for x that approach the value 4, you will get answers other than divided by zero. And by taking values for x approaching 4 from both directions, you start to narrow down the value of limit. It becomes apparent that as x approaches the value 4 from either direction, the limit of the function approaches the value of 8.

A limit can also result in a value of infinity. For example, if the function is

$$f(x) = \frac{3}{x}$$

then the limit of $f(x)$ when x approaches zero would be:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{3}{x}\right),$$

obviously will approach infinity as x becomes closer and closer to zero. It cannot be directly calculated but can be determined by observation of the fact that the function would be a division by zero at the limit conditions, which is by definition equal to infinity.

finition of limit: Let $f(x)$ be defined for all x in some open interval ntaining the number a, with the possible exception that $f(x)$ may not defined at a. We will write

$$\lim_{x\to a} f(x) = L$$

given any number $\varepsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x)-L| < \varepsilon \text{ if } x \text{ satisfies } 0 < |x-a| < \delta.$$

ample: Prove that $\lim_{x\to 2}(3x-5) = 1$

lution: We must show that given any positive number ε, we can find positive number δ such that

$$\left|\underbrace{(3x-5)}_{f(x)} - \underbrace{1}_{L}\right| < \varepsilon \text{ if } x \text{ satisfies } 0 < \left|x - \underbrace{2}_{a}\right| < \delta$$

ut this " if statement " can be rewritten as

$$|3x-6| < \varepsilon \quad \text{if} \quad 0 < |x-2| < \delta$$

$$3|x-2| < \varepsilon \quad \text{if} \quad 0 < |x-2| < \delta$$

$$|x-2| < \varepsilon/3 \quad \text{if} \quad 0 < |x-2| < \delta$$

ne choice of δ that makes the " if statement " true for any ε > 0 is = ε /3.The value δ= ε /3 is not the only value that will make " if atement " true. Any smaller δ will do as well.

2.2. Computations of limits.

Let f and g be two functions and assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}(k\cdot f(x)) = k\cdot \lim_{x\to a} f(x)$, for any constant k
4. $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$

6. $\lim_{x\to a}(f(x))^{g(x)} = \left(\lim_{x\to a} f(x)\right)^{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} f(x) > 0$

7. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ if $\lim_{x\to a} f(x) \geq 0$

Example: Find $\lim_{x\to 5}(x^2 - 4x + 3)$

Solution: $\lim_{x\to 5}(x^2 - 4x + 3) = \lim_{x\to 5} x^2 - \lim_{x\to 5} 4x + \lim_{x\to 5} 3 =$

$= \lim_{x\to 5} x^2 - 4\lim_{x\to 5} x + \lim_{x\to 5} 3 = 5^2 - 4\cdot 5 + 3 = 8$

Example: Find $\lim_{x\to 3}\frac{x^2 - 6x + 9}{x - 3}$

Solution: The numerator and denominator both have a limit of zero as x approaches 3, so there is a common factor of (x-3). We proceed as follows:

$$\lim_{x\to 3}\frac{x^2 - 6x + 9}{x - 3} = \lim_{x\to 3}\frac{(x-3)^2}{x - 3} = \lim_{x\to 3}(x - 3) = 0.$$

Example: Find $\lim_{x\to -4}\frac{2x + 8}{x^2 + x - 12}$

Solution: The numerator and denominator both have a limit of zero as x approaches -4, so there is a common factor of (x-(-4))=x+4. We proceed as follows:

$$\lim_{x\to -4}\frac{2x + 8}{x^2 + x - 12} = \lim_{x\to -4}\frac{2(x+4)}{(x+4)(x-3)} = \lim_{x\to -4}\frac{2}{x - 3} = -2/7.$$

Example: Find $\lim_{x\to 1}\frac{3x^2 - 4x + 7}{2x^2 - 5x + 6}$

Solution:

$$\lim_{x\to 1}\frac{3x^2 - 4x + 7}{2x^2 - 5x + 6} = \frac{\lim_{x\to 1}(3x^2 - 4x + 7)}{\lim_{x\to 1}(2x^2 - 5x + 6)} =$$

$$= \frac{\lim_{x\to 1} 3x^2 - \lim_{x\to 1} 4x + \lim_{x\to 1} 7}{\lim_{x\to 1} 2x^2 - \lim_{x\to 1} 5x + \lim_{x\to 1} 6} = \frac{3\cdot\lim_{x\to 1} x^2 - 4\lim_{x\to 1} x + \lim_{x\to 1} 7}{2\cdot\lim_{x\to 1} x^2 - 5\lim_{x\to 1} x + \lim_{x\to 1} 6} =$$

$$= \frac{3\cdot 1^2 - 4\cdot 1 + 7}{2\cdot 1^2 - 5\cdot 1 + 6} = 6/3 = 2.$$

Example:

Find $\lim\limits_{x \to 4} \frac{x^2-6x+8}{x-4}$

Solution: The numerator and denominator both have a limit of zero as x approaches 4. Let us rewrite x^2-$6x$+8 using $x^2+px+q=(x-x_1)(x-x_2)$ where x_1 and x_2 are roots of equation $x^2+px+q=0$. An equation $x^2-6x+8=0$ has roots $x_1=2$ and $x_2=4$, so $x^2-6x+8=(x-2)(x-4)$.
After substituting we get

$$\lim_{x \to 4} \frac{x^2-6x+8}{x-4} = \lim_{x \to 4} \frac{(x-2)(x-4)}{x-4} = \lim_{x \to 4}(x-2) = 4-2=2.$$

Example:

Find $\lim\limits_{x \to 1} \frac{3x^2-x-2}{4x^2-5x+1}$

Solution: Again when x=1 both numerator and denominator have a limit of zero. Let us factor given function using identity

$ax^2+bx+c=a(x-x_1)(x-x_2)$

where x_1 and x_2 are roots of equation $ax^2+bx+c=0$. The limit can be obtained as follows:

$$\lim_{x \to 1} \frac{3x^2-x-2}{4x^2-5x+1} = \lim_{x \to 1} \frac{3(x-1)(x+2/3)}{4(x-1)(x-1/4)} = \lim_{x \to 1} \frac{3(x+2/3)}{4(x-1/4)} =$$

$$= \frac{3(1+2/3)}{4(1-1/4)} = 5/3.$$

2.3. Limits of polynomials as $x \to +\infty$ or $x \to -\infty$.

Sometimes it is useful to know how $f(x)$ behaves when x is a large positive (or a negative number of large absolute value).

Rather than writing "as x gets arbitrary large through positive values, $f(x)$ approaches the number L", it is customary to use the notation $\lim\limits_{x \to \infty} f(x) = L$. This is read "as x approaches infinity, $f(x)$ approaches L", or "the limit of $f(x)$ as x approaches infinity is L". Similarly, $\lim\limits_{x \to -\infty} f(x) = L$ means that "the limit of $f(x)$ as x approaches minus infinity is L".

Remark: Notations $x \to \infty$ and $x \to +\infty$ are equivalent to each othe and we will use both of these notations.

Remark: All properties of limits stated above hold when $x \to a$ is replaced by

$x \to +\infty$ or by $x \to -\infty$.

$$\lim_{x\to+\infty} x^n = +\infty \quad, n=1,2,3,\ldots\ldots;$$

$$\lim_{x\to-\infty} x^n = \begin{cases} +\infty & ,\text{if } n = 2,4,6,\ldots\ldots \\ -\infty & ,\text{if } n = 1,3,5,\ldots\ldots \end{cases};$$

$$\lim_{x\to+\infty} \frac{1}{x^n} = \left(\lim_{x\to+\infty} \frac{1}{x}\right)^n = 0;$$

$$\lim_{x\to-\infty} \frac{1}{x^n} = \left(\lim_{x\to-\infty} \frac{1}{x}\right)^n = 0.$$

A polynomial $P(x) = c_0+c_1x+c_2x^2+\ldots\ldots+c_nx^n$ behaves like it term of highest degree as $x \to \infty$ or $x \to -\infty$. If $c_n \neq 0$, then

$$\lim_{x\to+\infty} (c_0 + c_1x + c_2x^2 + \ldots\ldots + c_nx^n) = \lim_{x\to+\infty} c_nx^n$$

$$\lim_{x\to-\infty} (c_0 + c_1x + c_2x^2 + \ldots\ldots + c_nx^n) = \lim_{x\to-\infty} c_nx^n.$$

Example: Find $\lim_{x\to+\infty} (7x^5 - 8x^3 + 12x - 8)$

Solution: $\lim_{x\to+\infty} (7x^5 - 8x^3 + 12x - 8) = \lim_{x\to+\infty} 7x^5 = +\infty$.

Example: Find $\lim_{x\to-\infty} (-4x^6 + 14x^2 - 4x + 3)$

Solution: $\lim_{x\to-\infty} (-4x^6 + 14x^2 - 4x + 3) = \lim_{x\to-\infty} (-4x^6) = -\infty$.

Remark: It is important to keep in mind that "$+\infty$" or "$-\infty$" is not number. The last two limits above do not exist.

2.4. Limits of rational functions as $x \to \infty$ or $x \to -\infty$.

To find limits of rational functions as $x \to \infty$ or $x \to -\infty$ we ide the numerator and denominator of a rational function by the hest power of x that occurs in the denominator. What happens then ɪends on the degrees of the polynomials involved.

ample: Find $\lim_{x\to+\infty} \frac{5x-3}{7x+6}$

lution: Divide the numerator and denominator by the highest wer of x that occurs in the denominator; this is $x^1=x$. e obtain

$$\lim_{x\to+\infty} \frac{5x-3}{7x+6} = \lim_{x\to+\infty} \frac{5-3/x}{7+6/x} = \frac{\lim_{x\to+\infty}(5-3/x)}{\lim_{x\to+\infty}(7+6/x)} =$$

$$= \frac{\lim_{x\to+\infty} 5 - \lim_{x\to+\infty} 3/x}{\lim_{x\to+\infty} 7 + \lim_{x\to+\infty} 6/x} = \frac{5-0}{7+0} = \frac{5}{7}.$$

ample: Find $\lim_{x\to-\infty} \frac{5x^2-x}{3x^3-4}$

lution: Divide the numerator and denominator by the highest power x that occurs in the denominator, namely x^3. We obtain

$$\lim_{x\to-\infty} \frac{5x^2-x}{3x^3-4} = \lim_{x\to-\infty} \frac{5/x-1/x^2}{3-4/x^3} =$$

$$= \frac{\lim_{x\to-\infty}(5/x-1/x^2)}{\lim_{x\to-\infty}(3-4/x^3)} = \frac{0-0}{3-0} = 0.$$

ample: Find $\lim_{x\to-\infty} \frac{7x^3-4x^2+1}{2x-5}$

lution: Divide the numerator and denominator by x to obtain

$$\lim_{x\to-\infty} \frac{7x^3-4x^2+1}{2x-5} = \lim_{x\to-\infty} \frac{7x^2-4x+1/x}{2-5/x} = +\infty,$$

ɪce $7x^2-4x \to +\infty$, $\frac{5}{x} \to 0$, and $\left(2-\frac{5}{x}\right) \to 2$ as $x \to -\infty$.

2.5. A quick method for finding limits of rational functions as $x \to \infty$ or $x \to -\infty$.

Let $f(x)$ be a polynomial and let ax^n be its term of highest degree. Let $g(x)$ be another polynomial and let bx^m be its term of highest degree. Then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{ax^n}{bx^m} \text{ and } \lim_{x\to-\infty}\frac{f(x)}{g(x)}=\lim_{x\to-\infty}\frac{ax^n}{bx^m}$$

Example: Evaluate the following limits:

a) $\lim\limits_{x\to\infty}\frac{3x^4+5x^2}{-x^4+10x+5}$; b) $\lim\limits_{x\to\infty}\frac{x^3-16x}{15x^4+x^3-5x}$; c) $\lim\limits_{x\to-\infty}\frac{x^4+x}{6x^3-x^2}$

Solution: By the preceding observations,

a) $\lim\limits_{x\to\infty}\frac{3x^4+5x^2}{-x^4+10x+5}=\lim\limits_{x\to\infty}\frac{3x^4}{-x^4}=\lim\limits_{x\to\infty}(-3)=-3$;

b) $\lim\limits_{x\to\infty}\frac{x^3-16x}{15x^4+x^3-5x}=\lim\limits_{x\to\infty}\frac{x^3}{15x^4}=\lim\limits_{x\to\infty}\frac{1}{15x}=0$;

c) $\lim\limits_{x\to-\infty}\frac{x^4+x}{6x^3-x^2}=\lim\limits_{x\to-\infty}\frac{x}{6}=-\infty$.

Exercises.

In exercises 1-5 use definition of limit to prove that the given limit statement is correct.

1. $\lim\limits_{x\to5}3x=15$
2. $\lim\limits_{x\to2}(2x-7)=-3$
3. $\lim\limits_{x\to-1}(2-3x)=5$
4. $\lim\limits_{x\to1/3}\frac{1}{x}=3$
5. $\lim\limits_{x\to6}\sqrt{x+3}=3$

In exercises 6-20 find the limits.

6. $\lim\limits_{x\to3}(2x^2-7x+6)$
7. $\lim\limits_{x\to3}(3x^4-5x^3+6x^2-4x+7)$
8. $\lim\limits_{x\to2}\frac{4x^2-5x+2}{3x^2-6x+4}$
9. $\lim\limits_{x\to4}\frac{x^2-5x+4}{x^2-7x+6}$

10. $\lim\limits_{x\to6}\frac{x^2-8x+12}{x^2-7x+6}$
11. $\lim\limits_{x\to2}\frac{3x^2-7x+2}{4x^2-5x-6}$
12. $\lim\limits_{y\to6}\frac{y+6}{y^2-36}$
13. $\lim\limits_{x\to-1}\frac{x^2+6x+5}{x^2-3x-4}$
14. $\lim\limits_{t\to1}\frac{t^3+t^2-5t+3}{t^3-3t+2}$
15. $\lim\limits_{x\to\infty}\frac{5x^2+7}{3x^2-x}$
16. $\lim\limits_{x\to\infty}\frac{x^7+8x^6-5x^4}{10x^6+7x^5-4x+17}$
17. $\lim\limits_{x\to\infty}(4x^2-x+3)$
18. $\lim\limits_{x\to-\infty}(6x^5+12x^3)$
19. $\lim\limits_{x\to\infty}\frac{6x^3-x}{2x^{10}+5x+8}$
20. $\lim\limits_{x\to-\infty}\frac{5x^3+2x}{x^2+x+7}$

Answers.

6. 3; **7.** 157; **8.** 2; **9.** 0; **10.** 4/5; **11.** 5/11; **12.** Does not exist; **13.** –4/5; **14.** 4/3; **15.** 5/3; **16.** ∞; **17.** ∞; **18.** $-\infty$; **19.** 0; **20.** $-\infty$.

2.6. Limits involving radicals.

Example: Find $\lim\limits_{x\to+\infty}\sqrt[3]{\frac{3x+5}{8x-3}}$

Solution:

Using the property of limit

$$\lim_{x\to+\infty}\sqrt[3]{\frac{3x+5}{8x-3}}=\sqrt[3]{\lim_{x\to+\infty}\frac{3x+5}{8x-3}}=\sqrt[3]{\lim_{x\to+\infty}\frac{3x}{8x}}=\sqrt[3]{\frac{3}{8}}=\frac{\sqrt[3]{3}}{2}.$$

Example: Find:

a) $\lim\limits_{x\to+\infty}\frac{\sqrt{3x^2+2x}}{x}$; and b) $\lim\limits_{x\to-\infty}\frac{\sqrt{3x^2+2x}}{x}$.

Solution:

Before beginning the solution, note that if x is positive, $\sqrt{x^2}=x$, but if x is negative, $\sqrt{x^2}=-x$.

a) $\lim\limits_{x\to+\infty}\dfrac{\sqrt{3x^2+2x}}{x}=\lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2(3+1/x)}}{x}=\lim\limits_{x\to+\infty}\dfrac{x\sqrt{3+1/x}}{x}=$

$=\lim\limits_{x\to+\infty}\sqrt{3+1/x}=\sqrt{3}$.

b) $\lim\limits_{x\to-\infty}\dfrac{\sqrt{3x^2+2x}}{x}=\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2(3+1/x)}}{x}=\lim\limits_{x\to-\infty}\dfrac{-x\sqrt{3+1/x}}{x}=$

$=\lim\limits_{x\to-\infty}\left(-\sqrt{3+1/x}\right)=-\sqrt{3}$.

Example: Find $\lim\limits_{x\to0}\dfrac{\sqrt[3]{1+x}-1}{\sqrt{1+x}-1}$

Solution:
Let us substitute $z^6=1+x$. We choose z^6 in order to take roots easily. From substitution it is easy to see that if $x\to0$ then $z\to1$. Using substitution, we get

$$\lim_{x\to0}\frac{\sqrt[3]{1+x}-1}{\sqrt{1+x}-1}=\lim_{z\to1}\frac{\sqrt[3]{z^6}-1}{\sqrt{z^6}-1}=\lim_{z\to1}\frac{z^2-1}{z^3-1}=$$

$$=\lim_{z\to1}\frac{(z-1)(z+1)}{(z-1)(z^2+z+1)}=\lim_{z\to1}\frac{z+1}{z^2+z+1}=2/3.$$

Example: Find $\lim\limits_{x\to+\infty}(\sqrt{x^2+6x+5}-x)$.

Solution:
When $x\to\infty$ we get $(\infty-\infty)$. Let us multiply and divide given function to $(\sqrt{x^2+6x+5}+x)$. We will get

$$\lim_{x\to+\infty}(\sqrt{x^2+6x+5}-x)=$$

$$=\lim_{x\to+\infty}\frac{(\sqrt{x^2+6x+5}-x)\cdot(\sqrt{x^2+6x+5}+x)}{\sqrt{x^2+6x+5}+x}=$$

$$=\lim_{x\to+\infty}\frac{(x^2+6x+5)-x^2}{\sqrt{x^2+6x+5}+x}=\lim_{x\to+\infty}\frac{6x+5}{\sqrt{x^2+6x+5}+x}=$$

$$=\lim_{x\to+\infty}\frac{6+5/x}{\sqrt{1+6/x+5/x^2}+1}=\frac{6}{1+1}=3.$$

Example: Find $\lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}$.

Solution:

As $x\to+\infty$, the values of x are eventually positive, so we can replace $|x|$ by x where desirable. We obtain

$$\lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}=\lim_{x\to+\infty}\frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|}=\lim_{x\to+\infty}\frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/x}=$$

$$=\lim_{x\to+\infty}\frac{\sqrt{1+2/x^2}}{3-6/x}=\frac{\lim_{x\to+\infty}\sqrt{1+2/x^2}}{\lim_{x\to+\infty}(3-6/x)}=1/3.$$

Exercises.

In exercises 1-21 find the limits.

1. $\lim_{x\to0}\frac{\sqrt{x+4}-2}{x}$
2. $\lim_{x\to0}\frac{\sqrt{5x+9}-3}{x}$
3. $\lim_{x\to0}\frac{\sqrt{x^2+4}-2}{x}$
4. $\lim_{x\to3}\frac{1-\sqrt{x-2}}{x-3}$
5. $\lim_{x\to+\infty}\left(\sqrt{x^2+3}-x\right)$
6. $\lim_{x\to+\infty}\left(\sqrt{2x^2+5}-x\right)$
7. $\lim_{x\to+\infty}\left(\sqrt{x^2+5x}-x\right)$
8. $\lim_{x\to+\infty}\left(\sqrt{x^2-3x}-x\right)$
9. $\lim_{x\to+\infty}\left(\sqrt{x^2+ax}-x\right)$
10. $\lim_{x\to5}\frac{x^2-25}{2-\sqrt{x-1}}$
11. $\lim_{n\to\infty}\frac{\sqrt{4n^2-3}}{2n+1}$
12. $\lim_{x\to\pi}\frac{\sqrt{1-\tan x}-\sqrt{1+\tan x}}{\sin 2x}$
13. $\lim_{x\to-\infty}\frac{x-2}{x^2+2x+1}$
14. $\lim_{x\to-\infty}\frac{\sqrt{5x^2-2}}{x+3}$
15. $\lim_{y\to-\infty}\frac{2-y}{\sqrt{7+6y^2}}$
16. $\lim_{x\to-\infty}\frac{\sqrt{3x^4+x}}{x^2-8}$

17. $\lim_{x\to 3}\frac{x}{x-3}$ 18. $\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}$

19. $\lim_{n\to\infty}\frac{12n+5}{\sqrt[3]{27n^3+6n^2+8}}$ 20. $\lim_{x\to\infty}\frac{6x-5}{1+\sqrt{x^2+3}}$

21. $\lim_{x\to 1}\frac{\sqrt[4]{x}-1}{\sqrt[3]{x}-1}$ $\left(\text{Hint: Let } x=t^{12}\right)$

Answers.

1. 1/4; **2.** 5/6; **3.** 0; **4.** –1/2; **5.** 0; **6.** $+\infty$; **7.** 5/2; **8.** –3/2; **9.** $a/2$; **10.** **11.** 1; **12.** –1/2; **13.** 0; **14.** $-\sqrt{5}$; **15.** $1/\sqrt{6}$; **16.** $\sqrt{3}$; **17.** does not e **18.** 6; **19.** 4; **20.** 6; **21.** 3/4.

2.7. One sided limits.

Let $f(x)=\frac{x}{|x|}$. If x approaches 0 from the right, $f(x)$ is alwa

If x approaches 0 from the left, $f(x)$ is always -1. This introduces notion of one-sided limits.

Definition: Right-hand limit of $f(x)$ at a. Let f be a function and a some fixed number. Assume that the domain of f contains an op interval (a, b). If, as x approaches a from the right, $f(x)$ approache specific number L, then L is called the right-hand limit $f(x)$ as x approaches a.

This is written:

$$\lim_{x\to a^+} f(x)=L \quad \text{or as } x\to a^+, f(x)\to L.$$

The assertion that $\lim_{x\to a^+} f(x)=L$ is read : " the limit of f as x approaches a from the right is L", or " as x approaches a from the $f(x)$ approaches L".

The left-hand limit is defined similarly. The only differen are that the domain of f must contain an open interval of the form and $f(x)$ is examined as x approaches a from the left. The notation the left-hand limits are: $\lim_{x\to a^-} f(x)=L$ or as $x\to a^-, f(x)\to$

$$\lim_{x\to 0^+}\frac{1}{x}=+\infty\ ; \quad \lim_{x\to 0^-}\frac{1}{x}=-\infty;$$

$$\lim_{x\to a^+}\frac{1}{x-a}=+\infty; \qquad \lim_{x\to a^-}\frac{1}{x-a}==-\infty.$$

xample: Find: a) $\lim_{x\to 4^+}\frac{2-x}{(x-4)(x+2)}$; b) $\lim_{x\to 4^-}\frac{2-x}{(x-4)(x+2)}$

lution: In both examples the limit of the numerator is –2 and nominator is 0, so the limit of the ratio does not exist. We need to alyze the sign of the ratio. As x approaches 4 from the right, the ratio always negative, and as x approaches 4 from the left the sign of the tio is eventually positive (after x exceeds 2), so

$$\lim_{x\to 4^+}\frac{2-x}{(x-4)(x+2)}=-\infty \quad \text{and} \quad \lim_{x\to 4^-}\frac{2-x}{(x-4)(x+2)}=+\infty$$

xample: Find $\lim_{x\to -1^+}\frac{1}{(x+1)^2}$

olution: As x approaches -1 from the right, $(x+1)$ approaches 0 from ght. The reciprocal $\frac{1}{(x+1)^2}$ stays positive and increases beyond all ounds.

xample: Find: a) $\lim_{x\to 0^-} 3^{\frac{1}{x}}$ and b) $\lim_{x\to 0^+} 3^{\frac{1}{x}}$

olution: a) When x approaches 0 from the left, then $\frac{1}{x}$ is large nd negative. We can write: $3^{\frac{1}{x}}=3^{-\frac{1}{|x|}}=\frac{1}{3^{\frac{1}{|x|}}}$.

Since $\frac{1}{|x|}$ is large when $x\to 0$, then $3^{\frac{1}{|x|}}$ is also large, and its reciprocal $\frac{1}{3^{\frac{1}{|x|}}}$ **is a small** number. Consequently

$$\lim_{x\to 0^-} 3^{\frac{1}{x}}=\lim_{x\to 0^-} 3^{-\frac{1}{|x|}}=\lim_{x\to 0^-}\frac{1}{3^{\frac{1}{|x|}}}=0$$

b) When x approaches 0 from the right, then $\frac{1}{x}$ is large and positive.

The value $3^{\frac{1}{x}}$ will be large and positive.

$$\lim_{x\to 0^+} 3^{\frac{1}{x}} = \lim_{x\to 0^+} 3^{\frac{1}{|x|}} = +\infty .$$

2.8. Existence of limits.

In general, no guarantee that a function $f(x)$ actually has a limit as $x \to x_0^+, x \to x_0^-$ or $x \to x_0$. If there is no limit, then we say that the **limit does not exist.**

Theorem: A function $f(x)$ has a limit as x approaches x_0 if and only if the right-hand and left-hand limit at x_0 exist and are equal. In symbols,

$$\lim_{x\to x_0} f(x) = L \Leftrightarrow \lim_{x\to x_0^+} f(x) = L \text{ and } \lim_{x\to x_0^-} f(x) = L$$

Remark: Keep in mind that the symbols $+\infty$ and $-\infty$ are simply descriptions of limits that fail to exist. These symbols do not represent real numbers and consequently they can not be manipulated using rules of algebra. For example, it is not correct to write $+\infty - \infty = 0$.

Example:

Figure 2.1 shows the graph of function f whose domain is the closed interval $[0, 5]$.

a) Does $\lim_{x\to 1} f(x)$ exist?

b) Does $\lim_{x\to 2} f(x)$ exist?

c) Does $\lim_{x\to 3} f(x)$ exist?

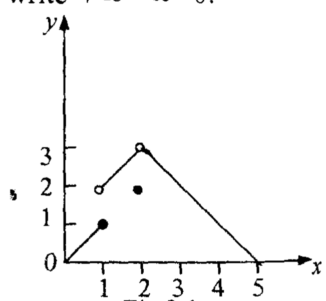


Fig.2.1

Solution:

a) Inspection of the graph shows that

$$\lim_{x\to 1^-} f(x) = 1 \text{ and } \lim_{x\to 1^+} f(x) = 2$$

Although the two one-sided limits exist, they are not equal. Thus, $\lim_{x\to 1} f(x)$ does not exist. In short, "f does not have a limit as $x \to 1$".

Inspection of the graph shows that

$$\lim_{x\to 2^-} f(x) = 3 \text{ and } \lim_{x\to 2^+} f(x) = 3$$

us, $\lim_{x\to 2} f(x)$ exists and is 3. The solid dot at (2,2) shows that 2)=2. This information, however, plays no role in our examination of limit of $f(x)$ as $x \to 2$.

Inspection of the graph shows that $\lim_{x\to 3^-} f(x) = 2$ and $\lim_{x\to 3^+} f(x) = 2$

us, $\lim_{x\to 3} f(x)$ exists and is 2. Incidentally, the fact that $f(3)$ is equal 2 is irrelevant in determining $\lim f(x)$.

2.9. Continuity.

finition: A function f is said to be continuous at a point c if the lowing conditions are satisfied:

1. $f(c)$ is defined
2. $\lim_{x\to c} f(x)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

If one or more of the conditions in this definition fails to hold, en f is called discontinuous at c and c is said to be a point of scontinuity of f. If f is continuous at all points of an open interval , b), then f is said to be continuous on (a, b).

ample: $f(x) = \dfrac{x^2-4}{x-2}$ is discontinuous at 2, because (2) is undefined.

ample:

$$g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & , \quad x \neq 2 \\ 3 & , \quad x = 2 \end{cases}$$ is also discontinuous

at 2 because $g(2) = 3$, and

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4, \text{ so that } \lim_{x\to 2} g(x) \neq g(2).$$

ample: Show that $f(x) = |x|$ is a continuous function.

Solution: We can write $f(x)$ as

$$f(x)=|x|=\begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -x & \text{if } x<0 \end{cases}$$

$f(x)=|x|$ is continuous if $x>0$ or $x<0$. $|x|$ is identical to the polynom and all polynomials are continuous functions. Thus, x=0 is the only point that remains to be considered. At this point $f(0)=|0|=0$, so it remains to show that $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}|x|=0$

Because the formula for f changes at 0, it will be helpful to consider one-sided limits at 0 rather than the two- sided limit. We obtain:

$$\lim_{x\to 0^+}|x|=\lim_{x\to 0^+} x=0 \text{ and } \lim_{x\to 0^-}|x|=\lim_{x\to 0^-}(-x)=0$$

Thus, (2) holds and $|x|$ is continuous at x=0.

Theorem: If functions f and g are continuous at c, then

a) $f+g$ is continuous at c;
b) $f-g$ is continuous at c;
c) $f\cdot g$ is continuous at c;
d) f/g is continuous at c if $g(c)\neq 0$ and is discontinuous at c if $g(c)=0$.

Theorem: A rational function is continuous everywhere except at th points where the denominator is zero.

Example: Where is $h(x)=\dfrac{x^2-9}{x^2-5x+6}$ continuous?

Solution: By theorem, the ratio is continuous everywhere except at t points where the denominator is zero. Since solution of $x^2-5x+6=0$ are x=2 and x=3, $h(x)$ is continuous everywhere excep at these two points.

Exercises.

In exercises 1-13 find the limits.

1. $\lim\limits_{x\to 3^-}\dfrac{x}{x-3}$
2. $\lim\limits_{x\to 2^+}\dfrac{x}{x^2-4}$

3. $\lim\limits_{x\to 2^-}\dfrac{x}{x^2-4}$
4. $\lim\limits_{x\to 0^+}\dfrac{7}{3x}$
5. $\lim\limits_{t\to -5^-}\dfrac{2t}{t+5}$
6. $\lim\limits_{t\to -5^+}\dfrac{2t}{t+5}$
7. $\lim\limits_{x\to 3^-}\dfrac{6}{x-3}$
8. $\lim\limits_{x\to 3^+}\dfrac{6}{x-3}$
9. $\lim\limits_{x\to 2^+}\dfrac{4}{(x-2)^2}$
10. $\lim\limits_{x\to 2^+} 2^{\frac{1}{x-2}}$
11. $\lim\limits_{x\to 2^-} 2^{\frac{1}{x-2}}$
12. $\lim\dfrac{x^2-1}{2x+4}$ as a) $x\to -2^+$ and b) $x\to -2^-$
13. $\lim\dfrac{1}{x^2-4}$ as $a)\ x\to 2^+$ $b)\ x\to 2^-$ $c)\ x\to -2^+$ $d)\ x\to -2^-$

In each of exercises 14 and 15 there is a graph of functions.

14. (See Fig.2.2). Decide which of given limits exist, and evaluate those which do.
$a)\ \lim\limits_{x\to 0^+} f(x)$; $b)\ \lim\limits_{x\to 1} f(x)$;
$c)\ \lim\limits_{x\to 2^-} f(x)$; $d)\ \lim\limits_{x\to 2^+} f(x)$;

15. (See Fig.2.3)
$a)\ \lim\limits_{x\to 1} f(x)$; $b)\ \lim\limits_{x\to 2} f(x)$;
$c)\ \lim\limits_{x\to 3} f(x)$; $d)\ \lim\limits_{x\to 4^-} f(x)$

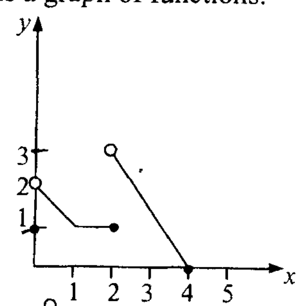


Fig.2.2

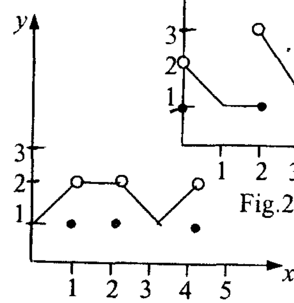


Fig.2.3

16. **Graph** $f(x)=\begin{cases} x^3, & x\neq 1 \\ 0, & x=1 \end{cases}$

a) Find $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1^+} f(x)$.

b) Does $\lim\limits_{x\to 1} f(x)$ exist? If so, what is that? If not, why not?

In exercises 17-22 find points of discontinuity, if any.

17. $f(x)=x^3-2x+3$

18. $f(x)=\dfrac{x}{x^2+1}$

19. $f(x)=\dfrac{x-4}{x^2-16}$

20. $f(x)=\dfrac{x}{|x|-3}$

21. $f(x)=|x^3-2x^2|$

22. $f(x)=\begin{cases}2x+3, & x\le 4\\ 7+\dfrac{16}{x}, & x>4\end{cases}$

23. Find a value for the constant *k*, if possible, that will make the function continuous.

a) $f(x)=\begin{cases}7x-2, & x\le 1\\ kx^2, & x>1\end{cases}$; b) $f(x)=\begin{cases}kx^2, & x\le 2\\ 2x+k, & x>2\end{cases}$

24. Let $f(x)$ equal the least integer that is greater than or equal to x. For instance, $f(3)=3, f(3.4)=4, f(3.9)=4$. This function is sometimes denoted $\lceil x\rceil$ and called the " ceiling of *x*". Graph the function and answer the questions.

a) Does $\lim\limits_{x\to 4^-} f(x)$ exist? If so, what is it?

b) Does $\lim\limits_{x\to 4^+} f(x)$ exist? If so, what is it?

c) Does $\lim\limits_{x\to 4} f(x)$ exist? If so, what is it?

d) Is *f* continuous at 4?

e) Where is *f* continuous?

f) Where is *f* not continuous?

Answers.

1. $-\infty$; 2. $+\infty$; 3. $-\infty$; 4. $+\infty$; 5. $+\infty$; 6. $-\infty$; 7. $-\infty$; 8. $+\infty$; 9. $+\infty$; 10. $+\infty$; 11. 0; 12. a) $+\infty$; b) $-\infty$; 13. a) $+\infty$; b) $-\infty$; c) $-\infty$; d) $+\infty$; 14. a) 2; b) 1; c) 1; d) 3; 15. a) 2; b) 2; c) 1; d) 2; 16. a) 1, 1; b) 1; 17. none; 18. none; 19. $x=\pm 4$; 20. $x=\pm 3$; 21. none; 22. none; 23. a) 5; b) 4/3; 24. a) yes; 4; b) yes; 5; c) no; d) no; e) all nonintegers; f) all integers.

2.10. The limit of trigonometric functions. The first remarkable limit

st of all, let us consider principle called the **squeeze principle.**

e squeeze principle:

If $g(x)\le f(x)\le h(x)$ and $\lim\limits_{x\to a} g(x)=L=\lim\limits_{x\to a} h(x)$

then $\lim\limits_{x\to a} f(x)=L$

eorem 1: Let $\sin\theta$ denote the sine of an angle of θ radians. Then

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

metimes this limit is also called 'the first remarkable limit'.

eorem 2: Let $\cos\theta$ denote the cosine of an angle of θ radians. en

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0$$

$x\to+\infty$ or $x\to-\infty$, the values of $\sin x$ and $\cos x$ oscillate eatedly between -1 and 1 without approaching any fixed real value. us, the limits $\lim\limits_{x\to+\infty}\sin x$, $\lim\limits_{x\to-\infty}\sin x$, $\lim\limits_{x\to+\infty}\cos x$, $\lim\limits_{x\to-\infty}\cos x$ do not st. We shall say that they fail to exist due to oscillation.

ample: Find $\lim\limits_{x\to 0}\dfrac{\sin ax}{x}$

lution: Let $ax=\theta$, as $x\to 0, \theta\to 0$. Thus,

$$\lim_{x\to 0}\frac{\sin ax}{x}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta/a}=\lim_{\theta\to 0}a\cdot\frac{\sin\theta}{\theta}=a\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=a\cdot 1=a.$$

$$\lim_{x\to 0}\frac{\sin ax}{x}=a.$$

particular, if $a=2$, then $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}=2$.

ample: Find $\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}$

lution: $\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}=\lim\limits_{x\to 0}\dfrac{1}{5}\cdot\dfrac{\sin 5x}{x}=\dfrac{1}{5}\lim\limits_{x\to 0}\dfrac{\sin 5x}{x}=\dfrac{1}{5}\cdot 5=1.$

Example: Find $\lim\limits_{x\to 0}\frac{\sin ax}{\sin bx}$

Solution: Let us divide numerator and denominator by x

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\frac{\frac{\sin ax}{x}}{\frac{\sin bx}{x}}=\frac{\lim\limits_{x\to 0}\frac{\sin ax}{x}}{\lim\limits_{x\to 0}\frac{\sin bx}{x}}=\frac{a}{b}.$$

Example: Find $\lim\limits_{x\to 0}\frac{\tan x}{x}$

Solution: $$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}(\frac{\sin x}{\cos x}\cdot\frac{1}{x})=\lim_{x\to 0}(\frac{1}{\cos x}\cdot\frac{\sin x}{x})=$$

$$=\lim_{x\to 0}\frac{1}{\cos x}\cdot\lim_{x\to 0}\frac{\sin x}{x}=1\cdot 1=1.$$

Example: Find $\lim\limits_{x\to 0}\frac{\sin^2(x/3)}{x^2}$

Solution: $$\lim_{x\to 0}\frac{\sin^2\frac{x}{3}}{x^2}=\lim_{x\to 0}\left(\frac{\sin\frac{x}{3}}{x}\right)^2=\left[\lim_{x\to 0}\frac{\sin\frac{x}{3}}{x}\right]^2=\left(\frac{1}{3}\right)^2=\frac{1}{9}.$$

Example: Find $\lim\limits_{x\to 0}\frac{\sin x}{\sqrt{x+9}-3}$

Solution: As $x\to 0$ then numerator and denominator approaches
Let us multiply numerator and denominator by the conjugate of denominator:

$$\lim_{x\to 0}\frac{\sin x}{\sqrt{x+9}-3}=\lim_{x\to 0}\frac{\sin x(\sqrt{x+9}+3)}{(\sqrt{x+9}-3)(\sqrt{x+9}+3)}=$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}(\sqrt{x+9}+3)=1\cdot(3+3)=6.$$

Example: Find $\lim\limits_{x\to 2}(2-x)\cdot\tan\frac{\pi x}{4}$

Solution: Observe that as $x\to 2$, we shall have $(0\cdot\infty)$.

Let $x=2-\alpha$. We obtain

$$\lim_{x\to 2}(2-x)\cdot\tan\frac{\pi x}{4}=\lim_{\alpha\to 0}\alpha\cdot\tan\frac{\pi}{4}(2-\alpha)=\lim_{\alpha\to 0}\alpha\cdot\tan\left(\frac{\pi}{2}-\frac{\pi}{4}\alpha\right)=$$

$$=\lim_{\alpha\to 0}\alpha\cdot\cot\frac{\pi}{4}\alpha=\lim_{\alpha\to 0}\alpha\cdot\frac{\cos\frac{\pi}{4}\alpha}{\sin\frac{\pi}{4}\alpha}=$$

$$=\lim_{\alpha\to 0}\frac{\frac{\pi}{4}\alpha}{\frac{\pi}{4}\cdot\sin\frac{\pi}{4}\alpha}\cdot\lim_{\alpha\to 0}\cos\frac{\pi}{4}\alpha=\frac{1}{\frac{\pi}{4}}\cdot 1=\frac{4}{\pi}.$$

Exercises.

In exercises 1-18 find the limits.

1. $\lim\limits_{\theta\to 0}\frac{\sin 3\theta}{\theta}$
2. $\lim\limits_{x\to 0^+}\frac{\sin x}{5\sqrt{x}}$
3. $\lim\limits_{x\to 0}\frac{\tan 7x}{\sin 3x}$
4. $\lim\limits_{h\to 0}\frac{h}{\tan h}$
5. $\lim\limits_{\theta\to 0}\frac{\theta^2}{1-\cos\theta}$
6. $\lim\limits_{h\to 0}\frac{1-\cos 5h}{\cos 7h-1}$
7. $\lim\limits_{x\to 0}\frac{2x+\sin x}{x}$
8. $\lim\limits_{x\to\infty}x\cdot\sin\frac{1}{x}$ $\left[\text{let } t=\frac{1}{x}\right]$
9. $\lim\limits_{x\to 0}\frac{\sin ax}{\tan bx}$
10. $\lim\limits_{x\to 0}\frac{\sin^3\frac{x}{2}}{x^3}$
11. $\lim\limits_{x\to 1}\frac{\sin(x-1)}{x^3-1}$
12. $\lim\limits_{x\to 0}\frac{\sin x}{\sqrt{x+4}-2}$
13. $\lim\limits_{x\to 0}\frac{1-\cos x-\tan^2 x}{x\cdot\sin x}$
14. $\lim\limits_{x\to 0}x\cdot\cot\frac{x}{3}$
15. $\lim\limits_{x\to 0}\frac{\sqrt{x+9}-3}{\sin 6x}$
16. $\lim\limits_{x\to 0}\frac{1+x^2-\cos x}{\sin^2 x}$
17. $\lim\limits_{x\to\frac{\pi}{2}}(x-\frac{\pi}{2})\cdot\tan x$ $\left[\text{let } \theta=x-\frac{\pi}{2}\right]$

18. $\lim_{x\to 2}\frac{\cos\left(\frac{\pi}{x}\right)}{x-2}$ $\left[\text{let } t=\frac{\pi}{2}-\frac{\pi}{x}\right]$

19. Find a nonzero value for the constant k so that

$$f(x)=\begin{cases}\frac{\tan kx}{x} & \text{if } x<0\\ 3x+2k^2 & \text{if } x\ge 0\end{cases}$$ will be continuous at x=0.

Answers.

1. 3; **2.** 0; **3.** 7/3; **4.** 1; **5.** 2; **6.** –25/49; **7.** 3; **8.** 1; **9.** a/b; **10.** 1/8; **11.** 1/3; **12.** 4; **13.** –1/2; **14.** 3; **15.** 1/36 ; **16.** 3/2; **17.** –1; **18.** $\pi/4$; **19.** 1/2.

2.11. The number *e*. Second remarkable limit.

Number e is the limit

(1) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$ or

(2) $\lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e$

Limits (1) and (2) are equivalent and called the second remarkabl limits.

To evaluate $\lim_{x\to a}[\phi(x)]^{\varphi(x)}=C$ there are following possible cases.

a) If $\lim_{x\to a}\phi(x)=A$ and $\lim_{x\to a}\varphi(x)=B$ then $C=A^B$

b) If $\lim_{x\to a}\phi(x)=A\neq 1$ and $\lim_{x\to a}\varphi(x)=B$ then we apply

$$\lim_{x\to\infty}C^x=\begin{cases}0 & \text{if } 0<C<1\\ +\infty & \text{if } C>1\end{cases}$$

or

$$\lim_{x\to-\infty}C^x=\begin{cases}+\infty & \text{if } 0<C<1\\ 0 & \text{if } C>1\end{cases}$$

) If $\lim_{x\to a}\phi(x)=1$ and $\lim_{x\to a}\varphi(x)=\infty$ then we assume

$(x)=1+\alpha(x)$, where $\alpha(x)\to 0$ as $x\to a$ and

$$C=\lim_{x\to a}\left\{[1+\alpha(x)]^{\frac{1}{\alpha(x)}}\right\}^{\alpha(x)\cdot\varphi(x)}=e^{\lim_{x\to a}\alpha(x)\cdot\varphi(x)}=e^{\lim_{x\to a}[\phi(x)-1]\varphi(x)}.$$

xample: Find $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x$

olution: As $x\to\infty$, expression $\left(1+\frac{k}{x}\right)\to 1$ and we get indeterminate orm 1^∞. Let us introduce α by $\frac{k}{x}=\alpha\ \Rightarrow\ x=\frac{k}{\alpha}$.

f $x\to\infty$ then $\alpha\to 0$. Thus,

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x=\lim_{\alpha\to 0}(1+\alpha)^{\frac{k}{\alpha}}=\lim_{\alpha\to 0}\left[(1+\alpha)^{\frac{1}{\alpha}}\right]^k$$

Jsing (2) we obtain

$$\lim_{\alpha\to 0}\left[(1+\alpha)^{\frac{1}{\alpha}}\right]^k=\left[\lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}\right]^k=e^k$$

(3) $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x=e^k$;

In particular, if k=3, then $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x=e^3$

Example: Find $\lim_{x\to 0}\frac{\ln(1+x)}{x}$

Solution: Since $\frac{\ln(1+x)}{x}=\frac{1}{x}\ln(1+x)=\ln(1+x)^{\frac{1}{x}}$

Using (2) we obtain

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\left[\ln(1+x)^{\frac{1}{x}}\right]=\ln\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]=\ln e=1.$$

Example: Find $\lim\limits_{x\to\infty}\left(\frac{x+2}{x-3}\right)^x$

Solution:

Let us divide numerator and denominator by x, and then use

$$\lim_{x\to\infty}\left(\frac{x+2}{x-3}\right)^x=\lim_{x\to\infty}\left(\frac{1+\frac{2}{x}}{1+\frac{-3}{x}}\right)^x=\frac{e^2}{e^{-3}}=e^5.$$

Example: Find $\lim\limits_{x\to\infty}\left(1-\frac{2}{3x-1}\right)^{1-4x}$

Solution:

$$\lim_{x\to\infty}\left(1-\frac{2}{3x-1}\right)^{1-4x}=\lim_{x\to\infty}\left(1+\frac{-2}{3x-1}\right)^{1-4x}$$

Let $-\frac{2}{3x-1}=\frac{1}{y}$. Then $x=-\frac{2}{3}y+\frac{1}{3}$.

As $x\to\infty$, then $y\to-\infty$.

We obtain

$$\lim_{x\to\infty}\left(1+\frac{-2}{3x-1}\right)^{1-4x}=\lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^{-\frac{8}{3}y-\frac{1}{3}}=$$

$$=\lim_{y\to-\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^{-\frac{8}{3}}\cdot\left(1+\frac{1}{y}\right)^{-\frac{1}{3}}=e^{-\frac{8}{3}}.$$

Example: Find $\lim\limits_{x\to\infty}\left(\frac{2x+4}{2x-4}\right)^{x-3}$

Solution: $\lim\limits_{x\to\infty}\left(\frac{2x+4}{2x-4}\right)^{x-3}=\lim\limits_{x\to\infty}\left(1+\frac{8}{2x-4}\right)^{x-3}$

Let $\frac{8}{2x-4}=\frac{1}{y}$. Then

$$\lim_{x\to\infty}\left(1+\frac{8}{2x-4}\right)^{x-3}=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{4y-1}=$$

$$=\lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^4\cdot\left(1+\frac{1}{y}\right)^{-1}=e^4.$$

Exercises.

In exercises 1-12 find the limits.

$\text{im}_{\to\infty}\left(1+\frac{2}{n}\right)^n$

2. $\lim\limits_{x\to0}(1-3x)^{\frac{1}{x}}$

$\text{im}_{\to0}\frac{\ln(1+4x)}{x}$

4. $\lim\limits_{x\to\infty}x\cdot[\ln(x+1)-\ln x]$

$\text{im}_{\to\infty}\left(\frac{x+1}{x-3}\right)^x$

6. $\lim\limits_{x\to\infty}\left(\frac{x+n}{x+m}\right)^x$

$\lim\limits_{y\to\infty}\left(1-\frac{4}{3y-1}\right)^{y+2}$

8. $\lim\limits_{x\to\infty}\left(\frac{2x-3}{2x+1}\right)^{5-2x}$

$\lim\limits_{x\to\infty}\left(\frac{3x-1}{3x+2}\right)^{2x-4}$

10. $\lim\limits_{x\to\infty}\ln\left(\frac{2x+3}{2x-1}\right)^x$

$\lim\limits_{x\to\infty}\ln\left(\frac{5x+3}{5x-1}\right)^{x-4}$

12. $\lim\limits_{x\to1}(3x-2)^{\frac{x}{(x^2-1)}}$

Answers.

e^2; **2.** e^{-3}; **3.** 4; **4.** 1; **5.** e^4; **6.** e^{n-m}; **7.** $e^{-4/3}$; **8.** e^4; **9.** e^{-2};
: 2; **11.** 4/5; **12.** $e^{3/2}$.

Chapter 3.
Derivatives.
3.1. Definition of derivatives.

Definition: Let f be a function that is defined in at least in some op interval that contains the number x. If

(1) $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exists, it is called the **derivative of f at x** and is denoted $f'(x)$.
The function f is said to be **differentiable** at x.

Example: Find the derivative of the function x^2 at any number x.

Solution: By the definition of the derivative

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h}=$$

$$=\lim_{h\to 0}\frac{2hx+h^2}{h}==\lim_{h\to 0}(2x+h)=2x.$$

The fact that the derivative of the function x^2 is $2x$ is recorded in the notation $(x^2)'=2x$ or $\frac{d}{dx}(x^2)=2x$.

3.2. Geometric interpretation of derivatives.

Definition: If $P(x_0,y_0)$ is a point on the graph of a function f, the the **tangent** line to the graph of f at P is defined to be the line through P with slope

(1) $$m_{\tan}=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

provided this limit exists.

For brevity, the tangent line at $P(x_0,y_0)$ is often called the tangent line at x_0.(Fig.3.1)

It follows from the definition that point-slope form of the tangent line at x_0 is:

(2) $$y-y_0=m_{\tan}(x-x_0)$$

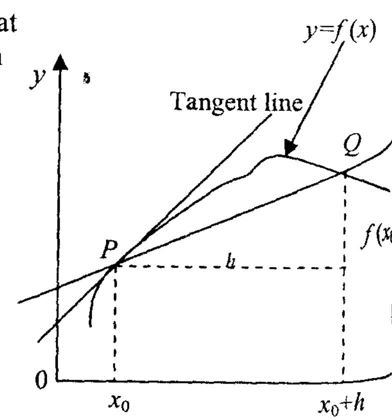


Fig.3.1

Geometric interpretation of the derivative:

is the function whose value at x is the slope of the tangent e to the graph of f at x.

ample: Find the slope and an equation of the tangent line to the ph of $f(x)=x^2$ at the point $P(3,9)$.

lution: We have $x_0=3$ and $y_0=9$, so from (1)

$$m_{\tan}=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{(3+h)^2-(3)^2}{h}=$$

$$=\lim_{h\to 0}\frac{9+6h+h^2-9}{h}=\lim_{h\to 0}\frac{6h+h^2}{h}=\lim_{h\to 0}(6+h)=6$$

us, from (2) the point-slope form of the tangent line is $-9=6(x-3)$ and the slope-intercept form is $y=6x-9$.

ample: Let $f(x)=x^2+1$

a) Find $f'(x)$

b) Use the result in part (a) to find the slope of the tangent line to $y=x^2+1$ at $x=2, x=0,$ and $x=-2$

lution:

a) From (1) of (3.1)

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[(x+h)^2+1]-[x^2+1]}{h}=$$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2+1-x^2-1}{h}=\lim_{h\to 0}\frac{2xh+h^2}{h}=$$

$$=\lim_{h\to 0}(2x+h)=2x$$

b) From part (a) the slope of the tangent line at any point x is $f'(x)=2x$.

Thus, at $x=2, x=0,$ and $x=-2$ the slopes are $f'(2)=4$; $f'(0)=0$ and $f'(-2)=-4$.

ample: Find

a) the derivative with respect to x of $f(x)=\sqrt{x}$

b) the slope of the tangent line to the graph of $y=\sqrt{x}$ at $x=9$.

Solution: a) From definition

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=$$

$$=\lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}=$$

$$=\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}.$$

b) The slope of the tangent line at $x=9$ is $f'(9)$, and from part (a) we have

$$f'(9)=\frac{1}{2\sqrt{9}}=\frac{1}{6}.$$

Example: Find the derivative of $f(x)=\dfrac{x}{x-9}$

Solution:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x+h}{x+h-9}-\frac{x}{x-9}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x+h)(x-9)-x(x+h-9)}{(x+h-9)(x-9)}\right]=$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-9x+x\cdot h-9h-x^2-x\cdot h+9x}{(x+h-9)(x-9)}\right]=$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-9h}{(x+h-9)(x-9)}\right]=\lim_{h\to 0}\frac{-9}{(x+h-9)(x-9)}=-\frac{9}{(x-9)^2}.$$

3.3. Derivative notation.

The process of finding a derivative is called differentiation. I[t is] often useful to think of differentiation as an operation that, when applied to a function f, produces a new function f'. In the case when the independent variable is x, the differentiation operation is often denoted by the symbol $\frac{d}{dx}[f(x)]$, which is read, " the derivative of $f(x)$ with respect to x". Thus,

$$\frac{d}{dx}[f(x)]=f'(x)$$

mark: $\dfrac{dy}{dx}$ should not be regarded as a ratio, it should be considered a single symbol denoting the derivative.
he independent variable is not x, then appropriate adjustments in the tation have to be made. For example, if $y=f(u)$ then

$$\frac{d}{du}[f(u)]=f'(u).$$

particular, adjusting the notation in last two examples above yields

$$\frac{d}{du}[\sqrt{u}]=\frac{1}{2\sqrt{u}}\ \text{ and }\ \frac{d}{du}\left[\frac{u}{u-9}\right]=-\frac{9}{(u-9)^2}.$$

3.4. Existence of derivatives.

The derivative of a function f is defined at those points where e limit in (1) exists. If x_0 is such a point, then we say that **f is fferentiable at x_0** or **f has a derivative at x_0**. Stated another way, the main of f' consists of those points where f is differentiable. We say at **f is differentiable on an open interval (a, b)** if it is differentiable each point in (a, b), and we say that f is a **differentiable function** if is differentiable on $(-\infty,+\infty)$. At points where f is not differentiable e say that **the derivative of f does not exist.**

xample:

ne function $f(x)=|x|$ is continuous for x and consequently is continuous $x=0$.(Fig.3.2)

) Show that $f(x)=|x|$ is not differentiable $x=0$

) Find $f'(x)$.

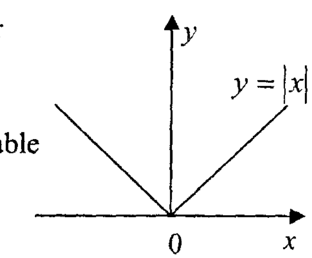


Fig.3.2

lution: a) From definition:

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=$$

$$= \lim_{h\to 0}\frac{|h|-|0|}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

But $\frac{|h|}{h} = \begin{cases} 1 & \text{if } h>0 \\ -1 & \text{if } h<0 \end{cases}$,so that $\lim_{h\to 0^-}\frac{|h|}{h} = -1$

and $\lim_{h\to 0^+}\frac{|h|}{h} = 1$. Thus $f'(0) = \lim_{h\to 0}\frac{|h|}{h}$ does not exist because the one sided limits are not equal. Consequently, $f(x)=|x|$ is not differenti at x=0.

b) If x>0, then $f(x)=|x|=x$, so $f'(x)=1$ and

if x<0, then $f(x)=|x|=-x$, so $f'(x)=-1$

$$f'(x) = \frac{d}{dx}\left[|x|\right] = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

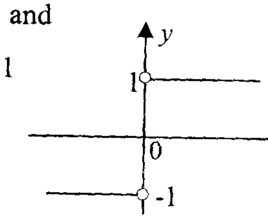


Fig.3.3

As we see in Fig.3.3. f' is not a continuous function. So, this example shows that a continuous function can have a derivative that is not continuous.

Exercises.

In exercises 1-8 use definition of derivative to find $f'(x)$

1. $f(x)=3x^2$
2. $f(x)=x^2-x$
3. $f(x)=x^3$
4. $f(x)=2x^3+1$
5. $f(x)=\sqrt{x+1}$
6. $f(x)=1/x$
7. $f(x)=ax^2+b$ (a , b constants)
8. $f(x)=\frac{1}{\sqrt{x}}$

In exercises 9-12 use definition of derivative (with the appropriate change in notation) to obtain the derivative requested.

9. Find $f'(t)$ if $f(t)=4t^2+t$

10. Find $g'(u)$ if $g(u)=5u+3$

11. Find $\frac{dA}{d\lambda}$ if $A=3\lambda^2-\lambda$

Find $\frac{dV}{dr}$ if $V=\frac{4}{3}\pi r^3$.

In exercises 13-17 find $f'(a)$ and the equation of the tangent to the graph of f at the point where x= a.

f is the function in Exercise 1.; a=3
f is the function in Exercise 2.; a=2
f is the function in Exercise 3.; a=0
f is the function in Exercise 4.; a=-1
f is the function in Exercise 5.; a=8

Given that $f(3)$=-1and $f'(3)=5$, find an equation for the tangent to the graph of $y=f(x)$ at the point where x=3.

Show that $f(x)=\begin{cases} x^2+1 & , x\le 1 \\ 2x & , x>1 \end{cases}$ is continuous and erentiable at x=1. Sketch the graph of f.

Show that $f(x)=\begin{cases} x^2+2 & , x\le 1 \\ x+1 & , x>1 \end{cases}$ is continuous but not erentiable at x=1. Sketch the graph of f.

Let $f(x)=\begin{cases} 3x^2 & , x\le 1 \\ ax+b & , x>1 \end{cases}$

d the values of a and b so that f will be differentiable at x=1.

Answers.

x ; **2.** $2x$-1; **3.** $3x^2$; **4.** $6x^2$; **5.** $\frac{1}{2\sqrt{x+1}}$; **6.** $-\frac{1}{x^2}$; **7.** $2ax$; $-\frac{1}{2}\frac{1}{(\sqrt{x})^3}$; **9.** $8t+1$; **10.** 5 ; **11.** $6\lambda-1$; **12.** $4\pi r^2$. **13.** 18; $=18x-27$; **14.** 3; $y=3x-4$; **15.** 0; $y=0$; **16.** 6; $y=6x+5$; $\frac{1}{6}$; $y=\frac{x}{6}+\frac{5}{3}$; **18.** $y=5x-16$; **21.** a=6, b=-3.

3.5. Techniques of differentiation.

Rule 1: If f is a constant function, say $f(x)= c$ for all x, then $f'(x)$
that is,

$$\frac{d}{dx}[c]=0$$

Example: If $f(x)=5$ for all x, then $f'(x)=0$ for all x; that is $\frac{d}{dx}[5]=$

Rule 2: (The power rule) For any fixed exponent n

$$\frac{d}{dx}\left[x^n\right]=n\cdot x^{n-1}$$

through any interval where x^n and x^{n-1} are both defined.

Example:

$$\frac{d}{dx}\left[x^5\right]=5\cdot x^4;\ \frac{d}{dx}[x]=1\cdot x^0=1;$$

$$\frac{d}{dx}\left[x^{12}\right]=12\cdot x^{11}.$$

Rule 3: Let c be a constant. If f is differentiable at x, then so is cf,

$$\frac{d}{dx}[c\cdot f(x)]=c\cdot\frac{d}{dx}[f(x)]$$

Remark: In words, a constant factor can be moved through a deriv
sign. In function notation, rule 3 states $(c\cdot f)'=c\cdot f'$

Example:

$$\frac{d}{dx}\left[4x^8\right]=4\cdot\frac{d}{dx}\left[x^8\right]=4\cdot(8x^7)=32x^7;$$

$$\frac{d}{dx}\left[-x^{12}\right]=(-1)\cdot\frac{d}{dx}\left[x^{12}\right]=-12x^{11}$$

Rule 4: If f and g are differentiable at x, then so is $f+g$, and

$$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)]$$

Rule 4 can be written in function notation as $(f+g)'=f'+g'$

Example: $\frac{d}{dx}\left[x^4+x^7\right]=\frac{d}{dx}\left[x^4\right]+\frac{d}{dx}\left[x^7\right]=4x^3+7x^6.$

By writing $f-g=f+(-1)g$ and applying rule 4 it follows that

$$\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}[f(x)]-\frac{d}{dx}[g(x)]$$

in function notation as $(f-g)'=f'-g'$.

le 4 can be extended to any finite number of functions.
$f_1, f_2, \ldots\ldots f_n$ are all differentiable at x, then their sum is differentiable
x and:

$$\frac{d}{dx}[f_1(x)+f_2(x)+\ldots\ldots f_n(x)]=$$

$$=\frac{d}{dx}[f_1(x)]+\frac{d}{dx}[f_2(x)]+\ldots\ldots+\frac{d}{dx}[f_n(x)].$$

xample:

$$\frac{d}{dx}\left[3x^8+7x^5+6x-7\right]=\frac{d}{dx}\left[3x^8\right]+\frac{d}{dx}\left[7x^5\right]+\frac{d}{dx}[6x]+\frac{d}{dx}[-7]=$$

$$=24x^7+35x^4+6.$$

ule 5: (The product rule) If f and g are differentiable at x, then so is
e product $f\cdot g$, and

$$\frac{d}{dx}[f(x)\cdot g(x)]=f(x)\cdot\frac{d}{dx}[g(x)]+g(x)\cdot\frac{d}{dx}[f(x)]$$

he product rule can be written in function notation as

$$(f\cdot g)'=f\cdot g'+g\cdot f'.$$

Varning: Note that it is **not** true in general that $(f\cdot g)'=f'\cdot g'$; that
derivative of a product is **not** generally the product of the derivatives.

xample:

Find $\frac{dy}{dx}$ if $y=(4x^2-7)(7x^3+x)$

olution: There are two methods that can be used to find dy/dx. We can
ither use the product rule or we can multiply out the factors in y and
hen differentiate.

Method 1. (Using product rule)

$$\frac{dy}{dx}=\frac{d}{dx}[(4x^2-7)(7x^3+x)]=$$

$$=(4x^2-7)\frac{d}{dx}[7x^3+x]+(7x^3+x)\frac{d}{dx}[4x^2-7]=$$

$$=(4x^2-7)(21x^2+1)+(7x^3+x)(8x)=$$

$$=84x^4+4x^2-147x^2-7+56x^4+8x^2=140x^4-135x^2-7.$$

Method 2. (Multiplying first)

$$y=(4x^2-7)(7x^3+x)=$$

$$=28x^5+4x^3-49x^3-7x=28x^5-45x^3-7x.$$

Thus,

$$\frac{dy}{dx}=\frac{d}{dx}[28x^5-45x^3-7x]=140x^4-135x^2-7,$$

which agrees with the result obtained using the product rule.

The product rule can be extended to any finite number of differentiable functions :

$$(f_1\cdot f_2\cdot\ldots\cdot f_n)'=f_1'\cdot f_2\cdot\ldots f_n+f_2'\cdot f_1\cdot\ldots f_n+\ldots\ldots$$

$$+f_n'\cdot f_1\cdot f_2\cdot\ldots f_{n-1}$$

Rule 6: (The quotient rule) If f and g are differentiable at x and $g(x)$ then f/g is differentiable at x and and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\cdot\frac{d}{dx}[f(x)]-f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

The quotient rule can be written in function notation as

$$\left(\frac{f}{g}\right)'=\frac{f'\cdot g-g'\cdot f}{g^2}.$$

Example: Let $y=\frac{x^2-1}{x^4+1}$. Find $\frac{dy}{dx}$.

Solution: $$\frac{dy}{dx}=\frac{(x^4+1)(x^2-1)'-(x^2-1)(x^4+1)'}{(x^4+1)^2}=$$

$$=\frac{(x^4+1)(2x)-(x^2-1)(4x^3)}{(x^4+1)^2}=$$

$$=\frac{2x^5+2x-4x^5+4x^3}{(x^4+1)^2}=\frac{-2x^5+4x^3+2x}{(x^4+1)^2}=-\frac{2x(x^4-2x^2-1)}{(x^4+1)^2}.$$

emark: Since it is needed often, ... worth memorizing that if $f(x)=c$ then

$$\left(\frac{f}{g}\right)'=\left(\frac{c}{g}\right)'=-\frac{c\cdot g'}{g}$$

xample: Find $\left(\frac{1}{2x^3+x+5}\right)'$.

olution: By the formula for $\left(\frac{1}{g}\right)'$,

$$\left(\frac{1}{2x^3+x+5}\right)'=-\frac{1(2x^3+x+5)'}{(2x^3+x+5)^2}=-\frac{6x^2+1}{(2x^3+x+5)^2}.$$

xample: $\frac{d}{dx}[x^{-9}]=-9\cdot x^{-9-1}=-9x^{-10}$;

$$\frac{d}{dx}\left[\frac{1}{x}\right]=\frac{d}{dx}[x^{-1}]=(-1)x^{-1-1}=-\frac{1}{x^2}.$$

xample: Find y' if $y=\frac{1}{5}x^5-\frac{2}{3}x^3+x$

olution: Applying rule 2, rule 3, and rule 4 where it is necessary, we obtain

$$y'=(\frac{1}{5}x^5)'-(\frac{2}{3}x^3)'+(x)'=\frac{1}{5}(x^5)'-\frac{2}{3}(x^3)'+(x)'=$$

$$=\frac{1}{5}5x^4-\frac{2}{3}3x^2+1=x^4-2x^2+1=(x^2-1)^2.$$

xample: Find $f'(x)$ if $f(x)=\frac{x^2-2}{x^2+2}$

olution: Using rule 6 we get

$$f'(x)=\left(\frac{x^2-2}{x^2+2}\right)'=\frac{(x^2-2)'(x^2+2)-(x^2-2)(x^2+2)'}{(x^2+2)^2}=$$

$$=\frac{2x(x^2+2)-2x(x^2-2)}{(x^2+2)^2}=\frac{2x(x^2+2-x^2+2)}{(x^2+2)^2}=\frac{8x}{(x^2+2)^2}.$$

Example: Find y' if $y=(2x^7-x^2)\cdot\frac{x-1}{x+1}$

Solution:

$$y'=\left[(2x^7-x^2)\cdot\frac{x-1}{x+1}\right]'=(2x^7-x^2)'\cdot\frac{x-1}{x+1}+$$

$$+\left(\frac{x-1}{x+1}\right)'\cdot(2x^7-x^2)=(14x^6-2x)\cdot\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\cdot(2x^7-x^2).$$

Example: Find y' if $y=(x^5+2x)^2$

Solution: $y=(x^5+2x)^2=x^{10}+4x^6+4x^2$

$$y'=(x^{10}+4x^6+4x^2)'=10x^9+24x^5+8x.$$

Example: Find y' if $y=\frac{\lambda\cdot\lambda_0+\lambda^6}{2-\lambda_0}$ (λ_0 is constant)

Solution: $y'=\left(\frac{\lambda\cdot\lambda_0+\lambda^6}{2-\lambda_0}\right)'=\frac{1}{2-\lambda_0}(\lambda\cdot\lambda_0+\lambda^6)'=\frac{1}{2-\lambda}(\lambda_0+6$

3.6. Higher order derivatives.

If the derivative of y' of a function y is itself differentiabl

then the derivative of y' is denoted by y'' (or $\frac{d^2y}{dx^2}$) and is called t

second derivative of *y*.

$$y''=\frac{dy'}{dx}=\frac{d}{dx}\left[\frac{dy}{dx}\right]=\frac{d^2y}{dx^2}.$$

As long as we have differentiability, we can continue the process differentiating derivatives to obtain third, fourth, fifth, and even hi derivatives of *y*.

$$y'''=\frac{dy''}{dx};\ldots\ldots\ldots\ldots;\ y^{(n)}=\frac{d}{dx^n}[y^{(n-1)}].$$

y' -read as *y* prime

y'' -read as *y* double prime

$\frac{d^2y}{dx^2}$ -read as *d* squared *y dx* squared

$y^{(n)}$ -read as *y* super *n*.

ple: If $y=3x^4-2x^3+x^2+4x+2$, then

$\frac{y}{x}=12x^3-6x^2+2x+4$;

$\frac{^2y}{x^2}=36x^2-12x+2$

$\frac{d^3y}{dx^3}=72x-12$;

$\frac{d^4y}{dx^4}=72$

$=0$.

Exercises.

exercises 1-15 use the techniques of differentiation to find *dy/dx*.

$=4x^7$ 2. $y=3x^8+2x+1$

$=\pi^3$ 4. $y=-\frac{1}{3}(x^7+2x-9)$

$\sqrt{2}x+\left[1/\sqrt{2}\right]$ 6. $y=ax^3+bx^2+cx+d$

$\frac{1}{a}x^2+\frac{1}{b}x+c$ 8. $y=x^{-3}+\frac{1}{x^7}$

$(3x^2+6)(2x-\frac{1}{4})$ 10. $y=(x^3+7x^2-8)(2x^{-3}+x^{-4})$

$=(3x^2+1)^2$ 12. $y=\frac{1}{5x-3}$

$=\frac{3x}{2x+1}$ 14. $y=\frac{2x-1}{x+3}$

$=\frac{1+1/x}{1-1/x}$

$f(3)=-2$ and $f'(3)=4$, find $g'(3)$

a) $g(x)=3x^2-5\cdot f(x)$ b) $g(x)=\dfrac{2x+1}{f(x)}$

17. If $f(2)=-3$ and $f'(2)=4$, $g(2)=1$ and $g'(2)=-5$ then find $F'(2)$

a) $F(x)=5\cdot f(x)+2\cdot g(x)$ b) $F(x)=f(x)-3\cdot g(x)$

c) $F(x)=f(x)\cdot g(x)$ d) $F(x)=\dfrac{f(x)}{g(x)}$

18. Find y''

a) $y=4x^7-5x^3+2x$ b) $y=3x+2$

c) $y=\dfrac{3x-2}{5x}$ d) $y=(x^3-5)(2x+3)$

19. Find y'''

a) $y=x^{-5}+x^5$ b) $y=\dfrac{1}{x}$

c) $y=ax^3+bx+c$ (*a*, *b*, *c* constant)

20. Find

a) $f'''(2)$, where $f(x)=3x^2-2$

b) $\left.\dfrac{d^2y}{dx^2}\right|_{x=1}$, where $y=6x^5-4x^2$

c) $\left.\dfrac{d^4}{dx^4}\left[x^{-3}\right]\right|_{x=1}$

21. Find $\dfrac{d^2y}{dx^2}$

a) $y=7x^3-5x^2+x$ b) $y=12x^2-2x+3$

c) $y=\dfrac{x+1}{x}$ d) $y=(5x^2-3)(7x^3+x)$

In each of exercises 22-25 find an equation of the tangent to th given curve at the given point.

22. $y=x^3-x^2+2x$ at (1, 2)

$y=\dfrac{1}{2x+1}$ at (2, 1/5)

. $y=\sqrt{x}(x^2+2)$ at (4, 36)

. $y=\dfrac{x+1}{x+2}$ at (-1, 0)

Answers.

$28x^6$; **2.** $24x^7+2$; **3.** 0 ; **4.** $-\frac{7}{3}x^6-\frac{2}{3}$; **5.** $\sqrt{2}$;
$3ax^2+2bx+c$; **7.** $\frac{2x}{a}+\frac{1}{b}$; **8.** $-3x^{-4}-\frac{7}{x^8}$; **9.** $18x^2-\frac{3}{2}x+12$;
. $-15x^{-2}-14x^{-3}+48x^{-4}+32x^{-5}$;**11.** $12x(3x^2+1)$;
. $-\frac{5}{(5x-3)^2}$; **13.** $\frac{3}{(2x+1)^2}$; **14.** $\frac{7}{(x+3)^2}$; **15.** $\frac{-2}{(x-1)^2}$;
. a) -2; b) –8; **17.** a) 10; b) 19; c) 19; d) -11; **18.** a) $168x^5-30x$;
0; c) $\frac{-4}{5x^3}$; d) $6x(4x+3)$; **19.** a) $-210x^{-8}+60x^2$; b) $-6x^{-4}$;
$6a$; **20.** a) 0; b) 112; c) 360 ; **21.** a) $42x-10$; b) 24; c) $2/x^3$;
$700x^3-96x$; **22.** $y=3x-1$; **23.** $y=-\frac{2x}{25}+\frac{9}{25}$;
. $y=20.5x-46$; **25.** $y=x+1$.

3.7. The derivatives of the trigonometric functions.

For the purpose of finding derivatives of the trigonometric ınctions sin*x*,cos*x*, tan*x*, cot*x* , sec*x* , and csc*x*, we shall assume that *x* measured in radians. In order to find $\frac{d}{dx}(\sin x)$ and $\frac{d}{dx}(\cos x)$, it will e necessary to make use of the limits

$$\lim_{h\to 0}\frac{\sin h}{h}=1 \quad\text{and}\quad \lim_{h\to 0}\frac{1-\cos h}{h}=0.$$

Let us first consider the problem of differentiating sin *x*. Let *x* e any real number. From the definition of a derivative,

$$\frac{d}{dx}[\sin x]=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=$$

$$=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}=$$

$$=\lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{\text{h}}\right)+\cos x\left(\frac{\sin h}{h}\right)\right]=$$

$$=\lim_{h\to 0}\left[\cos x\left(\frac{\sin h}{h}\right)-\sin x\left(\frac{1-\cos h}{h}\right)\right]$$

Since sin$x$ and cos$x$ do not involve h, they remain constant as $h\to 0$; thus

$$\lim_{h\to 0}(\sin x)=\sin x \text{ and } \lim_{h\to 0}(\cos x)=\cos x.$$

Consequently,

$$\frac{d}{dx}[\sin x]=\cos x\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right)-\sin x\cdot\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)=$$

$$=\cos x\cdot(1)-\sin x\cdot 0=\cos x.$$

Thus, we have shown that

(1) $\frac{d}{dx}[\sin x]=\cos x$;

or in function notation $(\sin x)'=\cos x$

The derivative of cos$x$ can be obtained similarly, resulting in the formula

(2) $\frac{d}{dx}[\cos x]=-\sin x$;

or in function notation $(\cos x)'=-\sin x$

The derivatives of remaining trigonometric functions can be obtained using the relationships

$$\tan x=\frac{\sin x}{\cos x};\quad \cot x=\frac{\cos x}{\sin x};\quad \sec x=\frac{1}{\cos x};\quad \csc x=\frac{1}{\sin x}.$$

For example,

$$\frac{d}{dx}[\tan x]=\frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]=\frac{\cos x\cdot\frac{d}{dx}[\sin x]-\sin x\cdot\frac{d}{dx}[\cos x]}{\cos^2 x}=$$

$$\frac{x\cdot\cos x-\sin x\cdot(-\sin x)}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\sec^2 x.$$

(3) $\frac{d}{dx}[\tan x]=\sec^2 x$;

or in function notation $(\tan x)'=\sec^2 x$

remaining formulas are

(4) $\frac{d}{dx}[\cot x]=-\csc^2 x$;

or in function notation $(\cot x)'=-\csc^2 x$

(5) $\frac{d}{dx}[\sec x]=\sec x\cdot\tan x$;

or in function notation $(\sec x)'=\sec x\cdot\tan x$

(6) $\frac{d}{dx}[\csc x]=-\csc x\cdot\cot x$;

or in function notation $(\csc x)'=-\csc x\cdot\cot x$

mple: Find $f'(x)$ if $f(x)=x^2\cdot\tan x$

tion: Using the product rule and formula (3), we obtain

$$f'(x)=x^2\cdot\frac{d}{dx}[\tan x]+\tan x\cdot\frac{d}{dx}[x^2]=$$

$$=x^2\cdot\sec^2 x+2x\cdot\tan x.$$

mple: Find dy/dx if $y=\sin 2x$

tion:

$\sin 2x=2\sin x\cdot\cos x$ trigonometric identity

s

$$(\sin 2x)'=(2\sin x\cos x)'=2(\sin x\cos x)'=$$

$$=2\cdot\left[\sin x(\cos x)'+\cos x(\sin x)'\right]=$$

$$\cdot[\sin x(-\sin x)+\cos x(\cos x)]=2(\cos^2 x-\sin^2 x)=2\cos 2x.$$

ample: Find $(x^3\cdot\sec x)'$

tion: $(x^3\cdot\sec x)'=$

$$^3\cdot(\sec x)'+\sec x\cdot(x^3)'=x^3\sec x\cdot\tan x+\sec x\cdot(3x^2)$$

Example:

Find dy/dx if $y = \dfrac{\sin x}{1+\cos x}$

Solution: Using the quotient rule together with formulas (1) and (
we obtain

$$\frac{dy}{dx} = \frac{(1+\cos x)\dfrac{d}{dx}[\sin x] - \sin x \cdot \dfrac{d}{dx}[1+\cos x]}{(1+\cos x)^2} =$$

$$= \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} =$$

$$= \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}.$$

Example: Find $y''\left(\dfrac{\pi}{4}\right)$ if $y(x) = \sec x$

Solution:

$$y' = \sec x \tan x;$$

$$y'' = \sec x \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[\sec x] =$$

$$= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \cdot \tan x = \sec^3 x + \sec x \cdot \tan^2 x.$$

Thus,

$$y''\left(\frac{\pi}{4}\right) = \sec^3\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right) \cdot \tan^2\left(\frac{\pi}{4}\right) =$$

$$= (\sqrt{2})^3 + (\sqrt{2}) \cdot (1)^2 = 3\sqrt{2}.$$

Exercises.

In exercises 1-12 find $f'(x)$

1. $f(x) = 2\cos x - 3\sin x$
2. $f(x) = \dfrac{\sin x}{x}$
3. $f(x) = x^3 \sin x - 5\cos x$
4. $f(x) = \sec x - \sqrt{2}\tan x$
5. $f(x) = \sec x \cdot \tan x$
6. $f(x) = x - 4\csc x + 2\cot x$

$f(x) = \csc x \cdot \cot x$

8. $f(x) = \dfrac{\cot x}{1+\csc x}$

$f(x) = \sin^2 x + \cos^2 x$

0. $f(x) = \dfrac{\sin x \sec x}{1 + x \cdot \tan x}$

1. $f(x) = \dfrac{1+3\sec x}{\tan x}$

2. $f(x) = \dfrac{\cot x}{1+x^2}$

In exercises 13-17 find $\dfrac{d^2 y}{dx^2}$

13. $y = x \cdot \cos x$
14. $y = \csc x$
15. $y = x \cdot \sin x - 3\cos x$
16. $y = x^2 \cdot \cos x + 4\sin x$
17. $y = \sin x \cdot \cos x$

Answers.

1. $-2\sin x - 3\cos x$; **2.** $\dfrac{x\cos x - \sin x}{x^2}$; **3.** $x^3\cos x + (3x^2+5)\sin x$; **4.** $\sec x \cdot \tan x - \sqrt{2}\sec^2 x$; **5.** $\sec^3 x + \sec x \cdot \tan^2 x$; **6.** $1 + 4\csc x \cot x - 2\csc^2 x$; **7.** $-\dfrac{1+\cos^2 x}{\sin^3 x}$; **8.** $-\dfrac{1}{1+\sin x}$; **9.** 0; **10.** $\dfrac{1}{(1+x\cdot\tan x)^2}$ **11.** $\dfrac{-1-3\cos x}{\sin^2 x}$; **12.** $\dfrac{-(1+x^2)\csc^2 x - 2x\cdot\cot x}{(1+x^2)^2}$;

13. $-x\cdot\cos x - 2\sin x$; **14.** $\dfrac{1+\cos^2 x}{\sin^3 x}$; **15.** $-x\cdot\sin x + 5\cos x$;

16. $2\cos x - 4x\cdot\sin x - x^2\cdot\cos x - 4\sin x$; **17.** $-4\sin x\cdot\cos x$.

Example:

Find f' if $f(x)=(x^2-x+1)^{23}$

Solution: Let $u=x^2-x+1$, so $f(u)=u^{23}$, then apply (3) to obta

$$\frac{d}{dx}\left[(x^2-x+1)^{23}\right]=\frac{d}{dx}[u^{23}]=23u^{22}\cdot\frac{du}{dx}=$$

$$=23(x^2-x+1)^{22}\cdot\frac{d}{dx}[x^2-x+1]=$$

$$=23(x^2-x+1)^{22}(2x-1).$$

More generally, if u were any other differentiable function of x, the pattern of computation would be the same. For example, if $u=\cos$ then

$$\frac{d}{dx}\left[\cos^{23}x\right]=\frac{d}{dx}[u^{23}]=23u^{22}\cdot\frac{du}{dx}=$$

$$=23\cos^{22}x\cdot\frac{d}{dx}[\cos x]=-23\sin x\cdot\cos^{22}x.$$

Generalized derivative formulas.

$$\frac{d}{dx}[u^n]=n\cdot u^{n-1}\cdot\frac{du}{dx}\quad (n \text{ an integer})$$

$$\frac{d}{dx}\left[\sqrt{u}\right]=\frac{1}{2\sqrt{u}}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[\sin u]=\cos u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u]=-\sin u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u]=\sec^2 u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[\cot u]=-\csc^2 u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[\sec u]=\sec u\cdot\tan u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[\csc u]=-\csc u\cdot\cot u\cdot\frac{du}{dx}.$$

ımple: Find $\frac{d}{dx}[\sin(2x)]$

ution: Taking $u=2x$ in the generalized derivative formula sin u yields

$$\frac{d}{dx}[\sin(2x)]=\frac{d}{dx}[\sin u]=\cos u\cdot\frac{du}{dx}=$$

$$=\cos(2x)\cdot\frac{d}{dx}[2x]=\cos(2x)\cdot 2=2\cos(2x).$$

ample:

Find $\frac{d}{dx}[\tan(x^2+1)]$

ution: Taking $u=x^2+1$ in the generalized derivative formula for tan u yields

$$\frac{d}{dx}[\tan(x^2+1)]=\frac{d}{dx}[\tan u]=\sec^2 u\cdot\frac{du}{dx}=$$

$$=\sec^2(x^2+1)\cdot\frac{d}{dx}[x^2+1]=2x\cdot\sec^2(x^2+1).$$

ample: Find $\frac{d}{dx}\left[(1+x^5\cdot\cot x)^{-8}\right]$

ution: Taking $u=1+x^5\cdot\cot x$ in the generalized derivative mula for u^{-8} yields

$$\frac{d}{dx}\left[(1+x^5\cdot\cot x)^{-8}\right]=\frac{d}{dx}[u^{-8}]=-8u^{-9}\frac{du}{dx}=$$

$$=-8(1+x^5\cdot\cot x)^{-9}\cdot\frac{d}{dx}[x^5\cdot(-\csc^2 x)+5x^4\cdot\cot x]=$$

$$=(1+x^5\cdot\cot x)^{-9}\cdot(8x^5\cdot\csc^2 x-40x^4\cdot\cot x)$$

; sometimes necessary to apply the Chain rule more than once to l a derivative.

ample: Find $\frac{d}{dx}\left[\cos^2(\pi x)\right]$

ution: Taking $u=\cos(\pi x)$ in the generalized derivative formula for u^2 yields

$$\frac{d}{dx}\left[\cos^2(\pi x)\right]=\frac{d}{dx}[u^2]=2u\frac{du}{dx}=2\cos(\pi x)\cdot\frac{d}{dx}[\cos(\pi x)]$$

Taking $u=\pi x$ in the generalized derivative formula for $\cos u$ yiel

$$2\cos(\pi x)\cdot\frac{d}{dx}[\cos(\pi x)]=2\cos(\pi x)\cdot(-\pi\cdot\sin(\pi x))=$$

$$=-2\pi\cdot\sin(\pi x)\cdot\cos(\pi x)=-\pi\sin(2\pi x).$$

Example:

Find $\frac{d}{dx}\left[\sin\sqrt{1+\cos x}\right]$

Solution: $\frac{d}{dx}\left[\sin\sqrt{1+\cos x}\right]=\cos\sqrt{1+\cos x}\cdot\frac{d}{dx}\left[\sqrt{1+\cos x}\right]=$

$$=\cos\sqrt{1+\cos x}\cdot\frac{-\sin x}{2\sqrt{1+\cos x}}=\frac{-\sin x\cdot\cos\sqrt{1+\cos x}}{2\sqrt{1+\cos x}}.$$

As you become more comfortable with using the Chain rule, you may want to dispense with actually writing out the expression fo in your computations. To accomplish this, it is helpful to express formula (3) in words. If we call *u* the " inside function " and *f* the "outside function " in the composition $f(u)$, then (3) states:
" To find *dy/dx* , differentiate the " outside function " *f* and leave the "inside function" *u* alone; then multiply by the derivative of " inside function ".

Example: Find $\frac{d}{dx}\left[\cos(x^2+9)\right]=\underbrace{-\sin(x^2+9)}_{\text{derivative of the outside function}}\cdot\underbrace{2x}_{\text{derivative of the inside function}}$

Example: Find $\frac{d}{dx}\left[\tan^2 x\right]=\frac{d}{dx}\left[\tan x\right]^2=\underbrace{2\tan x}_{\text{derivative of the outside function}}\cdot\underbrace{\sec^2 x}_{\text{derivative of the inside function}}$

Example: Find $\frac{d\mu}{dt}$ if $\mu=t^2\cdot\sec\sqrt{\omega t}$

Solution: Because the independent variable is *t* rather than *x*, appropriate adjustments in notation have to be made.

$$\frac{d\mu}{dt}=\frac{d}{dt}\left[t^2\cdot\sec\sqrt{\omega t}\right]=$$

$$=2t\cdot\sec\sqrt{\omega t}+t^2\cdot\sec\sqrt{\omega t}\cdot\tan\sqrt{\omega t}\cdot\frac{d}{dt}\left[\sqrt{\omega t}\right]=$$

$$=2t\cdot\sec\sqrt{\omega t}+t^2\cdot\sec\sqrt{\omega t}\cdot\tan\sqrt{\omega t}\cdot\frac{\omega}{2\sqrt{\omega t}}.$$

Exercises.

In exercises 1-20 find $f'(x)$

1. $f(x)=(x^3+2x)^{37}$
2. $f(x)=\left(x^3-\frac{7}{x}\right)^{-2}$
3. $f(x)=\frac{4}{(3x^2-2x+1)^3}$
4. $f(x)=\sqrt{4+3\sqrt{x}}$
5. $f(x)=\sin(x^3)$
6. $f(x)=\tan(4x^2)$
7. $f(x)=4\cos^5 x$
8. $f(x)=\sin\left(\frac{1}{x^2}\right)$
9. $f(x)=2\sec^2(x^7)$
10. $f(x)=\sqrt{\cos(5x)}$
11. $f(x)=\left[x+\csc(x^3+3)\right]^{-3}$
12. $f(x)=x^2\cdot\sqrt{5-x^2}$
13. $f(x)=x^3\cdot\sin^2(5x)$
14. $f(x)=x^5\cdot\sec\left(\frac{1}{x}\right)$
15. $f(x)=\cos(\cos x)$
16. $f(x)=\cos^3(\sin 2x)$
17. $f(x)=(5x+8)^{13}\cdot(x^3+7x)^{12}$
18. $f(x)=\left(\frac{x-5}{2x+1}\right)^3$
19. $f(x)=\frac{(2x+3)^3}{(4x^2-1)^8}$
20. $f(x)=\left(x\cdot\sin(2x)+\tan^4(x^7)\right)^5$
21. **Find** $\frac{d^2y}{dx^2}$ if $y=x\cdot\cos(5x)-\sin^2 x$

In exercises 22-23 find an equation for the tangent line to the **graph at the** specified point.

22. $y=x\cdot\cos(3x)$; $x=\pi$

23. $y=\sec^3\left(\frac{\pi}{2}-x\right);\quad x=-\frac{\pi}{2}$

In exercises 24-25, find the indicated derivative

24. $y=\cot^3(\pi-\theta)$; find $\frac{dy}{d\theta}$

25. $\frac{d}{dw}\left[a\cdot\cos^2(\pi w)+b\cdot\sin^2(\pi w)\right]$; ($a$, b constants)

26. Given that $f'(0)=2$, $g(0)=0$, and $g'(0)=3$, find $(f\circ g)'($

27. Given that $f'(x)=\sqrt{3x+4}$ and $g(x)=x^2-1$, find $F'(x)$ if $F(x)=f(g(x))$

28. Given that $f'(x)=\frac{x}{x^2+1}$ and $g(x)=\sqrt{3x-1}$, find $F'(x)$ if $F(x)=f(g(x))$

In exercises 29-30 find the value of $(f\circ g)'$ at the given value

29. $f(u)=\cot\frac{\pi u}{10}$; $u=g(x)=5\sqrt{x}$; $x=1$.

30. $f(u)=\frac{2u}{u^2+1}$; $u=g(x)=10x^2+x+1$; $x=0$.

Answers.

1. $37\cdot(x^3+2x)^{36}\cdot(3x^2+2)$; **2.** $-2\left(x^3-\frac{7}{x}\right)^{-3}\left(3x^2+\frac{7}{x^2}\right)$;

3. $\frac{24\cdot(1-3x)}{(3x^2-2x+1)^4}$; **4.** $\frac{3}{4\sqrt{x}\cdot\sqrt{4+3\sqrt{x}}}$; **5.** $3x^2\cdot\cos(x^3)$;

6. $8x\cdot\sec^2(4x^2)$; **7.** $-20\cdot\cos^4x\cdot\sin x$; **8.** $-\frac{2}{x^3}\cdot\cos(\frac{1}{x^2})$;

9. $28x^6\cdot\sec^2(x^7)\cdot\tan(x^7)$; **10.** $-\frac{5\cdot\sin(5x)}{2\sqrt{\cos(5x)}}$;

11. $-3\cdot\left[x+\csc(x^3+3)\right]^{-4}\cdot\left[1-3x^2\cdot\csc(x^3+3)\cdot\cot(x^3+3)\right]$;

12. $\frac{x\cdot(10-3x^2)}{\sqrt{5-x^2}}$; **13.** $10x^3\cdot\sin(5x)\cdot\cos(5x)+3x^2\cdot\sin^2(5x)$;

$-x^3\cdot\sec\left(\frac{1}{x}\right)\cdot\tan\left(\frac{1}{x}\right)+5x^4\cdot\sec\left(\frac{1}{x}\right)$; **15.** $\sin x\cdot\sin(\cos x)$;

$-6\cdot\cos^2(\sin 2x)\cdot\sin(\sin 2x)\cdot\cos 2x$;

$12\cdot(5x+8)^{13}\cdot(x^3+7x)^{11}\cdot(3x^2+7)+65\cdot(x^3+7x)^{12}\cdot(5x+8)^{12}$;

$\frac{33(x-5)^2}{(2x+1)^4}$; **19.** $-\frac{2\cdot(2x+3)^2\cdot(52x^2+96x+3)}{(4x^2-1)^9}$;

$5\cdot\left[x\cdot\sin 2x+\tan^4(x^7)\right]^4\cdot[2x\cos(2x)+\sin(2x)+28x^6\cdot$

n$^3(x^7)\cdot\sec^2(x^7)]$; **21.** $-25x\cdot\cos(5x)-10\cdot\sin(5x)-2\cdot\cos(2x)$;

$y=-x$; **23.** $y=-1$; **24.** $3\cdot\cot^2\theta\cdot\csc^2\theta$;

$\pi\cdot(b-a)\cdot\sin(2\pi w)$; **26.** 6; **27.** $2x\cdot\sqrt{3x^2+1}$; **28.** $\frac{1}{2x}$; **29.** $-\frac{\pi}{4}$;

0.

3.9. Implicit differentiation.

Consider the equation

(1) $xy=1$

e way to obtain $\frac{dy}{dx}$ is to rewrite this equation as

(2) $y=\frac{1}{x}$

which it follows that

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{1}{x}\right]=-\frac{1}{x^2}$$

vever, there is another possibility. We can differentiate both sides of before solving for y in terms of x, treating y as a differentiable ction of x. With this approach we obtain

$$\frac{d}{dx}[xy]=\frac{d}{dx}[1]$$

$$x\frac{d}{dx}[y]+y\frac{d}{dx}[x]=0$$

$$x\frac{dy}{dx}+y=0 \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{y}{x}$$

If we now substitute (2) into the last expression, we obtain

$$\frac{dy}{dx}=-\frac{1}{x^2}$$

which agrees with the previous computation.

The equation $xy=1$ is said to describe the function $y=f(x)$ **implicitly.** The equation $y=\frac{1}{x}$ describes the function $y=f(x)$ **explicitly**.

The second method of obtaining derivatives is called implici differentiation. It is especially useful when it is inconvenient or impossible to solve explicitly for y in terms of x.

Example: By implicit differentiation find dy/dx if $5y^2+\sin y=x^2$

Solution: Differentiating both sides with respect to x and treating y a differentiable function of x, we obtain

$$\frac{d}{dx}[5y^2+\sin y]=\frac{d}{dx}[x^2]$$

$$5\frac{d}{dx}[y^2]+\frac{d}{dx}[\sin y]=2x$$

$$5\cdot(2y\frac{dy}{dx})+(\cos y)\frac{dy}{dx}=2x$$

$$10y\frac{dy}{dx}+(\cos y)\frac{dy}{dx}=2x$$

Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx}=\frac{2x}{10y+\cos y}$$

Note that this formula for dy/dx involves both the variables x and y. In order to obtain a formula involving x alone we would have t solve the original equation for y in terms of x and substitute it in dy/d However, it is impossible to do this, so the formula for dy/dx must be left in terms of x and y.

xample: Find $\frac{dy}{dx}$ if $3x+y^3=y^2+4$

olution: $\frac{d}{dx}[3x+y^3]=\frac{d}{dx}[y^2+4]$

$$3+3y^2\frac{dy}{dx}=2y\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx}-2y\frac{dy}{dx}=-3$$

$$\frac{dy}{dx}=\frac{3}{2y-3y^2}$$

ne **important** thing to remember when doing implicit differentiation:

The derivative of y is y'. The derivative of x is taken in the ormal way of a variable, so that the derivative of x is 1, as an example. he reason for this is that y is a function of x, and does not exist without eing dependant on the equation on the other side. The derivative of, or example, y^2 is $2y\cdot y'$. The reason is that we actually use Chain ule to solve it. So the " outside function" has a derivative $2y$, and the inside function" has derivative y', we multiply them together and btain $2y\cdot y'$. So the derivative of $3y^2$ is $6y\cdot y'$.

xample: Assume that equation $2xy+\pi\cdot\sin y=2\pi$ defines $y=f(x)$.

Find $\frac{dy}{dx}$ when $x=1$ and $y=\frac{\pi}{2}$

olution: Implicit differentiation yields

$$\frac{d}{dx}[2xy+\pi\cdot\sin y]=\frac{d}{dx}[2\pi]$$

$$2x\frac{dy}{dx}+2y+\pi\cdot(\cos y)\frac{dy}{dx}=0$$

olving for the derivative, dy/dx, we get

$$\frac{dy}{dx}=-\frac{2y}{2x+\pi\cdot\cos y}$$

In particular, when $x = 1$ and $y = \frac{\pi}{2}$,

$$\frac{dy}{dx} = -\frac{2 \cdot \frac{\pi}{2}}{2 \cdot 1 + \pi \cdot \cos\frac{\pi}{2}} = -\frac{\pi}{2}$$

Implicit differentiation takes four steps:

Step 1. Differentiate both sides of the equation.

Step 2. Send all terms with y' to one side of the equation, and all t without to the other side.

Step 3. On the side with y' terms, factor out the y'.

Step 4. Solve for y' by dividing.

Example:

Find the slope of the tangent line at the point (4,0) on the g of $7y^4 + x^3y + x = 4$

Solution: It is difficult to solve given equation for y in terms of x, we shall differentiate implicitly. We obtain

$$\frac{d}{dx}[7y^4 + x^3y + x] = \frac{d}{dx}[4]$$

$$28y^3\frac{dy}{dx} + x^3\frac{dy}{dx} + 3x^2y + 1 = 0$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = -\frac{1 + 3x^2y}{28y^3 + x^3}$$

At the point (4,0) we have $x = 4$, and $y = 0$, so

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=4\\y=0}} = -\frac{1}{64}$$

Example: Find the tangent and normal to the curve $y^2 - x + 1 = 0$ at the point (2, -1).

Solution: We first use implicit differentiation to find $\frac{dy}{dx}$:

$$\frac{d}{dx}[y^2 - x + 1] = \frac{d}{dx}[0]$$

$$2y\frac{dy}{dx} - 1 = 0$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

: then evaluate the derivative at $x = 2$, $y = -1$, to obtain

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-1}} = -\frac{1}{2}$$

e tangent to the curve at (2, -1) is

$$y - (-1) = -\frac{1}{2}(x - 2)$$

$y = -\frac{x}{2} + 1 - 1$ and in the end $y = -\frac{x}{2}$

e slope of normal line can be found from the condition $m_1 \cdot m_2 = -1$

$$m_{nor} = 2$$

e normal to the curve at (2, -1) is

$$y - (-1) = 2(x - 2)$$

$$y = 2x - 3$$

ample: Find slope of the curve $x^2 + y^3 = 2x + y$ at (2, 1).

lution:

$$\frac{d}{dx}[x^2 + y^3] = \frac{d}{dx}[2x + y]$$

$$2x + 3y^2\frac{dy}{dx} = 2 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2 - 2x}{3y^2 - 1}$$

ibstitute (2, 1) into $\frac{dy}{dx}$ to find the slope at that point

$\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=1}} = \frac{2 - 2 \cdot 2}{3 \cdot 1^2 - 1} = -1$ is the slope of the curve.

Example: Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$ if $4x^2-2y^2$

Solution: Differentiating both sides of equation implicitly yields

$$8x-4y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{2x}{y}$$

We now apply the quotient rule to find y''

$$y''=\frac{d}{dx}\left(\frac{2x}{y}\right)=\frac{2\cdot y-2x\cdot\dfrac{dy}{dx}}{y^2}$$

Finally we substitute $y'=\dfrac{2x}{y}$ into y'' to express y'' in terms of x and y

$$y''=\frac{d^2y}{dx^2}=\frac{2y-2x\cdot\dfrac{2x}{y}}{y^2}=\frac{2y^2-4x^2}{y}\cdot\frac{1}{y^2}=\frac{2y^2-4x^2}{y^3}.$$

Exercises.

In exercises 1-12 find dy/dx by implicit differentiation.

1. $x^2+y^2=25$
2. $x^2y+3xy^3-x=7$
3. $\dfrac{1}{x}+\dfrac{1}{y}=2$
4. $\sqrt{x}+\sqrt{y}=9$
5. $(x^2+3y^2)^{35}=x$
6. $3xy=(x^3+y^2)^{3/2}$
7. $\sin(x^2y^2)=x$
8. $\tan^3(xy^2+y)=x$
9. $\sqrt{1+\sin^3(xy^2)}=y$
10. $xy+\sin y=0$
11. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ (a, b constants)
12. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

In exercises 13-17 use the differentiation to find the slope the tangent line to the given curve at the given point.

13. $\dfrac{2xy}{\pi}+\sin y=2$ at $\left(1,\ \pi/2\right)$

14. $2y^3+4xy+x^2=7$ at (1, 1)
15. $x^5+y^3x+yx^2+y^5=4$ at (1, 1)
16. $x+\tan xy=2$ at $\left(1,\ \pi/4\right)$
17. $x^3y+y^3x=10$ at (1, 2)

In exercises 18-21 find $\dfrac{d^2y}{dx^2}$ by implicit differentiation.

18. $3x^2-4y^2=6$
19. $x^3+y^3=27$
20. $x^3y^3-6=0$
21. $2xy-y^2=13$

In exercises 22-23 find the lines that are a) tangent and b) normal to the curve at the given point.

22. $x^2+xy^2-2y^2=0$ at (1, 1)
23. $x^2y^2=9$ at (-1, 3)
24. At what point(s) is the tangent line to the curve $y^2=2x^3$ perpendicular to the line $4x-3y+1=0$?
25. Find the values of a and b for the curve $x^2y+ay^2=b$ if the point (1, 1) is on its graph and the tangent line at (1, 1) has the equation $4x+3y=7$.

Answers.

1. $-\dfrac{x}{y}$; **2.** $\dfrac{1-2xy-3y^3}{x^2+9xy^2}$; **3.** $-\dfrac{y^2}{x^2}$; **4.** $-\sqrt{\dfrac{y}{x}}$;

5. $\dfrac{1-70x(x^2+3y^2)^{34}}{210y(x^2+3y^2)^{34}}$; **6.** $\dfrac{\frac{3}{2}x^2(x^3+y^2)^{1/2}-y}{x-y(x^3+y^2)^{1/2}}$;

7. $\dfrac{1-2xy^2\cos(x^2y^2)}{2x^2y\cdot\cos(x^2y^2)}$; **8.** $\dfrac{1-3y^2\tan^2(xy^2+y)\cdot\sec^2(xy^2+y)}{3(2xy+1)\tan^2(xy^2+y)\sec^2(xy^2+y)}$;

9. $\dfrac{3y^2\sin^2(xy^2)\cos(xy^2)}{2\sqrt{1+\sin^3(xy^2)}-6xy\sin^2(xy^2)\cos(xy^2)}$; **10.** $-\dfrac{y}{x+\cos y}$;

11. $\frac{b^2x}{a^2y}$; **12.** $-\left(\frac{y}{x}\right)^{\frac{1}{3}}$; **13.** $-\frac{\pi}{2}$; **14.** $-\frac{3}{5}$; **15.** $-\frac{8}{9}$; **16.** $-\frac{2+\pi}{4}$;

17. $-\frac{14}{13}$; **18.** $\frac{12y^2-9x^2}{16y^3}$; **19.** $-\frac{2xy^3+2x^4}{y^5}$; **20.** $\frac{2y}{x^2}$;

21. $\frac{y^2-2xy}{(y-x)^3}$; **22.** $y_{\tan}=\frac{3x}{2}-\frac{1}{2}$; $y_{nor}=-\frac{2x}{3}+\frac{5}{3}$; **23.** $y_{\tan}=3x+$|

$y_{nor}=-\frac{1}{3}x+\frac{8}{3}$; **24.** for all (x, y) which satisfies $y+4x^2=0$;

25. $a=\frac{1}{4}$; $b=\frac{5}{4}$.

3.10. The linearization.

Fig. 3.4 suggests that if f is differentiable at the given point a, then the tangent line to the curve $y=f(x)$ at a is a reasonably good approximation to the curve $y=f(x)$ for vales x near a. Since the tangent line passes through $(a, f(a))$ and has slope $f'(a)$, the point- slope form of its equation is

$y-f(a)=f'(a)(x-a)$ or

$y=f(a)+f'(a)(x-a)$

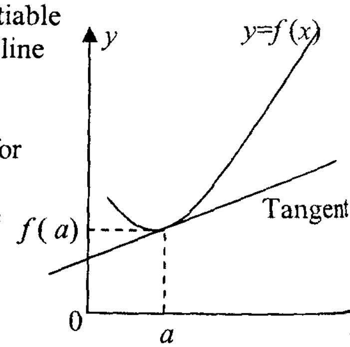


Fig.3.4

Definition: If $y=f(x)$ is a differentiable at $x=a$, then

$$(1) \qquad L(x)=f(a)+f'(a)(x-a)$$

is the linearization of f at a. The approximation $f(x)\approx L(x)$ is the linear approximation of f near $x=a$.

ample: Find the linearization of $f(x)=\tan x$ at $\pi/4$.

lution: In this case $f'(x)=\sec^2 x$. We compute

$$f\left(\frac{\pi}{4}\right)=\tan\frac{\pi}{4}=1$$

$$f'\left(\frac{\pi}{4}\right)=\sec^2\frac{\pi}{4}=(\sqrt{2})^2=2$$

us the linearization of tan x at $\frac{\pi}{4}$ is $L(x)=1+2(x-\frac{\pi}{4})$

ample: Find the linearization of $f(x)=\frac{1}{x^2}$ at $a=2$.

lution: We evaluate (1) for $f(x)=\frac{1}{x^2}$

$$f'(x)=(x^{-2})'=-2\cdot x^{-3}=-\frac{2}{x^3}$$

r $a=2$ we obtain $f(2)=\frac{1}{4}$; and $f'(2)=-\frac{1}{4}$

uation (1) gives $L(x)=\frac{1}{4}-\frac{1}{4}(x-2) \Rightarrow L(x)=-\frac{1}{4}x+\frac{3}{4}$.

3.11. The differential.

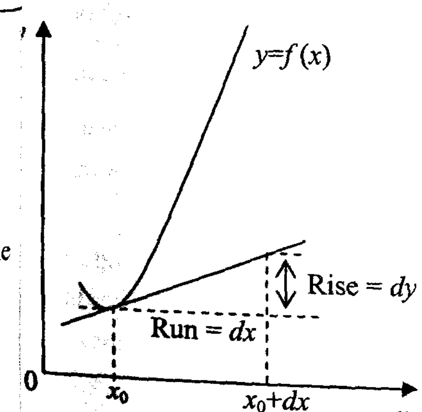


Fig. 3.5

Up to now we have been viewing the expression dy/dx as a single symbol for the derivative. "dy" and "dx" are called differentials. Regard x as fixed and define dx to be an independent variable that can be assigned an arbitrary value.

If f is differentiable at x_0, then we define dy by the formula

$$(1) \qquad dy = f'(x_0)\cdot dx$$

If $dx \neq 0$, then we can divide both sides of (1) by dx to obtain $\frac{dy}{dx} = f'(x_0)$. Since $\frac{dy}{dx} = f'(x_0) = m_{\tan}$, where $m_{\tan}$ is the slope of tangent to $y = f(x)$ at x_0, the differentials dy and dx can be viewe[d] a corresponding **rise** and **run** of this tangent line (Fig. 3.5).

It is important to understand the distinction between the increment Δy and differential dy.

To see the difference, let us represent the change in y that occurs when we start at x_0 and travel along the curve $y = f(x)$ until we moved Δx (=dx) units in the x-direction, while dy represents the change in y that occurs if we started at x_0 and travel along the tangent line until we have moved dx (=Δx) units in the x-direction (Fig. 3.6)

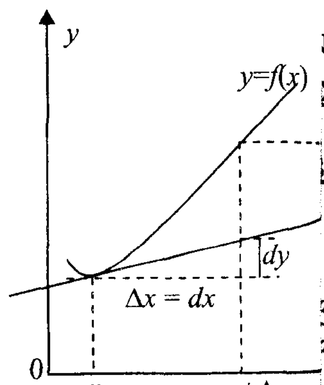


Fig.3.6

Definition: Let $y = f(x)$ be a differentiable function and x_0 be a number in the domain of f.
Then $f'(x_0)dx$ is called the differential of f at x and denoted df or

Example: If $y = x^3$, then the relation $\frac{dy}{dx} = 3x^2$ can be written in differential form $\qquad dy = 3x^2 dx$

For example, when $x_0 = 2$, this becomes $dy = 12dx$.

This tells us that if we have travel along the tangent to the curve $y = x^3$ at $x_0 = 2$, then a change of dx units in x produces a change 12dx units in y. For example, if the change in x is dx =3 then the ch in y along the tangent is $dy = 12\cdot 3 = 36$ units.

xample:

Let $y = \sqrt{x}$. Find dy and Δy if $x_0 = 4$ and $\Delta x = 3$.

olution: From $\Delta y = f(x_0 + \Delta x) - f(x_0)$ with $f(x) = \sqrt{x}$,

$$\Delta y = \sqrt{x_0 + \Delta x} - \sqrt{x_0} = \sqrt{7} - \sqrt{4} \approx 0.65$$

ince $y = \sqrt{x}$, then $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, $dy = \frac{1}{2\sqrt{x}}dx = \frac{1}{2\sqrt{4}}\cdot 3 = \frac{3}{4} = 0.75$.

Iotice that dy is very close to Δy, as was to be expected.

xample: Find dy and Δy for $y = x^2 - 2x$ when $x_0 = 2$ and $dx = \Delta x = 1$.

olution: $\Delta y = f(x_0 + \Delta x) - f(x_0) = f(3) - f(2) =$

$= (3^2 - 2\cdot 3) - (2^2 - 2\cdot 2) = 3$.

$y = x^2 - 2x$;

$\frac{dy}{dx} = 2x - 2$; and $dy = (2x-2)dx = (2\cdot 2 - 2)\cdot 1 = 2$.

3.12. Using the differential.

The linearization of $y = f(x)$ at x_0 is

$$y - f(x_0) = f'(x_0)\cdot(x - x_0) \text{ or}$$
$$y = f(x_0) + f'(x_0)(x - x_0)$$

or values of x close to x_0, the height y of tangent line will closely pproximate the height $f(x)$ of the curve, which yields approximation

$$(1) \qquad f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

or x near x_0. If we let $\Delta x = x - x_0$, so that $x = x_0 + \Delta x$, then (1) can e written in the alternative form

$$(2) \qquad f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

hich is a good approximation when Δx is near zero.

he output of f at $x_0 + \Delta x$ is approximated by the output of f at x_0 plus e differential $f'(x_0)\cdot\Delta x$.

Example: Use approximation formula to estimate $\sqrt[3]{29}$

Solution: Because we wish to estimate $\sqrt[3]{29}$, we introduce the cub root function $f(x)=\sqrt[3]{x}$. We know the exact value of $\sqrt[3]{x}$ when $x = 27$, which is near 29, so we use (2) with $x_0 = 27$, and $\Delta x = 29-27=2$

$$\sqrt[3]{29}=\sqrt[3]{27+2}\approx\sqrt[3]{27}+f'(27)\cdot(29-27)$$

Since $f(x)=x^{\frac{1}{3}}$; $\quad f'(x)=\dfrac{1}{3\cdot\sqrt[3]{x^2}}$.

Thus we have

$$\sqrt[3]{29}\approx 3+\frac{1}{3\cdot\sqrt[3]{27^2}}\cdot 2 \text{ or}$$

$$\sqrt[3]{29}\approx 3+\frac{2}{27}\approx 3+0.0741\approx 3.0741.$$

To estimate $f(b)$ take the following steps:

1. Find a number x_0 near b at which $f(x_0)$ and $f'(x_0)$ are easy to calculate
2. Find $\Delta x = b - x_0$. (Δx may be positive or negative)
3. Compute $f(x_0)+f'(x_0)\cdot\Delta x$. This is an estimate of $f(b)$.

In short, $f(b)\approx f(x_0)+f'(x_0)\cdot(b-x_0)$

Example: Use a differential to estimate $\sqrt{61}$

Solution: Let us take the point $x_0=64$, since $f(64)$ is known. We h

$$f(64)=\sqrt{64}=8.$$

$$f'(64)=\frac{1}{2\cdot\sqrt{64}}=\frac{1}{16}$$

Since 61=64-3, $\Delta x=-3$. Therefore

$$\sqrt{61}=f(64\text{-}3)\approx f(64)+f'(64)\cdot(-3)=8+\frac{1}{16}(-3)=7.81.$$

Example: Use (2) to approximate $\cos 62^\circ$

Solution: We shall take advantage of the fact that 62° is close to 60 at which point the trigonometric functions are easy to estimate

$$62^\circ=\frac{31\pi}{90} \text{ and } 60^\circ=\frac{\pi}{3}.$$

$$x_0+\Delta x=\frac{31\pi}{90} \text{ and } x_0=\frac{\pi}{3}$$

ollows that $\Delta x=\dfrac{\pi}{90}$, so (2) yields

$$f\left(\frac{31\pi}{90}\right)\approx f\left(\frac{\pi}{3}\right)+f'\left(\frac{\pi}{3}\right)\cdot\frac{\pi}{90};$$

$$\cos\frac{31\pi}{90}\approx\cos\frac{\pi}{3}-\sin\frac{\pi}{3}\cdot\frac{\pi}{90}\approx\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\pi}{90}\approx 0.4698.$$

$d[\ ]$ denotes the differential of the expression in the brackets.

r example:

$$d[x^2]=2xdx$$

$$d[f(x)]=f'(x)dx$$

e basic rules of differentiation can be expressed in terms of fferentials.

fferential formulas:

$$d[c]=0$$

$$d[cf]=c\cdot df$$

$$d[f+g]=df+dg$$

$$d[f\cdot g]=f\cdot dg+g\cdot df$$

$$d\left[\frac{f}{g}\right]=\frac{g\cdot df-f\cdot dg}{g^2}$$

xample: Find dy if $y=x\cdot\cos x$

lution:

$$dy=d[x\cdot\cos x]=x\cdot d[\cos x]+\cos x\cdot d[x]=$$

$$=x(-\sin x dx)+\cos x dx=(\cos x-x\sin x)dx$$

Exercises.

In exercises 1-4 find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = \sqrt{x}$ at $a = 1$ 2. $f(x) = \sqrt[3]{x}$ at $a = 8$
3. $f(x) = x^3 - x$ at $a = 2$ 4. $f(x) = \dfrac{x}{x+1}$ at $a = 1$

In exercises 5-8 compute df and Δf of the given functions, and values of x_0 and dx

5. $f(x) = x^2$ at $x_0 = 1$ and $dx = 0.3$
6. $f(x) = \sqrt{x}$ at $x_0 = 9$ and $dx = -2$
7. $f(x) = \tan x$ at $x_0 = \dfrac{\pi}{6}$ and $dx = \pi/12$
8. $f(x) = x^{-1}$ at $x_0 = 0.5$ and $dx = 0.1$

In exercises 9-14 use differentials to estimate the given quantiti

9. $\sqrt{119}$ 10. $\sqrt[3]{25}$
11. $\tan\left(\dfrac{\pi}{4} - 0.01\right)$ 12. $\sin\left(\dfrac{\pi}{3} - 0.02\right)$
13. $\sin 0.13$
14. $\sin 32^\circ$ [Hint: first translate into radians]

In exercises 15-19 find dy

15. $y = 4x^3 - 7x^2 + 2x - 1$ 16. $y = \sin^3(x/2)$
17. $d[\tan^3 x]$ 18. $d[x^3 \cdot \sec^2 5x]$
19. $y = \dfrac{1 - x^3}{2 - x}$ 20. $y = \dfrac{1}{x^3 - 1}$

Answers.

$L(x) = \dfrac{x+1}{2}$; **2.** $L(x) = \dfrac{x}{12} + \dfrac{4}{3}$; **3.** $L(x) = 11x - 16$;

$L(x) = \dfrac{x}{4} + \dfrac{1}{4}$; **5.** $dy = 0.6$; $\Delta y = 0.69$; **6.** $dy = -\dfrac{1}{3}$; $\Delta y = \sqrt{7} - 3$;

$dy = \dfrac{\pi}{9}$; $\Delta y = 1 - \dfrac{1}{\sqrt{3}}$; **8.** $dy = -\dfrac{2}{5}$; $\Delta y = -\dfrac{1}{3}$; **9.** 10.91; **10.** 2.9259;

0.98; **12.** 0.856; **13.** 0.13; **14.** 0.5302; **15.** $dy = (12x^2 - 14x + 2)dx$;

$dy = \dfrac{3}{2}\sin^2\dfrac{x}{2}\cos\dfrac{x}{2}dx$; **17.** $3\tan^2 x \cdot \sec^2 x dx$;

$[3x^2 \cdot \sec^2 5x + 10x^3 \cdot \sec^2 5x \cdot \tan 5x]dx$; **19.** $\dfrac{2x^3 - 6x^2 + 1}{(2-x)^2}dx$;

$-\dfrac{3x^2}{(x^3-1)^2}dx$.

Chapter 4.
Applications of derivatives.

4.1. Relative maxima and minima. The first and second derivative tests.

Definition: A function f is said to have a relative maximum at x_0 if $f(x_0) \geq f(x)$ for all x in some open interval containing x_0.

Definition: A function f is said to have a relative minimum at x_0 if $f(x_0) \leq f(x)$ for all x in some open interval containing x_0.

Definition: A function f has a global maximum at the number x_0 if $f(x_0) \geq f(x)$ for all x in the domain of f.

Definition: A function f has a global minimum at the number x_0 if $f(x_0) \leq f(x)$ for all x in the domain of f.

Definition: **A critical point** for a function f is any value of x in the domain of f at which $f'(x) = 0$ or at which f is not differentiable; the critical points where $f'(x) = 0$ are called **stationary** points of f.

The first derivative theorem for local extreme value:

Let f be a function defined at least on the open interval (a, b). If f takes on an extreme value at a number c in this interval and if $f'(c)$ exists, then

$$f'(c) = 0.$$

If an extreme value occurs within an open interval and derivative exists there, the derivative must be 0 at that point.

Warning 1: The theorem is not necessarily true if the open interval (a, b) replaced by a closed interval $[a, b]$.
As Fig. 4.1 shows, the maximum occurs at b where the derivative is not zero.

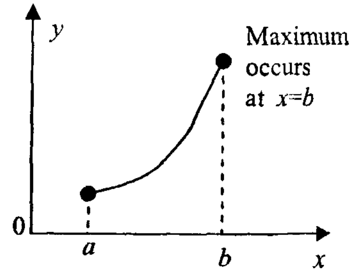


Fig. 4.1

Warning 2: The converse of this theorem is not true. Having the derivative equals to 0 at a point does not guarantee that there is an extremum at that point.

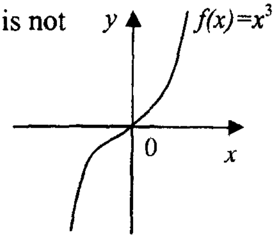


Fig.4.2

Let us consider $f(x) = x^3$.

$f'(x) = 3x^2;\ f'(0) = 0$

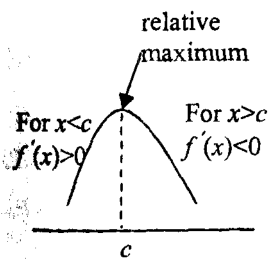
As we see $f(x)$ has $f'(0) = 0$, but $f(x) = x^3$ has neither maximum nor minimum at the point $x = 0$.

After finding a number c such that $f'(c) = 0$, we would like to know whether there may be a relative maximum or relative minimum at c. The following test describes a way to get the answer.

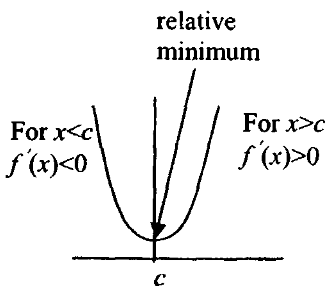
The first derivative test for local maximum at point c:

Suppose f is continuos at a critical point c.

a) If $f'(c) > 0$ on an open interval extending left from c and $f'(c) < 0$ on an open interval extending right from c, then f has a relative maximum at c.(Fig.4.3)



a critical point
Fig.4.3



a critical point
Fig.4.4

b) If $f'(c) < 0$ on an open interval extending left from c and $f'(c) > 0$ on an open interval extending right from c, then f has a relative minimum at c.(Fig.4.4).

c) If f has the same sign [either $f'(c) > 0$ or $f'(c) < 0$] on an open interval extending left from c and on an open interval extending right from c, then f does not have a relative extremum at c.

Briefly, the relative extrema, on an open interval where f is continuous occur at those critical points where f' changes the sign.

Example: Locate the relative extrema of $f(x) = x^2 - 2x - 1$.

Solution:

$$f'(x) = 2x - 2$$

Solving $f'(x) = 0$ we obtain $x = 1$. To see whether the critical point is relative maximum, relative minimum or neither, check the sign of

$$f'(x) = 2x - 2.$$

If $x > 1$ then $(2x - 2) > 0$, and

If $x < 1$ then $(2x - 2) < 0$, so the first derivative test tells us that the critical point $x = 1$ is a relative minimum.

Example: Locate the relative extrema of $f(x) = x^3 - 3x^2 + 3x - 1$.

Solution:

$$f'(x) = 3x^2 - 6x + 3$$

$f'(x) = 0 \Rightarrow 3(x-1)^2 = 0$ and $x = 1$ is only critical point.

Since $(x-1)^2 \geq 0$ for all x and $f'(x)$ does not change sign at $x = 1$, we make conclusion:

In spite the fact that $x = 1$ is a critical point but f does not have a relative extrema at $x = 1$. Thus f has no extrema.

Example: Locate the relative extrema of $f(x) = \sqrt[3]{x-1}$.

Solution:

$$f'(x) = \frac{1}{3 \cdot \sqrt[3]{(x-1)^2}}.$$

Since $f'(x)$ does not exist when $x = 1$, so this point is a critical point. But, as we see, f' does not change the sign. We make conclusion that f does not have a relative extrema at $x = 1$.

Example: Locate the relative extrema using first derivative test of

$$f(x) = 2x^3 - 9x^2 + 12x.$$

Solution: $f'(x) = 6x^2 - 18x + 12$

$f'(x) = 0 \Rightarrow 6x^2 - 18x + 12 = 0$ yields $x_1 = 1;\quad x_2 = 2$

The sign of f' changes from "+" to "-" at $x = 1$ and from "-" to "+" at the point $x = 2$. Thus there is a relative maximum at $x = 1$ and relative minimum at $x = 2$.

```
+ + + | - - - - | + + +
      1         2
```

Sign of $f' = 6x^2 - 18x + 12$

Relative maximum equals 5 at $x = 1$ and relative minimum is 4 at $x = 2$.

Second derivative test:

Suppose that f is twice differentiable at a stationary point c.

a) If $f''(c) > 0$ then f has a relative minimum at c.

b) If $f''(c) < 0$ then f has a relative maximum c.

Example: Locate the relative extrema of $f(x) = x^4 - 2x^2$.

Solution: $f'(x) = 4x^3 - 4x$

$$f''(x) = 12x^2 - 4$$

Solving $f'(x) = 0$ yields the stationary points $x = 0$, $x = 1$, $x = -1$.

Since

$$f''(0) = -4 < 0$$

$$f''(1) = 8 > 0$$

$$f''(-1) = 8 > 0$$

there is a relative maximum at $x = 0$, relative minimum at $x = 1$; $x = -1$.

Example: Find all relative extrema of the function $f(x) = x^4 + 2x^3$.

Solution: $f'(x) = 4x^3 + 6x^2$

$$f''(x) = 12x^2 + 12x$$

$f'(x) = 0 \Rightarrow 2x^2(x+3) = 0$;

$x_1 = 0;\quad x_2 = -3$.

```
- - - - | + + + | + + +
       -3       0
```

Sign of $f'(x) = 4x^3 + 6x^2$

$f''(0) = 0;\qquad f''(-3) = 72 > 0$

Since $f''(0) = 0$, then second derivative test fails. Let us use first derivative test. Since at $x = 0$, f' does not change the sign then we say that $x = 0$ is not extreme point.

Function $f(x) = x^4 + 2x^3$ has a relative minimum at $x = -3$.

Example: Find relative extrema of $f(x)=\frac{1}{2}x-\sin x$, $0<x<2\pi$ using second derivative test.

Solution: $f'(x)=\frac{1}{2}-\cos x$

$f''(x)=\sin x$

From $f'(x)=0$ we obtain $\cos x=\frac{1}{2}$; $x=\frac{\pi}{3}$.

$f''\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}>0$, so $f(x)$ has a relative minimum at $x=\frac{\pi}{3}$ and its value is

$f\left(\frac{\pi}{3}\right)=\frac{\pi}{6}-\frac{\sqrt{3}}{2}=\frac{\pi-3\sqrt{3}}{6}$.

Exercises.

In exercises 1-6 locate the critical points and classify them as stationary points or points of nondifferentiability.

1. $f(x)=x^2-5x+6$
2. $f(x)=x^3+3x^2-9x+1$
3. $f(x)=x^4-6x^2-3$
4. $f(x)=\frac{x}{x^2+2}$
5. $f(x)=\sin^2 2x$; $0<x<2\pi$;
6. $f(x)=x^{1/3}\cdot(x+4)$

In exercises 7-16 use any method to find the relative extrema.

7. $f(x)=x^5$
8. $f(x)=x^3+5x-2$
9. $f(x)=x(x-1)^2$
10. $f(x)=2x^2-x^4$
11. $f(x)=x^{4/5}$
12. $f(x)=\tan(x^2+1)$
13. $f(x)=\sqrt{2x-x^2}$
14. $f(x)=\frac{x^2-3x+2}{x^2+2x+1}$
15. $f(x)=(x-1)^3$
16. $f(x)=1-(x-2)^{4/5}$

In exercises 17-18 $f'(x)$ is given. Find all critical points and determine whether a relative maximum, relative minimum or neither occurs there.

17. $f'(x)=x^3(x^2-5)$ 18. $f'(x)=\dfrac{9-4x^2}{\sqrt[3]{x+1}}$

19. Find a value of k so that $x^2+\dfrac{k}{x}$ will have a relative extremum at $x=3$.

20. Find a value of k so that $\dfrac{x}{x^2+k}$ will have a relative extremum at $x=2.5$

Answers.

1. x=5/2 (stationary); **2.** x=-3;1 (stationary); **3.** x =0; $\pm\sqrt{3}$ (stationary); **4.** x=$\pm\sqrt{2}$ (stationary); **5.** $x=\dfrac{\pi n}{4}$; n=1,2,3,4,5,6,7 (stationary); **6.** x=-1 (stationary); x =0 (not differentiable); **7.** Critical point is x=0, neither maximum nor minimum; **8.** none; **9.** Relative min. of 0 at x=1; Relative max. of $\dfrac{4}{27}$ at $x=\dfrac{1}{3}$; **10.** Relative min. of 0 at x=1; Relative max. of 1 at x =1; -1; **11.** Relative min. of 0 at x=0; **12.** Relative min. of tan(1) at x=0; **13.** Relative min. of 0 at x=0; Relative max. of 1 at x =1; **14.** Relative min.of $-\dfrac{1}{24}$ at $x=\dfrac{7}{5}$; **15.** none; **16.** Relative max. of 1 at x =2; **17.** Relative maximum at x =0; Relative minimum at $x=\pm\sqrt{5}$; **18.** Relative maximum at $x=\pm\dfrac{3}{2}$; Relative minimum at $x=-1$; **19.** 54 ; **20.** 6.25.

4.2. Maximum and minimum values of a function on a closed interval.

In many applied problems we are interested in finding global maximum or global minimum of function over some closed interval $[a, b]$. To find the extreme values of continuous function f on a closed interval $[a, b]$ we use following steps:

Step 1: Find the critical points of f.

Step 2: Evaluate f at the endpoints a and b, and at those critical points which lie in $[a, b]$.

Step 3: Select the largest of the values in step 2 as the global (absolute) maximum value of f on $[a, b]$ and the smallest value as the global (absolute) minimum.

Example:

Find the maximum and minimum value of

$$f(x) = x^3 - 3x^2 + 3x \text{ on } [0, 2]$$

Solution:

Step 1: Let us find critical points

$$f'(x) = 3x^2 - 6x + 3$$

$f'(x) = 0$ is equivalent to $3x^2 - 6x + 3 = 0$ or $3(x-1)^2 = 0$.

Thus 1 is the only critical number, and it lies in the interval [0, 2].

Step 2: Evaluating f at critical point $x = 1$ and the endpoints, we have

$$f(0) = 0;$$

$$f(1) = 1^3 - 3\cdot 1^2 + 3\cdot 1 = 1;$$

$$f(2) = 2^3 - 3\cdot 2^2 + 3\cdot 2 = 2$$

Thus, the maximum value is 2 and the minimum value is 0. The maximum occurs at $x = 2$ and the minimum occurs at $x = 0$.

Example:

Find the maximum and minimum value of

$$f(x) = 2x^3 - 3x^2 - 12x \text{ on the interval } [0, 3]$$

Solution:

Step 1: $f'(x) = 6x^2 - 6x - 12$

The equation $f'(x) = 0$ becomes $6x^2 - 6x - 12 = 0$, which simplifies to $x^2 - x - 2 = 0$ or $(x-2)(x+1) = 0$.

There are two critical points $x_1 = -1$ and $x_2 = 2$. But only $x_2 = 2$ lies on [0, 3]. So $x_1 = -1$ is not a point of interest.

Step 2:

$$f(0) = 0;$$

$$f(2) = 2\cdot 2^3 - 3\cdot 2^2 - 12\cdot 2 = -20;$$

$$f(3) = 2\cdot 3^3 - 3\cdot 3^2 - 12\cdot 3 = -9$$

Step 3:

Thus, the minimum value of f on [0, 3] is –20, which occurs at $x = 2$, and the maximum value of f on [0, 3] is 0, which occurs at $x = 0$.

Example:

Find the maximum and minimum value of $f(x) = x^{\frac{2}{3}}(20 - x)$ on [-1, 20]

Solution:

Differentiating, we obtain

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(20-x) - x^{\frac{2}{3}} = x^{-\frac{1}{3}}\left(\frac{2}{3}(20-x) - x\right) = \frac{40-5x}{3\sqrt[3]{x}}$$

Thus, $f'(x) = 0$ at $x = 8$ and $f'(x)$ does not exist at $x = 0$. It follows that both of critical points lie on the [-1, 20]. Let us evaluate f at these and at endpoints:

$$f(x) = \sqrt[3]{x^2}\cdot(20 - x)$$

$$f(-1) = \sqrt[3]{(-1)^2}\cdot(20 - (-1)) = 21$$

$$f(0) = 0$$

$$f(8) = \sqrt[3]{8^2}\cdot(20 - 8) = 4\cdot 12 = 48$$

$$f(20) = \sqrt[3]{20^2}\cdot(20 - 20) = 0.$$

Thus, the maximum value of f on [-1, 20] is 48, which occurs at $x = 8$ and the minimum value is 0, which occurs at two points $x = 0$ and $x = 20$.

4.3. Concavity.

Definition: The curve is called **concave down** if it lies below its tangent lines and above its chords.(Fig. 4.5)

Definition: The curve is called **concave up** if it lies above its tangent lines and below its chords. (Fig. 4.6)

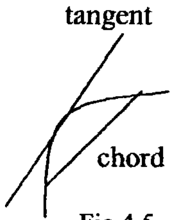


Fig.4.5

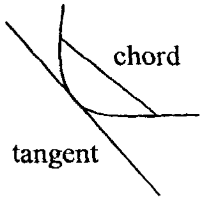


Fig.4.6

Since f' is the slope of a tangent line to the graph of f, it brings the following definitions.

Definition: A function f whose derivative is increasing through the open interval (a, b) is **called concave up** in that interval.

Definition: A function f whose derivative is decreasing through the open interval (a, b) is called **concave down** in that interval.

Theorem:

a) If $f''(x)>0$ on an open interval (a, b), then f is concave up on (a, b)

b) If $f''(x)<0$ on an open interval (a, b), then f is concave down on (a, b).

Example: Find open interval on which following functions are concave up and open intervals on which they are concave down.

a) $f(x)=x^2+4x-7$; b) $f(x)=x^5$; c) $f(x)=x(x-1)^3$

Solution:

a) Calculating f' and f'' we obtain

$$f'(x)=2x+4 \quad \text{and} \quad f''(x)=2$$

Since $f''(x)>0$ for all x, the function is concave up on $(-\infty;+\infty)$.

b) Calculating f' and f'' we obtain

$$f'(x)=5x^4 \quad \text{and} \quad f''(x)=20x^3$$

Since $f''(x)<0$ if $x<0$ and $f''(x)>0$ if $x>0$, the function $f(x)$ is concave down on $(-\infty;+0)$ and concave up on $(0;+\infty)$.

c) $f'(x) = (x-1)^2(4x-1)$

$$f''(x) = 12(x - \frac{1}{2})(x-1).$$

$$f'' = 0 \text{ if } x_1 = \frac{1}{2} \text{ and } x_2 = 1.$$

$$f''(x) > 0 \text{ if } x \in (-\infty; 1/2) \text{ and } x \in (1; +\infty),$$

so function is concave up on these intervals.

$f''(x) < 0$ if $x \in (1/2; 1)$ so f is concave down on this interval.

4.4. Inflection points.

Definition: If function f is continuous on (a, b) containing x_0 and if f changes the direction of concavity at x_0, then the point $(x_0, f(x_0))$ is called an inflection point of f, and we say that function f has an inflection point at x_0.

The simplest way to look for an inflection point is to use the second derivative.

To find inflection point of $f(x)$ use following steps:

1) Compute $f''(x)$;
2) Look for numbers x_0 such that $f''(x)$=0 or $f''(x)$ is not defined at x_0;
3) Check whether $f''(x)$ changes the sign.

Example: Find the inflection point of $f(x) = x^4 - 6x^3 + 12x^2$

Solution: Let us find $f''(x)$

$f'(x) = 4x^3 - 18x^2 + 24x$ and

$f''(x) = 12x^2 - 36x + 24$

Let us find points where $f''(x)$ =0.

$12x^2 - 36x + 24 = 0 \Leftrightarrow$

$x^2 - 3x + 2 = 0$;

Roots of equation are: $x_1 = 1$; $x_2 = 2$.

+ + + + | - - - - - | + + +

-1 2

Sign of $f''(x)$=(x-2)(x-1)

As we see $f''(x)$ changes sign at $x = 1$ and $x = 2$, so both of these numbers are inflection numbers. The graph of $f(x)$ has two inflection points namely $(1, f(1))=(1, 7)$ and $(2, f(2))=(2, 16)$.

Example:

Find the inflection point of $f(x)=\sqrt[3]{x-1}$

Solution: $$f'(x)=\frac{1}{3}(x-1)^{-\frac{2}{3}};$$

$$f''(x)=-\frac{2}{9}(x-1)^{-\frac{5}{3}}=\frac{-2}{9\cdot\sqrt[3]{(x-1)^5}}$$

$f''(x)$ is never zero. $f''(x)$ is not defined at $x=1$. Let us check if $f''(x)$ changes the sign at $x=1$.

If $x<1$ then $f''(x)>0$ and if $x>1$ then $f''(x)<0$ so point $(1, f(1))=(1, 0)$ is an inflection point.

Exercises.

In exercises 1-7 find the maximum and minimum values of$f$ on the given closed interval and state where these values occur.

1. $f(x)=x^2-x^4$; $[0, 1]$
2. $f(x)=4x-x^2$; $[0, 1]$
3. $f(x)=x^3-2x^2+5x$; $[-1, 3]$
4. $f(x)=(x-1)^3$; $[0, 4]$
5. $f(x)=\frac{3x}{\sqrt{4x^2+1}}$; $[-1, 1]$
6. $f(x)=(x^2+x)^{2/3}$; $[-2,3]$
7. $f(x)=x-\tan x$; $\left[-\frac{\pi}{4};\frac{\pi}{4}\right]$.

In exercises 8-15 find the intervals where the function is concave up or concave down and give x coordinate of inflection points.

8. $f(x)=x^3-3x^2+2$
9. $f(x)=x^2+x+1$
10. $f(x)=x^4-4x^3$
11. $f(x)=\frac{1}{1+x^2}$
12. $f(x)=3x^4-4x^3$
13. $f(x)=\cos x$; $0<x<2\pi$
14. $f(x)=2x^3-3x^2+15$

Answers.

1. $\max f(x) = f\left(\sqrt{2}/2\right) = 1/4$; $\min f(x) = f(0) = f(1) = 0$;

2. $\max f(x) = f(1) = 3$; $\min f(x) = f(0) = 0$;

3. $\max f(x) = f(3) = 24$; $\min f(x) = f(-1) = -8$;

4. $\max f(x) = f(4) = 27$; $\min f(x) = f(0) = -1$;

5. $\max f(x) = f(1) = 3/\sqrt{5}$; $\min f(x) = f(-1) = -3/\sqrt{5}$;

6. $\max f(x) = f(3) = \sqrt[3]{144}$; $\min f(x) = f(-1) = f(0) = 0$;

7. $\max f(x) = f\left(-\frac{\pi}{4}\right) = 1 - \frac{\pi}{4}$; $\min f(x) = f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - 1$;

8. concave up: $x > 1$, concave down $x < 1$, inflection point: $x = 1$; **9.** concave up for all x; **10.** concave up for $x < 0$ and $x > 2$; concave down for $0 < x < 2$; inflection points: $x = 0, 2$; **11.** concave up for $|x| > 1/\sqrt{3}$, concave down for $|x| < 1/\sqrt{3}$; inflection points: $x = \pm 1/\sqrt{3}$; **12.** concave up for $x < 0$ and $x > 2/3$, concave down for $0 < x < 2/3$; inflection points: $x = 0$; $2/3$; **13.** concave up for $\pi/2 < x < 3\pi/2$; concave down for $(0 < x < \pi/2)$ and $(3\pi/2 < x < 2\pi)$; inflection points: $\pi/2$; $3\pi/2$; **14.** concave up for $x > 1/2$; concave down for $x < 1/2$; inflection point: 1/2.

4.5. Asymptotes.

The line is an asymptote of the graph of $y = f(x)$ if the distance between the line and graph approaches zero as we move farther from the origin. There are three kinds of asymptotes: vertical asymptote, **horizontal asymptote and an oblique asymptote.**

Definition:

The line $x = x_0$ is called a vertical asymptote for the graph of $f(x)$ if

$$\lim_{x \to x_0^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to x_0^-} f(x) = \pm\infty$$

The line $y = y_0$ is called a horizontal asymptote for the graph of $f(x)$ if

$$\lim_{x \to +\infty} f(x) = y_0 \quad \text{or} \quad \lim_{x \to -\infty} f(x) = y_0$$

If $\lim\limits_{x \to +\infty} \frac{f(x)}{x} = k$ and $\lim\limits_{x \to +\infty} [f(x) - kx] = b$ exists then the line $y = kx + b$ is called an oblique asymptote. In other words, rational function has an oblique asymptote if the degree of numerator is one more greater than degree of denominator.

Example: Find vertical and horizontal asymptote of $f(x) = \frac{1}{(x-1)^2}$.

Solution: To search for vertical asymptote we set denominator equals to zero.

$(x-1)^2 = 0$, we find $x = 1$.

From definition:

$\lim\limits_{x \to 1^+} \frac{1}{(x-1)} = +\infty$;

$\lim\limits_{x \to 1^-} \frac{1}{(x-1)} = +\infty$

so $x = 1$ is a vertical asymptote.

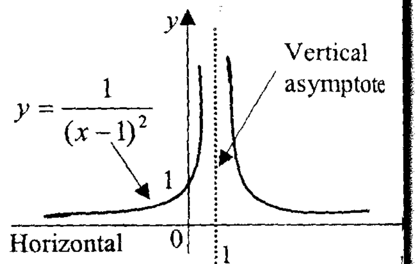


Fig.4.7

To search for horizontal asymptote we examine $\lim\limits_{x \to +\infty} \frac{1}{(x-1)^2} = 0$ and $\lim\limits_{x \to -\infty} \frac{1}{(x-1)^2} = 0$. The line $y = 0$ is a horizontal asymptote.(Fig. 4.7).

Example:

Find vertical and horizontal asymptote of $f(x) = \frac{x}{x-2}$.

Solution: Vertical asymptotes occurs at $x = 2$;

$\lim\limits_{x \to 2^+} \frac{x}{x-2} = +\infty$; and $\lim\limits_{x \to 2^-} \frac{x}{x-2} = -\infty$.

The line $x = 2$ is a vertical asymptote.

The horizontal asymptote:

$\lim\limits_{x \to +\infty} \frac{x}{x-2} = 1$ and $\lim\limits_{x \to -\infty} \frac{x}{x-2} = 1$,

so the graph approaches the line $y = 1$ as $x \to +\infty$ and $x \to -\infty$.

The line $y = 1$ is the horizontal asymptote.

Example:

Find asymptotes of $f(x) = \frac{x^3}{x^2+1}$.

Solution: As we see the graph of function has neither a vertical ($x^2 + 1 \neq 0$ for all x), nor a horizontal ($\lim\limits_{x \to +\infty} \frac{x^3}{x^2+1} = \infty$) asymptote.

Let us find an oblique asymptote:

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^3}{x^2+1} \div x = 1.$$

$$b = \lim_{x \to \pm\infty} [f(x) - 1 \cdot x] = \lim_{x \to \pm\infty} \left[\frac{x^3}{x^2+1} - x \right] = 0.$$

So, an oblique asymptote is the line $y = x$.

Exercises.

In exercises 1-13 find asymptotes and sketch the graph of the given rational functions.

1. $y = \frac{2x}{x-3}$;
2. $y = \frac{x^2}{x^2-1}$;
3. $y = \frac{x}{1+x^2}$;
4. $y = \frac{3-4x}{2+5x}$;
5. $y = \frac{1-x^2}{1+x^2}$;
6. $y = \frac{1+x^2}{1-x^2}$;
7. $y = \frac{3x^5}{2+x^4}$;
8. $y = \frac{1}{x(x-1)(x-2)}$;
9. $y = \frac{x-1}{x^2-4}$;
10. $y = \frac{(x-1)^2}{x^2}$;
11. $y = 3 - \frac{4}{x} - \frac{4}{x^2}$;
12. $y = \frac{(x-2)^3}{x^2}$;
13. $y = x + 1 - \frac{1}{x} - \frac{1}{x^2}$.

Answers.

<u>**1.**</u> $y = 2; x = 3$; <u>**2.**</u> $y = 1; x = 1; x = -1$; <u>**3.**</u> $y = 0$; <u>**4.**</u> $x = -2/5; y = -4/5$; <u>**5.**</u> $y = -1$; <u>**6.**</u> $y = -1, x = 1; x = -1$; <u>**7.**</u> $y = 3x$; <u>**8.**</u> $y = 0, x = 2; x = 0; x = 1$; <u>**9.**</u> $y = 0, x = -2; x = 2$; <u>**10.**</u> $y = 1, x = 0$; <u>**11.**</u> $y = 3$, ;$x = 0$; <u>**12.**</u> $y = x - 6$; $x = 0$; <u>**13.**</u> $y = x + 1$; $x = 0$.

4.6. The derivative and sketching the graph.

For graphing the function it is advisable to follow the scheme:

1). Find the domain of the function.
2). Check for symmetry.
3). Find the vertical asymptote.
4). Investigate the behavior of the function in $+\infty$ and $-\infty$ find horizontal or oblique asymptotes.
5). Find extreme values and intervals where function increase or decrease.
6). Find point of inflection, intervals of concavity.
7). Find x and y intercepts.
8). Sketch the graph.

<u>**Example:**</u> Sketch the graph of $y = \dfrac{1+x^2}{1-x^2}$.

<u>**Solution:**</u>

1) Domain of the function is all $x \neq \pm 1$

2) The function is odd, since

$$f(x) = \frac{1+x^2}{1-x^2} \text{ and } f(-x) = \frac{1+(-x)^2}{1-(-x)^2} = \frac{1+x^2}{1-x^2} = f(x),$$

and the graph is symmetric about the y-axis.

3) $\lim\limits_{x\to 1^+} \dfrac{1+x^2}{1-x^2} = -\infty$ and $\lim\limits_{x\to 1^-} \dfrac{1+x^2}{1-x^2} = +\infty$,

so $x = 1$ is a vertical asymptote.

$\lim\limits_{x\to -1^+} \dfrac{1+x^2}{1-x^2} = +\infty$ and $\lim\limits_{x\to -1^-} \dfrac{1+x^2}{1-x^2} = -\infty$,

so $x = -1$ is a vertical asymptote.

4) $\lim\limits_{x\to+\infty}\frac{1+x^2}{1-x^2}=-1$ and $\lim\limits_{x\to-\infty}\frac{1+x^2}{1-x^2}=-1$,

so $y=-1$ is a horizontal asymptote.

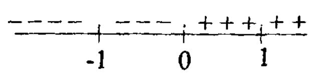
5) Let us find the critical points:

$$y'=\frac{2x(1-x^2)+2x(1+x^2)}{(1-x^2)^2}=\frac{4x}{(1-x^2)^2}$$

There is only one critical point $x=0$; since $x=\pm1$ are not from domain of function, we omit them. If $x<0$, then $f'(x)<0$, if $x>0$, then $f'(x)>0$ .So point $x=0$ is a point of relative minimum.

$f_{min}=f(0)=1$

On $(-\infty,-1)$ and (-1,0) function is decreasing and on $(0,1)$ and $(1,\infty)$ it is increasing.



Sign of f'

6) Concavity:

$$y''=\frac{4(1+3x^2)}{(1-x^2)^3}$$

Since $y''>0$ on (-1,1), the graph is concave up on this interval. $y''<0$ on $(-\infty,1)$ and $(1,+\infty)$, on these intervals graph of function is concave down. $y''\neq0$ so there is no inflection point.

7) y intercept is 1, since $f(0)=1$.

The graph does not cross Ox- axis, because $\frac{1+x^2}{1-x^2}=0$ does not have a solution. The final graph is shown in Fig.4.8.

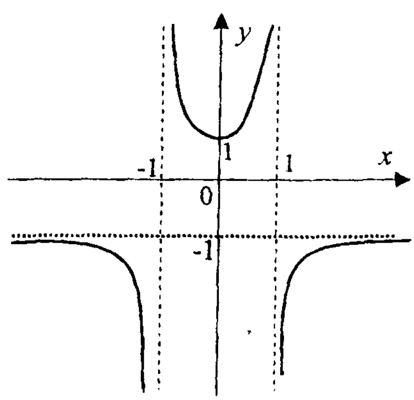


Fig. 4.8

Example: Sketch the graph of $y=\frac{x^2-1}{x^3}$.

Solution:

1) Domain of function is $(-\infty,0)\cup(0,+\infty)$.

2) Symmetries: Replacing x by $-x$ and y by $-y$ yields an equation that simplifies back to the original equation, so the graph is symmetric about the origin.

3) $\lim_{x\to 0^+} \frac{x^2-1}{x^3} = -\infty$ and $\lim_{x\to 0^-} \frac{x^2-1}{x^3} = +\infty$ yield the vertical asymptote $x = 0$.

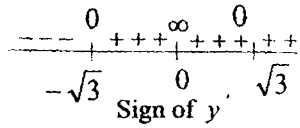
4) $\lim_{x\to +\infty} \frac{x^2-1}{x^3} = 0$ and $\lim_{x\to -\infty} \frac{x^2-1}{x^3} = 0$ yield the horizontal asymptote $y = 0$.

5) $\frac{dy}{dx} = \frac{2x\cdot x^3 - 3x^2(x^2-1)}{x^6} = \frac{3-x^2}{x^4}$

$x = \pm\sqrt{3}$ are stationary points.

Function is decreasing on $(-\infty,-\sqrt{3})$ and $(\sqrt{3},+\infty)$.

And function is increasing on $(-\sqrt{3},0)$ and $(0,\sqrt{3})$

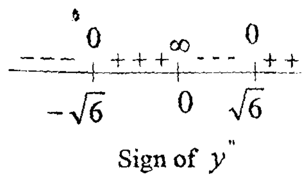


At $x = -\sqrt{3}$ there is a relative minimum $f(-\sqrt{3}) = -\frac{2\sqrt{3}}{9}$;

at $x = \sqrt{3}$ there is a relative maximum $f(\sqrt{3}) = \frac{2\sqrt{3}}{9}$;

6) Concavity: $y'' = \frac{2(x^2-6)}{x^5}$

This analysis show that a change in concavity occurs at the vertical asymptote $x = 0$ and at the points $x = -\sqrt{6}$ and $x = \sqrt{6}$.



7) x intercepts are $x = 1; x = -1$. There is no y intercept, since setting $x = 0$ leads to a division by zero. The final graph is sketched in Fig. 4.9.

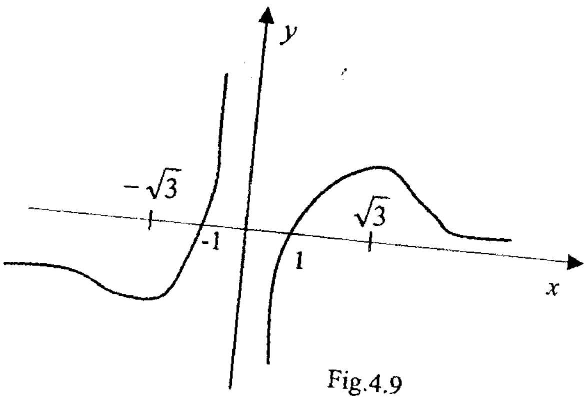


Fig.4.9

Exercises.

In exercises 1-15 use the techniques illustrated in this section to sketch the graph of given functions.

1. $y = \dfrac{3x+1}{3x-1}$

2. $y = \dfrac{x}{x^2+1}$

$y = \dfrac{x}{x^2-1}$

4. $y = x^2 + \dfrac{1}{x^2}$

$y = x^3 - 12x^2 + 36x$

6. $y = x + \dfrac{27}{x^3}$

$y = \dfrac{(x-1)^3}{(x+1)^2}$

8. $y = \dfrac{x^2-2}{x^2+1}$

$= 2 + \dfrac{3}{x} - \dfrac{1}{x^3}$

10. $y = \dfrac{x+1}{x^2-16}$

$= \dfrac{x-3}{x+4}$

12. $y = 3x^4 + 4x^3$

$= \dfrac{x^2-2x-3}{x+2}$

14. $y = \dfrac{4-x^3}{x^2}$

$= \dfrac{1}{(x-1)^2(x-2)^2}$.

4.7. Rolle's theorem; Mean-value theorem.

Rolle's theorem: Let f be a continuous function on the closed interval $[a, b]$ and have a derivative at all x in the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

Example: Verify Rolle's theorem for the case $f(x) = \cos x$ and $[a,b] = [\pi, 5\pi]$.

Solution: Note that $f(\pi) = -1 = f(5\pi)$. $f(x) = \cos x$ is differentiable for all x and it is continuous on $[\pi, 5\pi]$. According to Rolle's, there must be at least one number c in $(\pi, 5\pi)$ for which $(\cos x)' = 0$. So $(\cos x)' = -\sin x = 0$. As can be checked, the equation has three such solutions, $2\pi, 3\pi$ and 4π.

Example: Verify Rolle's theorem for the case $f(x) = x^3 - 3x^2 + 2x$ and $[a,b] = [0, 2]$.

Solution: $f(0) = 0$

$$f(2) = 2^3 - 3\cdot 2^2 + 2\cdot 2 = 0, \quad f(a) = f(b) \text{ satisfies.}$$

Given function is continuous on [0, 2] and differentiable on (0, 2), so according to Rolle's theorem there must be at least one number c in [0, 2] such that $f'(c) = 0$.

Let us find that number

$$f'(x) = 3x^2 - 6x + 2$$

$$f'(x) = 0 \Rightarrow 3x^2 - 6x + 2 = 0 \text{ and we obtain}$$

$$x_1 = 1 + \frac{\sqrt{3}}{3} \text{ and } x_2 = 1 - \frac{\sqrt{3}}{3}$$

As we see equation has two solutions in (0, 2), we make conclusion that at two points in (0, 2) given function has derivative equals zero.

Example: The function $f(x) = |x| - 1$ has the property that $f(-1)=0$ and $f(1)=0$, but there is no point in (-1, 1) where $f'(x) = 0$. This fact that $f'(x)$ is never zero does not contradict Rolle's theorem, because $f(x) = |x| - 1$ does not satisfy all conditions required for Rolle's theorem, namely the function f is not differentiable at every point of the interval (-1, 1).

The Mean- value theorem:

Let f be a continuous function on the closed interval [a, b] and have a derivative at every x in the interval (a, b). Then there is at least one number c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example:

Verify the Mean-value theorem for $f(x) = 2x^2 + x + 1$; a =-2; b =3

Solution:

Because f is a polynomial, it is differentiable and continuous everywhere, hence it is continuous on [-2,3] and differentiable on (-2, 3). The hypothesis of Mean-value theorem are satisfied.

$$f(a) = f(-2) = 2(-2)^2 + (-2) + 1 = 7$$

$$f(b) = f(3) = 2(3)^2 + (3) + 1 = 22$$

$$f'(x) = 4x + 1$$

$f'(c) = 4c + 1$, so the equation

$f'(c) = \frac{f(b) - f(a)}{b - a}$ becomes

$$4c + 1 = \frac{22 - 7}{3 - (-2)} = \frac{15}{5} = 3, \Rightarrow c = \frac{1}{2}.$$

We see $c \in (-2,3)$ and it is only number whose existence is guaranteed by the Mean-value theorem.

Example:

Verify the Mean-value theorem for $f(x) = \frac{1}{x - 1}$ on [2, 5] and find all values of c that satisfy the conclusion of the theorem.

Solution:

$f(x) = \frac{1}{x - 1}$ is continuous on [2, 5] and differentiable on (2, 5). The hypothesis of the theorem are satisfied.

$$f(a) = f(2) = 1; \qquad f(b) = f(5) = \frac{1}{4}$$

$$f'(x) = \frac{-1}{(x-1)^2}; \qquad f'(c) = \frac{-1}{(c-1)^2},$$

according to the theorem $\dfrac{-1}{(c-1)^2} = \dfrac{\frac{1}{4} - 1}{5-2}$;

$(c-1)^2 = 4$ and we find $c_1 = 3$ and $c_2 = -1$.

Only the c_1 is in the interval (2, 5), so c =3 is the only number in (2, 5) that satisfies the conclusion of the Mean- value theorem.

Exercises.

In exercises 1-4 verify that the given function satisfies the hypothesis of Rolle's theorem for the given interval. Find all numbers c that satisfy the conclusion of the theorem.

1. $f(x) = x^2 - 2x - 3$ and [0, 2]
2. $f(x) = x^4 - 2x^2 + 1$ and [-2, 2]
3. $f(x) = x^2 - 6x + 8$ and [2, 4]
4. $f(x) = \frac{1}{2}x - \sqrt{x}$ and [0, 4]

In exercises 5-8 verify that the hypothesis of Mean- value theorem are satisfied on the given interval and find all values of c that satisfies the conclusion of the theorem.

5. $f(x) = x^2 - 4x$ and [0, 5]
6. $f(x) = x^3 + 2x - 4$ and [-1; 4]
7. $f(x) = \sqrt{x+3}$ and [1, 6]
8. $f(x) = \sqrt{36 - x^2}$ and [-6, 0]
9. Let function $f(x) = x^{2/3}$ be given.
   a) Graph given function for x in [-1, 1]
   b) Show that $f(-1)=f(1)$

c) Is there any number c in $(-1, 1)$ such that $f'(c) = 0$?
d) Why does this function not contradict Rolle's theorem?

Answers.

1. 1; 2. 0; 1; -1; 3. 3; 4. 1; 5. 2.5; 6. $\sqrt{\frac{13}{3}}$; 7. 13/4 ; 8. $-3\sqrt{2}$.

4.8. Indeterminate forms and L'Hopital's rule.

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then finding $\lim_{x\to a} \frac{f(x)}{g(x)}$ by simply substituting the limits of numerator and denominator produces $\left(\frac{0}{0}\right)$, a meaningless expression traditionally called an **indeterminate form**. Because geometric arguments and the technique of canceling factors apply only to limited range of problems it is desirable to have a general method for handling indeterminate forms. This is provided by L'Hopital's rule.

Theorem: (L'Hopital's rule for $\left(\frac{0}{0}\right)$ case)

Let a be a number and let f and g be differentiable over some open interval that contains a. Assume also that $g'(x)$ is not 0 for any x in that interval except perhaps at a. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \left(\frac{0}{0}\right) = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example: Find $\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1}$

Solution: We can find this limit using factorization:

$$\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x\to 1} \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)} = \frac{3}{2}$$

Now let us use L'Hopital's rule to evaluate it.

In this case, $a=1$, $f(x)=x^3-1$, $g(x)=x^2-1$. All assumptions of L'Hopital's rule are satisfied. In particular,

$$\lim_{x\to 1}(x^3-1)=0 \text{ and } \lim_{x\to 1}(x^2-1)=0.$$

According to L'Hopital's rule,

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}=\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{(x^3-1)'}{(x^2-1)'}=\lim_{x\to 1}\frac{3x^2}{2x}=3/2.$$

Example: Use L'Hopital's rule to evaluate $\lim_{x\to 0}\frac{\sin 3x}{x}$

Solution: Since $\lim_{x\to 0}\sin 3x=0$ and $\lim_{x\to 0}x=0$ the given limit is an indeterminate form of type $\left(\frac{0}{0}\right)$. Thus, we can apply L'Hopital's rule:

$$\lim_{x\to 0}\frac{\sin 3x}{x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(\sin 3x)'}{(x)'}=\lim_{x\to 0}\frac{3\cos 3x}{1}=3.$$

Example: Find $\lim_{x\to 0}\frac{\sin x-x}{x^3}$

Solution: As $x\to 0$, both numerator and denominator approach zero. According to L'Hopital's rule

$$\lim_{x\to 0}\frac{\sin x-x}{x^3}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(\sin x-x)'}{(x^3)'}=\lim_{x\to 0}\frac{\cos x-1}{3x^2}=\left(\frac{0}{0}\right)$$

Let us use L'Hopitals rule for the second time:

$$\lim_{x\to 0}\frac{\cos x-1}{3x^2}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(\cos x-1)'}{(3x^2)'}=\lim_{x\to 0}\frac{-\sin x}{6x}=\left(\frac{0}{0}\right)$$

Using L'Hopital's rule again we obtain

$$\lim_{x\to 0}\frac{-\sin x}{6x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(-\sin x)'}{(6x)'}=\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}.$$

Warning: When applying L'Hopita's rule to $\lim_{x\to a}\frac{f(x)}{g(x)}$, the derivative of $f(x)$ and $g(x)$ are taken **separately.** $\lim_{x\to a}\frac{f'(x)}{g'(x)}$. Do not make the mistake of differentiating $\frac{f(x)}{g(x)}$ according to quotient rule.

Theorem: (L'Hopital's rule for $\left(\frac{\infty}{\infty}\right)$ case)

Let f and g be defined and differentiable functions of x. If $f(x)$ and $g(x)$ both approaches infinity as x approaches a, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

A similar result holds for $x\to a^-, x\to a^+, x\to +\infty$ or $x\to -\infty$.
Moreover, $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ could both be $-\infty$, or one could be $+\infty$ and the other $-\infty$.

Example: Find $\lim_{x\to\infty}\frac{4x^2-3x}{5x^2-7}$

Solution: Both numerator and denominator approach ∞ as $x\to\infty$.
Trying L'Hopital's rule we obtain

$$\lim_{x\to\infty}\frac{4x^2-3x}{5x^2-7}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{(4x^2-3x)'}{(5x^2-7)'}=\lim_{x\to\infty}\frac{8x-3}{10x}=\left(\frac{\infty}{\infty}\right)=$$

$$=\lim_{x\to\infty}\frac{(8x-3)'}{(10x)'}=\lim_{x\to\infty}\frac{8}{10}=\frac{4}{5}.$$

Example: Find $\lim_{x\to\pi/2}\frac{\tan x}{\tan 3x}$

Solution:

$$\lim_{x\to\pi/2}\frac{\tan x}{\tan 3x}=\left(\frac{\infty}{-\infty}\right);$$ Let us apply L'Hopital's rule

$$\lim_{x\to\pi/2}\frac{\tan x}{\tan 3x}=\left(\frac{\infty}{-\infty}\right)=\lim_{x\to\pi/2}\frac{(\tan x)'}{(\tan 3x)'}=\lim_{x\to\pi/2}\frac{\frac{1}{\cos^2 x}}{\frac{3}{\cos^2 3x}}=$$

$$=\lim_{x\to\pi/2}\frac{\cos^2 3x}{3\cos^2 x}=\left(\frac{0}{0}\right)=\lim_{x\to\pi/2}\frac{-6\cos 3x\sin 3x}{-6\cos x\sin x}=\lim_{x\to\pi/2}\frac{\sin 6x}{\sin 2x}=$$

$$=\left(\frac{0}{0}\right)=\lim_{x\to\pi/2}\frac{6\cos 6x}{2\cos 2x}=\frac{6}{2}=3.$$

Indeterminate form of type $(0\cdot\infty)$ or $(\infty-\infty)$.

If $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$ then $\lim_{x\to a}[f(x)\cdot g(x)]$ is called an indeterminate form of type $(0\cdot\infty)$.

Similarly if $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$ then $\lim_{x\to a}[f(x)-g(x)]$ produces an indeterminate form of $(\infty-\infty)$. The limits of these types can be converted to the form of $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ and then can be evaluated by L'Hopital's rule.

Example: Find $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

Solution:

Since $\lim_{x\to 0}\frac{1}{x}=\infty$ and $\lim_{x\to 0}\frac{1}{\sin x}=\infty$ the given problem is an indeterminate form of type $(\infty-\infty)$. Let us convert it to $\left(\frac{0}{0}\right)$ form and apply L'Hopital's rule

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\frac{\sin x-x}{x\cdot\sin x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(\sin x-x)'}{(x\cdot\sin x)'}=$$

$$=\lim_{x\to 0}\frac{\cos x-1}{\sin x+x\cos x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(\cos x-1)'}{(\sin x+x\cos x)'}=$$

$$=\lim_{x\to 0}\frac{-\sin x}{\cos x+\cos x-x\sin x}=0.$$

Example: Evaluate $\lim_{x\to\pi/4}(1-\tan x)\cdot\sec 2x$

Solution:

The given problem is an indeterminate form of type $(0\cdot\infty)$. We can convert it to type $\left(\frac{0}{0}\right)$ and then apply L'Hopital's rule

$$\lim_{x\to\pi/4}(1-\tan x)\cdot\sec 2x=\lim_{x\to\pi/4}\frac{1-\tan x}{\cos 2x}=\left(\frac{0}{0}\right)=\lim_{x\to\pi/4}\frac{(1-\tan x)'}{(\cos 2x)'}=$$

$$=\lim_{x\to\pi/4}\frac{-\sec^2 x}{-2\sin 2x}=\frac{-2}{-2}=1.$$

Although the L'Hopital's rule is very useful and widely used method, there are some limits for which L'Hopital's rule does not help.

Example: Evaluate $\lim_{x\to\infty}\frac{\sqrt{4x+1}}{\sqrt{x+3}}$

Solution:

$\lim_{x\to\infty}\sqrt{4x+1}=\infty$ and $\lim_{x\to\infty}\sqrt{x+1}=\infty$, so the given limit is an indeterminate form of $\left(\frac{\infty}{\infty}\right)$. Thus,

$$\lim_{x\to\infty}\frac{\sqrt{4x+1}}{\sqrt{x+3}}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{(\sqrt{4x+1})'}{(\sqrt{x+3})'}=\lim_{x\to\infty}\frac{4\sqrt{x+3}}{\sqrt{4x+1}}=\left(\frac{\infty}{\infty}\right)$$

Repeated application of L'Hopital's rule simply will produce another indeterminate form of $\left(\frac{\infty}{\infty}\right)$. We must try something else.

The given limit can be rewritten as

$$\lim_{x\to\infty}\frac{\sqrt{4x+1}}{\sqrt{x+3}}=\lim_{x\to\infty}\sqrt{\frac{4x+1}{x+3}}=\lim_{x\to\infty}\sqrt{\frac{4+1/x}{1+3/x}}=\lim_{x\to\infty}\sqrt{4}=2.$$

Exercises.

In exercises 1-16 use L'Hopital's rule to find limits if it applies.

1. $\lim_{x\to 0}\frac{\tan x-\sin x}{x-\sin x}$

2. $\lim_{x\to 0}\frac{\pi/x}{\cot\frac{\pi x}{2}}$

3. $\lim_{x\to\pi/2}\frac{\tan x}{\tan 5x}$

4. $\lim_{x\to 0}(1-\cos x)\cot x$

5. $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$

6. $\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{5}{x^2-x-6}\right)$

7. $\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}$

8. $\lim_{x\to 1}\frac{4x^3-2x^2-2}{4x^2-5x+1}$

9. $\lim_{x\to \pi/6}\frac{1-2\sin x}{\cos 3x}$

10. $\lim_{x\to 0}(x\cdot\cot \pi x)$

11. $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$

12. $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)$

13. $\lim_{x\to 0}\frac{\sin x^2}{(\sin x)^2}$

14. $\lim_{x\to 2}\frac{x^3+8}{x^2+5}$

15. $\lim_{x\to\infty}\frac{4x^2+3\cos 3x}{2x^2-2\sin 2x}$

16. $\lim_{x\to 0}\frac{x-\sin x}{\tan x-\sin x}$

In exercises 17-20 verify that L'Hopital's rule is no help in finding limit, then find limit by some other method.

17. $\lim_{x\to+\infty}\frac{x+\sin 2x}{x}$

18. $\lim_{x\to+\infty}\frac{x(2+\sin 2x)}{x+1}$

19. $\lim_{x\to\infty}\frac{\sqrt{16x-1}}{\sqrt{2x+5}}$

20. $\lim_{x\to+\infty}\frac{2x-\sin x}{3x+\sin x}$

Answers.

1. 3; **2.** $\frac{\pi^2}{2}$; **3.** 5; **4.** 0; **5.** $\frac{2}{\pi}$; **6.** $\frac{1}{5}$; **7.** $-\frac{1}{3}$; **8.** $\frac{8}{3}$; **9.** $\frac{\sqrt{3}}{3}$;

10. $\frac{1}{\pi}$; **11.** 2/3; **12.** 0; **13.** 1; **14.** 16/9; **15.** 2; **16.** 1/3; **17.** 1;

18. does not exist; **19.** $2\sqrt{2}$; **20.** 2/3.

Chapter 5.
Integration.

5.1. Antiderivatives. The indefinite integral. Properties and some integration formulas.

Definition: A function $F(x)$ is called an antiderivative of the function $f(x)$ on a given interval if

$$F'(x) = f(x)$$

for all x in that interval.

Example: The functions $\frac{1}{4}x^4$; $\frac{1}{4}x^4 + 7$; $\frac{1}{4}x^4 - \pi$; $\frac{1}{4}x^4 + C$ (C-is any constant) are antiderivatives of $f(x) = x^3$, because derivatives of all above functions are x^3. A function can have many antiderivatives.

If $F(x)$ is antiderivative of any function $f(x)$, and C is any constant, then $F(x) + C$ is also antiderivative of $f(x)$.

$$\frac{d}{dx}[F(x) + C] = f(x)$$

We will denote it by

$$(1) \qquad \int f(x)dx = F(x) + C$$

The symbol $\int$ is called an integral sign, $f(x)$ is the integrand.

(1) is read as " the definite integral of $f(x)$ equals $F(x)$ plus C "

For example above:

$$\int x^3 dx = \frac{x^4}{4} + C.$$

Properties of indefinite integrals. Integration formulas.

An indefinite integral has the following properties:

1. A constant factor can be moved through an integral sign

$$\int c \cdot f(x)dx = c \cdot \int f(x)dx$$

2. An antiderivative of a sum is the sum of the antiderivatives

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

3) An antiderivative of a difference is the difference of the antiderivatives

$$\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx .$$

List of some basic integration formulas.

(1) $\int 0 \cdot dx = C$

(2) $\int dx = x + C$

(3) $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$; $\quad (n \neq -1)$

(4) $\int \sin x dx = -\cos x + C$

(5) $\int \cos x dx = \sin x + C$

(6) $\int \sec^2 x dx = \tan x + C$

(7) $\int \csc^2 x dx = -\cot x + C$

(8) $\int \sec x \cdot \tan x dx = \sec x + C$

(9) $\int \csc x \cdot \cot x dx = -\csc x + C$

Correctness of integration formulas can be checked by differentiation. For example (3) is correct, since

$$\left(\frac{x^{n+1}}{n+1} + C\right)' = \frac{(n+1)x^n}{n+1} = x^n$$

__**Example:**__ Evaluate $\int 8\cos x dx$

__**Solution:**__

$\int 8\cos x dx =$ Apply the first property$= 8\int \cos x dx =$

=Apply formula (5)$= 8\sin x + C$.

Example: Evaluate

a) $\int \frac{dx}{x^5}$; b) $\int \frac{dx}{\sqrt[3]{x}}$ c) $\int \sqrt[4]{x}dx$

Solution:

For all integrals we will apply (3) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for different n:

a) $$\int \frac{dx}{x^5} = \int x^{-5} dx = \frac{x^{-5+1}}{-5+1} + C = -\frac{1}{4x^4} + C;$$

b) $$\int \frac{dx}{\sqrt[3]{x}} = \int x^{-\frac{1}{3}} dx = \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C = \frac{3}{2}\sqrt[3]{x^2} + C;$$

c) $$\int \sqrt[4]{x}dx = \int x^{\frac{1}{4}} dx = \frac{x^{\frac{1}{4}+1}}{\frac{1}{4}+1} + C = \frac{4}{5}\sqrt[4]{x^5} + C$$

Second and third properties of indefinite integrals can be extended to more than two functions:

$$\int [f(x) - g(x) + h(x)]dx = \int f(x)dx - \int g(x)dx + \int h(x)dx$$

Example: Evaluate $\int (6x^3 - 2x^2 + 7x - 4)dx$

Solution:

$$\int (6x^3 - 2x^2 + 7x - 4)dx = \int 6x^3 dx -$$

$$- \int 2x^2 dx + \int 7x dx - \int 4dx = 6\int x^3 dx - 2\int x^2 dx + 7\int x dx -$$

$$-4\int dx = 6 \cdot \left(\frac{x^4}{4} + C_1\right) - 2 \cdot \left(\frac{x^3}{3} + C_2\right) +$$

$$+7\left(\frac{x^2}{2} + C_3\right) - 4(x + C_4) = \frac{3}{2}x^4 - \frac{2}{3}x^3 + \frac{7}{2}x^2 - 4x +$$

$$+(6C_1 - 2C_2 + 7C_3 - 4C_4) = \frac{3}{2}x^4 - \frac{2}{3}x^3 + \frac{7}{2}x^2 - 4x + C$$

Sometimes it is useful to rewrite an integrand (simplify) in a different form before performing integration.

Example: Evaluate

a) $\int \frac{\sin x}{\cos^2 x} dx$; b) $\int \frac{t^4 - 3t^6}{t^6} dt$

Solution:

a) $\int \frac{\sin x}{\cos^2 x} dx = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx = \int \tan x \cdot \sec x dx = \sec x + C$

b) $\int \frac{t^4 - 3t^6}{t^6} dt = \int \left(\frac{1}{t^2} - 3 \right) dt = \int t^{-2} dt - \int 3 dt =$

$= \frac{t^{-1}}{-1} - 3t + C = \frac{-1}{t} - 3t + C$.

Example: Find $\int \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2 dx$

Solution:

$$\int \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2 dx = \int \left(\sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2} + \cos^2 \frac{x}{2} \right) dx =$$

$$= \int (1 + \sin x) dx = x - \cos x + C$$

Example: Evaluate $\int \frac{1}{1 + \sin x} dx$

Solution:

Let us multiply numerator and denominator by (1-sin$x$).

We obtain

$$\int \frac{1}{1 + \sin x} dx = \int \frac{1 - \sin x}{(1 + \sin x)(1 - \sin x)} dx = \int \frac{1 - \sin x}{\cos^2 x} dx =$$

$$= \int \frac{1}{\cos^2 x} dx - \int \tan x \cdot \sec x dx = \tan x - \sec x + C$$

Example: Evaluate $\int \sin^2 \frac{x}{2} dx$

Solution:

Let us use identity $\sin^2 x = \frac{1 - \cos 2x}{2}$

$\int \sin^2\frac{x}{2}dx = \int\frac{1-\cos x}{2}dx = \int\frac{1}{2}dx - \int\frac{\cos x}{2}dx = \frac{x}{2} - \frac{\sin x}{2} + C$.

Exercises.

In exercises 1-15 evaluate the integrals and check each answer by differentiating it.

1. $\int 5x^2 dx$
2. $\int (2x - x^3 + x^5)dx$
3. $\int \frac{4}{\sqrt{t}}dt$
4. $\int x\sqrt[4]{x}dx$
5. $\int (2x^{2/3} - 8x^{-2/5})dx$
6. $\int (2x + x^2)^2 dx$
7. $\int \frac{x^5 + 2x^2 - 1}{x^4}dx$
8. $\int [4\sin x + 2\cos x]dx$
9. $\int \sec x(\sec x + \tan x)dx$
10. $\int [\sqrt{\theta} - \csc^2\theta]d\theta$
11. $\int \frac{\sin 2x}{\cos x}dx$
12. $\int \frac{\cos^3\theta - 5}{\cos^2\theta}d\theta$
13. $\int \left(\sqrt[3]{x^2} + \frac{1}{\sqrt[3]{x}}\right)dx$
14. $\int (ax - b)^3 dx$
15. $\int \sin^2\frac{x}{2}dx$

16. Find the antiderivative $F(x)$ of $f(x) = \sqrt[3]{x}$ that satisfies $F(1) = 2$.

17. Find a function *f* such that $f'(x) + \cos x = 0$ and $f\left(\frac{\pi}{2}\right) = 2$.

Answers.

1. $\frac{5}{3}x^3 + C$; **2.** $x^2 - \frac{x^4}{4} + \frac{x^6}{6} + C$; **3.** $8\sqrt{t} + C$; **4.** $\frac{4}{9}x^{\frac{9}{4}} + C$;
5. $\frac{6}{5}x\sqrt[3]{x^2} - \frac{40}{3}\sqrt[5]{x^3} + C$; **6.** $\frac{x^5}{5} + x^4 + \frac{4}{3}x^3 + C$; **7.** $\frac{1}{2}x^2 - \frac{2}{x} + \frac{1}{3x^3} + C$; **8.** $-4\cos x + 2\sin x + C$; **9.** $\tan x + \sec x + C$;

10. $\frac{2}{3}\theta\sqrt{\theta}+\cot\theta+C$; **11.** $-2\cos x+C$; **12.** $\sin\theta-5\tan\theta+C$;

13. $\frac{3}{5}x\sqrt[3]{x^2}+\frac{3}{2}\sqrt[3]{x^2}+C$; **14.** $\frac{a^3}{4}x^4-a^2bx^3+\frac{3}{2}ab^2x^2-$
$-b^3x+C$; **15.** $\frac{1}{2}(x-\sin x)+C$; **16.** $F(x)=\frac{3}{4}x^{4/3}+\frac{5}{4}$;
17. $f(x)=3-\sin x$.

5.2. Integration by substitution.

We use substitution method of integration to transform an integral not listed in an integral table to one that is listed.

Example: Find $\int(\sin x^2)\cdot 2x\,dx$

Solution:

Note that $2x$ is the derivative of x^2. Make substitution $u=x^2$. Then $du=2xdx$ and

$$\int(\sin x^2)\cdot 2x\,dx=\int\sin u\,du=-\cos u+C$$

Replacing u by x^2 in $-\cos u$ yields $-\cos x^2$. Thus

$$\int(\sin x^2)\cdot 2x\,dx=-\cos x^2+C$$

The answer can be checked by differentiation:

$$\frac{d}{dx}[-\cos x^2+C]=(\sin x^2)\cdot 2x.$$

Example: Evaluate $\int(x^3+1)^{60}\cdot 3x^2dx$

Solution:

If we let $u=x^3+1$, then $du=3x^2dx$, so

$$\int(x^3+1)^{60}\cdot 3x^2dx=\int u^{60}du=\frac{u^{61}}{61}+C=\frac{(x^3+1)^{61}}{61}+C.$$

Example: Evaluate $\int\sin 5xdx$

Solution:

$$\int \sin 5x dx = \left| \begin{array}{l} u = 5x \\ du = 5dx \\ dx = \dfrac{du}{5} \end{array} \right| = \int \sin u \cdot \frac{du}{5} = -\frac{1}{5}\cos u + C = -\frac{1}{5}\cos 5x + C.$$

Remark: There are no hard and fast rules for choosing u – making an appropriate choice will come with experience.

Example: Evaluate $\int \sin^3 x \cos x\, dx$

Solution:

$$\int \sin^3 x \cos x dx = \left| \begin{array}{l} u = \sin x \\ du = \cos x dx \end{array} \right| = \int u^3 du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C$$

Example: Evaluate $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

Solution:

If we let $u = \sqrt{x}$, then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, so

$du = \frac{dx}{2\sqrt{x}}$ and $\frac{dx}{\sqrt{x}} = 2du$. Then

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = \int \sin u \cdot 2du = -2\cos u + C = -2\cos \sqrt{x} + C.$$

Example: Evaluate $\int \frac{x^2}{\sqrt{1+x^3}} dx$

Solution:

$$\int \frac{x^2}{\sqrt{1+x^3}} dx = \left| \begin{array}{l} u = 1 + x^3 \\ du = 3x^2 dx \\ \dfrac{du}{3} = x^2 \mathrm{dx} \end{array} \right| = \int \frac{du/3}{\sqrt{u}} = \frac{1}{3} \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C =$$

$$=\frac{1}{3}\cdot 2\sqrt{u}+C=\frac{2}{3}\sqrt{1+x^3}+C.$$

Example: Evaluate $\int 7t^3\cdot\sqrt[3]{4-7t^4}\,dt$

Solution:

$$\int 7t^3\cdot\sqrt[3]{4-7t^4}\,dt=$$

$$=\left|\begin{array}{l} u=4-7t^4 \\ du=-28t^3dt \\ \dfrac{du}{-4}=7t^3\text{dt}\end{array}\right|=\int\sqrt[3]{u}\cdot\left(-\frac{du}{4}\right)=-\frac{1}{4}\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1}+C=$$

$$=-\frac{1}{4}\cdot\frac{3}{4}\sqrt[3]{u^4}+C=-\frac{3}{16}\sqrt[3]{(4-7t^4)^4}+C.$$

Example: Evaluate $\int x^2\sqrt{x+1}\,dx$

Solution:

$$\int x^2\sqrt{x+1}\,dx=\left|\begin{array}{l} u=x+1 \\ du=dx \\ x=u-1\end{array}\right|=\int(u-1)^2\sqrt{u}\,du=$$

$$=\int(u^2-2u+1)\sqrt{u}\,du=\int\left(u^{\frac{5}{2}}-2u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)du=$$

$$=\frac{7}{2}u^{\frac{7}{2}}-\frac{4}{5}u^{\frac{5}{2}}+\frac{2}{3}u^{\frac{3}{2}}+C=$$

$$=\frac{7}{2}(x+1)^{\frac{7}{2}}-\frac{4}{5}(x+1)^{\frac{5}{2}}+\frac{2}{3}(x+1)^{\frac{3}{2}}+C.$$

Exercises.

In exercises 1-20 use appropriate substitutions to evaluate integrals.

1. $\int(1+x^2)^6\,x\,dx$
2. $\int\sqrt[3]{(1+x^2)}\cdot xdx$
3. $\int\cos 7\theta\,d\theta$
4. $\int(x-3)^5\,dx$

$\int x^5 \cdot \sin x^6 \, dx$

6. $\int \frac{2x+7}{(x^2+7x+6)^4} dx$

$\int t\sqrt{8t^2+14}\, dt$

8. $\int \sqrt{x+4}\, dx$

$\int \cos\frac{x}{4}\, dx$

10. $\int \frac{dx}{\sin^2 3x}$

$\int x\sqrt{x-2}\, dx$

12. $\int \frac{\cos x}{\sqrt{1+4\sin x}} dx$

$\int \frac{x}{\sqrt{1-x^2}} dx$

14. $\int \frac{x^2}{\sqrt{x^3+1}} dx$

$\int \frac{\sin(5/x)}{x^2} dx$

16. $\int x^2 \sec^2(x^3)\, dx$

$\int \sin^5 3t \cdot \cos 3t \, dt$

18. $\int \cos 4\theta \cdot \sqrt{2-\sin 4\theta}\, d\theta$

$\int \sec^3 2x \cdot \tan 2x \, dx$

20. $\int [\sec^2(\cos 3\theta)] \cdot \sin 3\theta \, d\theta$

Answers.

$\frac{}{4}(1+x^2)^7 + C$; **2.** $\frac{3}{8}(1+x^2)^{4/3} + C$; **3.** $\frac{\sin 7\theta}{7} + C$; **4.** $\frac{(x-3)^6}{6} + C$;

$-\frac{1}{6}\cos x^6 + C$; **6.** $-\frac{1}{3}\cdot\frac{1}{(x^2+7x+6)^3} + C$; **7.** $\frac{1}{24}\sqrt{(8t^2+14)^3} + C$;

$\frac{2}{3}\sqrt{(x+4)^3} + C$; **9.** $4\sin\frac{x}{4} + C$; **10.** $-\frac{1}{3}\cot 3x + C$;

$\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C$; **12.** $\frac{1}{2}\sqrt{1+4\sin x} + C$;

$-\sqrt{1-x^2} + C$; **14.** $\frac{2}{3}\sqrt{x^3+1} + C$; **15.** $\frac{1}{5}\cos(5/x) + C$;

$\frac{1}{3}\tan(x^3) + C$; **17.** $\frac{1}{18}\sin^6 3t + C$; **18.** $-\frac{1}{6}(2-\sin 4\theta)^{3/2} + C$;

$\frac{1}{6}\sec^3 2x + C$; **20.** $-\frac{1}{3}\tan(\cos 3\theta) + C$.

5.3. Sigma notation.

Definition: Let $a_1, a_2, a_3, \ldots\ldots, a_n$ be n numbers. The sum $a_1 + a_2 + a_3 + \ldots\ldots + a_n$ will be denoted in sigma notation by the symbol $\sum_{k=1}^{n} a_k$, which is read as " the sum of a sub k as k runs from 1 to n.

Example: Write $1^3 + 2^3 + 3^3 + 4^3$ in the sigma notation.

Solution:

$$1^3 + 2^3 + 3^3 + 4^3 = \sum_{k=1}^{4} k^3$$

Example: Compute $\sum_{k=1}^{5} 2^k$

Solution: $\sum_{k=1}^{5} 2^k = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 2+4+8+16+32=62.$

Properties of sigma notation:

a) $\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k$

b) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

c) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

Some useful sums:

(1) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(2) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example: Evaluate $\sum_{k=1}^{10} k(k+3)$

Solution:

$$\sum_{k=1}^{10} k(k+3) = \sum_{k=1}^{10} k^2 + \sum_{k=1}^{10} 3k = \sum_{k=1}^{10} k^2 + 3\sum_{k=1}^{10} k =$$

$$= \frac{10\cdot 11\cdot 21}{6} + 3\cdot\frac{10\cdot 11}{2} = 385 + 165 = 550.$$

Some sums are formed by adding up differences. When each term of a sum cancels part of the next term, leaving only portions of the first and last terms at the end of the sum is said to **telescope**.

Example: Evaluate $\sum_{k=1}^{50}\left(\frac{1}{k} - \frac{1}{k+1}\right)$

Solution:

$$\sum_{k=1}^{50}\left(\frac{1}{k} - \frac{1}{k+1}\right) = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) + \ldots..$$

$$+ (\frac{1}{50}-\frac{1}{51}) = 1 - \frac{1}{51} = \frac{50}{51}.$$

Example: Evaluate $\sum_{k=1}^{n}(a_k - a_{k+1})$

Solution:

$$\sum_{k=1}^{n}(a_k - a_{k+1}) = (a_1 - a_2) + (a_2 - a_3) + \ldots.. + (a_n - a_{n+1}) = a_1 - a_{n+1}$$

Exercises.

In exercises 1-3 evaluate the sums.

1\. a) $\sum_{k=1}^{3} k$; b) $\sum_{k=1}^{4} 2k$; c) $\sum_{n=1}^{3} n^2$

2\. a) $\sum_{i=1}^{4} 1^i$; b) $\sum_{k=2}^{6} (-1)^k$; c) $\sum_{j=1}^{150} 3$

3\. a) $\sum_{j=2}^{6} (3j-1)$; b) $\sum_{n=0}^{10} 1$; c) $\sum_{m=3}^{5} 2^{m+1}$

In exercises 4-10 write is sigma notation, but do not evaluate.

4. $x^3 + x^4 + x^5 + x^6 + x^7$

5. $\frac{1}{3} + \frac{1}{4} + ... + \frac{1}{102}$

6. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + 49 \cdot 50$

7. 1-3+5-7+9-11

8. $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}$

9. $-b_0 + b_1 - b_2 + b_3 - b_4 + b_5$

10. $a^5 + a^4 b + a^3 b^2 + a^2 b^3 + ab^4 + b^5$

In exercises 11-13 evaluate sum using useful sums formulas.

11. $\sum_{k=1}^{20} k^2$; 12. $\sum_{k=1}^{6} (4k^3 - 2k + 1)$; 13. $\sum_{k=1}^{30} k(k-2)(k+2)$

In exercises 14-16 evaluate telescoping sums.

14. $\sum_{i=1}^{100} (2^i - 2^{i-1})$; 15. $\sum_{i=1}^{50} \left(\frac{1}{2i+1} - \frac{1}{2(i-1)+1} \right)$

16. $\sum_{k=2}^{20} \left(\frac{1}{k^2} - \frac{1}{(k-1)^2} \right)$

17. Evaluate

a) $\sum_{i=1}^{100} i$; b) $\sum_{k=1}^{n} n$; c) $\sum_{k=1}^{n} kx$

Answers.

1. a) 6; b) 20; c)14; **2.** a) 4;b) 1; c) 450; **3.** a) 55; b) 11; c) 112;

4. $\sum_{i=3}^{7} x^i$; **5.** $\sum_{k=3}^{102} \frac{1}{k}$; **6.** $\sum_{n=1}^{49} n(n+1)$; **7.** $\sum_{k=1}^{6} (-1)^{k+1} (2k-1)$; **8.** $\sum_{k=1}^{5} (-1)^k \cdot \frac{1}{k}$;

9. $\sum_{k=0}^{5} (-1)^{k+1} \cdot b_k$; **10.** $\sum_{k=0}^{5} a^{5-k} \cdot b^k$; **11.** 2870; **12.** 1728;

13. 214365; **14.** $2^{100} - 1$; **15.** $-\frac{100}{101}$; **16.** $-\frac{399}{400}$; **17.** a) 5050; b) n^2;

c) $\frac{1}{2} n(n+1)x$.

5. 4. The definite integral and its properties.

t function f be continuous on $[a, b]$. t us divided $[a, b]$ into n bintervals. If an interval $[a, b]$ vided into n subintervals, and if is an arbitrary point in th subinterval, the $f(x_k^*)\Delta x_k$ is e area of a rectangle of height (x_k^*) and width Δx_k, so then the ea under the curve $y = f(x)$ ver the interval $[a , b]$ is defined

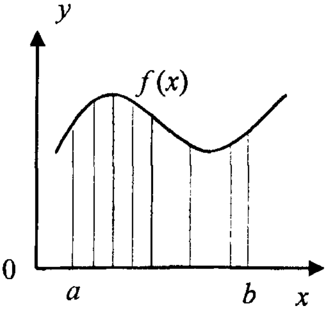


Fig. 5.1

y $\lim\limits_{\max \Delta x_k \to 0} \sum\limits_{k=1}^{n} f(x_k^*)\Delta x_k$

We write

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k .$$

Briefly, the definite integral f over $[a, b]$ is the area under the curve $y = f(x)$ over $[a, b]$.

$\int_a^b f(x)dx$ is called the definite integral of f from a to b.

a and b are called the **lower** and **upper** limits of integration, respectively.

Example: Express $\lim\limits_{\max \Delta x_k \to 0} \sum\limits_{k=1}^{n} (x_k^*)^4 \Delta x_k$; as a definite integral for $a = 1$, $b = 2$.

Solution: The function being evaluated at x_k^* in each term of the sum is $f(x) = x^4$. The partition interval is [1, 2], so the limit is the integral of f from 1 to 2

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^4 \Delta x_k = \int_1^2 x^4 dx$$

$$A = \begin{bmatrix} \text{area under} \\ y = f(x) \text{ over} \\ [a,b] \end{bmatrix} = \int_a^b f(x)dx .$$

Example: Evaluate $\int_3^5 x\, dx$

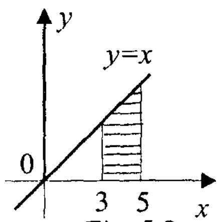


Fig. 5.2

Solution: The integral represents the area under the graph of $y = x$ over [3, 5]. (Fig.5.2). This region is a trapezoid whose parallel sides are have length 3 and 5, and whose height is 2.
Thus

$$\int_3^5 x\, dx = \frac{3+5}{2} \cdot 2 = 8$$

Example: Evaluate $\int_2^6 (2-x)\, dx$

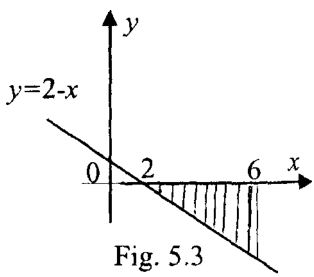


Fig. 5.3

Solution:
The integrand is negative over [2, 6], so the integral is the negative of the area of the triangle (Fig. 5.3). The area of triangle is 8, so

$$\int_2^6 (2-x)\, dx = -8 .$$

Properties of definite integrals.

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a f(x)dx = 0$

3. $\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx$

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^b [f(x)+g(x)-h(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx - \int_a^b h(x)dx$$

If a, b, and c are numbers then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \geq 0$$

9. If f and g are integrable functions on $[a, b]$, where $a < b$ and $f(x) \leq g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

10. If m and M are numbers and $m \leq f(x) \leq M$ for all x in $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \text{ if } a < b \qquad \text{and}$$

$$m(b-a) \geq \int_a^b f(x)dx \geq M(b-a) \text{ if } a > b$$

Example: Suppose that

$$\int_1^6 f(x)dx = -1; \quad \int_3^6 f(x)dx = 3 \text{ and } \int_3^6 g(x)dx = 4.$$

Find a) $\int_3^6 [3f(x) - g(x)]dx$; b) $\int_1^3 f(x)dx$

Solution:

a) From properties of definite integral

$$\int_3^6 [3f(x) - g(x)]dx = 3 \cdot \int_3^6 f(x)\,dx - \int_3^6 g(x)dx = 3 \cdot 3 - 4 = 5.$$

b) From 7 th property of definite integral with a=1, b=6, c=3

$$\int_1^6 f(x)dx = \int_1^3 f(x)dx + \int_3^6 f(x)dx \text{ , so}$$

$$\int_1^3 f(x)dx = \int_1^6 f(x)dx - \int_3^6 f(x)dx = -1 - 3 = -4.$$

Example:

Find $\int_3^{-2} f(x)dx$ if $\int_{-2}^1 f(x)dx = 2$ and $\int_1^3 f(x)dx = -6$.

Solution:

From 1 st property it follows that

$$\int_3^{-2} f(x)dx = -\int_{-2}^3 f(x)dx$$

From 7 th property we obtain

$$\int_{-2}^3 f(x)dx = \int_{-2}^1 f(x)dx + \int_1^3 f(x)dx = 2 + (-6) = -4 \text{ and}$$

$$\int_3^{-2} f(x)dx = -(-4) = 4$$

It can be shown that

$$\int_a^b c\,dx = c(b-a) \text{ (}c\text{-const);}$$

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}; \quad \text{and}$$

$$\int_0^b x^2\,dx = \frac{b^3}{3}.$$

Using these integrals we can evaluate other integrals.

Example: Evaluate $\int_0^3 x^2 dx$

Solution: $\int_0^3 x^2 dx = \frac{3^3}{3} = 9$

Example: Evaluate $\int_0^4 (3x^2 - 4x + 5)dx$

Solution: $\int_0^4 (3x^2 - 4x + 5)dx = 3 \cdot \int_0^4 x^2 dx - 4\int_0^4 xdx + 5\int_0^4 dx =$

$$= 3(\frac{4^3}{3}) - 4(\frac{4^2}{2} - \frac{0^2}{2}) + 5(4-0) = 64 - 4 \cdot 8 + 20 = 52.$$

Example: Evaluate $\int_2^3 x^2 dx$

Solution: $\int_2^3 x^2 dx = \int_0^3 x^2 dx - \int_0^2 x^2 dx = \frac{3^3}{3} - \frac{2^3}{3} = 6\frac{1}{3}.$

Exercises.

In exercises 1-3 use the given values of a and b to express the following limits as a definite integral. (Do not evaluate the integrals).

1. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^3 \Delta x_k$; $a = 1$; $b = 4$

2. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \{4(x_k^*)^2 - 3x_k^* + 3\}\Delta x_k$; $a = -1$; $b = 2$

3. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (\cos^2 x_k^*)\Delta x_k$; $a = -\frac{\pi}{2}$; $b = \frac{\pi}{3}$

In exercises 4-10 evaluate definite integrals by using area formulas from geometry, where needed.

4. $\int_0^2 (1-\frac{1}{2}x)\,dx$

5. $\int_{-1}^1 (1-\frac{1}{2}x)\,dx$

6. $\int_7^9 6\,dx$

7. $\int_0^5 (3-4x)\,dx$

8. $\int_0^3 \sqrt{9-x^2}\,dx$

9. $\int_{-2}^3 |3x-3|\,dx$

10. $\int_0^{\pi} \cos x\,dx$

11. Find $\int_{-1}^2 [f(x)-3g(x)]\,dx$ if $\int_{-1}^2 f(x)\,dx = 7$ and $\int_{-1}^2 g(x)\,dx = -5$.

12. Find $\int_1^5 f(x)\,dx$ if $\int_0^1 f(x)\,dx = -2$ and $\int_0^5 f(x)\,dx = 1$.

In exercises 13-16 use formulas for $\int_a^b c\,dx$; $\int_a^b x\,dx$ and $\int_0^b x^2\,dx$ to evaluate the given integrals:

13. a) $\int_2^5 x^2\,dx$; b) $\int_5^2 x^2\,dx$; c) $\int_5^5 x^2\,dx$

14. a) $\int_1^2 x\,dx$; b) $\int_2^1 x\,dx$; c) $\int_3^3 x\,dx$

15. $\int_0^2 (x+2)^2\,dx$

16. $\int_{\sqrt{3}}^0 (2x^2 - 3x)dx$

17. $\int_1^2 (x^2 - 2x + 7)\,dx$

Answers.

1. $\int_1^4 x^3\,dx$; **2.** $\int_{-1}^2 (4x^2 - 3x + 3)\,dx$; **3.** $\int_{-\pi/2}^{\pi/3} \cos^2 x\,dx$; **4.** 1; **5.** 2; **6.** 12;

-35; **8.** $\frac{9\pi}{4}$; **9.** 39/2; **10.** 0; **11.** 22; **12.** 3; **13.** a) 39; b) –39; c) 0;

14. a) 3/2; b) –3/2; c) 0; **15.** 56/3; **16.** $\frac{9-4\sqrt{3}}{2}$; **17.** $6\frac{1}{3}$.

5. 5. The first fundamental theorem of Calculus.

Theorem: (The first fundamental theorem of Calculus)

Let f be continuous on an open interval containing the interval $[a, b]$. Let

(1) $$F(x) = \int_a^x f(t)dt \quad \text{for } a \le x \le b$$

then $F(x)$ is differentiable on $[a, b]$ and its derivative is f; that is

$$F'(x) = f(x).$$

It also can be expressed by the formula

(2) $$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

In words (2) states: " where the integrand is continuous, the derivative of integral with respect to upper limit is equal to the integrand evaluated at the upper limit ".

Example: Differentiate $y = \int_0^x \sqrt{1+t^3}\,dt$

Solution: By the fundamental theorem

$$\frac{dy}{dx} = \frac{d}{dx}\left[\int_0^x \sqrt{1+t^3}\,dt\right] = \sqrt{1+x^3}$$

Example: Find $\frac{d}{dx}\left[\int_\pi^x \sin t\,dt\right]$

Solution: $\frac{d}{dx}\left[\int_\pi^x \sin t\,dt\right] = \sin x$ [Equation (2) with $f(t)=\sin t$].

Example: Find $\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$

Solution: $\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = x^3$

Example: Differentiate $y = \int_x^{10} \sqrt{1+t^2}\,dt$

Solution:
The lower limit of integration is x, we change the limit of integration first:

$y = -\int_{10}^{x} \sqrt{1+t^2}\,dt$ then apply the first fundamental theorem

$$\frac{dy}{dx} = \frac{d}{dx}\left[-\int_{10}^{x} \sqrt{1+t^2}\,dt\right] = -\sqrt{1+x^2}\,.$$

Example: Differentiate $y = \int_0^{x^4} \sqrt{t+t^2}\,dt$

Solution:
The first fundamental theorem does not apply directly since the upper limit of integration is x^4, not x. In this case let $u = x^4$. Then

$y = \int_0^u \sqrt{t+t^2}\,dt$, where $u = x^4$

By the first fundamental theorem $\frac{dy}{du} = \sqrt{u+u^2}$

The Chain rule tells us that

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \sqrt{u+u^2}\cdot 4x^3 = \sqrt{x^4+x^8}\cdot 4x^3 = 4x^5\sqrt{1+x^4}\,.$$

Example: Differentiate $y = \int_0^{\sin^2 x} \sqrt{t+1}\,dt$

olution: Let $u = \sin^2 x$, then

$$\frac{dy}{du} = \frac{d}{du}\left[\int_0^u \sqrt{t+1}\,dt\right] = \sqrt{u+1}$$

y the Chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \sqrt{u+1}\cdot 2\sin x\cos x = \sqrt{\sin^2 x + 1}\cdot \sin 2x\,.$$

xample: Find $\frac{d}{dx}\left[\int_{-x}^{x}\frac{1}{1+t}\,dt\right]$

olution:

et us first rewrite it as

$$\int_{-x}^{x}\frac{1}{1+t}\,dt = \int_{-x}^{0}\frac{1}{1+t}\,dt + \int_{0}^{x}\frac{1}{1+t}\,dt\,.$$

hen

$$\frac{d}{dx}\left[\int_{-x}^{0}\frac{1}{1+t}\,dt\right] = \frac{d}{dx}\left[-\int_{0}^{-x}\frac{1}{1+t}\,dt\right] = \left|\begin{matrix} u=-x \\ du=-dx\end{matrix}\right| = \frac{1}{1-x}$$

$$\frac{d}{dx}\left[\int_{0}^{x}\frac{1}{1+t}\,dt\right] = \frac{1}{1+x}$$

$$\frac{d}{dx}\left[\int_{-x}^{x}\frac{1}{1+t}\,dt\right] = \frac{1}{1-x} + \frac{1}{1+x} = \frac{2}{1-x^2}\,.$$

Exercises.

In exercises 1-6 differentiate the given functions.

$\int_1^x t^4\,dt$ 2. $\int_2^x t^4\,dt$

$\int_7^{x^2}\sqrt[3]{1+\sin t}\,dt$ 4. $\int_{-1}^{x} 4^{-t}\,dt$

5. $\int_1^{x^3} \frac{1}{t}\,dt$ 6. $\int_4^{\sin x} \frac{1}{1+t^2}\,dt$

7. Let $F(x) = \int_0^x \frac{\cos t}{t^2+4}\,dt$. Find

a) $F(0)$ b) $F'(0)$ c) $F''(0)$

8. Let $F(x) = \int_0^x \sqrt{4t^2+5}\,dt$. Find

a) $F(0)$ b) $F'(0)$ c) $F''(0)$

9. Prove that

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)dt = f(g(x))\cdot g'(x) - f(h(x))\cdot h'(x)$$

10. Find $\frac{dy}{dx}$ if $y = \frac{d}{dx}\int_{x^3}^{x^4} \sin^2 t\,dt$

11. Find $\frac{dy}{dx}$ if $y = \frac{d}{dx}\int_{2x}^{3x} t\cdot\tan t\,dt$

Answers.

1. x^4; 2. x^4; 3. $2x\sqrt[3]{1+\sin(x^2)}$; 4. 4^{-x^b}; 5. $3/x$; 6. $\frac{\cos x}{1+\sin^2 x}$; 7. a) 0; b) 1/4; c) 0; 8. a) 0; b) $\sqrt{5}$; c) 0; 10. $4x^3\sin^2(x^4) - 3x^2\sin^2(x^3)$; 11. $9x\tan 3x - 4x\tan 2x$.

5. 6. The second fundamental theorem of Calculus.

Theorem: If f is continuous on $[a, b]$ and F is antiderivative of f then

$$(1) \qquad \int_a^b f(x)dx = F(b) - F(a)$$

The difference $F(b) - F(a)$ is commonly denoted by $F(x)\big|_a^b$ and

(1) can be written as

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example: Evaluate $\int_1^3 x^2 dx$

Solution:

ince $F(x) = \frac{x^3}{3}$ by 2^{nd} fundamental theorem we obtain

$$\int_1^3 x^2 dx = \frac{x^3}{3}\Big|_1^3 = \frac{1}{3}(3)^3 - \frac{1}{3}(1)^3 = 9 - \frac{1}{3} = \frac{26}{3}$$

xample: Evaluate $\int_1^0 x\, dx$

lution:

$$\int_1^0 x\, dx = \frac{x^2}{2}\Big|_1^0 = \frac{0}{2} - \frac{1}{2} = -\frac{1}{2}$$

ample: Evaluate $\int_0^2 (x^4 - 4x + 5)dx$

ution:

$$x^4 - 4x + 5)dx = \left(\frac{x^5}{5} - 4\frac{x^2}{2} + 5x\right)\Big|_0^2 = \left(\frac{32}{5} - 8 + 10\right) - 0 = \frac{42}{5}.$$

mple: Evaluate $\int_{-3}^2 |x| dx$

tion:

e $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$, then

$$dx = \int_{-3}^0 (-x)dx + \int_0^2 x dx = -\frac{x^2}{2}\Big|_{-3}^0 + \frac{x^2}{2}\Big|_0^2 = \frac{9}{2} + \frac{4}{2} = \frac{13}{2}.$$

Theorem: (The Mean Value theorem for integrals)
If f is continuous on a closed interval $[a, b]$ then there is a number c between a and b such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

Definition: The average or mean value of an integrable function $f(x)$ on a closed interval $[a, b]$ is

$$f_{ave} = \frac{1}{b-a} \cdot \int_a^b f(x)dx.$$

Example:

Verify the Mean Value theorem for $f(x) = x^2$ and $[a,b]=[0,3]$

Solution:

$$\int_0^3 x^2 dx = \left.\frac{x^3}{3}\right|_0^3 = 9$$

Since $f(x) = x^2$, we are looking for c in closed interval [0, 3], such that $\int_0^3 x^2 dx = 9 = f(c)(3-0)$ or $9 = c^2 \cdot 3$; since $c^2 = 3 \Rightarrow c = \pm\sqrt{3}$,thus $c = \sqrt{3}$ is the number in [0, 3], whose existence is guaranteed by Mean-Value theorem.

Example: Find the average value of x^3 over the interval [1, 3]

Solution:

$$f_{ave} = \frac{1}{b-a} \cdot \int_a^b f(x)dx = \frac{1}{3-1}\int_1^3 x^3 dx =$$

$$= \frac{1}{2}\left(\frac{x^4}{4}\right)\Bigg|_1^3 = \frac{1}{2}\left(\frac{81}{4} - \frac{1}{4}\right) = 10.$$

Exercises.

In exercises 1-13 use the second fundamental theorem to evaluate the given integrals.

1. $\int_{1}^{2} 5x^3\,dx$
2. $\int_{1}^{4} (x+5x^2)\,dx$
3. $\int_{\pi/6}^{\pi/3} 5\cos x\,dx$
4. $\int_{0}^{\pi/2} \sin 2x\,dx$
5. $\int_{4}^{9} 5\sqrt{x}\,dx$
6. $\int_{1}^{8} \sqrt[3]{x^2}\,dx$
7. $\int_{1}^{2} (t^2-2t+8)\,dt$
8. $\int_{1}^{9} \sqrt{x}\,dx$

$\int_{4}^{9} 2y\sqrt{y}\,dy$

10. $\int_{1}^{4} \left(\frac{3}{\sqrt{x}} - 5\sqrt{x} - x^{-\frac{3}{2}}\right)dx$

1. $\int_{2}^{4} (x^3+x)\,dx$

12. $\int_{0}^{\pi/4} \frac{d\alpha}{\cos^2\alpha}$

. $\int_{4}^{9} \left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$

Find the area of the region under the curve $3x^2$ and above [1, 4]. Make a sketch of the region.

Find the area of the region under the curve $6x^4$ and above [-1, 1]. Make a sketch of the region.

Find the total area between the curve $x^2-3x-10$ and the interval [-3, 8]. Make a sketch of the region.

In exercises 17-21 find the average value of the given function r the given interval.

x^2; [3, 5]

$3x$; [1, 3]

$\sqrt{x}$; [0, 9]

20. $\frac{2}{\sqrt{x}}$; [4, 9]

21. $\sec^2 x$; $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$

Answers.

1. 75/4; **2.** $112\frac{1}{2}$; **3.** $\frac{5}{2}(\sqrt{3}-1)$; **4.** 1; **5.** 190/3; **6.** 93/5; **7.** 22/3; **8.** 52/3; **9.** 844/5; **10.** -55/3; **11.** 66; **12.** 1; **13.** 40; **14.** 63; **15.** 12/5; **16.** 203/2; **17.** 49/3; **18.** 6; **19.** 2; **20.** 4/5; **21.** $\frac{12(\sqrt{3}-1)}{\sqrt{3}\pi}$.

5. 7. Substitution in a definite integral.

Method 1:

The substitution technique extends to definite integrals, $\int_a^b f(x)\,dx$, with one important remark:

When making the substitution from x to u, be sure to replace the integral [a, b] by the interval whose endpoints are u (a) and u (b). Examples will illustrate the necessary changes in the limits of integration.

Example: Evaluate $\int_2^3 2(1+x^2)^5 x\,dx$

Solution:

If we let $u = 1 + x^2$, then $du = 2x dx$.

As x goes from 2 to 3, $u = 1 + x^2$ goes from $1 + 2^2 = 5$ to $1 + 3^2 = 10$. Then

$$\int_2^3 2(1+x^2)^5 x\,dx = \int_5^{10} u^5\,du = \left.\frac{u^6}{6}\right|_5^{10} = \frac{10^6 - 5^6}{6}$$

Remark: Once you make the substitution, you work only with expression involving u. There is no need to bring back x again.

mple: Evaluate $\int_0^{\pi/4} \cos(\pi - x)\,dx$

tion:

$u = \pi - x$ so that $du = -dx$ and $dx = -du$.

h this substitution

when $x = 0$ then $u = \pi$

when $x = \dfrac{\pi}{4}$ then $u = \dfrac{3\pi}{4}$, [this is the last you see of x], so

$$\int_0^{\pi/4} \cos(\pi - x)\,dx = -\int_{\pi}^{3\pi/4} \cos u\,du = -\sin u\Big|_{\pi}^{3\pi/4} =$$

$$= -\left[\sin\frac{3\pi}{4} - \sin\pi\right] = -\left[1/\sqrt{2} - 0\right] = -\sqrt{2}/2 .$$

ple: Evaluate $\int_0^5 x\sqrt{x+4}\,dx$

on:

$= \sqrt{x+4}$. Now let us express x and dx by u and du.

ing square of $u = \sqrt{x+4}$ we obtain

$u^2 = x + 4$, $x = u^2 - 4$ and $dx = 2u\,du$

x runs from 0 to 5, $u = \sqrt{x+4}$ runs from 2 to 3. Thus

$$\overline{+4}\,dx = \int_2^3 (u^2 - 4)u \cdot 2u\,du = \int_2^3 \left(2u^4 - 8u^2\right)du =$$

$$= 2\cdot\frac{u^5}{5}\Bigg|_2^3 - 8\cdot\frac{u^3}{3}\Bigg|_2^3 = \frac{2}{5}(3^5 - 2^5) - \frac{8}{3}(3^3 - 2^3) =$$

$$= \frac{2}{5}\cdot 211 - \frac{8}{3}\cdot 19 = \frac{506}{15} = 33\frac{11}{15}.$$

Example: Evaluate $\int_0^{\pi/8} \sin^5 2x \cdot \cos 2x\, dx$

Solution:

Let $u = \sin 2x$ so that $du = 2\cos 2x$ or $\cos 2x dx = \frac{du}{2}$.

With the substitution $u = \sin 2x$ we obtain

$u = \sin 0 = 0 \quad$ if $\quad x = 0$

$u = \sin 2 \cdot \frac{\pi}{8} = \frac{\sqrt{2}}{2} \quad$ if $\quad x = \frac{\pi}{8}$. Thus

$$\int_0^{\pi/8} \sin^5 2x \cdot \cos 2x\, dx = \int_0^{\sqrt{2}/2} u^5 \frac{du}{2} = \frac{1}{2}\frac{u^6}{6}\Big|_0^{\sqrt{2}/2} = \frac{1}{12}\left(\frac{\sqrt{2}}{2}\right)^6 = \frac{1}{96}.$$

Example: Evaluate $\int_0^1 \frac{x}{\sqrt{4+5x}} dx$

Solution:

Let $u = \sqrt{4+5x}$. To express x and dx we have to take the square of u

$u^2 = 4 + 5x$; $x = \frac{u^2 - 4}{5}$ and $dx = \frac{2}{5} u\, du$

if $x = 0$ then $u = \sqrt{4 + 5 \cdot 0} = 2$

if $x = 1$ then $u = \sqrt{4 + 5 \cdot 1} = 3$, so

$$\int_0^1 \frac{x}{\sqrt{4+5x}} dx = \frac{1}{5}\int_2^3 \frac{u^2 - 4}{u} \frac{2}{5} u\, du = \frac{2}{25}\int_2^3 (u^2 - 4) du =$$

$$= \frac{2}{25}\left[\frac{u^3}{3} - 4u\right]_2^3 = \frac{2}{25}[(9-12) - (\frac{8}{3} - 8)] = \frac{14}{75}.$$

Method 2:

There is another method for evaluating a definite integral $\int_a^b f(x)\, dx$.

The idea of the method is that first we evaluate the indefinite integral $\int f(x)dx$ by substitution as it discussed in 5.3., and then change back to x, and use the original x- limits.

Example: Evaluate $\int_0^2 x(x^2+1)^3\,dx$

Solution:

Substitute $\left| \begin{array}{l} u = x^2+1 \\ du = 2xdx \\ xdx = \dfrac{du}{2} \end{array} \right|$ then we obtain

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int u^3 du = \frac{u^4}{4} + C = \frac{(x^2+1)^4}{8} + C$$

Thus

$$\int_0^2 x(x^2+1)^3\,dx = \left.\frac{(x^2+1)^4}{8}\right|_0^2 = \frac{625}{8} - \frac{1}{8} = 78.$$

The choice of methods for evaluating a definite integral by substitution is a matter of taste, but it is best to know both methods.

Exercises.

In exercises 1-14 evaluate the integral by any method.

1. $\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx$
2. $\int_{\sqrt{8}}^{\sqrt{15}} \sqrt{1+x^2}\;x\,dx$
3. $\int_{\pi^2/16}^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$
4. $\int_{-1}^{0} (1-2x)^3\,dx$
5. $\int_0^8 x\sqrt{1+x}\,dx$
6. $\int_0^{\pi/2} 4\sin\left(\frac{x}{2}\right)dx$

7. $\int_{-2}^{-1} \frac{x}{(x^2+2)^3}\,dx$

8. $\int_{1}^{3} \frac{x+2}{\sqrt{x^2+4x+7}}\,dx$

9. $\int_{0}^{\sqrt{\pi}} 5x\cdot\cos(x^2)\,dx$

10. $\int_{0}^{\pi/2} \sin^2 3x\cdot\cos 3x\,dx$

11. $\int_{\pi/12}^{\pi/9} \sec^2 3\theta\,d\theta$

12. $\int_{0}^{1} \frac{y^2\,dy}{\sqrt{4-3y}}$

13. $\int_{0}^{1} 15x^2\sqrt{5x^3+4}\,dx$

14. $\int_{-1}^{4} \frac{x\,dx}{\sqrt{5+x}}$

Answers.

1. $\sqrt{2}-1$; **2.** 37/3; **3.** $\sqrt{2}$; **4.** 10; **5.** 1192/15; **6.** $8-4\sqrt{2}$; **7.** -1/48; **8.** $2(\sqrt{7}-\sqrt{3})$; **9.** 0; **10.** -1/9; **11.** $\frac{1}{3}(\sqrt{3}-1)$; **12.** 106/405; **13.** 38/3; **14.** 8/3.

Chapter 6.
Logarithmic and exponential functions.

6.1. Logarithms (an overview).

inition: If a and x are positive numbers, $a \neq 1$, and $a^y = x$, then the nber y is the logarithm x to the base a and we write

$$y = \log_a x$$

$\log_a x$ is read as " the logarithm to the base a of x "

eneral, if $y = \log_a x$, then y is that power to which a must be raised

roduce x, so $\qquad x = a^y$

$y = \log_a x$ has the following properties:

1) $\log_a 1 = 0$
2) $\log_a a = 1$
3) $\log_a (b \cdot c) = \log_a b + \log_a c$
4) $\log_a \frac{b}{c} = \log_a b - \log_a c$
5) $\log_a b^x = x \cdot \log_a b$
6) $\log_a \frac{1}{b} = -\log_a b$
7) $\log_a b = \frac{\log_c b}{\log_c a}$

inition: The number e is $e = \lim_{x \to 0} (1+x)^{\frac{1}{x}} = 2.7182$

will denote the natural logarithm of x by $\ln x$ (read"ell n of x").
s $\ln x$ is that power to which e must be raised to produce x.
example:

$=0$; $\qquad \ln e = 1$; $\qquad \ln(1/e) = -1$; $\qquad \ln e^3 = 3$.

ark: All properties for $y = \log_a x$ are hold for $y = \ln x$ as well.

mple: Evaluate

$g_9 3^7$; $\qquad$ b) $\log_5 \sqrt[3]{25^2}$; $\qquad$ c) $\log_2 \frac{1}{16} + \log_{10} 100$

Solution: a) $\log_9 3^7 = 7 \cdot \log_9 3 = 7 \cdot \frac{1}{2} = 7/2$

b) $\log_5 \sqrt[3]{25^2} = \log_5 (25)^{\frac{2}{3}} = \frac{2}{3} \log_5 25 = 4/3$

c) $\log_2 \frac{1}{16} + \log_{10} 100 = \log_2 2^{-4} + \log_{10} 10^2 = -4 + 2 = -2$

Example:

Express $\ln \frac{x^2 y^4}{\sqrt[3]{z}}$ in terms of sums, differences and multiplies

Solution:

$$\ln \frac{x^2 y^4}{\sqrt[3]{z}} = \ln x^2 + \ln y^4 - \ln \sqrt[3]{z} = 2\ln x + 4\ln y - \frac{1}{3}\ln z$$

Example: Write expression $(5\ln 3 + 2\ln 4 - \ln 7)$ as a single logarithm

Solution:

$$(5\ln 3 + 2\ln 4 - \ln 7) = \ln 3^5 + \ln 4^2 - \ln 7 = \ln \frac{3^5 \cdot 4^2}{7}$$

Example: Solve for x if $\ln 2x = 3$

Solution:

$$\ln 2x = 3$$

$$2x = e^3 \Rightarrow x = \frac{e^3}{2}$$

Exercises.

1. Evaluate

a) $\log_3 \sqrt{3}$; b) $\log_3 (3^5)$; c) $\log_3 \left(\frac{1}{27}\right)$

2. If $\log_4 A = 2.1$, then evaluate

a) $\log_4 A^2$; b) $\log_4 \frac{1}{A}$; c) $\log_4 16A$

3. Evaluate

a) $\log_2 16$; b) $\log_2\left(\frac{1}{32}\right)$; c) $\log_9 3$

4. Evaluate

a) $2^{\log_2 16}$; b) $2^{\log_2(1/2)}$; c) $2^{\log_2 7}$

In exercises 5-7 expand the logarithm in terms of sums, differences and multiplies.

5. $\log_3\left(10x\sqrt{x-3}\right)$

6. $\ln\frac{x^2\cdot\sin^3 x}{\sqrt{x^2+1}}$

7. $\ln\frac{\sqrt{x^2-1}\cdot x^3}{\sin^2 x}$

In exercises 8-9 write the expression as a single logarithm.

8. $4\log_3(x-3)-5\log_3(\sin 3x)+2$

9. $3\ln(x+2)+\frac{1}{2}\ln x-\ln(\sin x)$

In exercises 10-14 solve for x.

10. $\log_{10}\sqrt{x}=-2$; 11. $\ln\frac{3}{x}=-3$

12. $\log_7(7^{3x})=9$; 13. $\ln x^{3/2}-\ln\sqrt{x}=5$

14. $\ln\left(\frac{1}{x}\right)+\ln\left(2x^3\right)=\ln 8$

Answers.

1. a) 1/2; b) 5; c) -3; **2.** a) 4.2; b) -2.1; c) 4.1; **3.** a) 4; b) -5; c) 1/2;

4. a) 16; b) 1/2; c) 7; **5.** $\log_3 10+\log_3 x+\frac{1}{2}\log_3(x-3)$;

6. $2\ln|x|+3\ln(\sin x)-\frac{1}{2}\ln(x^2+1)$; **7.** $\frac{1}{2}\ln\left|x^2-1\right|+3\ln|x|-2\ln(\sin x)$;

8. $\log_3\frac{9\cdot(x-3)^4}{\sin^5 3x}$; **9.** $\ln\frac{\sqrt{x}\cdot(x+2)^3}{\sin x}$; **10.** 10^{-4} **11.** $3e^3$; **12.** 3;

13. e^5; **14.** 2.

6.2. The derivatives $y = \ln x$ and $y = \log_a x$.

The derivative of natural logarithm function $y = \ln x$ is the function $\frac{1}{x}$;

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0.$$

If $u(x) > 0$, and if the function u is differentiable at x, then applying the Chain rule to the function $y = \ln u$ yields

$$(1) \qquad \frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$$

or in function notation

$$(\ln u)' = \frac{1}{u} \cdot u'$$

To find the derivative of a base a logarithm, we first convert it to a natural logarithm.

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{\ln a}\frac{d}{dx}[\ln x] = \frac{1}{x \cdot \ln a}$$

In general, if u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left[\frac{\ln u}{\ln a}\right] = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}$$

$$(2) \qquad \frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}$$

or, in function notation

$$(\log_a u)' = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot u'$$

Example: Find $\frac{d}{dx}\left[\ln(x^3 + 1)\right]$

Solution: From (1) with $u = x^3 + 1$,

$$\frac{d}{dx}\left[\ln(x^3 + 1)\right] = \frac{1}{x^3 + 1}\frac{d}{dx}[x^3 + 1] = \frac{3x^2}{x^3 + 1}.$$

Example: Find $\left(\ln\dfrac{\cos x}{2x}\right)'$

Solution:
According to (1)

$$\left(\ln\frac{\cos x}{2x}\right)' = \frac{2x}{\cos x}\left(\frac{\cos x}{2x}\right)' = \frac{2x}{\cos x}\,\frac{-\sin x\cdot 2x - 2\cos x}{4x^2} =$$
$$= -\frac{x\sin x + \cos x}{x\cos x}.$$

xample: Find $\dfrac{d}{dx}\left[\log_3(2x^2-7)\right]$

olution:
rom (2) with $u = 2x^2 - 7$ we obtain

$$\frac{d}{dx}\left[\log_3(2x^2-7)\right] = \frac{1}{\ln 3}\cdot\frac{1}{2x^2-7}\cdot\frac{d}{dx}[2x^2-7] = \frac{4x}{(2x^2-7)\ln 3}$$

ample: Find $\left(\log_5\left(\dfrac{x^2}{x+1}\right)\right)'$

lution:

$$\left(\log_5\left(\frac{x^2}{x+1}\right)\right)' = \frac{1}{\ln 5}\cdot\frac{x+1}{x^2}\cdot\left(\frac{x^2}{x+1}\right)' =$$
$$= \frac{1}{\ln 5}\cdot\frac{x+1}{x^2}\cdot\frac{(2x^2+2x-x^2)}{(x+1)^2} = \frac{x+2}{\ln 5\cdot x(x+1)}$$

ark: Before differentiating, if it is possible, the properties of rithms should be used to convert products, quotients and exponents sums, differences, and constant multiples. Let us apply it to the last ple.

$$\left(\log_5\left(\frac{x^2}{x+1}\right)\right)' = \left(\frac{\ln\dfrac{x^2}{x+1}}{\ln 5}\right)' = \frac{1}{\ln 5}\left[\ln x^2 - \ln(x+1)\right]' =$$

$$=\frac{1}{\ln 5}[2\ln x-\ln(x+1)]'=\frac{1}{\ln 5}\left[\frac{2}{x}-\frac{1}{x+1}\right]=\frac{x+2}{\ln 5\cdot x(x+1)}.$$

Example: Find $\frac{d}{dx}\left[\ln\frac{x^3\cdot\cos x}{\sqrt{1+x^2}}\right]$

Solution:

First of all, let us use properties of natural logarithm function to convert given expression into sums, differences and constant multiples:

$$\frac{d}{dx}\left[\ln\frac{x^3\cdot\cos x}{\sqrt{1+x^2}}\right]=\frac{d}{dx}\left[\ln x^3+\ln\cos x-\ln\sqrt{1+x^2}\right]=$$

$$=\frac{d}{dx}\left[3\ln x+\ln\cos x-\frac{1}{2}\ln(1+x^2)\right]=\frac{1}{3}\cdot\frac{1}{x}-\frac{\sin x}{\cos x}-\frac{1}{2}\cdot\frac{2x}{1+x^2}=$$

$$=\frac{1}{3x}-\tan x-\frac{x}{x^2+1}$$

Example: Find $\frac{d}{dx}\left[\log_5\left(\frac{\sqrt{x}\cdot\sqrt[3]{x+1}}{\sin x\cdot\sec x}\right)\right]$

Solution:

$$\frac{d}{dx}\left[\log_5\left(\frac{\sqrt{x}\cdot\sqrt[3]{x+1}}{\sin x\cdot\sec x}\right)\right]=\frac{d}{dx}\left[\frac{1}{\ln 5}\left(\ln\frac{\sqrt{x}\cdot\sqrt[3]{x+1}}{\sin x\cdot\sec x}\right)\right]$$

Since $\frac{1}{\ln 5}$ is a constant, we can move it outside the derivative symbol. Then

$$\frac{1}{\ln 5}\frac{d}{dx}[\ln\sqrt{x}+\ln\sqrt[3]{x+1}-\ln\sin x-\ln\sec x]=$$

$$=\frac{1}{\ln 5}\left(\frac{1}{2}\ln x+\frac{1}{3}\ln(x+1)-\ln\sin x-\ln\sec x\right)=$$

$$=\frac{1}{\ln 5}\left(\frac{1}{2x}+\frac{1}{3(x+1)}-\cot x-\tan x\right)$$

Example: Use implicit differentiation to find $\frac{dy}{dx}$ if $y^2+\ln(x\cdot y)=3$

Solution: Differentiating both sides with respect to x yields

$$\frac{d}{dx}[y^2 + \ln(x \cdot y)] = \frac{d}{dx}[3]$$

Since y is the function of x we get

$$2yy' + \frac{1}{xy} \cdot (y + x \cdot y') = 0$$

By solving for y'

$$2yy' + \frac{1}{x} + \frac{y'}{y} = 0$$

$$y' = -\frac{y}{x(2y^2 + 1)}$$

Example: Find tangent to $y + \ln x^2 y = 2$ at (1, 1).

Solution: The tangent line is

$$y - y_0 = \left.\frac{dy}{dx}\right|_{(x_0, y_0)} (x - x_0) \qquad x_0 = 1; \qquad y_0 = 1$$

nd all we need to write equation of tangent line is $\left.\frac{dy}{dx}\right|_{(x_0, y_0)}$.

fferentiating both sides of equation we obtain

$$y' + \frac{1}{x^2 y}(2xy + x^2 y') = 0$$

$$y' + \frac{2}{x} + \frac{y'}{y} = 0$$

$$y' = \frac{dy}{dx} = -\frac{2y}{x(y+1)}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{2 \cdot 1}{1 \cdot (1+1)} = -1$$

ation of tangent line is $y - 1 = -1(x - 1)$ or $y = -x + 2$

Exercises.

In exercises 1-14 find dy/dx.

1. $y = \ln 3x$
2. $y = (\ln x)^3$
3. $y = \ln \dfrac{x}{1+x^2}$
4. $y = \ln|x^3 - 4x^2 - 5|$
5. $y = \sqrt{\ln x}$
6. $y = \cos\left(\dfrac{3}{\ln x}\right)$
7. $y = x^3 \ln(3-2x)$
8. $y = \dfrac{x^2}{1+\ln x}$
9. $y = \ln(\sec x + \tan x)$
10. $y = x(\ln^3 x - 3\ln^2 x + 6\ln x - 6)$
11. $y = \log_4 x^3$
12. $y = x \cdot \cos(\log_4 x)$
13. $y = x^2 \ln(\log_5(3x))$
14. $y = x \cdot \log_6(1/x)$
15. Find dy/dx by implicit differentiation if $y^2 + \ln(xy) = 2$.

In exercises 16-22 first simplify by using laws of logarithms, then differentiate.

16. $y = \dfrac{1}{5}\ln\left(\dfrac{x}{3x+5}\right)$
17. $y = \dfrac{1}{10}\ln\left(\dfrac{5+x}{5-x}\right)$
18. $y = \ln\left[(x^2+1)^3(x^5+1)^4\right]$
19. $y = \ln \dfrac{\sqrt{2x+1}\,\sqrt[3]{3x+2}}{(x^2+1)^5}$
20. $y = \log_7 \dfrac{\sin x \cdot \cos 3x}{\sqrt[3]{x^2}}$
21. $y = \log_6(\log_4 7x^2)$
22. $y = \log_3 \dfrac{x^2 \cdot \sin 5x}{\cos x \cdot \sqrt{x+1}}$

Answers.

1. $\dfrac{1}{x}$; **2.** $3(\ln x)^2 \cdot \dfrac{1}{x}$; **3.** $\dfrac{1-x^2}{x(1+x^2)}$; **4.** $\dfrac{3x^2-8x}{x^3-4x^2-5}$; **5.** $\dfrac{1}{2\sqrt{\ln x}} \cdot \dfrac{1}{x}$; **6.** $\dfrac{3\sin(3/\ln x)}{x(\ln x)^2}$; **7.** $-\dfrac{2x^3}{3-2x} + 3x^2 \cdot \ln(3-2x)$; **8.** $\dfrac{x(1+2\ln x)}{(1+\ln x)^2}$;

9. $\sec x$;**10.** $\ln^3 x$;**11.** $\frac{3}{x\ln 4}$; **12.** $\cos(\log_4 x) - \frac{\sin(\log_4 x)}{\ln 4}$;

13. $2x\ln(\log_5(3x) + \frac{x}{\ln 3x}$; **14.** $-\frac{1+\ln x}{\ln 6}$;**15.** $-\frac{y}{x(2y^2+1)}$;

16. $\frac{1}{x(3x+5)}$; **17.** $\frac{1}{25-x^2}$;**18.** $\frac{6x}{x^2+1} + \frac{20x^4}{x^5+1}$;**19.** $\frac{1}{2x+1} + \frac{1}{3x+2} -$ $\frac{10x}{x^2+1}$; **20.** $\frac{1}{\ln 7}(\cot x - 3\tan 3x - \frac{2}{3x})$; **21.** $\frac{2}{x} \cdot \frac{1}{\ln 6 \cdot \ln 7x^2}$

22. $\frac{1}{\ln 3}[\frac{2}{x} + 5\cot 5x + \tan x - \frac{1}{2(x+1)}]$.

6.3. Logarithmic differentiation.

The logarithmic differentiation is the special case of implicit differentiation. This way of differentiation is useful for differentiating functions that are composed of products, quotients and powers.

Example: Differentiate $y = \frac{\sqrt[3]{x} \cdot \sqrt{(1+x^2)^3}}{x^{4/5}}$

Solution: If you were differentiating directly, you would run into many computations. Let us take logarithm of both sides of the equation. We can write

$$\ln y = \frac{1}{3}\ln x + \frac{3}{2}\ln(1+x^2) - \frac{4}{5}\ln x$$

Differentiating implicitly gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\frac{1}{x} + \frac{3}{2}\frac{2x}{1+x^2} - \frac{4}{5}\frac{1}{x}$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \left(\frac{1}{3x} + \frac{3x}{1+x^2} - \frac{4}{5x}\right) \cdot y = \frac{\sqrt[3]{x} \cdot \sqrt{(1+x^2)^3}}{x^{4/5}} \cdot \left(\frac{1}{3x} + \frac{3x}{1+x^2} - \frac{4}{5x}\right)$$

Remark: Notice that we replaced y back with what it is equal to. We do that by a regular implicit differentiation problem.

Example: Find $\frac{dy}{dx}$ if $y = \sqrt[5]{\frac{x-1}{x+1}}$

Solution:

$$\ln y = \ln\sqrt[5]{\frac{x-1}{x+1}} = \frac{1}{5}\ln(x-1) - \frac{1}{5}\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{5}\frac{1}{x-1} - \frac{1}{5}\frac{1}{x+1}$$

$$\frac{dy}{dx} = \frac{2}{5(x^2-1)}\sqrt[5]{\frac{x-1}{x+1}}$$

Logarithmic differentiation is also useful for differentiation of functions of the form $(f(x))^{g(x)}$

Example: Find $\frac{dy}{dx}$ if $y = x^{2x}$

Solution:

Again taking logarithms of both sides yields

$$\ln y = \ln x^{2x} = 2x\ln x$$

Differentiating implicitly we obtain

$$\frac{1}{y}\frac{dy}{dx} = 2\ln x + 2x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = 2(\ln x + 1)\cdot x^{2x}$$

Example: Differentiate $y = (\sin x)^{\cos x}$

Solution:

$$\ln y = \ln(\sin x)^{\cos x} = \cos x \cdot \ln(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = -\sin x\ln(\sin x) + \cos x \cdot \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = [\cos x \cdot \cot x - \sin x\ln(\sin x)]\cdot(\sin x)^{\cos x}$$

Exercises.

In exercises 1-12 obtain $\dfrac{dy}{dx}$ by logarithmic differentiation.

1. $y=(1+3x)^5(\sin 3x)^6$

2. $y=\dfrac{(\sec 4x)^{5/4}\cdot\sin^3 2x}{\sqrt{x}}$

3. $y=\sqrt[3]{\dfrac{x+1}{x-1}}$

4. $y=\dfrac{(x^2-8)^4\cdot\sqrt{x^2+1}}{\sin x}$

5. $y=x^5\cdot\sqrt[4]{x^3+1}$

6. $y=\sqrt[3]{\dfrac{x(x-1)(x^2+1)}{(x^3+2)(3x-2)}}$

7. $y=x^{6x}$

8. $y=(\sin x)^{\tan x}$

9. $y=x^{x+2}$

10. $y=(2x)^{\log_3 x}$

11. $y=(\cos x)^{\ln x}$

12. $y=(2-t)^{t+1}$

Answers.

$(1+3x)^5(\sin 3x)^6\cdot\left(\dfrac{15}{1+3x}+18\cot 3x\right)$; **2.** $\dfrac{(\sec 4x)^{5/4}\cdot\sin^3 2x}{\sqrt{x}}\cdot$
$5\tan 4x+6\cot 2x-\dfrac{1}{2x}\Big)$; **3.** $\sqrt[3]{\dfrac{x+1}{x-1}}\cdot\left(\dfrac{-2}{3(x^2-1)}\right)$;
$\dfrac{(x^2-8)^4\cdot\sqrt{x^2+1}}{\sin x}\cdot\left[\dfrac{8x}{x^2-8}+\dfrac{x}{x^2+1}-\cot x\right]$; **5.** $y=x^5\cdot\sqrt[4]{x^3+1}$
$\dfrac{5}{x}+\dfrac{3x^2}{4(x^3+1)}\Big)$; **6.** $\sqrt[3]{\dfrac{x(x-1)(x^2+1)}{(x^3+2)(3x-2)}}\cdot\dfrac{1}{3}\Big[\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{2x}{x^2+1}-$
$\dfrac{3x^2}{x^3+2}-\dfrac{3}{3x-2}\Big]$; **7.** $6\cdot x^{6x}(\ln x+1)$; **8.** $(\sin x)^{\tan x}(1+\sec^2 x\ln(\sin x))$;
$x^{x+2}\cdot(\ln x+1+2/x)$; **10.** $\dfrac{1}{x}\cdot(2x)^{\log_3 x}\cdot\log_3 2x^2$; **11.** $y=(\cos x)^{\ln x}\cdot$
$\ln(\cos x)-\ln x\cdot\tan x]$; **12.** $(2-t)^{t+1}\left[\ln(2-t)+\dfrac{t+1}{t-2}\right]$.

6.4. Integrals involving $\ln x$ and $\log_a x$.

$$\int \frac{1}{x}\, dx = \ln|x| + C \text{ for any } x \neq 0 .$$

If u is nonzero differentiable function, then

$$\int \frac{1}{u}\, du = \ln|u| + C$$

Example: Evaluate $\int \frac{4x^3}{x^4 + 3}\, dx$

Solution:

The numerator is exactly the derivative of the denominator. If we let $u = x^3 + 3$, then $du = 4x^3 dx$ so that

$$\int \frac{4x^3}{x^4 + 3}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln\left|x^4 + 3\right| + C$$

Example: Evaluate $\int \tan x\, dx$

Solution:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = \left| \begin{array}{l} u = \cos x \\ du = -\sin x dx \\ -du = \sin x dx \end{array} \right| =$$

$$= -\int \frac{1}{u}\, du = -\ln|u| + C = -\ln|\cos x| + C$$

Example: Evaluate $\int_1^4 \frac{1}{\sqrt{x}(1 + \sqrt{x})}\, dx$

Solution:

Let $u = 1 + \sqrt{x}$, so that $du = \frac{dx}{2\sqrt{x}}$ and $\frac{dx}{\sqrt{x}} = 2du$.

It is necessary to change limits of integration $x = 1, x = 4$ to the u-limits

if $x = 1$, then $u = 1 + \sqrt{1} = 2$

if $x = 4$, then $u = 1 + \sqrt{4} = 3$

$$\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})}dx = \int_2^3 \frac{2du}{u} = 2\ln|u|\Big|_2^3 = 2(\ln 3 - \ln 2) = 2\ln(3/2).$$

Integrals involving $\log_a x$.

To evaluate integrals involving base a logarithms, it is necessary to convert them to natural logarithm, then use usual way

Example: Evaluate $\int \frac{\log_5 x}{x}dx$

Solution:

$$\int \frac{\log_5 x}{x}dx = \int \frac{\frac{\ln x}{\ln 5}}{x}dx = \frac{1}{\ln 5}\int \frac{\ln x}{x}dx = \left| \begin{matrix} u = \ln x \\ du = \frac{1}{x}dx \end{matrix} \right| =$$

$$= \frac{1}{\ln 5}\int u\,du = \frac{1}{\ln 5}\frac{u^2}{2} + C = \frac{\ln^2 x}{2\ln 5} + C.$$

ample: Evaluate $\int_0^1 \frac{\log_3(x+2)}{x+2}dx$

lution:

$$\frac{\log_3(x+2)}{x+2}dx = \frac{1}{\ln 3}\int_0^1 \frac{\ln(x+2)}{x+2}dx = \left| \begin{matrix} u = \ln(x+2) \\ du = \frac{dx}{x+2} \end{matrix} \right| =$$

if $x = 0$, then $u = \ln 2$

if $x = 1$, then $u = \ln 3$, we get

$$\frac{1}{\ln 3}\int_0^1 \frac{\ln(x+2)}{x+2}dx = \frac{1}{\ln 3}\int_{\ln 2}^{\ln 3} u\,du = \frac{1}{\ln 3}\frac{u^2}{2}\Big|_{\ln 2}^{\ln 3} =$$

$$= \frac{1}{2\ln 3}\left(\ln^2 3 - \ln^2 2\right) = \frac{1}{2\ln 3}\cdot \ln 6 \cdot \ln\frac{3}{2}.$$

8. $\ln|\sin x| + C$; 9. $\frac{1}{6\ln 6}\ln^2 x + C$; 10. $\frac{\ln^2 x}{4} + C$;

11. $\frac{3\cdot\ln^2(2x+1)}{4\ln 10} + C$; 12. $\frac{1}{3}\ln\frac{5}{2}$; 13. $\frac{1}{2}\ln\frac{5}{6}$; 14. $-\frac{3}{2}\ln 7$; 15. $\ln 2$;

16. $\ln|\ln(\ln x)| + C$; 17. $2\ln 2$; 18. $\frac{\ln 3}{2}$; 19. $\ln 3\cdot\ln\left(\frac{\ln 5}{\ln 3}\right)$;

20. $-2\sqrt{\ln(\csc x + \cot x)}$.

6.5. Exponents (an overview).

Laws of exponents:

The bases are positive, the exponents are any real numbers.

1. $a^{x+y} = a^x \cdot a^y$
2. $a^{x-y} = \frac{a^x}{a^y}$
3. $(a^x)^y = a^{x\cdot y}$
4. $(a\cdot b)^x = a^x \cdot b^x$
5. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
6. $a^0 = 1$
7. $a^{\log_a x} = x$ (for $x > 0$)

aws above are true for the base e as well.

mple: $\ln e^3 = 3$

mple: $e^{\ln(x^2+3)} = x^2 + 3$

mple: $2^{\log_2 7} = 7$

mple: $4^{\log_2 x} = 2^{2\log_2 x} = 2^{\log_2 x^2} = x^2$

mple: $e^{x+\ln 5} = e^x \cdot e^{\ln 5} = 5e^x$

mple: Find y if $\ln(2y) = 4x + 3$

ion: Exponentiate both sides:

$$e^{\ln(2y)} = e^{4x+3}$$

$$(4)\ \frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}$$

Example: $\frac{d}{dx}[3^{\cos x}] = 3^{\cos x} \cdot \ln 3 \cdot \frac{d}{dx}[\cos x] = -3^{\cos x} \cdot \ln 3 \cdot \sin x$

Example: $\frac{d}{dx}[e^{3x+1}] = e^{3x+1} \cdot \frac{d}{dx}[3x+1] = 3e^{3x+1}$

Example: $\frac{d}{dx}[e^{x^5}] = e^{x^5} \cdot \frac{d}{dx}[x^5] = 5x^4 e^{x^5}$

Example: Find $\left(2^{3x^2} + e^{4x} \cdot x^2\right)'$

Solution:

$$\left(2^{3x^2} + e^{4x} \cdot x^2\right)' = 2^{3x^2} \cdot \ln 2 \cdot (3x^2)' + e^{4x} \cdot (4x)' \cdot x^2 + e^{4x} \cdot (x^2)' =$$

$$= 2x(3\ln 2 \cdot 2^{3x^2} + 2xe^{4x} + e^{4x})$$

Example: Find the extrema of $y = x \cdot e^{-x}$

Solution:

Let us examine $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x}{e^x}\right] = \frac{e^x - xe^x}{e^{2x}} = \frac{1-x}{e^x}$$

There is a critical point when $x = 1$.

When $x < 1, \frac{dy}{dx} > 0$ and when $x > 1$, $\frac{dy}{dx} < 0$. Hence there is a global maximum occurs when $x = 1$ and $f(\max) = f(1) = \frac{1}{e}$.

Exercises.

. Simplify

) $e^{-\ln 8}$ b) $e^{\ln 3 + \ln 5}$ c) $\log_9 3^{\sin x}$

. Solve for k

) $e^{3k} = 9$ b) $2^{k-2} = 7$ c) $e^{k/100} = t$

. Solve for y

a) $\ln(2-y)=k$ b) $\log_3(y+3)=4$ c) $e^{\sqrt{y}}=x^4$

In exercises 4-15 differentiate and simplify where it is necessary.

4. $y=e^{-7x^3}$
5. $y=e^{\frac{1}{x}}$
6. $y=\cos(e^x)$
7. $y=x^3e^x$
8. $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$
9. $y=e^{x\tan x}$
10. $y=e^{(x-e^{3x})}$
11. $y=\ln(1-xe^{-x})$
12. $y=\dfrac{e^{ax}(a\,x-1)}{a^2}$ (a is a constant)
13. $y=2^{\sec x}$
14. $y=3^{e^x}+\pi^{2x}$
15. $y=7^{\sqrt{x}}\cdot e^{4x}$
16. Find the minimum value of $y=x^2-\ln x$
17. Find the relative extrema of $\dfrac{(\ln x)^2}{x}$.

In exercises 18-19 find a) relative maxima or minima, b) inflection points of the given functions.

18. $f(x)=(1+x)\,e^{-x}$
19. $f(x)=x^3\cdot e^{-x}$
20. Let $f(x)=e^{kx}$ and $g(x)=e^{-kx}$. Find

a) $f^{(n)}(x)$ b) $g^{(n)}(x)$

21. Find $f'(x)$ if $f(x)=\dfrac{1}{\sqrt{2\pi}\cdot\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ where μ and σ are constants and $\sigma\neq 0$.

Answers.

1. a) $1/8$; b) 15; c) $(\sin x)/2$; **2.** a) $\ln 9/3$; b) $2+\log_2 7$; c) $100\ln t$; **3.** a) $2-e^k$; b) 78; c) $\ln^2 x^4$; **4.** $-21x^2e^{-7x^3}$; **5.** $-\dfrac{1}{x^2}e^{\frac{1}{x}}$;

6. $-e^x \sin(e^x)$; 7. $x^2 e^x (3+x)$; 8. $-\frac{4}{(e^x + e^{-x})^2}$;

9. $(x \cdot \sec^2 x + \tan x) \cdot e^{x \cdot \tan x}$; 10. $(1-3e^{3x})\, e^{x-e^{3x}}$; 11. $\frac{x-1}{e^x - x}$; 12. $x \cdot e^{ax}$;

13. $\sec x \cdot \tan x \cdot \ln 2 \cdot 2^{\sec x}$; 14. $3^{e^x} \cdot \ln 3 \cdot e^x + 2\ln\pi \cdot \pi^{2x}$;

15. $7^{\sqrt{x}} \cdot e^{4x}\left(\frac{\ln 7}{2\sqrt{x}} + 4\right)$; 16. $\frac{1}{2}(1+\ln 2)$; 17. Relative minimum (1,0), Relative maximum $(e^2, 4e^{-2})$; 18. a) global maximum (0, 1); b) $x = 1$;

19. a) global maximum $(3, \frac{27}{e^3})$; b) $x = 0; x = 3 \pm \sqrt{3}$; 20. a) $k^n e^{kx}$;

b) $(-1)^n k^n \cdot e^{-kx}$; 21. $-\frac{1}{\sqrt{2\pi\sigma^3}}(x-\mu) \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.

6.7. Integrals of the functions a^x and e^x.

$$\int e^x dx = e^x + C$$

$$\int e^u du = e^u + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int a^u dx = \frac{a^u}{\ln a} + C.$$

Example: Evaluate $\int e^{7x} dx$

Solution: Let $u = 7x$ so that $du = 7dx$ or $dx = \frac{du}{7}$, which yields

$$\int e^{7x} dx = \frac{1}{7}\int e^u du = \frac{1}{7}e^u + C = \frac{1}{7}e^{7x} + C.$$

Example: Evaluate $\int e^{-\sin x} \cos x\, dx$

Solution:

$$\int e^{-\sin x}\cos x\,dx = \left| \begin{array}{l} u = -\sin x \\ du = -\cos x dx \\ \cos x\,dx = -du \end{array} \right| = -\int e^{u}\,du = -e^{u} + C = -e^{-\sin x} + C.$$

Example: Evaluate $\int_{\ln 2}^{\ln 4} e^{x}(1+e^{x})^{2}\,dx$

Solution:

Make the u-substitution

$$u = 1 + e^{x} \qquad\qquad du = e^{x}dx$$

and change the x- limits of integration $(x = \ln 2, x = \ln 4)$ to the u-limits $(u = 1 + e^{\ln 2} = 3, u = 1 + e^{\ln 4} = 5)$. We obtain

$$\int_{\ln 2}^{\ln 4} e^{x}(1+e^{x})^{2}\,dx = \int_{3}^{5} u^{2}\,du = \left.\frac{u^{3}}{3}\right|_{3}^{5} = \frac{5^{3}-3^{3}}{3} = \frac{98}{3}.$$

Example: Evaluate $\int 2^{7x}\,dx$

Solution:

$$\int 2^{7x}\,dx = \left| \begin{array}{l} u = 7x \\ du = 7dx \\ dx = \dfrac{du}{7} \end{array} \right| = \frac{1}{7}\int 2^{u}\,du = \frac{1}{7}\frac{2^{u}}{\ln 2} + C = \frac{2^{7x}}{7\ln 2} + C.$$

Example: Evaluate $\int_{1}^{\sqrt{2}} x \cdot 4^{-x^{2}}\,dx$

Solution:

Let $u = -x^{2}$ so that $du = -2x\,dx$ or $x\,dx = -\dfrac{du}{2}$

if $x = 1$ then $u = -1$

if $x = \sqrt{2}$ then $u = -2$ and we obtain

$$\int_{1}^{\sqrt{2}} x \cdot 4^{-x^{2}}\,dx = -\frac{1}{2}\int_{-1}^{-2} 4^{u}\,du = -\frac{1}{2}\left.\frac{4^{u}}{\ln 4}\right|_{-1}^{-2} = -\frac{1}{2\ln 4}\left(\frac{1}{16} - \frac{1}{4}\right) = \frac{3}{64\ln 2}.$$

Example: Evaluate $\int_{-2}^{3} 3^{(x+1)}\,dx$

Solution:

$$\int_{-2}^{3} 3^{(x+1)}\,dx = \left| \begin{matrix} u = x+1 \\ du = dx \end{matrix} \right| = \int_{-1}^{4} 3^{u}\,du = \left.\frac{3^{u}}{\ln 3}\right|_{-1}^{4} =$$

$$= \frac{1}{\ln 3}(3^{4} - \frac{1}{3}) = \frac{242}{3\ln 3}.$$

Example: Evaluate $\int \frac{dx}{\sin x \cos x}$

Solution:

Let us divide numerator and denominator by $\cos^2 x$.

We obtain $$\frac{1}{\sin x \cos x} = \frac{\frac{1}{\cos^2 x}}{\tan x}.$$

Let $u = \tan x$ then $du = \frac{1}{\cos^2 x}\,dx$.

Then $$\int \frac{dx}{\sin x \cos x} = \int \frac{\frac{1}{\cos^2 x}}{\tan x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\tan x| + C.$$

Exercises.

In exercises 1-18 evaluate the integrals

$\int e^{-5x}\,dx$

2. $\int x^2 \cdot e^{-2x^3}\,dx$

$\int \frac{e^x}{1+e^x}\,dx$

4. $\int e^{2t}\sqrt{1+e^{2t}}\,dt$

$\int \sin x \cdot e^{\cos x}\,dx$

6. $\int e^{-x} \cdot \sec^2(2-e^{-x})\,dx$

$\int \pi^{\sin x} \cdot \cos x\,dx$

8. $\int (x\ln 3 - 4\pi \cdot e^2 \cdot \cos x)\,dx$

$\int (\ln(e^x) + \ln(e^{-x}))\,dx$

10. $\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy$

11. $\int_{0}^{\ln 5} e^x(3-4e^x)\,dx$

12. $\int_{1}^{3}(3-e^x)\,dx$

13. $\int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x+4}\,dx$

14. $\int \frac{e^{\sqrt{2x-1}}}{\sqrt{2x-1}}\,dx$

15. $\int \frac{x\,dx}{3-2x^2}$

16. $\int \frac{dx}{1+e^x}$

17. $\int_{1}^{2} 3x^{3x}(\ln x+1)\,dx$

18. $\int_{-2}^{-1} 8^{-2x}\cdot \ln 2\,dx$

19. Find the area of the region enclosed by $y=e^x$, $y=2$ and $x=0$.

20. Evaluate $\int \frac{e^{3x}}{e^x+3}\,dx$.

Answers.

1 $-\frac{1}{5}e^{-5x}+C$; **2.** $-\frac{1}{6}e^{-2x^3}+C$; **3.** $\ln(1+e^x)+C$; **4.** $\frac{1}{3}(1+e^{2t})^{3/2}+C$;

5. $-e^{\cos x}+C$; **6.** $\tan(2-e^{-x})+C$; **7.** $\frac{\pi^{\sin x}}{\ln \pi}+C$; **8.** $\frac{1}{2}x^2\ln 3-$

$-4\pi\cdot e^2\sin x+C$; **9.** C; **10.** $2e^{\sqrt{y}}+C$; **11.** -36; **12.** $6+e-e^3$;

13. $\ln\frac{21}{13}$; **14.** $e^{\sqrt{2x-1}}+C$; **15.** $-\frac{1}{4}\ln\left|3-2x^2\right|+C$;

16. $x-\ln(1+e^x)+C$; **17.** 63; **18.** $\frac{1}{6}(8^4-64)$; **19.** $2\ln 2-1$;

20. $\frac{e^{2x}}{2}-3e^x+9\ln(e^x+3)+C$.

6.8. Limits involving functions a^x, e^x and $\ln x$. L'Hopital's rule and the forms (1^∞), (0^0) and (∞^0).

We are familiar with the L'Hopital's rule, which is a technique for dealing with limits of the form "zero-over-zero" and " infinity-over-infinity". In both of these cases it asserts that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the latter limit exists.

Note that it concerns the quotients of two derivatives, not the derivatives of the quotient. Now let us consider some limits involving a^x, e^x and $\ln x$.

Example: Evaluate $\lim\limits_{x\to\infty}\frac{\ln x}{x^2}$

Solution:

First note that $\ln x\to\infty$ and $x^2\to\infty$ as $x\to\infty$. We may use L'Hopital's rule in the $\left(\frac{\infty}{\infty}\right)$ form. We have

$$\lim_{x\to\infty}\frac{\ln x}{x^2}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{(\ln x)'}{(x^2)'}=\lim_{x\to\infty}\frac{1/x}{2x}=\lim_{x\to\infty}\frac{1}{2x^2}=0.$$

Example: Evaluate $\lim\limits_{x\to+\infty}\frac{x}{e^x}$

Solution:

$\lim\limits_{x\to+\infty} x=\lim\limits_{x\to+\infty} e^x=+\infty$, so given limit is an indeterminate form of type $\left(\frac{\infty}{\infty}\right)$. Thus, by L'Hopital's rule:

$$\lim_{x\to+\infty}\frac{x}{e^x}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to+\infty}\frac{(x)'}{(e^x)'}=\lim_{x\to+\infty}\frac{1}{e^x}=0$$

Example: Evaluate $\lim\limits_{x\to 0}\dfrac{2^x-1}{x}$

Solution: Both (2^x-1) and x approach zero as $x \to 0$. The given limit is an indeterminate form of type $\left(\dfrac{0}{0}\right)$. Thus, by L'Hopital's rule

$$\lim_{x\to 0}\frac{2^x-1}{x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(2^x-1)'}{(x)'}=\lim_{x\to 0}\frac{2^x\ln 2}{1}=\ln 2$$

Example: Evaluate $\lim\limits_{x\to 0^+} x\cdot\ln x$

Solution:

Since $\lim\limits_{x\to 0^+} x=0$ and $\lim\limits_{x\to 0^+}\ln x=-\infty$, the given limit is an indeterminate form of type $(0\cdot\infty)$. Rewriting $x\cdot\ln x$ as $\dfrac{\ln x}{1/x}$ converts the problem to the form $\left(\dfrac{\infty}{\infty}\right)$ and by L'Hopital's rule

$$\lim_{x\to 0^+} x\cdot\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0^+}\frac{(\ln x)'}{(1/x)'}=$$

$$=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}(-x)=0$$

Remark: We could convert the problem to the form $\left(\dfrac{0}{0}\right)$ by writing $x\cdot\ln x=\dfrac{x}{1/\ln x}$. But it is less desirable than $\dfrac{\ln x}{1/x}$ because of the relatively complicated derivative of $\dfrac{1}{\ln x}$.

Example: Evaluate $\lim\limits_{x\to 1}\dfrac{\cos(\pi x/2)}{\ln x}$

Solution:

$$\lim_{x\to 1}\frac{\cos(\pi x/2)}{\ln x}=\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{(\cos(\pi x/2))'}{(\ln x)'}=\lim_{x\to 1}\frac{-\frac{\pi}{2}\sin(\pi x/2)}{1/x}=-\frac{\pi}{2}.$$

Example: Evaluate $\lim\limits_{x\to 0}\dfrac{x\cdot e^x\cdot\cos^2 6x}{e^{2x}-1}$

Solution:

$$\lim_{x\to 0}\frac{x\cdot e^x\cdot\cos^2 6x}{e^{2x}-1}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(x\cdot e^x\cdot\cos^2 6x)'}{(e^{2x}-1)'}=$$

$$=\lim_{x\to 0}\frac{e^x\cdot\cos^2 6x+x\cdot e^x\cdot\cos^2 6x-2\cos 6x\cdot\sin 6x\cdot 6\cdot x\cdot e^x}{2e^{2x}}=\frac{1}{2}.$$

Indeterminate forms of types (1^∞), (0^0) and (∞^0).

Limits of the form $\lim\limits_{x\to a} f(x)^{g(x)}$ gives an indeterminate forms of types (0^0), (∞^0) and (1^∞). All three types can be evaluated by introducing a dependent variable $y=f(x)^{g(x)}$. Then calculating

$$\lim_{x\to a}\ln y=\lim_{x\to a}\left[\ln(f(x))^{g(x)}\right]=\lim_{x\to a}g(x)\cdot\ln(f(x))$$

if $\lim\limits_{x\to a}\ln y=L$ then $\lim\limits_{x\to a}y=\lim\limits_{x\to a}e^{\ln y}=e^L$

Example: Find $\lim\limits_{x\to 0^+}x^x$

Solution:

The limit leads to indeterminate form of (0^0). But a little algebraic manipulation will change the problem, which obeys to L'Hopital's rule.

Let $y=x^x$. Then $\ln y=\ln x^x=x\ln x$. Taking limits of both sides yields

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}=\left(\frac{\infty}{\infty}\right)=$$

$$=\lim_{x\to 0^+}\frac{(\ln x)'}{(1/x)'}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=0;\quad L=0$$

and $\lim\limits_{x\to 0^+}y=e^0=1$.

Example: Find $\lim_{x\to 0^+}(e^{2x}-1)^{1/\ln x}$

Solution:
The limit leads us to the form (0^0). We let $y=(e^{2x}-1)^{1/\ln x}$ and find $\lim_{x\to 0^+}\ln y$. Since

$$\ln y = \ln(e^{2x}-1)^{1/\ln x} = \frac{\ln(e^{2x}-1)}{\ln x}$$

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(e^{2x}-1)}{\ln x} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to 0^+}\frac{(\ln(e^{2x}-1))'}{(\ln x)'} =$$

$$= \lim_{x\to 0^+}\frac{\frac{2e^{2x}}{e^{2x}-1}}{1/x} = \lim_{x\to 0^+}\frac{x\cdot 2e^{2x}}{e^{2x}-1} = \left(\frac{0}{0}\right) = \lim_{x\to 0^+}\frac{2e^{2x}+4xe^{2x}}{2e^{2x}} = 1$$

Therefore $\lim_{x\to 0^+} y = e^1 = e$ (With $L=1$)

Example: Show that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Solution:
The limit has indeterminate form of type (1^∞).

$$\ln y = x\ln(1+\frac{1}{x})$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{1/x} = \left(\frac{0}{0}\right) = \lim_{x\to\infty}\frac{\left(\ln(1+\frac{1}{x})\right)'}{(1/x)'} =$$

$$= \lim_{x\to\infty}\frac{\frac{x}{x+1}\cdot(-\frac{1}{x^2})}{(-\frac{1}{x^2})} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1.$$

Therefore $\ln y = e^1 = e$. (With $L=1$)

Example: Evaluate $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x^2}\right)^x$

Solution: The limit has indeterminate form of type $\left(1^\infty\right)$.

$$y=(1+\frac{1}{x^2})^x$$

$$\ln y=\ln(1+\frac{1}{x^2})^x=\frac{\ln(1+\frac{1}{x^2})}{1/x}$$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln(1+\frac{1}{x^2})}{1/x}=\left(\frac{0}{0}\right)=\lim_{x\to\infty}\frac{\left(\ln(1+\frac{1}{x^2})\right)'}{\left(1/x\right)'}=$$

$$=\lim_{x\to\infty}\frac{x^2}{x^2+1}\cdot\frac{-2}{x^3}\cdot(-x^2)=\lim_{x\to\infty}\frac{2x^4}{3x^5+3x^3}=0\text{, so}$$

$$\lim_{x\to\infty}\ln y=e^0=1.$$

xample: Evaluate $\lim\limits_{x\to+\infty}(3^x+5^x)^{1/x}$

lution: The limit leads to indeterminate form of type $\left(1^\infty\right)$

$$y=(3^x+5^x)^{1/x}$$

$$\ln y=\ln(3^x+5^x)^{1/x}=\frac{\ln(3^x+5^x)}{x}$$

$$\text{m}\ \ln y=\lim_{x\to+\infty}\frac{\ln(3^x+5^x)}{x}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to+\infty}\frac{(\ln(3^x+5^x))'}{(x)'}=$$

$$=\lim_{x\to+\infty}\frac{3^x\ln 3+5^x\ln 5}{3^x+5^x}$$

peated use of L'Hopital's rule does not help here. Let us divide merator and denominator by 5^x:

$$\text{n}\ \frac{(3/5)^x\ln 3+\ln 5}{(3/5)^x+1}=\ln 5\ \text{(since } (3/5)^x\to 0 \text{ as } x\to\infty)$$

refore $\lim\limits_{x\to+\infty}\ln y=e^{\ln 5}=5$.

Exercises.

In exercises 1-23 find the limits.

1. $\lim_{x\to\infty} \frac{x^4}{e^x}$

2. $\lim_{x\to\infty} \frac{(\ln x)^2}{x}$

3. $\lim_{x\to\infty} \frac{2^x}{3^x}$

4. $\lim_{x\to\infty} \frac{\log_2 x}{\log_3 x}$

5. $\lim_{x\to 0^+} \frac{\cot x}{\ln x}$

6. $\lim_{x\to+\infty} \frac{x\ln x}{x+\ln x}$

7. $\lim_{x\to+\infty} \frac{x^{101}}{e^x}$

8. $\lim_{x\to+\infty} x\cdot e^{-x}$

9. $\lim_{x\to\infty} x\cdot \sin\frac{\pi}{x}$

10. $\lim_{x\to\infty} x(e^{\sin(2/x)}-1)$

11. $\lim_{x\to+\infty}\left(1-\frac{3}{x}\right)^x$

12. $\lim_{x\to 0}(e^x+x)^{1/x}$

13. $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x^2}$

14. $\lim_{x\to 1}(2-x)^{\tan(\frac{\pi x}{2})}$

15. $\lim_{x\to 0^+} x^{\sin x}$

16. $\lim_{x\to 0^+}(\sin x)^{\frac{3}{\ln x}}$

17. $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

18. $\lim_{x\to+\infty}[x-\ln(x^2+1)]$

19. $\lim_{x\to+\infty}(\sqrt{x^2+x}-x)$

20. $\lim_{x\to 0}(\cos x)^{\frac{1}{x}}$

21. $\lim_{x\to 0}\frac{xe^x(1+x)^3}{e^x-1}$

22. $\lim_{x\to 0}(1+\sin 2x)^{\frac{1}{x}}$

23. $\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^{bx}$

Answers.

1 0; 2. 0; 3. 0; 4. $\frac{\ln 3}{\ln 2}$; 5. $-\infty$; 6. $+\infty$; 7. 0; 8. 0; 9. π; 10. 2; 11. e^{-3};
12. e^2; 13. $+\infty$; 14. $e^{2/\pi}$; 15. 1; 16. e^3; 17. 1/2; 18. $+\infty$; 19. 1/2; 20. 1;
21. 1; 22. e^2; 23. e^{ab}.

6.9. The hyperbolic functions.

Definition: The hyperbolic cosine and hyperbolic sine functions, denoted by **cosh** and **sinh** respectively, are defined by the formula

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sinh x = \frac{e^x - e^{-x}}{2}$$

The four other hyperbolic functions, namely, the hyperbolic tangent, the hyperbolic secant, the hyperbolic cotangent, the hyperbolic cosecant, are defined as follows:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \qquad \sec h\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}; \qquad \csc h\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$$

The hyperbolic functions satisfy various identities similar to the identities for the trigonometric functions. The most useful of these are:

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2\sinh x \cdot \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \sec h^2 x$$

$$\coth^2 x = 1 + \csc h^2 x.$$

Example: Simplify $\cosh 5x - \sinh 5x$

Solution:

$$\cosh 5x - \sinh 5x = \frac{e^{5x}+e^{-5x}}{2} - \frac{e^{5x}-e^{-5x}}{2} = e^{-5x}.$$

Example: Let $\sinh x = -2$. Find other five hyperbolic functions.

Solution:

From $\cosh^2 x - \sinh^2 x = 1$ we obtain $\cosh^2 x = 1 + \sinh^2 x = 1 + (-2) = 5 \Rightarrow \cosh x = \pm\sqrt{5}$. Since $\cosh x > 0$ for all x then the answer will be $\cosh x = \sqrt{5}$.

From $\tanh x = \dfrac{\sinh x}{\cosh x}$ we obtain $\tanh x = -\dfrac{2}{\sqrt{5}}$

From $\coth x = \dfrac{\cosh x}{\sinh x}$ we obtain $\coth x = -\dfrac{\sqrt{5}}{2}$

From $\operatorname{sec}h\, x = \dfrac{1}{\cosh x}$ we get $\operatorname{sec}h\, x = \dfrac{1}{\sqrt{5}}$ and

From $\operatorname{csc}h\, x = \dfrac{1}{\sinh x}$ we get $\operatorname{csc}h\, x = -\dfrac{1}{2}$.

Example: Simplify $\cosh(3\ln x)$

Solution: $\cosh(3\ln x) = \dfrac{e^{3\ln x} + e^{-3\ln x}}{2} = \dfrac{x^3 + x^{-3}}{2} = \dfrac{x^6+1}{2x^3}$.

Derivatives of hyperbolic functions.

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx} \qquad \frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}[\tanh u] = \operatorname{sec}h^2 u \frac{du}{dx} \qquad \frac{d}{dx}[\coth u] = -\operatorname{csc}h^2 u \frac{du}{dx}$$

$$\frac{d}{dx}[\operatorname{sec}h\, u] = -\operatorname{sec}hu \tanh u \frac{du}{dx} \qquad \frac{d}{dx}[\operatorname{csc}h\, u] = -\operatorname{csc}hu \coth u \frac{du}{dx}$$

Example: Differentiate $\frac{d}{dx}[\cosh(x^4)]$

Solution:

$$\frac{d}{dx}[\cosh(x^4)] = \sinh(x^4)\cdot\frac{d}{dx}[x^4] = 4x^3\cdot\sinh(x^4)$$

Example: Find $\frac{d}{dx}[\ln(\coth x)]$

Solution:

$$\frac{d}{dx}[\ln(\coth x)] = \frac{1}{\coth x}\frac{d}{dx}[\coth x] = -\frac{\csc h^2 x}{\coth x}$$

Example: Find $\frac{d}{dx}[\sinh^3(2x)]$

Solution:

$$\frac{d}{dx}[\sinh^3(2x)] = 3\sinh^2(2x)\frac{d}{dx}[\sinh(2x)] =$$

$$= 3\sinh^2(2x)\cdot\cosh(2x)\frac{d}{dx}[2x] = 6\sinh^2(2x)\cdot\cosh(2x).$$

Example: Find $\frac{dy}{dx}$ if $y = \sec h(e^{2x}) + \sinh(\cos 3x)$

Solution:

$$\frac{dy}{dx} = -\sec h(e^{2x})\cdot\tanh(e^{2x})\cdot(e^{2x})' + \cosh(\cos 3x)\cdot(\cos 3x)' =$$

$$= -2e^{2x}\cdot\sec h(e^{2x})\cdot\tanh(e^{2x}) - 3\sin 3x\cdot\cosh(\cos 3x).$$

Integral formulas for hyperbolic functions.

$\int \sinh u\, du = \cosh u + C$ $\quad$ $\int \cosh u\, du = \sinh u + C$

$\int \sec h^2 u\, du = \tanh u + C$ $\quad$ $\int \csc h^2 u\, du = \coth u + C$

$\int \sec hu \tanh u\, du = -\sec h u + C$ $\quad$ $\int \csc hu \coth u\, du = -\csc h u + C$

Example: Evaluate $\int \sinh^6 x \cosh x \, dx$

Solution:
Let $u = \sin x$ then $du = \cosh x dx$.
We obtain

$$\int \sinh^6 x \cosh x \, dx = \int u^6 \, du = \frac{u^7}{7} + C = \frac{\sinh^7 x}{7} + C$$

Example: Evaluate $\int \coth x \, dx$

Solution:

$$\int \coth x \, dx = \int \frac{\cosh x}{\sinh x} dx = \begin{vmatrix} u = \sinh x \\ du = \cosh x dx \end{vmatrix} =$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sinh x| + C$$

Example: Evaluate $\int_0^1 2e^x \cdot \cosh x \, dx$

Solution:

$$\int_0^1 2e^x \cdot \cosh x \, dx = \int_0^1 2e^x \cdot \frac{e^x + e^{-x}}{2} dx = \int_0^1 (e^{2x} + 1) \, dx =$$

$$= \left(\frac{e^{2x}}{2} + x \right) \Bigg|_0^1 = \frac{e^2}{2} + 1 - \frac{1}{2} = \frac{e^2 + 1}{2}.$$

Example: Evaluate $\int_1^4 \frac{\sinh \sqrt{x}}{2\sqrt{x}} dx$

Solution:

$$\int_1^4 \frac{\sinh \sqrt{x}}{2\sqrt{x}} dx = \begin{vmatrix} u = \sqrt{x} \\ du = \dfrac{dx}{2\sqrt{x}} \end{vmatrix} = \int_1^2 \sinh u \, du =$$

$$= \cosh u \Big|_1^2 = \cosh 2 - \cosh 1 = \frac{e^2 + e^{-2}}{2} - \frac{e + e^{-1}}{2}.$$

Exercises.

1. Let $\coth x = 2$. Find the values of the other five hyperbolic functions.

In exercises 2-8 simplify the expressions as much as you can.

2. $\sinh(4\ln x)$ 3. $\sinh(4x)+\cosh(4x)$

4. $(\sinh x+\cosh x)^2$ 5. $\ln(\cosh x+\sinh x)+\ln(\cosh x-\sinh x)$

6. $(\cosh x-\sinh x)^3$ 7. $\cosh(-x)$ 8. $\sinh(-x)$

In exercises 9-15 find $\dfrac{dy}{dx}$

9. $y=\sinh(7x-4)$ 10. $y=\coth(\ln x)$

11. $y=\csc h(1/x)$ 12. $y=\sqrt{4x+\cosh^2(5x)}$

13. $y=x^3\tanh^2\sqrt{x}$ 14. $y=e^{3x}\cdot\sinh x$

15. $y=x\cdot\sinh x-\cosh x$

In exercises 16-22 evaluate integrals.

16. $\int\sqrt{\tan x}\,\sec h^2 x\,dx$ 17. $\int\tanh x\sec h^3 x\,dx$

18 $\int\dfrac{\cosh\sqrt{x}}{\sqrt{x}}\,dx$. 19. $\int_0^1\sinh^2(2x)dx$

20. $\int\tanh(5x)\sec h^5(5x)\,dx$ 21. $\int_{-2}^{2}\sinh x\,dx$

22. $\int_{-\ln 3}^{\ln 3}\sqrt{\cosh 2x-1}\,dx$

Answers.

$\sinh x=1/\sqrt{3}$; $\cosh x=2/\sqrt{3}$; $\tanh x=1/2$; $\sec hx=\sqrt{3}/2$;
sc $hx=\sqrt{3}$; **2.** $\dfrac{x^8-1}{2x^4}$; **3.** e^{4x}; **4.** e^{2x}; **5.** 0; **6.** e^{-3x}; **7.** $\cosh x$; **8.** $-\sinh x$;
$7\cosh(7x-4)$; **10.** $[-\csc h^2(\ln x)]/x$; **11.** $[\csc(1/x)\cdot\coth(1/x)]/(x^2)$;

12. $[2+5\cosh(5x)\cdot\sinh(5x)]/\sqrt{4x+\cosh^2(5x)}$;

13. $x^{5/2}\cdot\tanh\left(\sqrt{x}\right)\sec h^2\left(\sqrt{x}\right)+3x^2\cdot\tanh^2\sqrt{x}$;

14. $e^{3x}(\cosh x+3\sinh x)$;**15.** $x\cosh x$ 1;**16.** $\frac{2}{3}(\tanh x)^{3/2}+C$;

17. $-\frac{1}{3}\sec h^3x+C$;**18.** $2\sinh\sqrt{x}+C$;**19.** $\frac{\sinh 4-4}{8}$;

20. $-\frac{\sec h^6 5x}{30}+C$;**21.** 0; **22.** 0.

Chapter 7.
Inverse trigonometric and hyperbolic functions.

7.1. Inverse trigonometric functions.

nition: The inverse sine function, denoted by $\sin^{-1}$, is defined to nverse of the function $\sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$$y = \sin^{-1} x \text{ is equivalent to } \sin y = x \quad \text{if} \quad \begin{cases} -1 \le x \le 1 \\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \end{cases}$$

finition: The inverse cosine function, denoted by $\cos^{-1}$, is defined be inverse of the function $\cos x$, $0 \le x \le \pi$

$$y = \cos^{-1} x \text{ is equivalent to } \cos y = x \quad \text{if} \quad \begin{cases} -1 \le x \le 1 \\ 0 \le y \le \pi \end{cases}$$

efinition: The inverse tangent function, denoted by $\tan^{-1}$, is defined be inverse of the function $\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$$y = \tan^{-1} x \text{ is equivalent to } \tan y = x \quad \text{if} \quad \begin{cases} -\infty \le x \le +\infty \\ -\frac{\pi}{2} < y < \frac{\pi}{2} \end{cases}$$

efinition: The inverse secant function, denoted by $\sec^{-1}$, is defined to e inverse of the function $\sec x$, $0 \le x < \frac{\pi}{2}$ or $\pi \le x < \frac{3\pi}{2}$

$$y = \sec^{-1} x \text{ is equivalent to } \sec y = x \quad \text{if} \quad \begin{cases} x \ge 1 \\ 0 \le y < \frac{\pi}{2} \end{cases} \text{or} \begin{cases} x \le -1 \\ \pi \le y < \frac{3\pi}{2} \end{cases}$$

he inverse cotangent and inverse cosecant functions are denoted by ot$^{-1} x$ and $\csc^{-1} x$, will not be needed. They may be defined as ollows:

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x \qquad \text{for all } x$$

$$\csc^{-1} x = \sin^{-1} \frac{1}{x} \qquad \text{for } |x| \geq 1$$

Example: Find $\cos^{-1}(-\sqrt{3}/2)$

Solution:

Let $y = \cos^{-1}(-\sqrt{3}/2)$. This is equivalent to

$$\cos y = -\sqrt{3}/2 \text{ and } y = \frac{5\pi}{6}$$

Example: Find $\sin^{-1}(-\sqrt{2}/2)$

Solution: Let $y = \sin^{-1}(-\sqrt{2}/2)$. Then

$$\sin y = -\sqrt{2}/2 \Rightarrow y = -\frac{\pi}{4}$$

The derivatives of the inverse trigonometric functions.

(1) $$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \qquad -1 < u < 1$$

(2) $$\frac{d}{dx}[\cos^{-1} u] = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \qquad -1 < u < 1$$

(3) $$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2} \frac{du}{dx}$$

(4) $$\frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2} \frac{du}{dx}$$

(5) $$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx} \qquad |u| > 1$$

(6) $$\frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx} \qquad |u| > 1$$

Example: Find the derivative of $\tan^{-1} \sqrt{x}$

Solution:

From (3) $$\frac{dy}{dx} = \frac{1}{1+x} \frac{d}{dx}\left[\sqrt{x}\right] = \frac{1}{2\sqrt{x}(1+x)}$$

Example: Find $\frac{dy}{dx}$ if $y=\sin^{-1}\left(\frac{3x}{4}\right)$

Solution:

From (1)

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(3x/4)^2}}\frac{d}{dx}\left[\frac{3x}{4}\right]=\frac{1}{\sqrt{1-9x^2/16}}\cdot\frac{3}{4}=\frac{3}{\sqrt{16-9x^2}}.$$

Example: Find $\frac{dy}{dx}$ if $y=\sec^{-1}(e^x)$

Solution:

From (5)

$$\frac{dy}{dx}=\frac{1}{e^x\sqrt{e^{2x}-1}}\frac{d}{dx}[e^x]=\frac{1}{\sqrt{e^{2x}-1}}$$

Example: Differentiate $y=x\sqrt{1-x^2}+\sin^{-1}x$

Solution:

$$\frac{dy}{dx}=\frac{d}{dx}[x\sqrt{1-x^2}+\sin^{-1}x]=\sqrt{1-x^2}+x\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)+$$

$$+\frac{1}{\sqrt{1-x^2}}=\frac{1-x^2}{\sqrt{1-x^2}}+\sqrt{1-x^2}=2\sqrt{1-x^2}$$

Example: Find $\frac{dy}{dx}$ if $y=(1+x\cdot\csc^{-1}x)^{10}$

Solution:

$$\frac{dy}{dx}=\frac{d}{dx}[(1+x\cdot\csc^{-1}x)^{10}]=$$

$$=10\cdot(1+x\cdot\csc^{-1}x)^9\cdot\frac{d}{dx}[1+x\cdot\csc^{-1}x]=$$

$$=10\cdot(1+x\cdot\csc^{-1}x)^9\cdot\left(\csc^{-1}x-x\cdot\frac{1}{x\sqrt{x^2-1}}\right).$$

With each derivative of (1)-(6) comes a corresponding antiderivative.

Most commonly needed integration formulas are

(7) $\displaystyle\int\frac{du}{\sqrt{1-u^2}}=\sin^{-1}u+C$ for $u^2<1$

(8) $\displaystyle\int\frac{du}{1+u^2}=\tan^{-1}u+C$ for all u

(9) $\displaystyle\int\frac{du}{|u|\sqrt{u^2-1}}=\sec^{-1}u+C=\cos^{-1}\left|\frac{1}{u}\right|+C$ for $u^2>1$

Example: Evaluate $\displaystyle\int\frac{dx}{1+6x^2}$

Solution:
Substituting $u=\sqrt{6}\,x$, $du=\sqrt{6}\,dx$ yields

$$\int\frac{dx}{1+6x^2}=\frac{1}{\sqrt{6}}\int\frac{du}{1+u^2}=\frac{1}{\sqrt{6}}\tan^{-1}u+C=$$

$$=\frac{1}{\sqrt{6}}\tan^{-1}(\sqrt{6}\,x)+C$$

Example: Evaluate $\displaystyle\int\frac{e^{2x}}{\sqrt{1-e^{4x}}}dx$

Solution:
Substituting $u=e^{2x}$, $du=2e^{2x}\,dx$ yields

$$\int\frac{e^{2x}}{\sqrt{1-e^{4x}}}dx=\frac{1}{2}\int\frac{du}{\sqrt{1-u^2}}=\frac{1}{2}\sin^{-1}u+C=\frac{1}{2}\sin^{-1}(e^{2x})+C.$$

Example: Evaluate $\displaystyle\int_{1/\sqrt{3}}^{1}\frac{dx}{x\sqrt{4x^2-1}}$

Solution:
Substituting $u=2x$, $du=2\,dx$ yields

$$\int_{1/\sqrt{3}}^{1}\frac{dx}{x\sqrt{4x^2-1}}=\int_{2/\sqrt{3}}^{2}\frac{du/2}{u/2\cdot\sqrt{u^2-1}}=\cos^{-1}\left|\frac{1}{u}\right|\Bigg|_{2/\sqrt{3}}^{2}=$$

$$=\cos^{-1}(1/2)-\cos^{-1}(\sqrt{3}/2)=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}.$$

Example: Evaluate $\int \frac{12\,dx}{\sqrt{e^{2x}-1}}$

Solution:

let us multiply and divide numerator and denominator by e^x.

$$\int \frac{12\,dx}{\sqrt{e^{2x}-1}} = \int \frac{12\cdot e^x\,dx}{e^x\cdot\sqrt{e^{2x}-1}}$$

Let $u = e^x$, $du = e^x dx$.

Then

$$\int \frac{12\cdot e^x\,dx}{e^x\cdot\sqrt{e^{2x}-1}} = 12\int \frac{du}{u\cdot\sqrt{u^2-1}} = 12\sec^{-1} u + C = 12\sec^{-1}(e^x) + C.$$

Exercises.

Evaluate without a calculator

$\sin^{-1}(1/2)$ b) $\tan^{-1}(1/\sqrt{3})$ c) $\sec^{-1}\sqrt{2}$

$\tan^{-1}(-\sqrt{3})$ e) $\sin^{-1}(\sqrt{2}/2)$ f) $\cos^{-1}(-1)$

$\sin^{-1}(\sin \pi/7)$ h) $\sin[2\cos^{-1}(3/5)]$ i) $\tan[2\sec^{-1}(3/2)]$

In exercises 2-13 find dy/dx

$y = \sin^{-1} 5x$ 3. $y = \sec^{-1} 3x$

$= \tan^{-1}\sqrt[3]{x}$ 5. $y = \frac{x\cdot\sec^{-1} 3x}{e^{2x}}$

$= \sin^{-1} x - \sqrt{1-x^2}$ 7. $y = (\tan^{-1} 2x)^3$

$= \frac{x}{2}\sqrt{2-x^2} + \sin^{-1}\frac{x}{\sqrt{2}}$ 9. $y = \frac{2}{3}\sec^{-1}\sqrt{3x^5}$

$= \tan^{-1}\left(\frac{1-x}{1+x}\right)$ 11. $y = e^x\cdot\sec^{-1} x$

$= \sin^{-1}(x^2 \ln x)$ 13. $y = \sqrt{x^2-1} - \sec^{-1} x$

In exercises 14-23 evaluate the integrals.

14. $\int_{-1}^{1}\frac{dx}{1+x^2}$

15. $\int_{-\sqrt{2}}^{-2\sqrt{2}}\frac{dx}{x\sqrt{x^2-1}}$

16. $\int\frac{dx}{1+16x^2}$

17. $\int\frac{e^x\,dx}{1+e^{2x}}$

18. $\int_{1}^{3}\frac{dx}{\sqrt{x}(x+1)}$

19. $\int\frac{\sec^2 x\,dx}{\sqrt{1-\tan^2 x}}$

20. $\int\frac{dx}{x\sqrt{1-(\ln x)^2}}$

21. $\int_{0}^{2}\frac{dy}{1+(y-1)^2}$

22. $\int_{\ln 2}^{\ln(2/\sqrt{3})}\frac{e^{-x}dx}{\sqrt{1-e^{-2x}}}$

23. $\int_{2}^{4}\frac{dx}{2x\sqrt{x-1}}$

Answers.

1. a) $\frac{\pi}{6}$; b) $\frac{\pi}{6}$; c) $\frac{\pi}{4}$; d) $-\frac{\pi}{3}$;e) $\frac{\pi}{4}$; f) π ; g) $\frac{\pi}{7}$; h) $\frac{24}{25}$; i) $-4\sqrt{5}$;

2. $\frac{5}{\sqrt{1-25x^2}}$;**3.** $\frac{1}{|x|\sqrt{9x^2-1}}$; **4.** $\frac{1}{3x^{2/3}(1+x^{2/3})}$;**5.** $e^{-2x}[\frac{x}{|x|\sqrt{9x^2-1}}+$

$+(1-2x)\sec^{-1}3x]$; **6.** $\sqrt{\frac{1+x}{1-x}}$; **7.** $\frac{6(\tan^{-1}2x)^2}{1+4x^2}$; **8.** $\sqrt{2-x^2}$;

9. $\frac{5}{3x\sqrt{3x^5-1}}$;**10.** $-\frac{1}{x^2+1}$;**11** $\frac{e^x}{x\sqrt{x^2-1}}+e^x\sec^{-1}x$;

12. $\frac{x+2x\ln x}{\sqrt{1-x^4\ln^2 x}}$; **13.** $\sqrt{1-\frac{1}{x^2}}$;**14.** $\pi/2$; **15.** $-\frac{\pi}{12}$;

16. $\frac{1}{4}\tan^{-1}4x+C$;

17. $\tan^{-1}(e^x)+C$;**18.** $\frac{\pi}{6}$;**19.** $\sin^{-1}(\tan x)+C$;**20.** $\sin^{-1}(\ln x)+C$;

21. $\frac{\pi}{2}$;**22.** $-\frac{\pi}{6}$; **23.** $\frac{\pi}{12}$.

7.2. The inverse hyperbolic functions.

$y=\sinh^{-1}x$, $y=\cosh^{-1}x$, $y=\tanh^{-1}x$, $y=\coth^{-1}x$, $y=\text{sech}^{-1}x$, $y=\text{csch}^{-1}x$ are inverses of $y=\sinh x$, $y=\cosh x$, $y=\tanh x$, $y=\coth x$, $y=\text{sech}x$ and $y=\text{csch}x$ respectively.

Some useful identities for inverse hyperbolic functions

$$\text{sec}h^{-1}x=\cosh^{-1}\frac{1}{x}$$

$$\text{csch}^{-1}x=\sinh^{-1}\frac{1}{x}$$

$$\coth^{-1}x=\tanh^{-1}\frac{1}{x}$$

Derivatives of inverse hyperbolic functions.

(1) $\dfrac{d}{dx}[\sinh^{-1}u]=\dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$

(2) $\dfrac{d}{dx}[\cosh^{-1}u]=\dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$; $u>1$

(3) $\dfrac{d}{dx}[\tanh^{-1}u]=\dfrac{1}{1-u^2}\dfrac{du}{dx}$; $|u|<1$

(4) $\dfrac{d}{dx}[\coth^{-1}u]=\dfrac{1}{1-u^2}\dfrac{du}{dx}$; $|u|>1$

(5) $\dfrac{d}{dx}[\text{sech}^{-1}u]=-\dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$; $0<u<1$

(6) $\dfrac{d}{dx}[\text{csch}^{-1}u]=-\dfrac{1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx}$; $u\neq 0$.

xample: Find $\dfrac{dy}{dx}$ if $y=\sinh^{-1}(2x)$

olution:

rom (1) $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1+4x^2}}\cdot\dfrac{d}{dx}(2x)=\dfrac{2}{\sqrt{1+4x^2}}$

xample: Find $\dfrac{dy}{dx}$ if $y=x^3(\cosh^{-1}x)^4$

Solution:

$$\frac{dy}{dx}=\left(x^3(\cosh^{-1}x)^4\right)'=3x^2\cdot(\cosh^{-1}x)^4+$$

$$+x^3\cdot 4\cdot(\cosh^{-1}x)^3\cdot\frac{1}{\sqrt{x^2-1}}=x^2(\cosh^{-1}x)^3\cdot\left[3\cosh^{-1}x+\frac{4x}{\sqrt{x^2-1}}\right]$$

Example: Find $(\operatorname{csch}^{-1}(\tan x))'$

Solution:

$$(\operatorname{csch}^{-1}(\tan x))'=-\frac{1}{|\tan x|\cdot\sqrt{1+\tan^2 x}}\cdot\frac{1}{\cos^2 x}=-\csc x.$$

Exercises.

In exercises 1-9 find dy/dx.

1. $y=\sinh^{-1}\left(\frac{1}{3}x\right)$
2. $y=\cosh^{-1}(2x+1)$
3. $y=\sec h^{-1}(x^7)$
4. $y=\csc h^{-1}(e^x)$
5. $y=e^x\cdot\sec h^{-1}x$
6. $y=x^2(\sinh^{-1}x)^3$
7. $y=(1+x\cdot\csc h^{-1}x)^{10}$
8. $y=\sec h^{-1}(\sin x)$
9. $y=\tanh^{-1}\left(\frac{1-x}{1+x}\right)$

Answers.

1. $\frac{1}{\sqrt{9+x^2}}$; **2.** $\frac{2}{\sqrt{(2x+1)^2-1}}$; **3.** $-\frac{7}{x\sqrt{1-x^{14}}}$; **4.** $-\frac{1}{\sqrt{1+e^{2x}}}$;

5. $-\frac{e^x}{x\sqrt{1-x^2}}+e^x\cdot\sec h^{-1}x$; **6.** $\frac{3x^2(\sinh^{-1}x)^2}{\sqrt{1+x^2}}+2x(\sinh^{-1}x)^3$;

7. $10(1+x\cdot\csc h^{-1}x)^9\cdot\left(-\frac{x}{|x|\cdot\sqrt{1+x^2}}+\csc h^{-1}x\right)$; **8.** $-\csc x$;

9. $-\frac{1}{2x}$.

Chapter 8
Techniques of integration.

8.1. Basic integration formulas.

The following integral table lists a few formulas that should be memorized. Each of them can be checked by differentiating the right-hand side of the equation.

1. $\int du = u + C$
2. $\int a\,du = a\int du = au + C$
3. $\int u^r\,du = \dfrac{u^{r+1}}{r+1} + C, \quad r \neq -1$
4. $\int \dfrac{du}{u} = \ln|u| + C$
5. $\int e^u\,du = e^u + C$
6. $\int a^u\,du = \dfrac{a^u}{\ln a} + C, \quad a > 0$
7. $\int \sin u\,du = -\cos u + C$
8. $\int \cos u\,du = \sin u + C$
9. $\int \sec^2 u\,du = \tan u + C$
10. $\int \csc^2 u\,du = -\cot u + C$
11. $\int \sec u \tan u\,du = \sec u + C$
12. $\int \csc u \cot u\,du = -\csc u + C$
13. $\int \tan u\,du = -\ln|\cos u| + C$
14. $\int \cot u\,du = \ln|\sin u| + C$
15. $\int \sinh u\,du = \cosh u + C$

16. $\int \cosh u \, du = \sinh u + C$

17. $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$

18. $\int \frac{du}{1+u^2} = \tan^{-1} u + C$

19. $\int \frac{du}{|u|\sqrt{u^2-1}} = \sec^{-1} u + C$

20. $\int u \cdot dv = u \cdot v - \int v \cdot du$ (Integration by parts)

Example: Find $\int x^3 dx$

Solution:
Using (3) with $r = 3$ we obtain

$$\int x^3 dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C$$

Example: Find $\int \frac{x^4 - 2x^3 + 3x^2}{x^2} dx$

Solution:
We divide the numerator by the denominator

$$\int \frac{x^4 - 2x^3 + 3x^2}{x^2} dx = \int (x^2 - 2x + 3) dx = \frac{x^3}{3} - x^2 + 3x + C$$

Example: Find $\int \left(1 - \frac{1}{x^2}\right)^2 dx$

Solution:
By expanding the integrand we get

$$\int \left(1 - \frac{1}{x^2}\right)^2 dx = \int \left(1 - \frac{2}{x^2} + \frac{1}{x^4}\right) dx = \int dx - 2\int x^{-2} dx + \int x^{-4} dx =$$

$$= x - 2\frac{x^{-2+1}}{-2+1} + \frac{x^{-4+1}}{-4+1} + C = x + \frac{2}{x} - \frac{1}{3x^3} + C$$

Example: Find $\int \frac{dx}{\sin^2 x \cos^2 x}$

Solution:

$$\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx =$$

$$= \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} = -\cot x + \tan x + C.$$

Example: Find $\int \sin^2 \frac{x}{2} dx$

Solution:

Since $\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$, we get

$$\int \sin^2 \frac{x}{2} dx = \int \frac{1-\cos x}{2} dx = \frac{1}{2}x - \frac{1}{2}\sin x + C$$

xample: Find $\int \sin x \cos 3x dx$

olution:

Since $\sin x \cdot \cos 3x = \frac{1}{2}(\sin 4x - \sin 2x)$, then we obtain

$$\int \sin x \cos 3x dx = \frac{1}{2}\int (\sin 4x - \sin 2x) dx = -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + C$$

8.2. The substitution method.

The substitution method changes the form of an integrand to that we can integrate more easily. Sometimes we use substitution to vert an integral not listed in an integral table to one that is listed.

mple: Find $\int \cos(x^3) \cdot 3x^2 dx$

tion:

Since $3x^2$ is the derivative of x^3, we make substitution $u = x^3$. Then $du = 3x^2 dx$ and

$$\int \cos(x^3) \cdot 3x^2 dx = \int \cos u \, du = \sin u + C = \sin(x^3) + C$$

Example: Find $\int 6e^{x^6} \cdot x^5 dx$

Solution:

Introduce $u = x^6$. Then $du = 6x^5 dx$ and

$$\int 6e^{x^6} \cdot x^5 dx = \int e^u du = e^u + C = e^{x^6} + C$$

Example: Find $\int \cos^3 x \cdot \sin x \, dx$

Solution:

Note that derivative of $\cos x$ is $-\sin x$. Let $u = \cos x$, then $du = -\sin x dx$ and $\sin x \, dx = -du$.

We obtain

$$\int \cos^3 x \cdot \sin x \, dx = -\int u^3 du = -\frac{u^4}{4} + C = -\frac{\cos^4 x}{4} + C.$$

Example: Evaluate $\int (1 + x^4)^5 x^3 \, dx$

Solution:

The derivative of $(1 + x^4)$ is $4x^3$, which differs from x^3 in the integrand only by constant factor 4.

If we let $u = 1 + x^4$,

then $du = 4x^3 dx$ and $x^3 dx = \dfrac{du}{4}$.

We obtain

$$\int (1 + x^4)^5 x^3 \, dx = \int u^5 \cdot \frac{du}{4} = \frac{1}{4}\frac{u^6}{6} + C = \frac{1}{24}(1 + x^4)^6 + C.$$

Check: $$\frac{d}{dx}\left(\frac{1}{24}(1 + x^4)^6 + C\right) =$$

$$= 6 \cdot \frac{1}{24}(1 + x^4)^5 \cdot \frac{d}{dx}[1 + x^4] = (1 + x^4)^5 \cdot x^3$$

Example: Find $\int \frac{x^2}{(1 + x)^3} dx$

Solution:

Try the substitution $u = 1 + x$. Then $du = dx$. Solving the equation $u = x + 1$ for x gives $x = u - 1$.

Thus

$$\int \frac{x^2}{(1+x)^3}\,dx = \int \frac{(u-1)^2}{u^3}\,du = \int \frac{u^2-2u+1}{u^3}\,du =$$

$$= \int \left(\frac{1}{u} - \frac{2}{u^2} + \frac{1}{u^3}\right) du = = \ln|u| + \frac{2}{u} - \frac{1}{2u^2} + C =$$

$$= \ln|1+x| + \frac{2}{1+x} - \frac{1}{2(1+x)^2} + C$$

Example: Evaluate $\int_2^3 \frac{e^{1/x}}{x^2}\,dx$

Solution:

Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}\,dx$ and $\frac{dx}{x^2} = -du$.

As x goes from 2 to 3, $u = \frac{1}{x}$ goes from $\frac{1}{2}$ to $\frac{1}{3}$

$$\int_2^3 \frac{e^{1/x}}{x^2}\,dx = -\int_{1/2}^{1/3} e^u\,du = -e^u\Big|_{1/2}^{1/3} = -(e^{1/3} - e^{1/2}) = \sqrt{e} - \sqrt[3]{e}.$$

Example: Evaluate $\int_0^1 \frac{2dx}{1+(4x+1)^2}$

Solution:

The nearest standard form is $\int \frac{du}{1+u^2} = \tan^{-1} u + C$,

so let $u = 4x+1$, $du = 4dx$ and $2dx = \frac{du}{2}$

if $x = 0$ then $u = 1$

if $x = 1$ then $u = 5$ and we obtain

$$\int_0^1 \frac{2dx}{1+(4x+1)^2} = \frac{1}{2}\int_1^5 \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1} u\Big|_1^5 =$$

$$= \frac{1}{2}\left(\tan^{-1} 5 - \tan^{-1} 1\right) = \frac{1}{2}\left(\tan^{-1} 5 - \frac{\pi}{4}\right)$$

Example: Evaluate $\int \frac{e^{3x}}{(1+e^{3x})^3}\,dx$

Solution:

Let $u = 1 + e^{3x}$, then $du = 3e^{3x}dx$ and $e^{3x}dx = \frac{du}{3}$.

We obtain

$$\int \frac{e^{3x}}{(1+e^{3x})^3}\,dx = \frac{1}{3}\int \frac{du}{u^3} = \frac{1}{3}\frac{u^{-3+1}}{-3+1} + C = -\frac{1}{6}\frac{1}{u^2} + C =$$

$$= -\frac{1}{6}\cdot\frac{1}{(1+e^{3x})^2} + C$$

Example: Evaluate $\int \frac{\sqrt[4]{2+\cot x}}{\sin^2 x}\,dx$

Solution:

Let $u = 2 + \cot x$, so that $du = -\frac{1}{\sin^2 x}dx$.

$$\int \frac{\sqrt[4]{2+\cot x}}{\sin^2 x}\,dx = -\int \sqrt[4]{u}\,du = -\frac{u^{\frac{1}{4}+1}}{\frac{1}{4}+1} + C = -\frac{4}{5}\sqrt[4]{u^5} + C =$$

$$= -\frac{4}{5}\sqrt[4]{(2+\cot x)^5} + C.$$

Exercises.

In exercises 1-9 evaluate the integrals.

1. $\int (x^3 - x^2 + 1)\,dx$
2. $\int \frac{x - 2\sqrt{x} + 2}{x^2 \cdot \sqrt[3]{x}}\,dx$
3. $\int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2 dx$
4. $\int \frac{3 - 4\cos^3 x}{\cos^2 x}\,dx$
5. $\int e^x (1 + \frac{e^{-x}}{x^3})\,dx$
6. $\int \frac{x^2}{1+x^2}\,dx$

7. $\int\left(\sin\frac{x}{3}+\cos\frac{x}{3}\right)dx$

8. $\int \sin x\cdot\cos 7x\,dx$

9. $\int \cos 4x\cdot\cos 6x\,dx$

In exercises 10-28 evaluate the integrals using appropriate substitutions.

10. $\int (1-x^2)^5 x\,dx$

11. $\int \sqrt[3]{1+x^2}\;x\,dx$

12. $\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

13. $\int \sin 3\theta\,d\theta$

14. $\int x^4\cdot\sin x^5\,dx$

15. $\int \frac{x}{1+x^4}\,dx$

6. $\int \frac{\ln 3x}{x}\,dx$

17. $\int \frac{x}{\sqrt{x+4}}\,dx$

8. $\int \sin^3 x\cdot\cos x\,dx$

19. $\int_0^1 x^2 e^{x^3}\,dx$

0. $\int \frac{\sin 2x}{\cos^4 x+1}\,dx$

21. $\int \frac{dx}{1+\sqrt{x}}$

2. $\int_{\pi^2/16}^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

23. $\int_{\pi/6}^{\pi/4} \tan\theta\cdot\sec^2\theta\,d\theta$

. $\int \frac{dx}{(1+x)\sqrt{x}}$

25. $\int_1^2 \frac{2^{\ln 2x}}{x}\,dx$

. $\int_0^1 5^{2\theta}\,d\theta$

27. $\int \frac{x^2\,dx}{ax+b}$; $a\neq 0$

$\int \frac{e^{x/2}dx}{\sqrt{1-e^x}}$

Answers.

1. $\frac{x^4}{4}-\frac{x^3}{3}+x+C$; **2.** $-\frac{3}{\sqrt[3]{x}}+\frac{12}{5\sqrt[6]{x^5}}-\frac{3}{2x\sqrt[3]{x}}+C$; **3.** $x-\cos x+C$;

4. $3\tan x-4\sin x+C$; **5.** $e^x-\frac{1}{2x^2}+C$; **6.** $x-\tan^{-1}x+C$;

7. $3(\sin\frac{x}{3}-\cos\frac{x}{3})+C$; **8.** $-\frac{1}{16}\cos 8x+\frac{1}{12}\cos 6x+C$;

9. $\frac{1}{20}\sin 10x+\frac{1}{4}\sin 2x+C$; **10.** $-\frac{1}{12}(1-x^2)^6+C$;

11. $\frac{3}{8}(1+x^2)^{4/3}+C$; **12.** $2e^{\sqrt{t}}+C$; **13.** $-\frac{1}{3}\cos 3\theta+C$;

14. $-\frac{1}{5}\cos x^5+C$; **15.** $\frac{1}{2}\tan^{-1}x^2+C$; **16.** $\frac{1}{2}(\ln 3x)^2+C$;

17. $\frac{2}{3}\sqrt{(x+4)^3}-8\sqrt{x+4}+C$; **18.** $\frac{\sin^4 x}{4}+C$; **19.** $\frac{e-1}{3}$;

20. $-\tan^{-1}(\cos^2 x)+C$; **21.** $2\sqrt{x}-2\ln(1+\sqrt{x})+C$; **22.** $\sqrt{2}$; **23.** $\frac{1}{3}$;

24. $2\tan^{-1}\sqrt{x}+C$; **25.** $\left(2^{\ln 4}-2^{\ln 2}\right)/\ln 2$; **26.** $\frac{12}{\ln 5}$;

27. $\frac{1}{a^3}\left(\frac{1}{2}a^2x^2-abx+b^2\ln|ax+b|\right)+C$; **28.** $2\sin^{-1}(e^{x/2})+C$;

8.3. Integration by parts.

Integration by parts is based on the equation

$$(1)\qquad \int u\cdot dv=u\cdot v-\int v\cdot du$$

where u and v are both differentiable functions of x.

Example: Find $\int x e^x dx$

Solution:

To use formula $\int u\cdot dv=u\cdot v-\int v\cdot du$ we must write $xe^x dx$ as $u\cdot dv$. One way to do this is

$u = x\,; dv = e^x dx$ so that $du = dx$ and $v = \int e^x dx = e^x$.

Thus from (1)

$$\int \underbrace{x}_{u} \cdot \underbrace{e^x dx}_{dv} = \underbrace{x \cdot e^x}_{u \cdot v} - \int \underbrace{e^x dx}_{v\,du} = xe^x - e^x + C$$

Remark: The key to applying integration by parts is the labeling of u and dv. Usually three conditions should be met:

1. v can be found by integration and should not be too messy.
2. du should not be more complicated than u
3. $\int v \cdot du$ should be easier than the original $\int u \cdot dv$

Application of integration by parts formula is a matter of experience that comes with lots of practice.

Example: Find $\int x \ln x \, dx$

Solution:

Letting $dv = \ln x \, dx$ is not a wise move, since $\int \ln x dx$ is not immediately apparent. But setting $u = \ln x$ is promising.

$u = \ln x$, $\qquad du = \frac{1}{x} dx$

$dv = xdx \qquad v = \int xdx = \frac{x^2}{2} \qquad$ Thus

$$\int x \ln x \, dx = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x} = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

You may check the result by differentiation.

Example: Evaluate $\int x^2 e^x dx$

Solution:

Let $\quad u = x^2 \qquad du = 2xdx$

$dv = e^x dx \qquad v = \int e^x dx = e^x \qquad$ so that

$$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx = x^2 e^x - 2\int xe^x dx$$

Another integration by parts applied to $\int x e^x dx$ will complete the problem.

We let

$$\begin{cases} u = x & du = dx \\ dv = e^x dx & v = e^x \end{cases} \text{ so that}$$

$$\int x e^x dx = xe^x - \int e^x dx = xe^x - e^x .$$

After substituting we obtain

$$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx = x^2 e^x - 2\int xe^x dx =$$

$$= x^2 e^x - 2(xe^x - e^x) = (x^2 - 2x + 2)e^x + C$$

For definite integrals the formula corresponding to $\int u dv = uv - \int v du$ is

$$(2) \qquad \int_a^b u \cdot dv = u \cdot v\Big|_a^b - \int_a^b v\, du$$

Remark: It is important to keep in mind that the variables u and v in (2) are functions of x, and that the limits of integration in (2) are limits on the variable x.

Example: Evaluate $\int_e^{e^3} \frac{\ln x}{x^3} dx$

Solution:

Let

$$u = \ln x \qquad du = \frac{1}{x} dx$$

$$dv = \frac{1}{x^3} dx \qquad v = -\frac{1}{2x^2} \text{. Thus}$$

$$\int_e^{e^3} \frac{\ln x}{x^3} dx = -\frac{\ln x}{2x^2}\Bigg|_e^{e^3} + \int_e^{e^3} \frac{1}{2x^3} dx \quad \text{But}$$

$$\int_e^{e^3} \frac{1}{2x^3} dx = -\frac{1}{4x^2}\Bigg|_e^{e^3} = -\frac{1}{4}(e^{-6} - e^{-2}) \text{, so}$$

$$\int_e^{e^3} \frac{\ln x}{x^3} dx = -\frac{\ln x}{2x^2}\Bigg|_e^{e^3} - \frac{1}{4}(e^{-6} - e^{-2}) =$$

$$=-\frac{1}{2}(3e^{-6}-e^{-2})- \quad -\frac{1}{4}(e^{-6}-e^{-2})=\frac{e^{-2}}{4}(3-7e^{-4})$$

Example: Evaluate $\int e^x \sin x dx$

Solution:

Let $u=e^x$, $du=e^x dx$

$dv=\sin x dx$, $v=\int \sin x dx=-\cos x$.

Thus

$$(3) \qquad \int e^x \sin x dx=-e^x \cos x+\int e^x \cos x dx$$

tegral $\int e^x \cos x dx$ is similar to the original integral; it seems that othing has been accomplished. Let us integrate new integral by parts; e let

$u=e^x$ $du=e^x dx$

$dv=\cos x dx$ $v=\int \cos x dx=\sin x$.

us

$$\int e^x \cos x dx= e^x \sin x-\int e^x \sin x dx.$$

stituting in (3) yields

$$\int e^x \sin x dx=-e^x \cos x+ e^x \sin x-\int e^x \sin x dx,$$

ch is an equation we can solve for the unknown integral. We obtain

$$2\int e^x \sin x dx= e^x \sin x- e^x \cos x \qquad \text{and hence}$$

$$\int e^x \sin x dx=\frac{1}{2}e^x \sin x-\frac{1}{2}e^x \cos x+C$$

Exercises.

In exercises 1-17 evaluate the integrals by integration by parts.

$e^{2x}dx$ 2. $\int x \sin 2x dx$

n$^{-1} x dx$ 4. $\int x^2 \sin x dx$

5. $\int \frac{\ln x}{\sqrt{x}}\,dx$

6. $\int \sin(\ln x)\,dx$

7. $\int_1^2 x^2 e^{-x}\,dx$

8. $\int_0^1 \sin^{-1} x dx$

9. $\int_2^3 (\ln x)^2\,dx$

10. $\int_1^e \frac{\ln x}{x^2}\,dx$

11. $\int \frac{\ln(1+x^2)}{x^2}\,dx$

12. $\int e^{ax}\sin bx\,dx$

13. $\int x^3 e^{x^2}\,dx$

14. $\int_{-2}^{2} \ln(x+3)dx$

15. $\int_0^{\pi/2} x\cdot \sin 4x dx$

16. $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$

17. $\int_0^{\pi/3} x\cdot \tan^2 x dx$

Answers.

1. $\frac{1}{4}e^{2x}(2x-1)+C$; **2.** $-\frac{x}{2}\cos 2x+\frac{1}{4}\sin 2x+C$; **3.** $x\tan^{-1} x-$ $-\frac{1}{2}\ln(1+x^2)+C$; **4.** $-x^2\cos x+2x\sin x+2\cos x+C$; **5.** $2\sqrt{x}\ln x-$ $-4\sqrt{x}+C$; **6.** $\frac{x}{2}(\sin(\ln x)-\cos(\ln x))+C$; **7.** $\frac{5e-10}{e^2}$; **8.** $\frac{\pi}{2}-1$;

9. $3(\ln 3)^2-6\ln 3-2(\ln 2)^2+4\ln 2+2$; **10.** $\frac{e-2}{e}$; **11.** $-\frac{\ln(1+x^2)}{x}+$ $+2\tan^{-1} x+C$; **12.** $\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$; **13.** $\frac{1}{2}x^2e^{x^2}-$ $-\frac{1}{2}e^{x^2}+C$; **14.** $5\ln 5-4$; **15.** $-\pi/8$; **16.** $\frac{1}{3}(2-\sqrt{2})$;

17. $\frac{\sqrt{3}}{3}\pi-\ln 2-\frac{\pi^2}{18}$;

8.4. Trigonometric integrals.

8.4.1. Integrating powers of sine and cosine functions.

Integrals of the form $\int \sin^m x dx$ and $\int \cos^n x dx$ can be evaluated by using so called **reduction formulas:**

$$\int \sin^m x\,dx = -\frac{1}{m}\sin^{m-1} x\cos x + \frac{m-1}{m}\int \sin^{m-2} x\,dx \qquad \text{and}$$

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx,$$

where n and m are integers and $m \geq 2$, $n \geq 2$.

But we will now give alternative methods for evaluating such integrals.

If m and n both positive integers, then evaluating integral $\int \sin^m x\cos^n x\,dx$ can be divided into three cases.

Case 1: m is odd

Case 2: n is odd

Case 3: m and n both even

Case1. If m is odd, we split of a factor of $\sin x$ as $m = 2k+1$ and apply identity $\sin^2 x = 1-\cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \cdot \sin x = (1-\cos^2 x)^k \cdot \sin x$$

Then we make the substitution $u = \cos x$, and solve as usual.

Example: Evaluate $\int \sin^3 x\cos x dx$

Solution:

$$\int \sin^3 x\cos x dx = \int \sin^2 x \cdot \sin x \cdot \cos x dx =$$

$$= \int (1-\cos^2 x)\cdot \cos x \cdot \sin x dx = \begin{vmatrix} u = \cos x \\ du = -\sin x dx \end{vmatrix} = -\int (1-u^2)u du =$$

$$= -\frac{u^2}{2} + \frac{u^4}{4} + C = \frac{\cos^4 x}{4} - \frac{\cos^2 x}{2} + C$$

Case2. If n is odd then in $\int \sin^m x \cos^n x\, dx$ we write n as $2k+1$ and use identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = \cos^{2k} x \cdot \cos x = (1-\sin^2 x)^k \cdot \cos x$$

Example: Evaluate $\int \sin^4 x \cos^5 x dx$

Solution:

$$\int \sin^4 x \cos^5 x dx = \int \sin^4 x \cdot \cos^4 x \cdot \cos x dx =$$

$$= \int \sin^4 x \cdot (1-\sin^2 x)^2 \cos x dx = \begin{vmatrix} u = \sin x \\ du = \cos x dx \end{vmatrix} = \int u^4 (1-u^2)^2 du =$$

$$= \int (u^4 - 2u^6 + u^8) du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{u^9}{9} + C =$$

$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$$

Case3. If m and n both are even, then we use identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

to reduce the powers on $\sin x$ and $\cos x$.

Example: Evaluate $\int \sin^2 x \cos^2 x dx$

Solution:

$$\int \sin^2 x \cos^2 x dx = \int \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2} dx =$$

$$= \int \frac{1-\cos^2 2x}{4} dx = \frac{1}{4} \int \sin^2 2x dx$$

For the $\sin^2 2x$ we again use identity

$$\sin^2 2x = \frac{1-\cos 4x}{2} \text{ and get}$$

$$\frac{1}{4} \int \sin^2 2x dx = \frac{1}{4} \int \frac{1-\cos 4x}{2} dx = \frac{1}{8} \int (1-\cos 4x) dx =$$

$$= \frac{1}{8}\left(x - \frac{\sin 4x}{4} \right) + C$$

Example: Evaluate $\int \cos^3 x dx$

Solution:

$$\int \cos^3 x dx = \int \cos^2 x \cdot \cos x dx = \int (1 - \sin^2 x) \cos x dx =$$

$$= \left| \begin{matrix} u = \sin x \\ du = \cos x dx \end{matrix} \right| = \int (1 - u^2) du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C$$

xample: Evaluate $\int \sin^4 x dx$

olution:

$$\int \sin^4 x dx = \int (\sin^2 x)^2 dx = \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx =$$

$$= \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) dx = \frac{1}{4} \int (1 - 2\cos 2x + \frac{1 + \cos 4x}{2}) dx =$$

$$= \frac{1}{4}\left(x - 2 \cdot \frac{\sin 2x}{2} + \frac{x}{2} + \frac{\sin 4x}{8} \right) + C = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

ntegrals of the form $\int \sin mx \cdot \sin nx dx$; $\int \sin mx \cdot \cos nx dx$;

$\int \cos mx \cdot \cos nx dx$ can be found using the product to sum formulas

$$\sin mx \cdot \sin nx = \frac{1}{2}[\cos(m - n)x - \cos(m + n)x]$$

$$\sin mx \cdot \cos nx = \frac{1}{2}[\sin(m - n)x + \sin(m + n)x]$$

$$\cos mx \cdot \cos nx = \frac{1}{2}[\cos(m - n)x + \cos(m + n)x]$$

ample: Evaluate $\int \sin 5x \cos 6x dx$

lution:

Since $\sin 5x \cos 6x = \frac{1}{2}[\sin(-x) + \sin(11x)] = \frac{1}{2}[\sin 11x - \sin x]$

We can write

$$\int \sin 5x \cos 6x dx = \frac{1}{2}\int (\sin 11x - \sin x)dx = -\frac{1}{2}\frac{\cos 11x}{11} + \frac{\cos x}{2} + C.$$

Example: Evaluate $\int \cos 2x \cos 8x dx$

Solution:

$$\int \cos 2x \cos 8x dx = \frac{1}{2}\int [\cos 6x + \cos 10x]dx =$$

$$= \frac{1}{2}\frac{\sin 6x}{6} + \frac{1}{2}\frac{\sin 10x}{10} + C = \frac{1}{4}\left(\frac{\sin 6x}{3} + \frac{\sin 10x}{5}\right) + C.$$

Example: Evaluate $\int \sin 7x \sin 3x dx$

Solution:

$$\int \sin 7x \sin 3x dx = \frac{1}{2}\int (\cos 4x - \cos 10x)dx =$$

$$= \frac{1}{8}\sin 4x - \frac{1}{20}\sin 10x + C.$$

Exercises.

In exercises 1-20 evaluate the integrals.

1. $\int \sin x \cdot \cos^4 x\, dx$
2. $\int \sin^3 x \cdot \cos^3 x\, dx$
3. $\int \sin^2 5x\, dx$
4. $\int \frac{\cos^5 x}{\sin^2 x} dx$
5. $\int \sin^2 x \cdot \cos^4 x\, dx$
6. $\int \sin x \cos^3 x\, dx$
7. $\int \cos^4 \frac{x}{4} dx$
8. $\int \sin^2 2t \cdot \cos^3 2t\, dt$
9. $\int \cos^4 x \cdot \sin^3 x\, dx$
10. $\int \frac{\sin x}{\cos^8 x} dx$
11. $\int_0^{\pi/3} \sin^4 3x \cdot \cos^3 3x\, dx$
12. $\int_0^{\pi/2} \sin^2 \frac{x}{2} \cdot \cos^2 \frac{x}{2} dx$
13. $\int \sin x \cdot \sin 3x\, dx$
14. $\int \sin 3x \cdot \cos 2x\, dx$

15. $\int \cos 4x \cdot \cos 2x \, dx$

16. $\int_0^{\pi/6} \sin 2x \cos 4x \, dx$

17. $\int \cos^5 x \cdot \sin x \, dx$

18. $\int_0^{2\pi} \sin^2 2x \cdot \cos^3 2x \, dx$

19. $\int \frac{\sin^4 x}{\cos^6 x} dx$

20. $\int \frac{\sin^3 x}{\cos^4 x} dx$

Answers.

1. $-\frac{\cos^5 x}{5} + C$; **2.** $-\frac{\cos^4 x}{4} + \frac{\cos^6 x}{6} + C$; **3.** $\frac{x}{2} - \frac{\sin 10x}{20} + C$;
4. $-\frac{1}{\sin x} - 2\sin x + \frac{\sin^3 x}{3} + C$; **5.** $\frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$;
6. $-\frac{1}{4}\cos^4 x + C$; **7.** $\frac{3}{8}x + \sin\frac{x}{2} + \frac{1}{8}\sin x + C$; **8.** $\frac{1}{6}\sin^3 2t -$
$-\frac{1}{10}\sin^5 2t + C$; **9.** $-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$; **10.** $\frac{1}{7\cos^7 x} + C$; **11.** 0;
12. $\frac{\pi}{16}$; **13.** $\frac{\sin 2x}{4} - \frac{\sin 4x}{8} + C$; **14.** $-\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$;
15. $\frac{1}{12}\sin 6x + \frac{1}{4}\sin 2x + C$; **16.** $\frac{1}{24}$; **17.** $-\frac{1}{6}\cos^6 x + C$; **18.** 0;
19. $\frac{\tan^5 x}{5} + C$; **20.** $\frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C$.

8.4.2. Trigonometric substitutions.

We will evaluate integrals of the form $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$ and $\sqrt{x^2 - a^2}$ by making substitutions involving trigonometric functions. One of the trigonometric identities $1 - \sin^2\theta = \cos^2\theta$, $\tan^2\theta + 1 = \sec^2\theta$, or $\sec^2\theta - 1 = \tan^2\theta$ will be used to convert a sum or difference of squares into a perfect square.

Case1: $\sqrt{a^2 - x^2}$; let $x = a\sin\theta$ $\quad (a > 0 \;, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$

Case2: $\sqrt{x^2 + a^2}$; let $x = a\tan\theta$ $\quad (a > 0 \;, -\frac{\pi}{2} < \theta < \frac{\pi}{2})$

Case3: $\sqrt{x^2 - a^2}$; let $x = a\sec\theta$ $\quad (a > 0 \;, 0 \le \theta \le \pi, \quad \theta \ne \frac{\pi}{2})$

For example, if we replace x in $\sqrt{a^2 - x^2}$ by $x = a\sin\theta$, we obtain

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = a\cos\theta$$

<u>**Example:**</u> Compute $\int\sqrt{1 + x^2}\,dx$

<u>**Solution:**</u>

To eliminate the radical we make the substitution

$x = \tan\theta$ (Fig.8.1) $\qquad dx = \sec^2\theta\, d\theta$

$\sqrt{1 + x^2} = \sqrt{1 + \tan^2\theta} = \sec\theta$.

Thus

$$\int\sqrt{1 + x^2}\,dx = \int\sec\theta\cdot\sec^2\theta\, d\theta$$

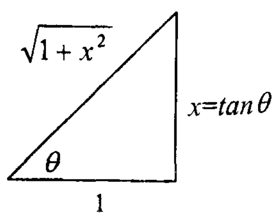


Fig.8.1

From the integral table,

$$\int\sec^3\theta\, d\theta = \frac{\sec\theta\cdot\tan\theta}{2} + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C\,.$$

To complete the solution we must express $\sec\theta$ and $\tan\theta$ in terms of x. Since $x = \tan\theta$ we obtain $\sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + x^2}$.

Thus

$$\int\sqrt{1+x^2}\,dx = \frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\ln\left(\sqrt{1+x^2}+x\right)+C.$$

Example: Compute $\int \frac{dx}{x^2\sqrt{4-x^2}}$

Solution:

Let $x = 2\sin\theta$ $\quad \frac{dx}{d\theta} = 2\cos\theta$ or $dx = 2\cos\theta\, d\theta$.

This yields

$$\int\frac{dx}{x^2\sqrt{4-x^2}} = \int\frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}} = \int\frac{2\cos\theta\, d\theta}{(2\sin\theta)^2(2\cos\theta)} =$$

$$= \frac{1}{4}\int\frac{d\theta}{\sin^2\theta} = \frac{1}{4}\int\csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C$$

From $x = 2\sin\theta$ we obtain $\sin\theta = \frac{x}{2}$ and $\cot\theta = \frac{\sqrt{4-x^2}}{x}$, (Draw triangle) so that

$$\int\frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\cot\theta + C = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

Example: Evaluate $\int\frac{\sqrt{x^2-25}}{x}dx$

Solution:

To eliminate the radical, we make substitution $x = 5\sec\theta$,

$\frac{dx}{d\theta} = 5\sec\theta\tan\theta$

or $dx = 5\sec\theta\tan\theta\, d\theta$.

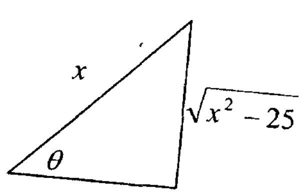


Fig.8.2

Thus

$$\int\frac{\sqrt{x^2-25}}{x}dx = \int\frac{\sqrt{25\sec^2\theta-25}}{5\sec\theta}\cdot 5\sec\theta\cdot\tan\theta\, d\theta =$$

$$\int\frac{5|\tan\theta|}{5\sec\theta}\cdot 5\sec\theta\cdot\tan\theta\, d\theta = 5\int\tan^2\theta\, d\theta = 5\int(\sec^2\theta-1)d\theta =$$

$$= 5\tan\theta - 5\theta + C$$

To express the solution in terms of x, from triangle (Fig.8.2) we obtain $\tan\theta = \frac{\sqrt{x^2-25}}{5}$, so that

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C$$

Example: Evaluate $\int \frac{dx}{x\sqrt{a^2+x^2}}$

Solution:

Let $x = a\tan t$, $dx = a\sec^2 t\,dt$. Then

$$\int \frac{dx}{x\sqrt{a^2+x^2}} = \int \frac{a\cdot\sec^2 t\,dt}{a\cdot\tan t\cdot\sqrt{a^2+a^2\tan^2 t}} = \frac{1}{a}\int \frac{\sec^2 t}{\tan t\cdot\sec t}\,dt =$$

$$= \frac{1}{a}\int \frac{\sec t}{\tan t}\,dt = \frac{1}{a}\int \frac{dt}{\sin t} = \frac{1}{a}\ln\left|\csc t - \cot t\right| + C$$

Since $\tan t = \frac{x}{a}$ we get $\cot t = \frac{a}{x}$ and $\csc t = \sqrt{1+\cot^2 t} = \frac{\sqrt{a^2+x^2}}{x}$.

Hence $$\int \frac{dx}{x\sqrt{a^2+x^2}} = \frac{1}{a}\ln\left|\frac{\sqrt{a^2+x^2}-a}{x}\right| + C$$

Exercises.

In exercises 1-10 find the integrals using trigonometric substitutions

1. $\int \frac{dx}{\sqrt{9+x^2}}$
2. $\int x^3\sqrt{1-x^2}\,dx$
3. $\int \sqrt{a^2-x^2}\,dx \quad (a>0)$
4. $\int \sqrt{a^2+x^2}\,dx$
5. $\int \frac{dx}{\sqrt{25x^2-16}}$
6. $\int \frac{\sqrt{x^2-9}}{x}\,dx$
7. $\int \frac{dx}{x^2\sqrt{9-4x^2}}$
8. $\int e^x\sqrt{1-e^{2x}}\,dx$

9. $\int_0^4 x^3\sqrt{16-x^2}\,dx$

10. $\int_{\sqrt{2}}^{2}\frac{dx}{x^2\sqrt{x^2-1}}$

Answers.

1. $\ln\left(\frac{\sqrt{x^2+9}+x}{3}\right)+C$; 2. $\frac{1}{5}(1-x^2)^{5/2}-\frac{1}{3}(1-x^2)^{3/2}+C$;
3. $\frac{1}{2}\left(a^2\sin^{-1}\frac{x}{a}+x\sqrt{a^2-x^2}\right)+C$; 4. $\frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}a^2\cdot$
$\ln(\sqrt{a^2+x^2}+x)+C$; 5. $\frac{1}{5}\ln\left|5x+\sqrt{25x^2-16}\right|+C$; 6. $\sqrt{x^2-9}-$
$3\sec^{-1}\frac{x}{3}+C$; 7. $-\frac{\sqrt{9-4x^2}}{9x}+C$; 8. $\frac{1}{2}\sin^{-1}(e^x)+\frac{1}{2}e^x\sqrt{1-e^{2x}}+C$;
$\frac{2048}{15}$; 10. $\frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$.

8.5. Integrals involving $ax^2+bx+c,\ a\neq 0$.

ume that the polynomial ax^2+bx+c is irreducible; that is , it not be factored into two first degree polynomials. This is the case n the discriminant (b^2-4ac) is negative. Integrals that involve a ratic expression can be evaluated by first completing the square, making an appropriate substitutions.

If integral is in the form $\int\frac{dx}{px^2+qx+r}$

completing the square reduces to one of the following integrals

(1) $\int\frac{du}{u^2+a^2}=\frac{1}{a}\tan^{-1}\frac{u}{a}+C$ or

(2) $\int\frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C$

If integral is in the form $\int \frac{mx+n}{px^2+qx+r}\,dx$

then completing the square converts given integral to (1), (2) or

(3) $$\int \frac{u\,du}{u^2 \pm a^2} = \frac{1}{2}\ln\left|u^2 \pm a^2\right| + C$$

If integral is in the form $\int \frac{dx}{\sqrt{px^2+qx+r}}$

then completing the square converts given integral to one of

(4) $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$ or

(5) $$\int \frac{du}{\sqrt{u^2+a}} = \ln\left|u+\sqrt{u^2+a}\right| + C$$

If integral is in the form $\int \sqrt{px^2+qx+r}\,dx$

then completing the square converts given integral to one of

(6) $$\int \sqrt{u^2+a}\,du = \frac{u}{2}\sqrt{u^2+a} + \frac{a}{2}\ln\left|u+\sqrt{u^2+a}\right| + C$$

(7) $$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C.$$

Example: Evaluate $\int \frac{dx}{x^2+4x+13}$

Solution:

In $x^2+4x+13$, $b^2-4ac = 4^2 - 4\cdot 1\cdot 13 = -36 < 0$.

Therefore $x^2+4x+13$ is irreducible. Completing the square yields

$$x^2+4x+13 = x^2+4x+2^2+13-2^2 = (x+2)^2+9.$$

Thus

$$\int \frac{dx}{x^2+4x+13} = \int \frac{dx}{(x+2)^2+9}.$$

Let $\left|\begin{matrix} 3u = x+2 \\ 3du = dx \end{matrix}\right|$ and

Hence $$\int\frac{dx}{(x+2)^2+9}=\int\frac{3du}{9u^2+9}=\frac{1}{3}\int\frac{du}{u^2+1}=\frac{1}{3}\tan^{-1}u+C$$

$$\int\frac{dx}{x^2+4x+13}=\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+C$$

Example: Evaluate $\int\frac{x+2}{x^2+2x+5}dx$

Solution:

It is the integral of the form $\int\frac{mx+n}{px^2+qx+r}dx$.

$$\int\frac{x+2}{x^2+2x+5}dx=\int\frac{x+2}{(x^2+2x+1)+4}dx=\int\frac{(x+1)+1}{(x+1)^2+4}dx=$$

$$=\left|\begin{matrix}u=x+1\\du=dx\end{matrix}\right|=\int\frac{u+1}{u^2+2^2}du=\int\frac{u}{u^2+2^2}du+\int\frac{du}{u^2+2^2}=$$

$$=\frac{1}{2}\ln(u^2+2^2)+\frac{1}{2}\tan^{-1}\frac{u}{2}+C=\frac{1}{2}\ln(x^2+2x+5)+\frac{1}{2}\tan^{-1}(\frac{x+1}{2})+C$$

Example: Evaluate $\int\frac{dx}{\sqrt{5-4x-x^2}}$

Solution:

Completing the square and using (4) yields

$$\int\frac{dx}{\sqrt{5-4x-x^2}}=\int\frac{dx}{\sqrt{-(x^2+4x+4-4-5)}}=$$

$$=\int\frac{dx}{\sqrt{9-(x+2)^2}}=\left|\begin{matrix}u=x+2\\du=dx\end{matrix}\right|=\int\frac{du}{\sqrt{3^2-u^2}}=$$

$$=\sin^{-1}\frac{u}{3}+C=\sin^{-1}\frac{x+2}{3}+C$$

ample: Evaluate $\int\frac{dx}{\sqrt{6-4x-2x^2}}$

Solution:

$$\int\frac{dx}{\sqrt{6-4x-2x^2}}=\int\frac{dx}{\sqrt{-2(x_2+2x-3)}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{-((x+1)^2-4)}}=$$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{4-(x+1)^2}}=\begin{vmatrix}u=x+1\\du=dx\end{vmatrix}=\frac{1}{\sqrt{2}}\int\frac{du}{\sqrt{2^2-u^2}}=$$

$$=\frac{1}{\sqrt{2}}\sin^{-1}\frac{u}{2}+C=\frac{1}{\sqrt{2}}\sin^{-1}\frac{x+1}{2}+C$$

Example: Evaluate $\int\frac{dx}{\sqrt{3x^2-6x+9}}$

Solution:

$$\int\frac{dx}{\sqrt{3x^2-6x+9}}=\int\frac{dx}{\sqrt{3(x^2-2x+3)}}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{(x-1)^2+2}}=$$

$=\begin{vmatrix}u=x-1\\du=dx\end{vmatrix}=\frac{1}{\sqrt{3}}\int\frac{du}{\sqrt{u^2+2}}$ Using (5) we obtain

$$\frac{1}{\sqrt{3}}\int\frac{du}{\sqrt{u^2+2}}=\frac{1}{\sqrt{3}}\ln\left|u+\sqrt{u^2+2}\right|+C=$$

$$=\frac{1}{\sqrt{3}}\ln\left|x-1+\sqrt{x^2-2x+3}\right|+C.$$

Example: Evaluate $\int\sqrt{x^2+8x+25}\,dx$

Solution:

$$\int\sqrt{x^2+8x+25}\,dx=\int\sqrt{(x^2+8x+16)-16+25}\,dx=$$

$=\int\sqrt{(x+4)^2+9}\,dx=\begin{vmatrix}u=x+4\\du=dx\end{vmatrix}=\int\sqrt{u^2+9}\,du=$ using (6) yields

$$\int\sqrt{u^2+9}\,du=\frac{x+4}{2}\sqrt{(x+4)^2+9}+\frac{9}{2}\ln\left|x+4+\sqrt{(x+4)^2+9}\right|+C=$$

$$=\frac{x+4}{2}\sqrt{x^2+8x+25}+\frac{9}{2}\ln\left|x+4+\sqrt{x^2+8x+25}\right|+C.$$

Example: Evaluate $\int\sqrt{8+2x-x^2}\,dx$

Solution:

Completing the square converts given integral to the integral of the form (7):

$$\int\sqrt{8+2x-x^2}\,dx = \int\sqrt{-(x^2-2x-8)}\,dx =$$

$$= \int\sqrt{-(x^2-2x+1-1-8)}\,dx = \int\sqrt{9-(x-1)^2}\,dx =$$

$$= \begin{vmatrix} u = x-1 \\ du = dx \end{vmatrix} = \int\sqrt{3^2-u^2}\,du = \frac{u}{2}\sqrt{9-u^2} + \frac{9}{2}\sin^{-1}\frac{u}{3} + C =$$

$$= \frac{x-1}{2}\sqrt{8+2x-x^2} + \frac{9}{2}\sin^{-1}\frac{x-1}{3} + C.$$

Exercises.

In exercises 1-20 evaluate the integrals.

1. $\int\frac{dx}{x^2+36}$
2. $\int\frac{dx}{2x^2+17}$
3. $\int\frac{dx}{3x^2-10}$
4. $\int\frac{dx}{4x^2+10x-24}$
5. $\int\frac{dx}{4x^2-5x+2}$
6. $\int\frac{dx}{x^2-x-1}$
7. $\int\frac{x-1}{x^2-x-1}dx$
8. $\int\frac{dx}{\sqrt{3x^2-6x+12}}$
9. $\int\frac{dx}{\sqrt{2+3x-2x^2}}$
10. $\int\frac{dx}{\sqrt{x^2+2x}}$
11. $\int\sqrt{x^2+4x+13}\,dx$
12. $\int\sqrt{5+4x-x^2}\,dx$
13. $\int\frac{dx}{x^2-4x+13}$
14. $\int\frac{dx}{\sqrt{8+2x-x^2}}$
15. $\int\frac{dx}{\sqrt{x^2-6x+10}}$
16. $\int\sqrt{3-2x-x^2}\,dx$

17. $\int \frac{2x+5}{x^2+2x+5}dx$

18. $\int \frac{dx}{x^2-2x+3}$

19. $\int \frac{2x}{x^2+2x+3}dx$

20. $\int \frac{3x+5}{3x^2+2x+1}dx$

Answers.

1. $\frac{1}{6}\tan^{-1}\frac{x}{6}+C$; **2.** $\frac{1}{\sqrt{34}}\tan^{-1}\sqrt{\frac{2}{17}}x+C$; **3.** $\frac{1}{2\sqrt{30}}\ln\left|\frac{\sqrt{3}x-\sqrt{10}}{\sqrt{3}x+\sqrt{10}}\right|+C$;

4. $\frac{1}{22}\ln\left|\frac{2x-3}{2x+8}\right|+C$; **5.** $\frac{2}{\sqrt{7}}\tan^{-1}\frac{8x-5}{\sqrt{7}}+C$; **6.** $\frac{1}{\sqrt{5}}\ln\left|\frac{2x-1-\sqrt{5}}{2x-1+\sqrt{5}}\right|+C$;

7. $\frac{1}{2}\ln\left|x^2-x-1\right|-\frac{1}{2\sqrt{5}}\ln\left|\frac{2x-1-\sqrt{5}}{2x-1+\sqrt{5}}\right|+C$; **8.** $\frac{1}{\sqrt{3}}\ln|x-1+$

$+\sqrt{x^2-2x+4}+C$; **9.** $\frac{1}{\sqrt{2}}\sin^{-1}\frac{4x-3}{5}+C$; **10.** $\ln\left|x+1+\sqrt{x^2+2x}\right|+$

$+C$; **11.** $\frac{x+2}{2}\sqrt{x^2+4x+13}+\frac{9}{2}\ln\left|x+2+\sqrt{x^2+4x+13}+C\right|$;

12. $\frac{x-2}{2}\sqrt{5+4x-x^2}+\frac{9}{2}\sin^{-1}\frac{x-2}{3}+C$; **13.** $\frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+C$;

14. $\sin^{-1}\left(\frac{x-1}{3}\right)+C$; **15.** $\ln\left|\sqrt{x^2-6x+10}+x-3\right|+C$;

16. $2\sin^{-1}\left(\frac{x+1}{2}\right)+\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+C$; **17.** $\ln\left(x^2+2x+5\right)+$

$+\frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$; **18.** $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-1}{\sqrt{2}}+C$; **19.** $\ln\left(x^2+2x+3\right)-$

$-\sqrt{2}\tan^{-1}\frac{x+1}{\sqrt{2}}+C$; **20.** $\frac{1}{2}\ln\left(3x^2+2x+1\right)+2\sqrt{2}\tan^{-1}\frac{3x+1}{\sqrt{2}}+C$.

8.6.1. Integration of rational functions by partial fractions.

A rational function $\frac{A(x)}{B(x)}$ in which the degree of $A(x)$ is less than the degree of $B(x)$ is called **proper** rational function. Otherwise the function is called **improper** rational function. An improper rational function can be expressed as the sum of a polynomial and a proper rational function.

First we will concentrate on the representation of a proper rational functions, and then improper rational functions.

To represent a proper rational function as a sum of rational functions we use following steps:

1. Write $B(x)$ as a product of first-degree polynomials and irreducible second degree polynomials.

2. If $(ax+b)$ appears exactly n times in the factorization of $B(x)$, form

(1) $$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\dots\dots\dots\dots+\frac{A_n}{(ax+b)^n}$$

where $A_1, A_2, \dots\dots\dots, A_n$ are constants to be determined.

3. If (ax^2+bx+c) appears exactly m times in the factorization of $B(x)$, then form the sum

(2) $$\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\dots\dots+\frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

where $A_1, A_2, \dots\dots\dots, A_m, B_1, B_2, \dots\dots\dots, B_m$ are constants to be determined.

4. Find all constants $A_i's$, $B_i's$ mentioned in steps 2 and 3 so that the sum of all expressions formed in steps 2 and 3 equals $\frac{A(x)}{B(x)}$.

Example: Carry out steps 2 and 3 for $\frac{A(x)}{B(x)}=\frac{x^2-3x+1}{(x+1)^3(x^2+2x+3)^2}$

Solution:

Degree of numerator is 2, degree of denominator is 7. Thus $\frac{A(x)}{B(x)}$ is a proper fraction. $x^2 + 2x + 3$ is irreducible, since $b^2 - 4ac = 4 - 12 = -8 < 0$. Therefore, we have

$$\frac{x^2 - 3x + 1}{(x+1)^3(x^2 + 2x + 3)^2} =$$

$$= \frac{A_1}{(x+1)} + \frac{A_2}{(x+1)^2} + \frac{A_3}{(x+1)^3} + \frac{A_4 x + A_5}{(x^2 + 2x + 3)} + \frac{A_6 x + A_7}{(x^2 + 2x + 3)^2}.$$

Remark: In example above the number of unknown constants is 7-equals the degree of $B(x)$. Always number of unknowns equals degree of polynomial in the denominator and this fact can be used as a check on your algebra.

Example: Express $\frac{2x-1}{x^2 - 3x + 2}$ in a partial fractions.

Solution:

The denominator $x^2 - 3x + 2$ is reducible, since $b^2 - 4ac = 9 - 8 = 1 > 0$. Its factorization is $x^2 - 3x + 2 = (x-1)(x-2)$.

Thus

$$\frac{2x-1}{x^2 - 3x + 2} = \frac{2x-1}{(x-1)(x-2)} = \frac{A_1}{x-1} + \frac{A_2}{x-2}$$

To find the constants A_1 and A_2, we multiply both sides of last equation by $(x-1)(x-2)$, obtaining

(3) $\quad 2x - 1 = A_1(x-2) + A_2(x-1)$

(3) is actually an identity that holds for all values of x. In particular it holds for $x = 1$ and $x = 2$.

Therefore:

if $x = 1$ then $2 \cdot 1 - 1 = A_1(1-2) + A_2(1-1)$

if $x = 2$ then $2 \cdot 2 - 1 = A_1(2-2) + A_2(2-1)$

These equations reduce to

$$\begin{cases} -A_1 + 0 \cdot A_2 = 1 \\ A_1 \cdot 0 + A_2 = 3 \end{cases}$$

from which we obtain that $A_1 = -1, A_2 = 3$. Then

$$\frac{2x-1}{x^2-3x+2} = \frac{-1}{x-1} + \frac{3}{x-2}$$

Example above can be solved by a completely different method, called **comparison of coefficients.** It is based on the fact that if two polynomials are equal, then their corresponding coefficients are equal.

$$2x - 1 = A_1(x-2) + A_2(x-1)$$

Let us multiply out and collect terms of like degree on the right side:

$$2x - 1 = A_1 x - 2A_1 + A_2 x - A_2 = (A_1 + A_2)x - (2A_1 + A_2)$$

and the equating the coefficients of the like powers of x on both sides to obtain

$$\begin{cases} A_1 + A_2 = 2 \\ 2A_1 + A_2 = 1 \end{cases}$$

The solution of this system of linear equations is $A_1 = -1, A_2 = 3$, which agrees with the results obtained above.

Example:

Find the partial-fraction representation of $\dfrac{3x^2 - 4x + 1}{(x-2)(x+1)(x-3)}$

Solution:

There are constants $A,\ B,\ C$ such that

$$(4) \qquad \frac{3x^2-4x+1}{(x-2)(x+1)(x-3)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3}$$

To find A, multiply both sides of (4) by $(x-2)$, obtaining

$$(5) \qquad \frac{3x^2-4x+1}{(x+1)(x-3)} = A + (x-2)\left(\frac{B}{x+1} + \frac{C}{x-3}\right)$$

Now replace x by 2, obtaining

$$\frac{3 \cdot 4 - 4 \cdot 2 + 1}{(2+1)(2-3)} = A + 0. \text{ Hence } A = -\frac{5}{3}$$

To obtain B, multiply both sides of (4) by $(x+1)$, obtaining

$$(6) \qquad \frac{3x^2-4x+1}{(x-2)(x-3)} = B+(x+1)\left(\frac{A}{x-2}+\frac{C}{x-3}\right)$$

replacing x by (-1) in (6) gives

$$\frac{3\cdot 1-4\cdot(-1)+1}{(-1-2)(-1-3)} = B+0. \qquad \text{Hence } B=\frac{2}{3}$$

To obtain C, multiply (4) by $(x-3)$, obtaining

$$(7) \qquad \frac{3x^2-4x+1}{(x-2)(x+1)} = C+(x-3)\left(\frac{A}{x-2}+\frac{B}{x+1}\right)$$

replacing x by 3 in (7) gives

$$\frac{3\cdot 9-4\cdot 3+1}{(3-2)(3+1)} = C+0. \text{ Hence } C=\frac{16}{4}=4.$$

Therefore

$$\frac{3x^2-4x+1}{(x-2)(x+1)(x-3)} = \frac{-5/3}{x-2}+\frac{2/3}{x+1}+\frac{4}{x-3}.$$

Example: Evaluate $\int \frac{x^2-x+2}{(x^2+1)(x+1)}dx$

Solution:

By the (1), factor $(x+1)$ introduces one term

$$\frac{A}{x+1} \qquad \text{and}$$

the quadratic factor (x^2+1) introduces

$$\frac{Bx+C}{x^2+1}.$$

Thus the partial fraction representation of integrand is

$$\frac{x^2-x+2}{(x^2+1)(x+1)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

Multiplying by $(x+1)(x^2+1)$ yields

$$x^2-x+2 = A(x^2+1)+(Bx+C)(x+1)$$

To determine A, B, and C we multiply out and collect like terms:

$$x^2-x+2 = (A+B)x^2+(B+C)x+(A+C)$$

Equating corresponding coefficients gives

$$\begin{cases} A+B=1 \\ B+C=-1 \\ A+C=2 \end{cases}$$ and we find that $A=2, B=-1, C=0$.

Thus integrand becomes

$$\frac{x^2-x+2}{(x^2+1)(x+1)}=\frac{2}{x+1}-\frac{x}{x^2+1} \quad \text{(Verify)}$$

and

$$\int\frac{x^2-x+2}{(x^2+1)(x+1)}dx=2\int\frac{dx}{x+1}-\int\frac{xdx}{x^2+1}=2\ln|x+1|-\frac{1}{2}\ln(x^2+1)+C$$

Example: Evaluate $\int\frac{3x+2}{x^3+2x^2}dx$

Solution:

The integrand can be rewritten as $\frac{3x+2}{x^3+2x^2}=\frac{3x+2}{x^2(x+2)}$

Although x^2 is a quadratic factor, it is not irreducible.

By the (1), x^2 introduces two terms of the form

$$\frac{A}{x}+\frac{B}{x^2}$$

and factor $(x+2)$ introduces one term

$$\frac{C}{x+2},$$

so partial representation is

$$\frac{3x+2}{x^2(x+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}$$

multiplying by $x^2(x+2)$ yields

$$3x+2=Ax(x+2)+B(x+2)+Cx^2$$

which after multiplying out and collecting like powers of x, becomes

$$3x+2=(A+C)x^2+(2A+B)x+2B$$

Equating corresponding coefficients gives

$$\begin{cases} A + C = 0 \\ 2A + B = 3 \\ 2B = 2 \end{cases}$$

which is true if $B = 1,\ A = 1,\ C = -1$ and

$$\frac{3x+2}{x^2(x+2)} = \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x+2}.$$

Thus

$$\int \frac{3x+2}{x^3+2x^2}\,dx = \int \frac{dx}{x} + \int \frac{dx}{x^2} - \int \frac{dx}{x+2} =$$

$$= \ln|x| - \frac{1}{x} - \ln|x+2| + C = \ln\left|\frac{x}{x+2}\right| - \frac{1}{x} + C$$

Example: Evaluate $\int \frac{x^2}{(x+2)(x-1)^2}\,dx$

Solution:

The factor $(x+2)$ introduces one term

$$\frac{A}{x+2}$$

The factor $(x-1)^2$ introduces two factors

$$\frac{B}{x-1} + \frac{C}{(x-1)^2}$$

So the partial fraction representation is

$$\frac{x^2}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiplying by $(x+2)(x-1)^2$ yields

$x^2 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$

if $x = 1$, then $3C = 1$ and $C = 1/3$

if $x = -2$, then $9A = 4$ and $A = 4/9$

if $x = 0$, then $0 = A(0-1)^2 + B(0+2)(0-1) + C(0+2)$

Substituting values of A and C into last equation we get $B = 5/9$.

Thus

$$\frac{x^2}{(x+2)(x-1)^2} = \frac{4/9}{x+2} + \frac{5/9}{x-1} + \frac{1/3}{(x-1)^2} \text{ and}$$

$$\int \frac{x^2}{(x+2)(x-1)^2}\,dx = \frac{4}{9}\int\frac{dx}{x+2} + \frac{5}{9}\int\frac{dx}{x-1} + \frac{1}{3}\int\frac{dx}{(x-1)^2} =$$

$$= \frac{4}{9}\ln|x+2| + \frac{5}{9}\ln|x-1| - \frac{1}{3}\frac{1}{(x-1)} + C =$$

$$= \frac{1}{9}\ln\left|(x+2)^4(x-1)^5\right| - \frac{1}{3(x-1)} + C.$$

8.6.2. Integrating improper rational functions .

If $\frac{A(x)}{B(x)}$ is improper, (degree of numerator greater than degree of denominator) first we use long division to divide $\frac{A(x)}{B(x)}$ and obtain

$$\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)}$$

where $Q(x)$ is a polynomial and $\frac{R(x)}{B(x)}$ is a proper (may even be zero).

Example: Evaluate $\int \frac{3x^3 + 2x^2 + x - 3}{x^2 - 1}\,dx$

Solution:

Since integrand is improper rational function, we first carry out he long division.

$$\begin{array}{r|l} 3x^3 + 2x^2 + x - 3 & x^2-1 \\ \hline -\;\; 3x^3 \quad\;\; - 3x & 3x+2 \\ \hline 2x^2+4x-3 & \\ -\;\; 2x^2-2 & \\ \hline 4x-1 & \end{array}$$

hus

$$\frac{x^3 + 2x^2 + x - 2}{x^2 - 1} = 3x + 2 + \frac{4x-1}{x^2-1}$$

nd

$$\int \frac{3x^3+2x^2+x-3}{x^2-1}dx = \int (3x+2)dx + \int \frac{4x-1}{x^2-1}dx$$

Second integral can be evaluated by a partial-fraction representation:

$$\int \frac{4x-1}{x^2-1}dx = \frac{5}{2}\int \frac{dx}{x+1} + \frac{3}{2}\int \frac{dx}{x-1}$$

In the end we obtain

$$\int \frac{3x^3+2x^2+x-3}{x^2-1}dx = 3\frac{x^2}{2} + 2x + \frac{5}{2}\ln|x+1| +$$

$$+\frac{3}{2}\ln|x-1| + C = \frac{3}{2}x^2 + 2x + \frac{1}{2}\ln\frac{|x+1|^5}{|x-1|^3} + C.$$

Exercises.

In exercises 1-4 indicate the form of the partial-fraction representation of the proper rational functions. (Do not find the numerical values of the coefficients).

1. $\frac{2x-1}{(x+1)(x-3)}$
2. $\frac{2-3x^4}{x^3(x^2+2)}$
3. $\frac{2x^2+1}{(x+1)^2(2x+2)(3x+3)}$
4. $\frac{2x+1}{(x+1)(x^2+x+1)^2}$

In exercises 5-11 express the rational function in terms of partial fractions.

5. $\frac{x-4}{x(x+2)}$
6. $\frac{5x^2-x-1}{x^2(x-1)}$
7. $\frac{2x^2+3}{x(x+1)(x+2)}$
8. $\frac{5x^2+9x+6}{(x+1)(x^2+2x+2)}$
9. $\frac{2x}{x^2-1}$
10. $\frac{3x^3+2x^2+3x+1}{x(x^2+1)}$
11. $\frac{x^3+2x^2+3x+1}{x(x+1)}$

In exercises 12-28 perform the integrations.

12. $\int \frac{x^2+2}{x^3+x^2-2x}\,dx$

13. $\int \frac{2x\,dx}{x^2+3x-4}$

14. $\int \frac{11x+17}{2x^2+7x-4}\,dx$

15. $\int \frac{dx}{(x-1)(x+2)(x-3)}$

16. $\int \frac{x^2+2}{x+2}\,dx$

17. $\int \frac{3x^2-10}{x^2-4x+4}\,dx$

18. $\int \frac{x^5+2x^2+1}{x^3-x}\,dx$

19. $\int \frac{x^2+x-16}{(x+1)(x-3)^2}\,dx$

20. $\int \frac{x^2}{(x+2)^2}\,dx$

21. $\int \frac{2x^2-1}{(4x-1)(x^2+1)}\,dx$

22. $\int \frac{x^3-3x^2+2x-3}{x^2+1}\,dx$

23. $\int \frac{\cos\theta}{\sin^2\theta+4\sin\theta-5}\,d\theta$

24. $\int \frac{dx}{1+e^x}$

25. $\int_0^1 \frac{x^2+2x+1}{(x^2+1)^2}\,dx$

26. $\int_0^{2\sqrt{2}} \frac{x^3\,dx}{x^2+1}$

27. $\int \frac{dx}{x^2(1+x^2)^2}$

28. $\int \frac{x^2\,dx}{(x+2)^2(x+1)}$

Answers.

1. $\frac{A}{x+1}+\frac{B}{x-3}$; **2.** $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{Dx+E}{x^2+2}$; **3.** $\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{D}{(x+1)^4}$; **4.** $\frac{A}{x+1}+\frac{Bx+C}{x^2+x+1}+\frac{Dx+E}{(x^2+x+1)^2}$; $-\frac{2}{x}+\frac{3}{x+2}$; **6.** $\frac{2}{x}+\frac{1}{x^2}+\frac{3}{x-1}$; **7.** $\frac{3}{2x}-\frac{5}{x+1}+\frac{11}{2(x+2)}$;

8. $\frac{2}{x+1}+\frac{3x+2}{x^2+2x+2}$; **9.** $\frac{1}{x-1}+\frac{1}{x+1}$; **10.** $3+\frac{1}{x}+\frac{x}{x^2+1}$;

11. $x-1+\frac{1}{x}+\frac{5}{x+1}$; **12.** $\ln\left|\frac{(x-1)(x+2)}{x}\right|+C$;

13. $\ln\sqrt[5]{(x-1)^2(x+4)^8}+C$; **14.** $\frac{5}{2}\ln|2x-1|+3\ln|x+4|+C$;

15. $-\frac{1}{6}\ln|x-1|+\frac{1}{15}\ln|x+2|+\frac{1}{10}\ln|x-3|+C$; **16.** $\frac{1}{2}x^2-2x+$

$+6\ln|x+2|+C$; **17.** $3x+12\ln|x-2|-\frac{2}{x-2}+C$; **18.** $\frac{1}{3}x^3+x+$

$+\ln\left|\frac{(x+1)(x-1)^2}{x}\right|+C$; **19.** $\ln\frac{(x-3)^2}{|x+1|}+\frac{1}{x-3}+C$; **20.** $\ln|x+2|+$

$+\frac{4}{x+2}-\frac{2}{(x+2)^2}+C$; **21.** $-\frac{7}{34}\ln|4x-1|+\frac{6}{17}\ln(x^2+1)+$

$+\frac{3}{17}\tan^{-1}x+C$; **22.** $\frac{1}{2}x^2-3x+\frac{1}{2}\ln(x^2+1)+C$;

23. $\frac{1}{6}\ln\left|\frac{\sin\theta-1}{\sin\theta+5}+C\right|$; **24.** $\ln\frac{e^x}{1+e^x}+C$; **25.** $\frac{\pi+2}{4}$; **26.** $4-\ln 3$;

27. $-\frac{1}{x}-\frac{1}{2}\cdot\frac{x}{1+x^2}-\frac{3}{2}\tan^{-1}x+C$; **28.** $\ln|x+1|+\frac{4}{x+2}+C$.

8.7. Special techniques of integration.

There are some integrals that do not fit into any of categories previously studied.

Example: Find $\int\sec x\,dx$

Solution:

$$\int\sec x\,dx=\int\sec x\left(\frac{\sec x+\tan x}{\sec x+\tan x}\right)dx=$$

$$=\int\frac{\sec^2 x+\sec x\cdot\tan x}{\sec x+\tan x}dx=\left|\begin{matrix}u=\sec x+\tan x\\ du=(\sec^2 x+\sec x\cdot\tan x)dx\end{matrix}\right|=$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

Integrals containing rational functions of $\sin x$ and $\cos x$, such as $\frac{\sin x + \cos^2 x}{\sin x + 5\cos x}, \frac{\cos x}{1 + \sin x - \cos^2 x}$ etc. can be evaluated by substitution $u = \tan\frac{x}{2}$.

$$u = \tan\frac{x}{2} \qquad -\pi < x < \pi \qquad x = 2\tan^{-1} u$$

$$\sin x = \frac{2u}{1+u^2} \qquad \cos x = \frac{1-u^2}{1+u^2} \qquad dx = \frac{2}{1+u^2}\,du.$$

Example: Evaluate $\int \frac{dx}{1+\sin x}$

Solution:

Let $u = \tan\frac{x}{2}$. Then $dx = \frac{2}{1+u^2}du$ and $\sin x = \frac{2u}{1+u^2}$

Hence

$$\int \frac{dx}{1+\sin x} = \int \frac{1}{1+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2}du = \int \frac{2}{1+2u+u^2}du =$$

$$= \int \frac{2}{(1+u)^2}du = -\frac{2}{1+u} + C = -\frac{2}{1+\tan\frac{x}{2}} + C.$$

Example: Evaluate $\int \frac{dx}{\sin x + \tan x}$

Solution:

$$u = \tan\frac{x}{2}, \qquad \sin x = \frac{2u}{1+u^2},$$

$$\tan x = \frac{2u}{1-u^2}, \qquad dx = \frac{2}{1+u^2}du$$

Then

$$\int \frac{dx}{\sin x + \tan x} = \int \frac{1}{\frac{2u}{1+u^2} + \frac{2u}{1-u^2}} \cdot \frac{2}{1+u^2} du =$$

$$= \int \frac{(1-u^2)(1+u^2)}{4u} \cdot \frac{2}{1+u^2} du = \frac{1}{2} \int \left(\frac{1}{u} + u \right) du =$$

$$= \frac{1}{2} \ln|u| + \frac{u^2}{2} + C = \frac{1}{2} \ln\left|\tan \frac{x}{2}\right| + \frac{1}{2} \tan^2 \frac{x}{2} + C$$

<u>Example:</u> Evaluate $\int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x(1+\sqrt[3]{x})} dx$

<u>Solution:</u>

The integrand involves $x^{\frac{2}{3}}$, $x^{\frac{1}{6}}$ and $x^{\frac{1}{3}}$. The least common multiple of denominator is 6, so we make substitution $x = u^6$.
Then

$dx = 6u^5 du$, $\sqrt[3]{x} = u^2$, $\sqrt[6]{x} = u$, $\sqrt[3]{x^2} = u^4$

We obtain

$$\int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x(1+\sqrt[3]{x})} dx = \int \frac{u^6 + u^4 + u}{u^6(1+u^2)} \cdot 6u^5 du =$$

$$= 6 \int \frac{u^5 + u^3 + 1}{1+u^2} du = 6 \int u^3 du + 6 \int \frac{du}{1+u^2} =$$

$$= \frac{3}{2} u^4 + 6 \tan^{-1} u + C = \frac{3}{2} \sqrt[3]{x^2} + 6 \tan^{-1} \sqrt[6]{x} + C$$

<u>Example:</u> Evaluate $\int \frac{dx}{2+\sqrt{x}}$

<u>Solution:</u>

The integrand contains $\sqrt{x} = x^{\frac{1}{2}}$, so we make substitution $x = u^2$ and $dx = 2udu$. This yields

$$\int \frac{dx}{2+\sqrt{x}} = \int \frac{2udu}{2+2u} = \int (1 - \frac{1}{1+u}) du = u - \ln|1+u| + C =$$

$$= \sqrt{x} - \ln\left|1 + \sqrt{x} + C\right|$$

Example: Evaluate $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx$

Solution:

First of all let us rewrite the given integral as

$$\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx = \int \left(1 + x^{\frac{1}{3}}\right)^{\frac{1}{2}} \cdot x^{-\frac{2}{3}} dx$$

and then introduce $(1 + x^{\frac{1}{3}}) = t^2$. It yields

$x^{\frac{1}{3}} = t^2 - 1$, $\quad \frac{1}{3} x^{-\frac{2}{3}} dx = 2t dt$, $\quad x^{-\frac{2}{3}} dx = 6t dt$

After substituting we obtain

$$\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx = \int t \cdot 6t\,dt = 6\int t^2 dt = 6 \cdot \frac{t^3}{3} + C = 2\left(1 + x^{\frac{1}{3}}\right)^{\frac{3}{2}} + C.$$

Example: Evaluate $\int \sqrt[3]{\frac{2-x}{2+x}} \cdot \frac{1}{(2-x)^2}\,dx$

Solution:

Let $\frac{2-x}{2+x} = t^3$. Solving for x yields

$$x = \frac{2-2t^3}{1+t^3}, \quad 2 - x = 2 - \frac{2-2t^3}{1+t^3} = \frac{4t^3}{1+t^3};$$

$$\frac{1}{(2-x)^2} = \frac{(1+t^3)^2}{16t^6} \quad \text{and} \quad dx = -\frac{12t^2}{(1+t^3)^2}\,dt$$

Thus

$$\int \sqrt[3]{\frac{2-x}{2+x}} \cdot \frac{1}{(2-x)^2}\,dx = \int t \cdot \frac{(1+t^3)^2}{16t^6} \cdot \frac{(-12t^2)}{(1+t^3)^2}\,dt =$$

$$= -\frac{3}{4}\int \frac{dt}{t^3} = \frac{3}{8t^2} + C = \frac{3}{8}\sqrt[3]{\left(\frac{2+x}{2-x}\right)^2} + C.$$

Exercises.

In exercises 1-17 evaluate the integrals.

1. $\int \frac{dx}{1+\sin x+\cos x}$

2. $\int_{\pi/2}^{\pi} \frac{dx}{1-\cos x}$

3. $\int \frac{\cos x}{2-\cos x}\,dx$

4. $\int \frac{dx}{4\sin x-3\cos x}$

5. $\int x\sqrt{x+5}\,dx$

6. $\int_{4}^{8} \frac{\sqrt{x-4}}{x}\,dx$

7. $\int_{0}^{4} \frac{1}{3+\sqrt{x}}\,dx$

8. $\int x^5\sqrt{x^3+1}\,dx$

9. $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

10. $\int \frac{dv}{v(1-v^{1/4})}$

11. $\int \frac{dt}{t^{\frac{1}{2}}-t^{\frac{1}{3}}}$

12. $\int \sin\sqrt{x}\,dx$

13. $\int \frac{dx}{\sqrt{e^x+1}}$

14. $\int \frac{x^3\,dx}{\sqrt{1+x^2}}$

15. $\int \frac{dx}{\sqrt[3]{x^2}\,(1+\sqrt[3]{x^2})}$

16. $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1-\sqrt[3]{x}\right)}\,dx$

Answers.

1. $\ln\left|\tan\left(\frac{x}{2}\right)+1\right|+C$; **2.** 1; **3.** $\frac{4}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{x}{2}\right)-x+C$;

4. $\frac{1}{5}\ln\left|\frac{\tan\frac{x}{2}-\frac{1}{3}}{\tan\frac{x}{2}+3}\right|+C$; **5.** $\frac{2}{5}\sqrt{(x+5)^5}-\frac{10}{3}\sqrt{(x+5)^3}+C$; **6.** $4-\pi$;

7. $4-6\ln\frac{5}{3}$; **8.** $\frac{2}{15}(x^3+1)^{5/2}-\frac{2}{9}(x^3+1)^{3/2}+C$; **9.** $2x^{1/2}-3x^{1/3}+$

$+6x^{1/6}-6\ln(x^{1/6}+1)+C$; **10.** $4\ln\frac{v^{1/4}}{\left|1-v^{1/4}\right|}+C$; **11.** $2t^{1/2}+3t^{1/3}+$

$+6t^{1/6}+6\ln\left|t^{1/6}-1\right|+C$; **12.** $-2\sqrt{x}\cos\sqrt{x}+2\sin\sqrt{x}+C$;

13. $\ln\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}+C$; **14.** $\frac{1}{3}(1+x^2)^{3/2}-(1+x^2)^{1/2}+C$;

15. $3\tan^{-1}\sqrt[3]{x}+C$; **16.** $\frac{3}{2}\sqrt[3]{x^2}+6\tan^{-1}\sqrt[6]{x}+C$.

Chapter 9.
Improper integrals.

9.1. Definition of improper integrals.

In the definition of $\int_a^b f(x)dx$ it is assumed that

1) the interval $[a,b]$ is finite
2) integrand $f(x)$ is defined and continuous on $[a, b]$

If at least one of two conditions above fails then integral

$\int_a^b f(x)dx$ is called **improper** integral.

Let us consider two cases

Case 1: If f is continuous on the interval $[a,\infty)$, then we define

improper integral $\int_a^{+\infty} f(x)dx$ as

$$(1) \qquad \int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx$$

If this limit exists, the improper integral is said to **converge**, and the value of the limit is the value assigned to the integral. If the limit does not exist, then the improper integral is said to **diverge**.

<u>**Example:**</u> Evaluate $\int_1^{+\infty}\frac{dx}{x^3}$

<u>**Solution:**</u>

Replacing the infinite upper limit with a finite upper limit b yields

$$\int_1^{+\infty}\frac{dx}{x^3} = \lim_{b\to+\infty}\int_1^b\frac{dx}{x^3} = \lim_{b\to+\infty}\left[-\frac{1}{2x^2}\right]_1^b = = \lim_{b\to+\infty}\left[-\frac{1}{2b^2}+\frac{1}{2}\right] = \frac{1}{2}$$

so the given integral converges to 1/2.

If f is continuous on the interval $(-\infty, b]$, then we define improper integral $\int_{-\infty}^{b} f(x)dx$ as

$$(2) \qquad \int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)dx$$

Again, the improper integral is said to converge if limit in (2) exists. Otherwise $\int_{-\infty}^{b} f(x)dx$ diverges.

Example: Evaluate $\int_{-\infty}^{1} e^{x} dx$

Solution:

Replacing the infinite lower limit with a finite lower limit a yields

$$\int_{-\infty}^{1} e^{x} dx = \lim_{a \to -\infty} \int_{a}^{1} e^{x} dx = \lim_{a \to -\infty} e^{x}\Big|_{a}^{1} = \lim_{a \to -\infty} [e^{1} - e^{a}] = e$$

If f is continuous on the interval $(-\infty, +\infty)$, then we define improper integral $\int_{-\infty}^{+\infty} f(x)dx$ as

$$(3) \qquad \int_{-\infty}^{+\infty} f(x)dx = \lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_{a}^{b} f(x)dx$$

If this limit exists, the improper integral is said to converge, if the limit does not exist, then the improper integral is said to diverge.

Example: Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$

Solution:

Replacing infinite limits with a finite limits yields

$$\int_{-\infty}^{+\infty}\frac{dx}{1+x^2} = \lim_{\substack{a\to-\infty \\ b\to+\infty}}\int_a^b\frac{dx}{1+x^2} = \lim_{\substack{a\to-\infty \\ b\to+\infty}}\tan^{-1}x\Big|_a^b = \frac{\pi}{2}+\frac{\pi}{2} = \pi$$

So $\int_{-\infty}^{+\infty}\frac{dx}{1+x^2}$ converges to π.

Remark: We may define $\int_{-\infty}^{+\infty}f(x)dx$ as

$$\int_{-\infty}^{+\infty}f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx + \lim_{b\to+\infty}\int_c^b f(x)dx$$

Case 2:

If f is continuous on $[a,b)$ but fails to have limit as x approaches b from the left, then we define improper integral $\int_a^b f(x)dx$ as

$$(4) \qquad \int_a^b f(x)dx = \lim_{l\to b^-}\int_a^l f(x)dx$$

Example: Evaluate $\int_0^2\frac{dx}{\sqrt{2-x}}$

Solution:

The integral is improper because the integrand approaches $+\infty$ as x approaches 2 from the left.

From (4)

$$\int_0^2\frac{dx}{\sqrt{2-x}} = \lim_{l\to2^-}\int_0^l\frac{dx}{\sqrt{2-x}} = \lim_{l\to2^-}[-2\sqrt{2-x}]\Big|_0^l =$$

$$= \lim_{l\to2^-}[-2\sqrt{2-l}+2\sqrt{2-0}] = 2\sqrt{2}$$

If f is continuous on $(a,b]$ but fails to have limit as x approaches a from the right, then we define improper integral $\int_a^b f(x)dx$ as

(5) $$\int_a^b f(x)dx = \lim_{l\to a^+}\int_l^b f(x)dx$$

Example: Evaluate $\int_2^3 \frac{dx}{2-x}$

Solution:

The integral is improper because the integrand approaches $-\infty$ as $x \to 2^+$. From (5) we obtain

$$\int_2^3 \frac{dx}{2-x} = \lim_{l\to 2^+}\int_l^3 \frac{dx}{2-x} = -\lim_{l\to 2^+}\ln|2-x|\Big|_l^3 =$$

$$= -\lim_{l\to 2^+}\left[\ln 1 - \ln|2-l|\right] = -\lim_{l\to 2^{+\infty}}\ln|2-l| = -\infty$$

so $\int_2^3 \frac{dx}{2-x}$ diverges.

Example: Evaluate $\int_0^{+\infty} xe^{-x^2}dx$

Solution:

$$\int_0^{+\infty} xe^{-x^2}dx = \lim_{b\to+\infty}\int_0^b xe^{-x^2}dx = \lim_{b\to+\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^b =$$

$$= \lim_{b\to+\infty}\left(\frac{1}{2}-\frac{1}{2}e^{-b^2}\right) = \frac{1}{2}$$

Example: Evaluate $\int_a^{+\infty}\frac{dx}{x^p}$

Solution:

$$\int_a^{+\infty}\frac{dx}{x^p} = \lim_{b\to+\infty}\int_a^b\frac{dx}{x^p} = \lim_{b\to+\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_a^b =$$

$$= \frac{1}{1-p}\lim_{b\to+\infty} b^{-p+1} - \frac{1}{1-p}\cdot a^{1-p}$$

if $p > 1$, then $\lim\limits_{b\to+\infty} b^{-p+1} = 0$, and $\int\limits_a^{+\infty}\frac{dx}{x^p}$ converges to $-\frac{1}{1-p}\cdot a^{1-p}$

if $p \le 1$, then $\lim\limits_{b\to+\infty} b^{-p+1} = +\infty$, and $\int\limits_a^{+\infty}\frac{dx}{x^p}$ diverges.

Example: Evaluate $\int\limits_1^{+\infty}\frac{dx}{e^x}$

Solution:

$$\int\limits_1^{+\infty}\frac{dx}{e^x} = \lim_{a\to+\infty}\int\limits_1^{a}\frac{dx}{e^x} = \lim_{a\to+\infty} -\left(e^{-x}\right)\Big|_1^a = \lim_{a\to+\infty}\left(-e^{-a}+e^{-1}\right) = \frac{1}{e},$$

so

$$\int\limits_1^{+\infty}\frac{dx}{e^x} \text{ converges to } \frac{1}{e}.$$

Warning: It is sometimes tempting to apply

$$\int\limits_a^b f(x)dx = F(x)\Big\|_a^b = F(b) - F(a)$$

directly to an improper integral without taking the appropriate limits. To illustrate what can go wrong let us suppose that we ignore the fact that integral $\int\limits_1^3\frac{dx}{(x-2)^2}$ is improper and write

$$\int\limits_1^3\frac{dx}{(x-2)^2} = -\frac{1}{x-2}\Big|_1^3 = -1-\frac{1}{3} = -\frac{4}{3}$$

Since (x^2-1) is always positive and $\int\limits_1^3\frac{dx}{(x-2)^2}$ can not be negative. To evaluate the given integral correctly we should write

$$\int\limits_1^3\frac{dx}{(x-2)^2} = \int\limits_1^2\frac{dx}{(x-2)^2} + \int\limits_2^3\frac{dx}{(x-2)^2}.$$

Then

$$\int_1^2 \frac{dx}{(x-2)^2} = \lim_{a\to 2^-} \int_1^a \frac{dx}{(x-2)^2} = \lim_{a\to 2^-}\left[-\frac{1}{x-2}\right]_1^a =$$

$$= \lim_{a\to 2^-}\left[-\frac{1}{a-2} - \frac{1}{2}\right] = +\infty,$$

so $\int_1^3 \frac{dx}{(x-2)^2}$ diverges.

9.2. Tests for convergence and divergence.

Theorem1: Let $f(x)$ and $g(x)$ be integrable functions over $[a,b]$ and $0 \le g(x) \le f(x)$ for all $x > a$, then

a) $\int_a^{+\infty} g(x)\,dx$ converges if $\int_a^{+\infty} f(x)\,dx$ converges

b) $\int_a^{+\infty} f(x)\,dx$ diverges if $\int_a^{+\infty} g(x)\,dx$ diverges.

Example:

Determine whether the integral $\int_1^{+\infty} \frac{dx}{1+x^{10}}$ diverges or converges.

Solution: $\int_1^{+\infty} \frac{dx}{1+x^{10}}$ converges

because $\frac{1}{1+x^{10}} < \frac{1}{x^{10}}$ for all x from $[1,+\infty)$ and $\int_1^{+\infty} \frac{dx}{x^{10}}$ converges

Theorem2: Let $f(x)$ and $g(x)$ be positive and integrable functions over $[a,b]$ and that

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty$$

Then $\int_{a}^{+\infty} f(x)\,dx$ and $\int_{a}^{+\infty} g(x)\,dx$ both converge or both diverge.

Example:

Determine whether the integral $\int_{1}^{+\infty} \frac{dx}{1+e^x}$ converges or diverges.

Solution:

With $f(x)=\frac{1}{1+e^x}$, $g(x)=\frac{1}{e^x}$ we have

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{e^x}{1+e^x} = 1 \text{ and } 1 \in (0,+\infty).$$

Therefore $\int_{1}^{+\infty} \frac{dx}{1+e^x}$ converges, because $\int_{1}^{+\infty} \frac{dx}{e^x}$ converges.

Example:

Determine whether the improper integral $\int_{1}^{+\infty} \frac{dx}{\sqrt{1+x^4}}$ is convergent or divergent.

Solution:

Let us solve the problem by both of two tests:

1) $$\frac{1}{\sqrt{1+x^4}} = \frac{1}{\sqrt{x^4\left(\frac{1}{x^4}+1\right)}} = \frac{1}{x^2\sqrt{\frac{1}{x^4}+1}},$$

and $$\frac{1}{x^2\sqrt{\frac{1}{x^4}+1}} < \frac{1}{x^2}$$

Since $\int_{1}^{+\infty} \frac{dx}{x^2}$ converges, so does $\int_{1}^{+\infty} \frac{dx}{\sqrt{1+x^4}}$.

2) Let $f(x)=\frac{1}{\sqrt{1+x^4}}$ and $g(x)=\frac{1}{\sqrt{x^4}}$

$\lim\limits_{x\to+\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to+\infty} \sqrt{\frac{x^4}{1+x^4}} = 1$. Limit has a positive finite value and we make conclusion that :

$\int\limits_1^{+\infty} \frac{dx}{\sqrt{1+x^4}}$ converges, because $\int\limits_1^{+\infty} \frac{dx}{\sqrt{x^4}} = \int\limits_1^{+\infty} \frac{dx}{x^2}$ converges.

Exercises.

In exercises 1-10 evaluate the integrals.

1. $\int\limits_0^{+\infty} e^{-x} dx$

2. $\int\limits_1^{+\infty} \frac{dx}{x^3}$

3. $\int\limits_e^{+\infty} \frac{1}{x \ln^3 x} dx$

4. $\int\limits_a^{+\infty} \frac{xdx}{(x^2+1)^2}$

5. $\int\limits_{-\infty}^{0} \frac{dx}{(2x-1)^3}$

6. $\int\limits_{-1}^{1} \frac{dx}{x^2}$

7. $\int\limits_{2/\pi}^{+\infty} \frac{1}{x^2} \cdot \sin\frac{1}{x} dx$

8. $\int\limits_0^{+\infty} e^{2x} dx$

9. $\int\limits_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$

10. $\int\limits_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-x}}} dx$

In exercises 11-25 determine whether the improper integrals diverge or converge.

11. $\int\limits_1^{+\infty} \frac{dx}{x^4+1}$

12. $\int\limits_1^{+\infty} \frac{dx}{x^{5/2}+1}$

13. $\int\limits_0^{+\infty} \frac{dx}{(x+2)^3}$

14. $\int\limits_1^{+\infty} x^{-0.999} dx$

15. $\int\limits_1^{+\infty} \frac{\ln x}{x} dx$

16. $\int\limits_0^{+\infty} \frac{xdx}{x^4+1}$

17. $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

18. $\int_2^{+\infty} \frac{2}{x^2-1}dx$

19. $\int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}}dx$

20. $\int_0^{+\infty} \frac{dx}{x^2}$

21. $\int_0^3 \frac{dx}{x-2}$

22. $\int_0^{+\infty} \frac{dx}{\sqrt{x^8+1}}$

23. $\int_1^{+\infty} \left(1-\cos\frac{x}{2}\right)dx$

24. $\int_1^{+\infty} \frac{1+x^2}{x^3}dx$

25. $\int_0^{+\infty} e^{-2x}\sin 3x dx$

Answers.

1. 1; **2.** $\frac{1}{2}$; **3.** $\frac{1}{2}$; **4.** $\frac{1}{2(a^2+1)}$; **5.** $-\frac{1}{4}$; **6.** $+\infty$; **7.** 1; **8.** $+\infty$; **9.** 2; **10.** 2; **11.** converges; **12.** converges; **13.** converges; **14.** diverges; **15.** diverges; **16.** converges; **17.** converges; **18.** converges; **19.** converges; **20.** diverges; **21.** diverges; **22.** converges; **23.** diverges; **24.** diverges; **25.** converges.

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Задачи и упражнения. Часть 1

Humbat Nabi oglu Aliyev

CALCULUS AND ANALYTIC GEOMETRY. SOLVED PROBLEMS AND EXERCISES. PART 1

Редактор Гаджиев Ф.А.

Подписано в печать 09.06.2005

Формат 60 х 84 1/16. Бумага тип. Ризограф.
Усл.печ.15,1 л. Уч.-изд. 15,3 л. Тираж 500.

Заказ № 501

Цена договорная

Издание Казахской головной архитектурно-строительной академии
Издательский дом КазГАСА «Строительство и архитектура»
480043, г.Алматы, ул. К.Рыскулбекова, 28