Remarks on weak o-minimality

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Some necessary and su±cient conditions for weak o-minimality are indicated.

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In this paper we consider the question of characterization of weak o-minimality of a model in terms of types realized in it. In particular, we present a simple proof of the main result of [1], which gives some necessary and su±cient conditions for weak o-minimality, and indicate two more conditions that are equivalent to weak o-minimality. Besides, we give a simpler and more conceptual proof of a theorem from [2], which characterizes o-minimality, and prove that every weakly o-minimal archimedean ordered ring is a real closed .eld.

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1 Preliminaries

In the following *M* is a model, which is an expansion of a linear order. A subset *A* of *M* is called *convex* if for all *a* < *b* < *c* in *M* one has *b 2 M* whenever *a; c 2 M*.

Model *M* is called *weakly o-minimal* if every *M*-de.nable subset of *M* is the union of .nitely many convex sets.

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Model *M* is called *o-minimal* if every *M*-de.nable subset of *M* is the union of .nitely many intervals (that is, convex sets with boundary points in *M*).

Let *L(M)* be the set of formulae of the language of *M* with parameters in *M*. For *A µ M*, let *S(A)* be the set of complete 1-types over *A*. For a formula *'(x) 2 L(M)*, we denote by *'(M)* the set of elements in *M* satisfying *'*. Similarly, for a type *p 2 S(A)*, we denote by *p(M)* the set of elements in *M* realizing *p*.

We will use the following obvious fact.

Lemma 1.1. *If model M is weakly o-minimal and A µ M, then every convex component of any A-de.nable subset of M is A-de.nable.*

A *cut* in *M* is a maximal consistent set of formulae of the form *x · m* or

m · *x* for *m 2 M*. If *C* is a cut in *M*, let *C*. = *fm 2 M* : (*m* · *x*) *2 Cg* and *C*+ = *M n C*..
2 Characterization of weak o-minimality
The following theorem characterizes weak o-minimality of a model in terms of types realized in it.
Theorem 2.1. *The following conditions are equivalent:*
(1) model M is weakly o-minimal;
(2) for every subset A µ M, type p 2 S(A), and model M0 A M, the set p(M0) is convex;
(3) for every type p 2 S(M0) and model M0 A M, the set p(M0) is convex;
(4) for every cut C in M, formula '(x) 2 L(M), and model M0 A M, the set C(M0) \ '(M0) is convex;
(5) for every (not necessarily complete) type p over M, which contains a cut in M, and every model M0 A M, the set p(M0) is convex.
Notice that the equivalence of conditions (1), (2), and (3) is the main result of [1]. The proof given in [1] is rather lengthy and complicated. Our proof is much simpler.
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Proof of Theorem 2.1. (1))(2). If *M* is weakly o-minimal, then, by Lemma 1.1, the type *p* contains exactly one convex component of each its formula. Therefore if *M0 A M*, the set *p*(*M0*) is convex as the intersection of convex sets.
(2))(3). This is obvious.
(3))(1). Suppose that (1) is false. Then for some formula '(*x*) *2 L*(*M*) the set '(*M*) has in.nitely many convex components. Choose one element in each component and denote the obtained set by *A*. Let us show that (3) is false. By Malcev's Compactness Theorem, it su±ces to prove that any .nite subset *Q0* of *Q* is consistent where
Q = Th(*M; a*)*a2M [fx* < *y* < *z;* '(*x*)*;:*'(*y*)*;* '(*z*)*g [S;*
S = *fA*(*x*) *$ A*(*z*) : *A*(*v*) *2 L*(*M*)*g:*
Let *Q0* \ *S* = *fAi*(*x*) *$ Ai*(*z*) : *i* < *ng*. Since the set . = *fAi*(*x*) : *i* < *ng* is .nite, there are only .nitely many complete .-types, that is, maximal consistent sets consisting of formulae from . and negations of formulae from .. Then in the in.nite set *A* we can .nd elements *a* < *b* that realize the same complete .-type. Since *a* and *b* belong to di®erent convex components of '(*M*), there is an element *c 2 :*'(*M*) between them. Then the triple (*a; c; b*) realizes *Q0*.
(1))(4). Suppose that (4) is false. Then there are a cut *C* in *M* and a formula '(*x*) *2 L*(*M*) such that the set
[f'[*d; e*] : *d 2 C.; e 2 C*+*g*
is consistent where
'[*d; e*] = *fd* < *x* < *y* < *z* < *e;* '(*x*)*;:*'(*y*)*;* '(*z*)*g:*
By induction on *n* < *!*, we de.ne elements *an; bn; cn; dn; en 2 M* such that
(i) *dn 2 C.*, *en 2 C*+;
(ii) (*an; bn; cn*) realizes '[*dn; en*];
(iii) *dn*+1 . *x* for all *x 2 C.* \ *fan; bn; cn; dng*;

(iv) en+1 · x for all x 2 C+ \ fan; bn; cn; eng.
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Since bn 2 C. [C+ for all n < !, one of the sets fn 2 ! : bn 2 C.g or fn 2 ! : bn 2 C+g is in.nite. If the .rst set is in.nite, we obtain an in.nite sequence an0 < bn0 < an1 < bn1 < : : : with M j= '(ani) ^ :'(bni) for all i < !. If the second set is in.nite, we obtain an in.nite sequence cn0 > bn0 > cn1 > bn1 > : : : with the same property. It follows that (1) is false.

(4))(5). Let C be the cut in M determined by the type p. Let M0 A M. Then p(M0) is convex as the intersection of convex sets C(M0) \ '(M0) for all '(x) 2 p.

(5))(3). This is obvious.

Theorem 2.1 is proven.

Corollary 2.2. *Model M is weakly o-minimal if and only if any two di®erent complete 1-types over M are distinguished by a convex formula.*

Corollary 2.3 [1]. *If model M is weakly o-minimal, then any cut C in M is contained in at most two complete 1-types over M.*

Proof. Assume the contrary. Then there are a model M0 A M and elements a < b < c in M0 that realize pairwise di®erent complete types over M containing C. Hence, there exist '(x) 2 tp(a=M) and A(x) 2 tp(c=M) with :'(x); :A(x) 2 tp(b=M). Let µ = '_A. Then M0 j= µ(a)^:µ(b)^µ(c), which implies that C(M0) \ µ(M0) is not convex. By Theorem 2.1, this contradicts weak o-minimality of M.

Corollary 2.3 is proven.

Remark. A cut C in M is called *rational* if C. has the least upper bound (equivalently, C+ has the greatest lower bound). If M is weakly o-minimal, any rational cut C in M is contained in a unique complete 1-type over M. Indeed, by weak o-minimality, every formula '(x) 2 L(M) partitions M into a .nite number of convex sets, each of which is a convex component of '(M) or :'(M). Since C is rational, only one of the sets is consistent with C.

3 Characterization of o-minimality

The following theorem was proven in [2]. We give a simpler and more conceptual proof.
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Theorem 3.1. *Model (M; <; : : :) is o-minimal if and only if every cut in M is contained in a unique complete 1-type over M.*

Proof. ()) By o-minimality, every formula '(x) 2 L(M) partitions M into a .nite number of intervals, each of which is a convex component of '(M) or :'(M). Since the intervals have boundary points in M, only one of them is consistent with the given cut.

(() Assume the contrary. Then there exists a formula '(x) 2 L(M) such that '(M) has in.nitely many convex components. The linear ordering of M linearly orders the components. It is known that every in.nite linear order contains either an in.nite increasing sequence or an in.nite decreasing sequence, so we can assume that there exists an in.nite increasing sequence

*I*0 < *I*1 < : : : of the components. Consider the cut

C = *fx* . *m* : (9*y* 2 [*i*<!

Ii)(*m* · *y*)*g* [*fx* · *m* : (8*y* 2 [*i*<!

Ii)(*y* < *m*)*g*:

Both *C* [*f*'(*x*)*g* and *C* [*f*:'(*x*)*g* are consistent, so *C* is contained in two di®erent complete 1-types over *M*.

Theorem 3.1 is proven.

4 Weakly o-minimal rings

In the following "a group" will mean "a group with an additional structure". Recall that an ordered group (*G*; <;+; 0) is called *archimedean* if for any positive elements *a*; *b* 2 *M* there exists *n* < ! such that *b* < *na*. A ring is called *archimedean* if its additive group is archimedean. In the following we consider rings with nonzero multiplication.

In [2] it was proven that every o-minimal ordered ring is a real closed .eld. In [3] this was generalized to quasi-o-minimal ordered rings. For weakly o-minimal ordered rings an analogous statement is false. So, the following theorem may be of some interest.

Theorem 4.1. *Every weakly o-minimal archimedean ordered ring is a real closed .eld.*

Proof. This follows from the following two assertions.

(1) Every weakly o-minimal .eld is real closed (this was proven in [4]).

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(2) Every weakly o-minimal archimedean ordered ring is a .eld.

In order to prove assertion (2), we need the following.

(3) Every de.nable subgroup of a weakly o-minimal ordered group is convex (this was proven in [4]).

(4) Let (*G*; <;+; 0) be a weakly o-minimal archimedean ordered group and *H* a nontrivial de.nable subgroup of *G*. Then *H* = *G*.

Indeed, let *a* 2 *H*, *a* 6= 0. Then *na* 2 *H* for all *n* < !. Since *G* is archimedean, *H* is co.nal in *G*. By (3), *H* is convex. Hence, *H* = *G*.

(5) Every ring without nontrivial de.nable proper subgroups is a .eld (this was proven in [3]).

Now assertion (2) follows from (4) and (5).

Theorem 4.1 is proven.

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