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« __________ 2023

Topic of the thesis:
Functional inequalities on Lie groups and applications

Thesis submitted as part of the requirements for the award of the MSc in "7M05401-Mathematics", SDU, 2021-2023

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Kaskelen, 2023

The Ministry of Science and Higher Education
of the Republic of Kazakhstan
Suleyman Demirel University

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Functional inequalities on Lie groups and applications

THESIS

Presented in Partial Fulfilment for the

Degree of Master of Science in Mathematics

(degree code: 7M05401)

Department of Mathematics and Natural Sciences
Faculty of Engineering and Natural Sciences

Supervisor: **Nurgissa Yessirkegenov**

Kaskelen, 2023

Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Madina Kalaman

Acknowledgements

I would like express my heartfelt appreciation to my supervisor Nurgissa Yessirkegenov for his support, patience, understanding and for making my research enjoyable. I also want to thank you for friendly advices, interesting discussions and cups of tea.

Dedication

This thesis is dedicated to my family for their unconditionally love and support. My endless love is their own.

I dedicate this thesis to memory of my friend Aldan, whose life was tragically cut short.

Abstract

In this thesis we discuss sharp remainder formulae for the cylindrical extensions of the improved Hardy inequalities. For more general p we obtain cylindrical improved L^p-Hardy identities for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$, while in L^2 case we have them for any complex-valued function $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$. Moreover, we show cylindrical L^p-Hardy inequalities for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$. As applications, we establish Heisenberg-Paul-Weyl type uncertainty principles and Caffarelli-Kohn-Nirenberg type inequalities. In particular cases, these inequalities imply new functional inequalities, which are not covered by the classical Caffarelli-Kohn-Nirenberg inequalities. Furthermore, the thesis contains L^2 and L^p identities with logarithmic type functions on the quasi-ball $B(0, R)$ with $R > 0$. In addition, we also discuss the results in the setting of homogeneous Lie groups.

Аңдатпа

Бұл тезисте біз жақсартылған Харди теңсіздіктерінің цилиндрлік кеңеюі үшін қалдықтардың нақты формулаларын талқылаймыз. Жалпы p үшін жақсартылған цилиндрлік L^p-Харди теңдігін барлық нақты мәнді функциялар үшін $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$, ал L^2 жағдайында комплекс мәнді функциялар $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$ үшін аламыз. Сонымен қатар, комплекс мәнді функциялар $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$ үшін цилиндірлік L^p-Харди теңсіздігін көрсетеміз. Алынған теңдіктердің қолданысы ретінде Гейзенберг-Пол-Вейл типті белгісіздік принципін және Каффарелли-Кон-Ниренберг типті теңсіздіктерді аламыз. Ерекше жағдайларда, бұл теңсіздіктер Каффарелли-Кон-Ниренбергтің классикалық теңсіздіктерімен қамтылмаған жаңа функционалдық теңсіздіктерді береді. Сонымен қатар, тезисте радиусы $R > 0$ болатын $B(0, R)$ квазишарда функциялары логарифм түрінде берілген L^2 және L^p теңдіктері көрсетілген. Онымен қоса, біз алынған нәтижелерді стратификацияланған Ли топтарында талқылаймыз.

Аннотация

В данном тезисе мы обсуждаем точные формулы остатков для цилиндрических расширений улучшенных неравенств Харди. Для более общего p мы получаем цилиндрические улучшенные L^p тождества Харди для всех действительнозначных функций $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$, в то время как в случае L^2 мы их получаем для комплекснозначных функции $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$. Более того, мы показываем цилиндрические неравенства L^p Харди для всех комплекснозначных функции $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$. В качестве применения мы устанавливаем принцип неопределенности типа Гейзенберга-Паули-Вейля и неравенства типа Каффарелли-Кона-Ниренберга. В частных случаях эти неравенства подразумевают новые функциональные неравенства, которые не охватываются классическими неравенствами Каффарелли-Кона-Ниренберга. Кроме того, тезис содержит тождества L^2 и L^p с функциями логарифмического типа на квазишаре $B(0, R)$ с радиусом $R > 0$. Кроме того, мы также обсуждаем результаты на однородных группах Ли.

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Chapter 1

Background and motivations

1.1 Introduction

Consider the following Hardy inequality:

$$\left\| \frac{f}{|x|_E} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \|\nabla f\|_{L^p(\mathbb{R}^n)}, n \geq 2, 1 \leq p < n, \tag{1.1.1}$$

where $f \in C_0^\infty(\mathbb{R}^n)$, $|x|_E = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ is the Euclidean norm on $\mathbb{R}^n$, ∇ is the usual gradient in $\mathbb{R}^n$ and the constant $\frac{p}{n-p}$ is best-possible. The one-dimensional version of the classical Hardy inequality (1.1.1) for $p = 2$ was obtained in [1]. The L^p version of the inequality goes back to [2]. The multidimensional version of the Hardy inequality (1.1.1) was shown by J. Leray in [3].

The Hardy inequality has great applications in studying partial differential equations. There are many well-known examples. Let us discuss a few of them. If we consider the following second order partial differential equations

$$\left.\begin{matrix} 0 \\ u_t \\ u_{tt} \end{matrix}\right\} - \Delta u = \lambda \frac{|u|^s}{|x|_E^2},$$

then existence of a solution depends on a relation between the constant λ and the sharp constant $\frac{2}{n-2}$ from the Hardy inequality, see, e.g., [4] and [5].

Furthermore, we shall include the uncertainty principle, which was introduced in connection with the study of quantum mechanics in [6], as one of the inequality's many uses. On the Euclidean space $\mathbb{R}^n$ the uncertainty principle says that

$$\left(\frac{n-2}{2}\right)^2 \left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^2 \leq \left(\int_{\mathbb{R}^n} |x|^2 |f(x)|^2 dx\right) \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2 dx\right), \tag{1.1.2}$$

where $u \in C_0^\infty(\mathbb{R}^n)$. Such type of the uncertainty principle is called Heisenberg-Pauli-Weyl type uncertainty principle in the literature. This uncertainty inequality can be derived easily from the L^2-Hardy and Schwartz inequality as follows

$$\left(\frac{n-2}{2}\right)\int_{\mathbb{R}^n}|f(x)|^2dx=\left(\frac{n-2}{2}\right)\int_{\mathbb{R}^n}|f(x)|^2\frac{1}{|x|}|x|dx$$
$$\leq\left(\frac{n-2}{2}\right)\left(\int_{\mathbb{R}^n}|f(x)|^2|x|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}\frac{|f(x)|^2}{|x|^2}dx\right)^{\frac{1}{2}}$$
$$\leq\left(\int_{\mathbb{R}^n}|f(x)|^2|x|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|\nabla f(x)|^2dx\right)^{\frac{1}{2}},$$

which is (1.1.2).

The Hardy inequality (1.1.1) has been analysed in many settings (see, for example, [7], [8] [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]).

One of the main aims of the thesis is inspired by the work of Badiale-Tarantello [19], namely by the following extended Hardy inequality: Let $x=(x',x'')\in\mathbb{R}^k\times\mathbb{R}^{n-k}$. Badiale and Tarantello proved that for $2\leq k\leq n$ and $1\leq p<k$ there exists a constant $C_{n,k,p}$ such that

$$\left\|\frac{1}{|x'|_k}f\right\|_{L^p(\mathbb{R}^n)}\leq C_{n,k,p}\|\nabla f\|_{L^p(\mathbb{R}^n)}, \tag{1.1.3}$$

where ∇ is the full gradient on $\mathbb{R}^n$ and $|x'|_k$ is the Euclidean norm on $\mathbb{R}^k$. Clearly, for $k=n$ (e.g. [3]) this gives the classical Hardy inequality with the best constant

$$C_{n,p}=\frac{p}{n-p}.$$

It was conjectured by Badiale and Tarantello that the best constant in (1.1.3) is given by

$$C_{k,p}=\frac{p}{k-p}$$

in [19], then it was proved in [20]. Moreover, new simple proof was obtained in [21].

In [19] Badiale and Tarantello mentioned importance of such cylindrical extensions of the classical functional inequalities in investigating well-posedness of the partial differential equations arised in astrophysics. As an example, they demonstrated an application of the extended Hardy inequality (1.1.3) in studying existence and non-existence of cylindrical solutions for the following nonlinear elliptic problem in $\mathbb{R}^3$:

$$\begin{cases}-\Delta u(x)=\phi(r)|u|^{p-2}u \text{ in } \mathbb{R}^3,\\ u(x)>0 \quad \text{in } \mathbb{R}^3\\ \int_{\mathbb{R}^3}\phi(r)u^{p-1}dx<+\infty\end{cases}$$

with $p>1$. For $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, we write $r=\sqrt{x_1^2+x_2^2}, z=x_3$ and $u=u(r,z)$ which is cylindrically symmetric function. The weight function ϕ is a non-negative continuous function, depending only on r. This equation has been proposed as a model describing the dynamics of galaxies.

In this direction, in particular we obtain a sharp remainder formula for the extended L^2-Hardy inequality (1.1.3) for complex-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{x'=0\})$

$$\left(\frac{k-2}{2}\right)^2 \left\|\frac{f}{|x'|_k}\right\|^2_{L^2(\mathbb{R}^n)} = \left\|\frac{x'\cdot\nabla_k}{|x'|_k}f\right\|^2_{L^2(\mathbb{R}^n)} - \left\|\frac{x'\cdot\nabla_k}{|x'|_k}f + \frac{k-2}{2|x'|_k}f\right\|^2_{L^2(\mathbb{R}^n)}, \quad (1.1.4)$$

where $|\cdot|_k$ is the Euclidean norm on $\mathbb{R}^k$.

If we drop the last term in the right-hand side of (1.1.4) then we get a refined version of (1.1.3) when $p=2$:

$$\left(\frac{k-2}{2}\right)^2 \left\|\frac{f}{|x'|_k}\right\|^2_{L^2(\mathbb{R}^n)} \le \left\|\frac{x'\cdot\nabla_k}{|x'|_k}f\right\|^2_{L^2(\mathbb{R}^n)} \le \|\nabla_k f\|^2_{L^2(\mathbb{R}^n)} \le \|\nabla f\|^2_{L^2(\mathbb{R}^n)}, \quad (1.1.5)$$

where in the last line we have used the Schwartz inequality.

Moreover, when $k=n$ taking into account

$$\frac{x\cdot\nabla_k f}{|x'|_k} = \frac{x\cdot\nabla f}{|x|_E} = \frac{df}{d|x|_E},$$

we derive from (1.1.4) the Hardy identity for all $f \in C_0^\infty(\mathbb{R}^n\backslash\{x'=0\})$ with the radial derivative operator

$$\left(\frac{n-2}{2}\right)^2 \left\|\frac{f}{|x|_E}\right\|^2_{L^2(\mathbb{R}^n)} = \left\|\frac{df}{d|x|_E}\right\|^2_{L^2(\mathbb{R}^n)} - \left\|\frac{df}{d|x|_E} + \frac{n-2}{2|x|_E}f\right\|^2_{L^2(\mathbb{R}^n)}, \quad (1.1.6)$$

which implies the following improved version of the classical Hardy inequality

$$\left(\frac{n-2}{2}\right)^2 \left\|\frac{f}{|x|_E}\right\|^2_{L^2(\mathbb{R}^n)} \le \left\|\frac{df}{d|x|_E}\right\|^2_{L^2(\mathbb{R}^n)} \le \|\nabla f\|^2_{L^2(\mathbb{R}^n)}. \quad (1.1.7)$$

We also discuss L^p versions of the above identities for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{x'=0\})$. In particular, on $\mathbb{R}^k\times\mathbb{R}^{n-k}$ we have

$$\left\|\frac{f}{|x'|_k}\right\|^p_{L^p(\mathbb{R}^n)} = \left(\frac{p}{k-p}\right)^p \left\|\frac{x'\cdot\nabla_k}{|x'|_k}f\right\|^p_{L^p(\mathbb{R}^n)} - p\int_{\mathbb{R}^n} I_p\left(\frac{f}{|x'|_k}, -\frac{p}{k-p}\frac{x'\cdot\nabla_k}{|x'|_k}f\right)\left|\frac{f}{|x'|_k} + \frac{p}{k-p}\frac{x'\cdot\nabla_k}{|x'|_k}f\right|^2 dx, \quad (1.1.8)$$

where $1<p<\infty$ and

$$I_p(h,g) = (p-1)\int_0^1 |\xi h + (1-\xi)g|^{p-2}\xi d\xi.$$

By dropping the last term in the right-hand side of (1.1.8) implies the inequality

$$\left\|\frac{f}{|x'|_k}\right\|_{L^p(\mathbb{R}^n)}^p \leq \left(\frac{p}{k-p}\right)^p \left\|\frac{x'\cdot\nabla_k}{|x'|_k}f\right\|_{L^p(\mathbb{R}^n)}^p \tag{1.1.9}$$

for every real-valued function $f \in C_0^\infty(\mathbb{R}^n\backslash\{x'=0\})$, where $1<p<\infty$. Then, as above using Schwartz's inequality on the right-hand side of (1.1.9) yields the L^p version of the discussion (1.1.5)

$$\left\|\frac{f}{|x'|_k}\right\|_{L^p(\mathbb{R}^n)}^p \leq \left(\frac{p}{k-p}\right)^p \|\nabla_k f\|_{L^p(\mathbb{R}^n)}^p$$

for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{x'=0\})$.

Actually, we also give a proof of the inequality (1.1.9) for complex-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{x'=0\})$ in Chapter 4.

As an application, we extend the classical Caffarelli-Kohn-Nirenberg type inequalities.

Let us begin by recalling the classical Caffarelli-Kohn-Nirenberg inequality from [22]:

Theorem 1. *Let $n \in \mathbb{N}$ and let p, q, r, a, b, d, $\delta \in \mathbb{R}$ such that $p,q \geq 1$, $r>0$, $0\leq\delta\leq 1$, and*

$$\frac{1}{p}+\frac{a}{n},\ \frac{1}{q}+\frac{b}{n},\ \frac{1}{r}+\frac{c}{n}>0, \tag{1.1.10}$$

where $c=\delta d+(1-\delta)b$. Then there exists a positive constant C such that

$$\||x|_E^c f\|_{L^r(\mathbb{R}^n)} \leq C\||x|_E^a|\nabla f|\|_{L^p(\mathbb{R}^n)}^\delta \||x|_E^b f\|_{L^q(\mathbb{R}^n)}^{1-\delta}, \tag{1.1.11}$$

holds for all $f \in C_0^\infty(\mathbb{R}^n)$, if and only if the following conditions hold:

$$\frac{1}{r}+\frac{c}{n}=\delta\left(\frac{1}{p}+\frac{a-1}{n}\right)+(1-\delta)\left(\frac{1}{q}+\frac{b}{n}\right), \tag{1.1.12}$$

$$a-d\geq 0 \quad \text{if} \quad \delta>0, \tag{1.1.13}$$

$$a-d\leq 1 \quad \text{if} \quad \delta>0 \quad \text{and} \quad \frac{1}{r}+\frac{c}{n}=\frac{1}{p}+\frac{a-1}{n}. \tag{1.1.14}$$

Many well-known inequalities such as Hardy-Sobolev inequalities, Sobolev inequalities, Gagliardo-Nirenberg inequalities, Nash inequalities are particular cases of the Caffarelli-Kohn-Nirenberg inequalities. Nowadays, there are many works concerning various versions of the Caffarelli-Kohn-Nirenberg type inequalities and applications, see e.g. [23, 24, 25, 26, 27] and the recent paper [28].

In order to compare with Theorem 1 let us state here our Caffarelli-Kohn-

Nirenberg inequality on $\mathbb{R}^k \times \mathbb{R}^{n-k}$.

$$\||x'|_k^c f\|_{L^r(\mathbb{R}^n)} \leq \left(\frac{2}{k-2}\right)^{\delta}$$
$$\times \left[\left\|\frac{x'\cdot\nabla_k}{|x'|_k}f\right\|^2_{L^2(\mathbb{R}^n)} - \left\|\frac{x'\cdot\nabla_k}{|x'|_k}f + \frac{k-2}{2|x'|_k}f\right\|^2_{L^2(\mathbb{R}^n)}\right]^{\frac{\delta}{2}} \||x'|_k^b f\|^{1-\delta}_{L^q(\mathbb{R}^n)}. \quad (1.1.15)$$

When $k = n$ if we drop the last term on the right-hand side of (1.1.15), then our result gives the following improvement of the classical Caffarelli-Kohn-Nirenberg inequality for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$:

$$\begin{aligned}\||x|_E^c f\|_{L^r(\mathbb{R}^n)} &\leq \left(\frac{2}{n-2}\right)^{\delta}\left\|\frac{x\cdot\nabla}{|x|_E}f\right\|^{\delta}_{L^2(\mathbb{R}^n)} \||x|_E^b f\|^{1-\delta}_{L^q(\mathbb{R}^n)} \\ &\leq \left(\frac{2}{n-2}\right)^{\delta}\|\nabla f\|^{\delta}_{L^2(\mathbb{R}^n)} \||x|_E^b f\|^{1-\delta}_{L^q(\mathbb{R}^n)},\end{aligned} \quad (1.1.16)$$

where in the last line we have used the Schwartz inequality.

In the special case $q = r = 2$, $b = -n/2$, $c = -\delta - n(1-\delta)/2$ the inequality (3.4.4) takes the form

$$\left\|\frac{f}{|x|_E^{\frac{2\delta+n(1-\delta)}{2}}}\right\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{2}{n-2}\right)^{\delta}\|\nabla f\|^{\delta}_{L^2(\mathbb{R}^n)}\left\|\frac{f}{|x|_E^{\frac{n}{2}}}\right\|^{1-\delta}_{L^2(\mathbb{R}^n)}. \quad (1.1.17)$$

Since we have $1/2 + b/n = 0$ here, we observe that the condition (1.1.10) is not satisfied, then the inequality (1.1.17) is not covered by the classical Caffarelli-Kohn-Nirenberg inequality, Theorem 1.

Also, taking into account when $k = n$

$$\frac{x'\cdot\nabla_k f}{|x'|_k} = \frac{x\cdot\nabla f}{|x|_E} = \frac{df}{d|x|_E},$$

we derive from (1.1.15) the Caffarelli-Kohn-Nirenberg type inequality for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{x'=0\})$ with the radial derivative operator

$$\||x|_E^c f\|_{L^r(\mathbb{R}^n)} \leq \left(\frac{2}{n-2}\right)^{\delta}$$
$$\times \left[\left\|\frac{df}{d|x|_E}\right\|^2_{L^2(\mathbb{R}^n)} - \left\|\frac{df}{d|x|_E} + \frac{n-2}{2|x|_E}f\right\|^2_{L^2(\mathbb{R}^n)}\right]^{\frac{\delta}{2}} \||x|_E^b f\|^{1-\delta}_{L^q(\mathbb{R}^n)}. \quad (1.1.18)$$

Furthermore, Caffarelli-Kohn-Nirenberg type inequalities with remainder terms and explicit constants are obtained for the horizontal gradient on stratified Lie groups. Stratified Lie groups are an important class of nilpotent Lie groups, which was investigated thoroughly by Folland [29]. There are many different, equivalent definitions of a stratified Lie group (see, for example, [30], [31], [32] or [33]). Also, in Section 2.1 we briefly recall the main concepts of stratified Lie groups.

Moreover, the thesis contains Hardy identities with logarithmic type functions on the quasi-ball $B(0,R) \subset \mathbb{G}$.

For every complex-valued functions $f \in C_0^\infty(B(0,R)\backslash\{0\})$

$$4\left\|\mathcal{R}_{|x|}f\right\|^2_{L^2(B_R)} - \left\|\frac{f}{|x|\left(log\frac{R}{|x|}\right)}\right\|^2_{L^2(B_R)} - 2(Q-2)\int_{B_R}\frac{|f|^2}{|x|^2\left(log\frac{R}{|x|}\right)}dx$$
$$= \int_{B_R}\left|\frac{f}{|x|\left(log\frac{R}{|x|}\right)} + 2\mathcal{R}_{|x|}f\right|^2 dx, \quad (1.1.19)$$

where $Q \geq 2, 1 < p < Q$.

For more general p we obtain the following identity for all real-valued functions $f \in C_0^\infty(B(0,R)\backslash\{0\})$:

$$\left(\frac{p}{p-1}\right)^p \left\|\mathcal{R}_{|x|}f\right\|^p_{L^p(B_R)} - \left\|\frac{f}{|x|\left(log\frac{R}{|x|}\right)}\right\|^p_{L^p(B_R)}$$
$$-p\left(\frac{Q-p}{p-1}\right)\int_{B_R}\frac{|f|^p}{|x|^p\left(log\frac{R}{|x|}\right)^{p-1}}dx = p\int_{B_R} I_p\left(\frac{f}{|x|\left(log\frac{R}{|x|}\right)}, -\frac{p}{p-1}\mathcal{R}_{|x|}f\right)$$
$$\times\left|\frac{f}{|x|\left(log\frac{R}{|x|}\right)} + \frac{p}{p-1}\mathcal{R}_{|x|}f\right|^2 dx, \quad (1.1.20)$$

where $Q \geq p, 1 < p < Q$.

In Chapter 2 we briefly recall all the necessary notions. L^2-Hardy type identities for complex-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{x'=0\})$ on $\mathbb{R}^k \times \mathbb{R}^{n-k}$, on stratified Lie groups and their applications are presented in Chapter 3. For more general p cylindrical L^p-Hardy inequality for complex-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{x'=0\})$ is showed in Chapter 4. Also, in Chapter (4) we establish L^p-Hardy identities for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{x'=0\})$. In Chapter 3 and Chapter 4 L^2 and L^p identities with logarithmic type functions on homogeneous Lie groups are studied, respectively.

Results of this thesis partly were announced in the works [34], [35] and [36].

Chapter 2

Preliminaries

In this section we briefly recall the necessary notations and definitions concerning the setting of homogeneous Lie groups following the books [31], [32] and [33]. Also, a few other facts needed for our analysis will be discussed.

2.1 Stratified Lie groups

In this subsection we discuss a popular subclass of the homogeneous Lie groups - stratified (or a homogeneous Carnot group) Lie groups, where a homogeneous second-order sub-Laplacian $\mathcal{L}$ can be defined.

Definition 1. *A Lie group $\mathbb{G} = (\mathbb{R}^n, +)$ is called stratified if it satisfies the conditions:*

- *For some natural numbers $N + N_2 + \cdots + N_r = n$, that is $N = N_1$, the decomposition $\mathbb{R}^n = \mathbb{R}^N \times \cdots \times \mathbb{R}^{N_r}$ is valid, and for every $\lambda > 0$ the dilation $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ given by*

$$\delta_\lambda(x) = \delta_\lambda \left(x', x^{(2)}, \ldots, x^{(r)}\right) := \left(\lambda x', \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)}\right)$$

  is an automorphism of the group $\mathbb{G}$. Here $x' \equiv x^{(1)} \in \mathbb{R}^N$ and $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 2, \ldots, r$.

- *Let N be as in above and let $X_1, \ldots, X_N$ be the left invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}\Big|_0$ for $k = 1, \ldots, N$. Then*

$$\operatorname{rank}(\operatorname{Lie}\{X_1, \ldots, X_N\}) = n,$$

  for every $x \in \mathbb{R}^n$, i.e. the iterated commutators of $X_1, \ldots, X_N$ span the Lie algebra of $\mathbb{G}$.

Thus, the triple $\mathbb{G} = (\mathbb{R}^n, +, \delta_\lambda)$ is a stratified Lie group.

The left invariant vector fields $X_1, \ldots, X_N$ are called the (Jacobian) generators of $\mathbb{G}$ and r is called a step of $\mathbb{G}$. The number

$$Q = \sum_{k=1}^{r} kN_k, \quad N_1 = N,$$

is called the homogeneous dimension of $\mathbb{G}$ and dx is the Haar measure on a group $\mathbb{G}$. The Haar measure on $\mathbb{G}$ is the standard Lebesgue measure for $\mathbb{R}^n$ (see, e.g. [32, Proposition 1.6.6]).

We also recall that the left invariant vector fields X_j have an explicit form and satisfy the divergence theorem, see e.g. [32, Section 3.1.5] and [37],

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{\ell=2}^{r} \sum_{m=1}^{N_1} a_{k,m}^{(\ell)} \left(x', \ldots, x^{\ell-1}\right) \frac{\partial}{\partial x_m^{(\ell)}}. \tag{2.1.1}$$

We will also need the following notations:

$$\nabla_H := (X_1, \ldots, X_N)$$

for the horizontal gradient,

$$\operatorname{div}_H v := \nabla_H \cdot v$$

for the horizontal divergence, and

$$|x'| = \sqrt{x_1'^2 + \cdots + x_N'^2}$$

for the Euclidean norm on $\mathbb{R}^N$.

The explicit representation of the left invariant vector fields X_j from (2.1.1) allows us to establish the identities

$$|\nabla_H |x'|^\gamma| = \gamma |x'|^{\gamma-1}, \tag{2.1.2}$$

and

$$\operatorname{div}_H \left(\frac{x'}{|x'|^\gamma}\right) = \frac{\sum_{j=1}^{N} |x'|^\gamma X_j x'_j - \sum_{j=1}^{N} x'_j \gamma |x'|^{\gamma-1} X_j |x'|}{|x'|^{2\gamma}} = \frac{N-\gamma}{|x'|^\gamma} \tag{2.1.3}$$

for any $\gamma \in \mathbb{R}$ and $|x'| \neq 0$.

2.2 Homogeneous Lie Groups

Now, in this section we recall basic necessary concepts and fix the notations of general homogeneous Lie groups in a very briefly manner.

Let us consider a family of dilations of a Lie algebra $\mathfrak{g}$, which is a family of lincar mappings of the following form

$$D_\lambda = \mathrm{Exp}(A \ln \lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} (\ln(\lambda) A)^k,$$

where A is a diagonalisable linear operator on the Lie algebra g with positive eigenvalues, and each D_λ is a morphism of $\mathfrak{g}$, that is, a linear mapping from $\mathfrak{g}$ to itself satisfying:

$$\forall X, Y \in \mathfrak{g}, \lambda > 0, [D_\lambda X, D_\lambda Y] = D_\lambda [X, Y],$$

where $[X, Y] := XY - YX$ is the Lie bracket. Then, a homogeneous Lie group is a connected simply connected Lie group whose Lie algebra is equipped with dilations. It induces the dilation structure on the homogeneous Lie group $\mathbb{G}$ which we denote by $D_{\lambda x}$ or just by λx.

Let dx be the Haar measure on $\mathbb{G}$ and let $|S|$ denote the volume of a measurable set $S \subset \mathbb{G}$. Then we have

$$|D_\lambda(S)| = \lambda^Q |S| \text{ and } \int_G f(\lambda x) dx = \lambda^{-Q} \int_G f(x) dx,$$

where Q is the homogeneous dimension of $\mathbb{G}$:

$$Q := \operatorname{Tr} A$$

Definition 2. *A homogeneous quasi-norm on $\mathbb{G}$ is a continuous non-negative function*

$$\mathbb{G} \ni x \mapsto |x| \in [0, \infty)$$

satisfying the following properties

- $|x^{-1}| = |x|$ *for all* $x \in \mathbb{G}$,
- $|\lambda x| = \lambda |x|$ *for all* $x \in \mathbb{G}$ *and* $\lambda > 0$,
- $|x| = 0$ *if and only if* $x = 0$.

The following polar decomposition on homogeneous Lie groups will be very useful for our analysis: there is a (unique) positive Borel measure σ on the unit sphere

$$\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\}$$

such that for all $f \in L^1(\mathbb{G})$ we have

$$\int_{\mathbb{G}} f(x) dx = \int_0^\infty \int_{\mathfrak{S}} f(ry) r^{Q-1} d\sigma(y) dr.$$

We use the notation

$$\mathcal{R}_{|x|} f := \frac{df(x)}{dr}. \tag{2.2.1}$$

Let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then the quasi-ball centred at $x \in \mathbb{G}$ with radius $R > 0$ is defined by

$$B(x, R) := y \in \mathbb{G} : |x^{-1}y| < R.$$

The following Lemma 1 from [38, Lemma 1.1] and Lemma 2 from [13, Lemma 1.1.] play important roles in obtaining Hardy's identities.

Lemma 1. *Let X be a scalar product space with scalar product $\langle \cdot, \cdot \rangle$. Let $c > 0$. Then the following statements are equivalent:*

- *The equality*
$$\|u\|^2 = -2c\,\mathrm{Re}\langle u, v\rangle \tag{2.2.2}$$
holds for all $u, v \in$ X.
- *The equality*
$$\|u\|^2 = 4c^2 \|v\|^2 - \|u + 2cv\|^2 \tag{2.2.3}$$
holds for all $u, v \in$ X.

Lemma 2. *Let (Ω, μ) be a measure space. Let $1 < p < \infty$ and let $L^p(\Omega, \mu)$ be the Banach space of p-th integrable real-valued functions on Ω with norm denoted by $\|\cdot\|_p$. Then for any $u, v \in L^p(\Omega, \mu)$ the following statements are equivalent:*

- *u and v satisfy*
$$\|u\|_p^p = \int_\Omega |u|^{p-2} uv d\mu;$$
- *u and v satisfy*
$$\|u\|_p^p = \|v\|_p^p - \int_\Omega \left(|v|^p + (p-1)|u|^p - p|u|^{p-2}uv\right) d\mu;$$
- *u and v satisfy*
$$\|u\|_p^p = \|v\|_p^p - p\int_\Omega I_p(u, v)|u - v|^2 d\mu,$$
where
$$I_p(h, g) = (p-1)\int_0^1 |\xi h + (1-\xi)g|^{p-2}\xi d\xi.$$

Proposition 1. *Let $p \in \mathbb{R}$ satisfy $p > 1$. Then I_p has the integral representation*

$$I_p(h, g) = (p-1)\int_0^1 |\xi h + (1-\xi)g|^{p-2}\xi d\xi.$$

Chapter 3

L^2-improved Hardy identities with remainder terms

In this chapter we discuss the cylindrical improved L^2-Hardy inequalities (1.1.3) with remainder terms. Actually, sharp remainder formulae for the cylindrical improved L^2-Hardy inequalities are obtained. Moreover, hypoelliptic extension of these results are discussed on stratified Lie groups. Also, we present L^2-Hardy type identities involving logarithmic functions on the quasi-ball $B(0,R)$ with a radius $R > 0$ on homogeneous Lie groups. As applications, we establish extended Caffarelli-Kohn-Nirenberg type inequalities with remainder terms.

For the L^p extensions of these results with some additional assumptions, we refer to the next Chapter 4.

3.1 L^2-identity on $\mathbb{R}^k \ltimes \mathbb{R}^{n-k}$

Theorem 2. *Let $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ and $k \geq 2$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$, we have*

$$\left(\frac{k-2}{2}\right)^2 \left\|\frac{f}{|x'|_k}\right\|^2_{L^2(\mathbb{R}^n)} = \left\|\frac{x' \cdot \nabla_k}{|x'|_k} f\right\|^2_{L^2(\mathbb{R}^n)} - \left\|\frac{x' \cdot \nabla_k}{|x'|_k} f + \frac{k-2}{2|x'|_k} f\right\|^2_{L^2(\mathbb{R}^n)}, \quad (3.1.1)$$

where $|\cdot|_k$ is the Euclidean norm on $\mathbb{R}^k$.

Remark 1. *By dropping the last term on the right-hand side of* (3.1.1), *we obtain the following inequality for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$:*

$$\left(\frac{k-2}{2}\right)^2 \left\|\frac{f}{|x'|_k}\right\|^2_{L^2(\mathbb{R}^n)} \leq \left\|\frac{x' \cdot \nabla_k}{|x'|_k} f\right\|^2_{L^2(\mathbb{R}^n)}. \quad (3.1.2)$$

Here, using Schwartz's inequality on the right-hand side of (3.1.2), *we obtain the* *following Hardy inequality:*

$$\left(\frac{k-2}{2}\right)^2 \left\| \frac{f}{|x|_k} \right\|^2_{L^2(\mathbb{R}^n)} \leq \|\nabla_k f\|^2_{L^2(\mathbb{R}^n)} \leq \|\nabla f\|^2_{L^2(\mathbb{R}^n)}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$ *yielding an improvement of Badiale-Tarantello's extended Hardy inequality* (1.1.3).

Remark 2. *When* $k = n$ *taking into account*

$$\frac{x \cdot \nabla_k f}{|x'|_k} = \frac{x \cdot \nabla f}{|x|_E} = \frac{df}{d|x|_E},$$

we derive from (3.1.1) *the Hardy identity for all complex-valued functions* $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ *with the radial derivative operator*

$$\left(\frac{n-2}{2}\right)^2 \left\| \frac{f}{|x|_E} \right\|^2_{L^2(\mathbb{R}^n)} = \left\| \frac{df}{d|x|_E} \right\|^2_{L^2(\mathbb{R}^n)} - \left\| \frac{df}{d|x|_E} + \frac{n-2}{2|x|_E} f \right\|^2_{L^2(\mathbb{R}^n)}, \qquad (3.1.3)$$

which was recently obtained by Ruzhansky and Suragan in [39].

Remark 3. *If we drop the last term on the right-hand side of* (3.1.3) *it implies the following improved version of the classical Hardy inequality:*

$$\left(\frac{n-2}{2}\right)^2 \left\| \frac{f}{|x|_E} \right\|^2_{L^2(\mathbb{R}^n)} \leq \left\| \frac{df}{d|x|_E} \right\|^2_{L^2(\mathbb{R}^n)} \leq \|\nabla f\|^2_{L^2(\mathbb{R}^n)}, \qquad (3.1.4)$$

where in the last line we have used Schwartz inequality.

Proof of Theorem 2. Since the case $k = 2$ is trivial, it is sufficient to prove the case $k > 2$.

By using the identity (2.1.3) and the divergence theorem one calculates

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x'|_k^2} dx = \frac{1}{k-2} \int_{\mathbb{R}^n} |f(x)|^2 \operatorname{div}_k \left(\frac{x'}{|x'|_k^2} \right) dx. \qquad (3.1.5)$$

Using the integration by parts, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x'|_k^2} dx &= \frac{1}{k-2} \int_{\mathbb{R}^n} |f(x)|^2 \operatorname{div}_k \left(\frac{x'}{|x'|_k^2} \right) dx \\ &= -\frac{2}{k-2} \operatorname{Re} \int_{\mathbb{R}^n} f(x) \frac{\overline{x' \cdot \nabla_k f}}{|x'|_k^2} dx \\ &= -\frac{2}{k-2} \operatorname{Re} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|_k} \frac{\overline{x' \cdot \nabla_k f}}{|x'|_k} dx. \end{aligned} \qquad (3.1.6)$$

Introducing the notations

$$c = \frac{1}{k-2}$$

and

$$u = \frac{f(x)}{|x'|_k}$$

and

$$v = \frac{x' \cdot \nabla_k}{|x'|_k} f$$

formula (3.1.6) can be rewritten as

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x'|_k^2} dx = -2c \operatorname{Re}\langle u, v\rangle,$$

which is (2.2.2). Then by Lemma 1 it is equivalent to the identity

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x'|_k^2} dx &= 4c^2\|v\|^2 - \|u + 2cv\|^2 \\ &= \left(\frac{2}{k-2}\right)^2 \left\| \frac{x' \cdot \nabla_k}{|x'|_k} f \right\|^2_{L^2(\mathbb{R}^n)} - \left\| \frac{f}{|x'|_k} + \frac{2}{k-2} \frac{x' \cdot \nabla_k}{|x'|_k} f \right\|^2_{L^2(\mathbb{R}^n)}, \end{aligned}$$

that is,

$$\left(\frac{k-2}{2}\right)^2 \left\| \frac{f}{|x'|_k} \right\|^2_{L^2(\mathbb{R}^n)} = \left\| \frac{x' \cdot \nabla_k}{|x'|_k} f \right\|^2_{L^2(\mathbb{R}^n)} - \left\| \frac{x' \cdot \nabla_k}{|x'|_k} f + \frac{k-2}{2|x'|_k} f \right\|^2_{L^2(\mathbb{R}^n)},$$

which is Hardy's identity (3.1.1).

The proof of Theorem 2 is completed.

3.2 L^2-identity on stratified Lie groups

In this section we adopt all the notation introduced in Chapter 2. We extend the L^2-Hardy identity from the previous section to stratified Lie groups.

Theorem 3. *Let $\mathbb{G}$ be a stratified Lie group with N being the dimension of the first stratum. We denote by x' the variables from the first stratum of $\mathbb{G}$. Let $N \geq 2$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$, we have*

$$\begin{aligned} \left(\frac{N-2}{2}\right)^2 \left\| \frac{f}{|x'|} \right\|^2_{L^2(\mathbb{G})} &= \left\| \frac{x' \cdot \nabla_H}{|x'|} f \right\|^2_{L^2(\mathbb{G})} \\ &\quad - \left\| \frac{x' \cdot \nabla_H}{|x'|} f + \frac{N-2}{2|x'|} f \right\|^2_{L^2(\mathbb{G})}, \end{aligned} \tag{3.2.1}$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$.

Remark 4. *By dropping the last term of (3.2.1), we obtain the following inequal-*

ity:

$$\left(\frac{N-2}{2}\right)^2 \left\|\frac{f}{|x'|}\right\|^2_{L^2(\mathbb{G})} \leq \left\|\frac{x'\cdot\nabla_H}{|x'|}f\right\|^2_{L^2(\mathbb{G})}, \tag{3.2.2}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$. *The inequality* (3.2.2) *was obtained in [21] and with more general weights in [40]. So, our identity* (3.2.1) *gives a sharp remainder formula for their horizontal Hardy inequalities.*

Remark 5. *In the Abelian case* $\mathbb{G} = (\mathbb{R}^n, +)$, *we have* $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, ..., \partial_{x_n})$ *and* $|\cdot|$ *being the Euclidean norm on* $\mathbb{R}^n$, *so* (3.2.1) *implies the following* L^2 *Hardy identity:*

$$\left(\frac{n-2}{2}\right)^2 \left\|\frac{f}{|x|_E}\right\|^2_{L^2(\mathbb{R}^n)} = \left\|\frac{x\cdot\nabla}{|x|_E}f\right\|^2_{L^2(\mathbb{R}^n)} - \left\|\frac{x\cdot\nabla}{|x|_E}f + \frac{n-2}{2|x|_E}f\right\|^2_{L^2(\mathbb{R}^n)}, \tag{3.2.3}$$

which is (3.1.3).

Remark 6. *By dropping the last term on the right-hand side of* (3.2.3) *we get*

$$\left(\frac{n-2}{2}\right)^2 \left\|\frac{f}{|x|_E}\right\|^2_{L^2(\mathbb{R}^n)} \leq \left\|\frac{x\cdot\nabla}{|x|_E}f\right\|^2_{L^2(\mathbb{R}^n)} \leq \|\nabla f\|^2_{L^2(\mathbb{R}^n)} \tag{3.2.4}$$

yielding the classical Hardy inequality, where in the last line we have used the Schwartz inequality.

Proof of Theorem 3. We may assume that $N > 2$ since for $N = 2$ the identity (3.2.1) is trivial. By using the identity

$$\mathrm{div}_H\left(\frac{x'}{|x'|^\gamma}\right) = \frac{N-\gamma}{|x'|^\gamma} \tag{3.2.5}$$

and the divergence theorem one calculates

$$\int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^2}dx = \frac{1}{N-2}\int_{\mathbb{G}} |f(x)|^2 \,\mathrm{div_H}\left(\frac{x'}{|x'|^2}\right) dx. \tag{3.2.6}$$

Using the integration by parts we have

$$\begin{aligned} \int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^2}dx &= \frac{1}{N-2}\int_{\mathbb{G}} |f(x)|^2 \,\mathrm{div_H}\left(\frac{x'}{|x'|^2}\right) dx \\ &= -\frac{2}{N-2}\,\mathrm{Re}\int_{\mathbb{G}} f(x)\frac{\overline{x'\cdot\nabla_H f}}{|x'|^2}dx \\ &= -\frac{2}{N-2}\,\mathrm{Re}\int_{\mathbb{G}} \frac{f(x)}{|x'|}\frac{\overline{x'\cdot\nabla_H f}}{|x'|}dx. \end{aligned} \tag{3.2.7}$$

Setting the notations

$$c = \frac{1}{N-2}$$

and

$$u = \frac{f(x)}{|x'|}$$

and

$$v = \frac{x' \cdot \nabla_H}{|x'|} f$$

formula (3.2.6) can be presented as

$$\int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^2} dx = -2c \operatorname{Re}\langle u, v \rangle,$$

which is (2.2.2). By virtue of Lemma 1 it means that

$$\int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^2} dx = 4c^2 \|v\|^2 - \|u + 2cv\|^2$$
$$= \left(\frac{2}{N-2}\right)^2 \left\| \frac{x' \cdot \nabla_H}{|x'|} f \right\|^2_{L^2(\mathbb{G})} - \left\| \frac{f}{|x'|} + \frac{2}{N-2} \frac{x' \cdot \nabla_H}{|x'|} f \right\|^2_{L^2(\mathbb{G})},$$

that is,

$$\left(\frac{N-2}{2}\right)^2 \left\| \frac{f}{|x'|} \right\|^2_{L^2(\mathbb{G})} = \left\| \frac{x' \cdot \nabla_H}{|x'|} f \right\|^2_{L^2(\mathbb{G})} - \left\| \frac{x' \cdot \nabla_H}{|x'|} f + \frac{N-2}{2|x'|} f \right\|^2_{L^2(\mathbb{G})},$$

which implies Hardy's identity (3.2.1).

The proof of Theorem 3 is finished.

3.3 Logarithmic Hardy identity on the quasi-ball $B(0, R)$

In this section, we present L^2-Hardy identity of logarithmic type on the quasi-ball $B(0, R)$ in the setting of a homogeneous Lie group $\mathbb{G}$.

Theorem 4. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q and let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$. Let $Q \geq 2, 1 < p < Q$ and $B(0, R) \subset \mathbb{G}$ be the quasi-ball with radius $R > 0$. Then for all complex-valued functions $f \in C_0^\infty(B(0, R)\backslash\{0\})$ we have*

$$4 \left\| \mathcal{R}_{|x|} f \right\|^2_{L^2(B(0,R))} - \left\| \frac{f}{|x| \left(log \frac{R}{|x|}\right)} \right\|^2_{L^2(B(0,R))} - 2(Q-2) \int_{B(0,R)} \frac{|f|^2}{|x|^2 \left(log \frac{R}{|x|}\right)} dx$$
$$= \int_{B(0,R)} \left| \frac{f}{|x| \left(log \frac{R}{|x|}\right)} + 2\mathcal{R}_{|x|} f \right|^2 dx, \quad (3.3.1)$$

where $\mathcal{R}_{|x|}$ *is the radial derivative defined by* (2.2.1).

Proof of Theorem 4. Note that

$$\mathcal{R}_r\left(\frac{r^{Q-2}}{log\frac{R}{r}}\right)=\frac{(Q-2)r^{Q-3}}{log\frac{R}{r}}+\frac{r^{Q-3}}{\left(log\frac{R}{r}\right)^2},$$

then introducing polar coordinates $(r,y)=\left(|x|,\frac{x}{|x|}\right)\in(0,\infty)\times\wp$ on $\mathbb{G}$, where the quasi-sphere $\wp$, we have

$$\int_{B(0,R)}\frac{|f(x)|^2}{|x|^2\left(log\frac{R}{|x|}\right)^2}dx=\int_0^\infty\int_{\mathfrak{S}}\frac{|f(ry)|^2r^{Q-3}}{\left(log\frac{R}{r}\right)^2}d\sigma(y)dr$$
$$=\int_0^\infty\int_{\mathfrak{S}}\left(\mathcal{R}_r\left(\frac{r^{Q-2}}{log\frac{R}{r}}\right)-\frac{(Q-2)r^{Q-3}}{log\frac{R}{r}}\right)|f(ry)|^2d\sigma(y)dr.$$

Then the integrating by parts implies that

$$\int_{B(0,R)}\frac{|f(x)|^2}{|x|^2\left(log\frac{R}{|x|}\right)^2}dx=$$
$$=\int_0^\infty\int_{\mathfrak{S}}\left(\mathcal{R}_r\left(\frac{r^{Q-2}}{log\frac{R}{r}}\right)-\frac{(Q-2)r^{Q-3}}{log\frac{R}{r}}\right)|f(ry)|^2d\sigma(y)dr$$
$$=-2\mathrm{Re}\int_0^\infty\int_{\mathfrak{S}}f(ry)\overline{\frac{df(ry)}{dr}}\frac{r^{Q-2}}{log\frac{R}{r}}d\sigma(y)dr-(Q-2)\int_0^\infty\int_{\mathfrak{S}}\frac{r^{Q-3}}{log\frac{R}{r}}|f(ry)|^2d\sigma(y)dr$$
$$=-2\mathrm{Re}\int_{B(0,R)}\frac{f(x)\overline{\mathcal{R}_{|x|}f}}{|x|\left(log\frac{R}{|x|}\right)}dx-(Q-2)\int_{B(0,R)}\frac{|f(x)|^2}{|x|^2\left(log\frac{R}{|x|}\right)}dx. \quad (3.3.2)$$

Applying the notations

$$u:=u(x)=-2\mathcal{R}_{|x|}f$$

and

$$v:=v(x)=\frac{f(x)}{|x|\left(log\frac{R}{|x|}\right)}$$

formula (3.3.2) can be rewritten as

$$\int_{B(0,R)}|v|^2dx=\mathrm{Re}\int_{B(0,R)}v\overline{u}dx-(Q-2)\int_{B(0,R)}\frac{|f|^2}{|x|^2\left(log\frac{R}{|x|}\right)}dx. \quad (3.3.3)$$

By Lemma 2 and Proposition 1 for all L^p-integrable real-valued functions u and v

for $p=2$ we

$$\|u\|^2_{L^2(B(0,R))} - \|v\|^2_{L^2(B(0,R))} + 2\int_{B(0,R)} \left(|v|^2 - \operatorname{Re} v\overline{u}\right) dx$$

$$= \int_{B(0,R)} \left(|u|^2 + |v|^2 - 2\operatorname{Re} v\overline{u}\right) dx$$

$$= 2\int_{B(0,R)} \frac{\frac{|u|^2}{2} + \frac{1}{2}|v|^2 - vu}{|v-u|^2}|v-u|^2 dx = 2\int_{B(0,R)} |v-u|^2 dx,$$

Combining this with (3.3.3) one obtains

$$4\left\|\mathcal{R}_{|x|}f\right\|^2_{L^2(B(0,R))} - \left\|\frac{f}{|x|\left(log\frac{R}{|x|}\right)}\right\|^2_{L^2(B(0,R))} - 2(Q-2)\int_{B(0,R)} \frac{|f|^2}{|x|^2\left(log\frac{R}{|x|}\right)}dx$$

$$= \int_{B(0,R)} \left|\frac{f}{|x|\left(log\frac{R}{|x|}\right)} + 2\mathcal{R}_{|x|}f\right|^2 dx. \quad (3.3.4)$$

The proof of Theorem 4 is finished.

3.4 Applications

In this section we establish Caffarelli-Kohn-Nirenberg type inequalities and their proofs. The proof is relying on Hardy identities and Hölder's inequality.

3.4.1 Caffarelli-Kohn-Nirenberg type inequalities on $\mathbb{R}^k \times \mathbb{R}^{n-k}$

Theorem 5. *Let $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $2 \le k \le n$, $1 < q < \infty$, $0 < r < \infty$, with $2 + q \ge r$, $\delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{2}{r}\right]$ and $b, c \in \mathbb{R}$. Assume that*

$$\frac{\delta r}{2} + \frac{(1-\delta)r}{q} = 1$$

and

$$c = -\delta + b(1-\delta).$$

Then for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x'=0\})$, we have

$$\left(\frac{k-2}{2}\right)^{\delta} \||x'|_k^c f\|_{L^r(\mathbb{R}^n)} \le$$

$$\left[\left\|\frac{x'\cdot\nabla_k}{|x'|_k}f\right\|^2_{L^2(\mathbb{R}^n)} - \left\|\frac{x'\cdot\nabla_k}{|x'|_k}f + \frac{k-2}{2|x'|_k}f\right\|^2_{L^2(\mathbb{R}^n)}\right]^{\frac{\delta}{2}} \||x'|_k^b f\|^{1-\delta}_{L^q(\mathbb{R}^n)}. \quad (3.4.1)$$

Remark 7. *By dropping the non-negative term from* (3.4.1), *we get the following inequality:*

$$\left(\frac{k-2}{2}\right)^{\delta} \left\| |x'|_k^c f \right\|_{L^r(\mathbb{R}^n)}^2 \leq \left\| \frac{x' \cdot \nabla_k}{|x'|_k} f \right\|_{L^2(\mathbb{R}^n)}^{\delta} \left\| |x'|_k^b f \right\|_{L^q(\mathbb{R}^n)}^{1-\delta}, \; k \geq 3 \tag{3.4.2}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$.
Using Schwartz's inequality on the right hand side of (3.4.2), *we get*

$$\left(\frac{k-2}{2}\right)^{\delta} \left\| |x'|_k^c f \right\|_{L^r(\mathbb{R}^n)}^2 \leq \|\nabla_k f\|_{L^2(\mathbb{R}^n)}^{\delta} \left\| |x'|_k^b f \right\|_{L^q(\mathbb{R}^n)}^{1-\delta} \tag{3.4.3}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$. *Note that the inequality* (3.4.3), *hence* (3.4.1), *can be thought as an cylindrical extension of the Caffarelli-Kohn-Nirenberg inequalities* (1.1.11).

Remark 8. *When* $k = n$ *if we drop the last term on the right-hand side of* (3.4.1), *then our result gives the following improvement of the classical Caffarelli-Kohn-Nirenberg inequality for all complex-valued functions* $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$*:*

$$\begin{aligned} \left(\frac{n-2}{2}\right)^{\delta} \left\| |x|_E^c f \right\|_{L^r(\mathbb{R}^n)} &\leq \left\| \frac{x \cdot \nabla}{|x|_E} f \right\|_{L^2(\mathbb{R}^n)}^{\delta} \left\| |x|_E^b f \right\|_{L^q(\mathbb{R}^n)}^{1-\delta} \\ &\leq \|\nabla f\|_{L^2(\mathbb{R}^n)}^{\delta} \left\| |x|_E^b f \right\|_{L^q(\mathbb{R}^n)}^{1-\delta}, \end{aligned} \tag{3.4.4}$$

where in the last line we have used the Schwartz inequality. Compared to the classical Caffarelli-Kohn-Nirenberg inequality (1.1.11), *our version* (3.4.4) *is with an explicit constant, and actually the general case of our result* (3.4.1) *is obtained with remainder terms.*

Proof of Theorem 5. Case $\delta = 0$. In this case, we have $q = r$ and $b = c$ by $\frac{\delta r}{2} + \frac{(1-\delta)r}{q} = 1$ and $c = -\delta + b(1-\delta)$, respectively. Then, the inequality (3.4.1) becomes the trivial estimate

$$\left\| |x'|_k^c f \right\|_{L^r(\mathbb{R}^n)} \leq \left\| |x'|_k^b f \right\|_{L^q(\mathbb{R}^n)}.$$

Case $\delta = 1$. Notice that in this case, $r = 2$ and $c = -1$. It means that

$$\left(\frac{k-2}{2}\right) \left\| |x'|_k^c f \right\|_{L^r(\mathbb{R}^n)} \leq \left[\left\| \frac{x' \cdot \nabla_k}{|x'|_k} f \right\|_{L^2(\mathbb{R}^n)}^2 - \left\| \frac{x' \cdot \nabla_k}{|x'|_k} f + \frac{k-2}{2|x'|_k} f \right\|_{L^2(\mathbb{R}^n)}^2 \right]^{\frac{1}{2}},$$

which is the inequality (3.1.1) that we already have proved in the previous Section 3.1.

Case $\delta \in (0,1) \cap \left[\frac{r-q}{r}, \frac{2}{r}\right]$. Taking into account $c = -\delta + b(1-\delta)$, a direct calculation gives

$$\||x'|_k^c f\|_{L^r(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |x'|_k^{cr}|f(x)|^r dx\right)^{\frac{1}{r}} = \left(\int_{\mathbb{R}^n} \frac{|f(x)|^{\delta r}}{|x'|_k^{\delta r}} \cdot \frac{|f(x)|^{(1-\delta)r}}{|x'|_k^{br(1-\delta)}} dx\right)^{\frac{1}{r}}.$$

Since we have $\delta \in (0,1) \cap \left[\frac{r-q}{r}, \frac{2}{r}\right]$ and $2+q \leq r$, then by using Hölder's inequality for $\frac{\delta r}{2} + \frac{(1-\delta)r}{q} = 1$, we get

$$\||x'|_k^c f\|_{L^r(\mathbb{R}^n)} \leq \left(\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x'|_{|k|}^2} dx\right)^{\frac{\delta}{2}} \left(\int_{\mathbb{R}^n} \frac{|f(x)|^q}{|x'|_k^{-2bq}} dx\right)^{\frac{(1-\delta)}{q}}$$
$$= \left\|\frac{f}{|x'|_k}\right\|_{L^2(\mathbb{R}^n)}^{\delta} \cdot \left\|\frac{f}{|x'|_k^{-b}}\right\|_{L^q(\mathbb{R}^n)}^{1-\delta}.$$

By Theorem 2 we get the following Caffarelli-Kohn-Nirenberg type inequality:

$$\left(\frac{k-2}{2}\right)^{\delta} \||x'|_k^c f\|_{L^r(\mathbb{R}^n)} \leq$$
$$\times \left[\left\|\frac{x' \cdot \nabla_k}{|x'|_k} f\right\|_{L^2(\mathbb{R}^n)}^2 - \left\|\frac{x' \cdot \nabla_k}{|x'|_k} f + \frac{k-2}{2|x'|_k} f\right\|_{L^2(\mathbb{R}^n)}^2\right]^{\frac{\delta}{2}} \||x'|_k^b f\|_{L^q(\mathbb{R}^n)}^{1-\delta},$$

yielding (3.4.1).

The proof of Theorem 5 is finished.

3.4.2 Caffarelli-Kohn-Nirenberg type inequalities on stratified Lie groups

Theorem 6. *Let $\mathbb{G}$ be a stratified Lie group with $N \geq 2$ being the dimension of the first stratum. We denote by x' the variables from the first stratum of $\mathbb{G}$. Let $1 < q < \infty$, $0 < r < \infty$ with $2+q \geq r$ and $\delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{2}{r}\right]$ and $b, c \in \mathbb{R}$. Assume that $\frac{\delta r}{2} + \frac{(1-\delta)r}{q} = 1$ and $c = -\delta + b(1-\delta)$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ we have the following Caffarelli-Kohn-Nirenberg type inequalities:*

$$\left(\frac{N-2}{2}\right)^{\delta} \||x'|^c f\|_{L^r(\mathbb{G})} \leq$$
$$\times \left[\left\|\frac{x' \cdot \nabla_H}{|x'|} f\right\|_{L^2(\mathbb{G})}^2 - \left\|\frac{x' \cdot \nabla_H}{|x'|} f + \frac{N-2}{2|x'|} f\right\|_{L^2(\mathbb{G})}^2\right]^{\frac{\delta}{2}} \||x'|^b f\|_{L^q(\mathbb{G})}^{1-\delta}. \quad (3.4.5)$$

Remark 9. *By dropping the last term on the right-hand side of* (3.4.5) *we get the* *following inequality:*

$$\left(\frac{N-2}{2}\right)^{\delta}\||x'|^{c}f\|^{2}_{L^{r}(\mathbb{G})}\leq\left\|\frac{x'\cdot\nabla_{H}}{|x'|}f\right\|^{\delta}_{L^{2}(\mathbb{G})}\||x'|^{b}f\|^{1-\delta}_{L^{q}(\mathbb{G})},\ N\geq 3$$

for all complex-valued functions $f\in C_0^{\infty}(\mathbb{G}\backslash\{x'=0\})$.

Remark 10. *In the Abelian case* $\mathbb{G}=(\mathbb{R}^n,+)$, *we have* $N=n,\nabla_H=\nabla=(\partial_{x_1},\ldots,\partial_{x_n})$, *so* (3.4.5) *implies the Caffarelli-Kohn-Nirenberg type inequality on* $\mathbb{R}^n$. *Let* $1<q<\infty$ *with* $2+q\geq r$ *and* $\delta\in[0,1]\cap\left[\frac{r-q}{r},\frac{2}{r}\right]$ *and* $b,c\in\mathbb{R}$. *Assume that* $\frac{\delta r}{2}+\frac{(1-\delta)r}{q}=1$ *and* $c=-\delta+b(1-\delta)$. *Then by dropping a non-negative term and applying the Schwarz inequality we obtain*

$$\begin{aligned}
&\||x|_E^{c}f\|_{L^r(\mathbb{R}^n)}\\
&\leq\left(\frac{2}{n-2}\right)^{\delta}\left[\left\|\frac{x\cdot\nabla}{|x|_E}f\right\|^{2}_{L^2(\mathbb{R}^n)}-\left\|\frac{x'\cdot\nabla}{|x|_E}f+\frac{n-2}{2|x|_E}f\right\|^{2}_{L^2(\mathbb{R}^n)}\right]^{\frac{\delta}{2}}\||x|_E^{b}f\|^{1-\delta}_{L^q(\mathbb{R}^n)}\\
&\leq\left(\frac{2}{n-2}\right)^{\delta}\left\|\frac{x\cdot\nabla}{|x|_E}f\right\|^{\delta}_{L^2(\mathbb{R}^n)}\||x|_E^{b}f\|^{1-\delta}_{L^q(\mathbb{R}^n)}\\
&\leq\left(\frac{2}{n-2}\right)^{\delta}\|\nabla f\|^{\delta}_{L^2(\mathbb{R}^n)}\||x|_E^{b}f\|^{1-\delta}_{L^q(\mathbb{R}^n)}
\end{aligned}\tag{3.4.6}$$

for all complex-valued functions $f\in C_0^{\infty}(\mathbb{R}^n\backslash\{0\})$, *where* $|x|_E=\sqrt{x_1^2+\ldots+x_n^2}$ *is the usual Euclidean norm on* $\mathbb{R}^n$. *In the special case* $q=r=2$, $b=-n/2$, $c=-\delta-n(1-\delta)/2$ *the inequality* (3.4.6) *takes the form*

$$\left(\frac{n-2}{2}\right)^{\delta}\left\|\frac{f}{|x|_E^{\frac{\delta+n(1-\delta)}{2}}}\right\|_{L^2(\mathbb{R}^n)}\leq\|\nabla f\|^{\delta}_{L^2(\mathbb{R}^n)}\left\|\frac{f}{|x|_E^{\frac{n}{2}}}\right\|^{1-\delta}_{L^2(\mathbb{R}^n)}.\tag{3.4.7}$$

Since we have $1/2+b/n=0$ *here, we observe that the condition* (1.1.10) *is not satisfied, then the inequality* (3.4.7) *is not covered by the classical Caffarelli-Kohn-Nirenberg inequality, Theorem 1. We also refer to [40, Theorem 4.1] for a similar type results on stratified Lie groups.*

Proof of Theorem 6. Case $\delta=0$. In this case, we have $q=r$ and $b=c$ by $\frac{\delta r}{2}+\frac{(1-\delta)r}{q}=1$ and $c=-\delta+b(1-\delta)$, respectively. Then, the inequality (3.4.5) becomes the trivial estimate

$$\||x'|^{c}f\|_{L^r(\mathbb{G})}\leq\||x'|^{b}f\|_{L^q(\mathbb{G})}.$$

Case $\delta=1$. Notice that in this case, $r=2$ and $c=-1$. It means that

$$\left(\frac{N-2}{2}\right)\||x'|^{c}f\|_{L^r(\mathbb{G})}\leq\left[\left\|\frac{x'\cdot\nabla_H}{|x'|}f\right\|^{2}_{L^2(\mathbb{G})}-\left\|\frac{x'\cdot\nabla_H}{|x'|}f+\frac{N-2}{2|x'|}f\right\|^{2}_{L^2(\mathbb{G})}\right]^{\frac{1}{2}},$$

which is (3.2.1) from the previous Section 3.2.

Case $\delta \in (0,1) \cap \left[\frac{r-q}{r}, \frac{2}{r}\right]$. Taking into account $c = -\delta + b(1-\delta)$, a direct calculation gives

$$\| |x'|^c f \|_{L^r(\mathbb{G})} = \left(\int_{\mathbb{G}} |x'|^{cr} |f(x)|^r dx \right)^{\frac{1}{r}} = \left(\int_{\mathbb{G}} \frac{|f(x)|^{\delta r}}{|x'|^{\delta r}} \cdot \frac{|f(x)|^{(1-\delta)r}}{|x'|^{-br(1-\delta)}} dx \right)^{\frac{1}{r}}.$$

Since we have $\delta \in (0,1) \cap \left[\frac{r-q}{r}, \frac{2}{r}\right]$ and $2+q \leq r$, then by using Hölder's inequality for $\frac{\delta r}{2} + \frac{(1-\delta)r}{q} = 1$, we get

$$\| |x'|^c f \|_{L^r(\mathbb{G})} \leq \left(\int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^2} dx \right)^{\frac{\delta}{2}} \left(\int_{\mathbb{G}} \frac{|f(x)|^q}{|x'|^{-bq}} dx \right)^{\frac{(1-\delta)}{q}} = \left\| \frac{f}{|x'|} \right\|^{\delta}_{L^2(\mathbb{G})} \cdot \left\| \frac{f}{|x'|^{-b}} \right\|^{1-\delta}_{L^q(\mathbb{G})}.$$

By Theorem 4 we get the following Caffarelli-Kohn-Nirenberg type inequality:

$$\left(\frac{N-2}{2} \right)^{\delta} \| |x'|^c f \|_{L^r(\mathbb{G})}$$
$$\leq \left[\left\| \frac{x' \cdot \nabla_H}{|x'|} f \right\|^2_{L^2(\mathbb{G})} - \left\| \frac{x' \cdot \nabla_H}{|x'|} f + \frac{N-2}{2|x'|} f \right\|^2_{L^2(\mathbb{G})} \right]^{\frac{\delta}{2}} \| |x'|^b f \|^{1-\delta}_{L^q(\mathbb{G})},$$

which is (3.4.5).

The proof of Theorem 6 is finished.

Chapter 4

L^p-improved Hardy identities with remainder terms

Here we discuss L^p extensions of the identities from Chapter 3 for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x'=0\})$. However, as functional inequalities we also obtain L^p versions of the inequalities from the previous chapter for any complex-valued function $f \in C_0^\infty(\mathbb{R}^n \backslash \{x'=0\})$. As consequences, we derive the Heisenberg-Pauli-Weyl type uncertainty principle and L^r-L^p-L^q Caffarelli-Kohn-Nirenberg type inequalities.

4.1 L^p inequality on $\mathbb{R}^k \times \mathbb{R}^{n-k}$

Theorem 7. *Let $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ and $1 < p < \infty$ and $k \geq p$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x'=0\})$, we have*

$$\left(\frac{k-p}{p}\right) \left\| \frac{f}{|x'|_k} \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{x' \cdot \nabla_k}{|x'|_k} f \right\|_{L^p(\mathbb{R}^n)}, \tag{4.1.1}$$

where $|\cdot|_k$ is the Euclidean norm on $\mathbb{R}^k$.

Proof of Theorem 7. We may assume that $k > p$ since for $k = p$ the inequality (4.1.1) is trivial. By using the identity (2.1.3) and the divergence theorem one calculates

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x'|^p} dx = \frac{1}{k-p} \int_{\mathbb{R}^n} |f(x)|^p \operatorname{div}_k \left(\frac{x'}{|x'|^p}\right) dx.$$

Integrating by parts implies that

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x'|^p} dx &= \frac{1}{k-p} \int_{\mathbb{R}^n} |f(x)|^p \operatorname{div_k} \left(\frac{x'}{|x'|^p} \right) dx \\ &= -\frac{p}{k-p} \operatorname{Re} \int_{\mathbb{R}^n} f(x)|f(x)|^{p-2} \frac{\overline{x' \cdot \nabla_k f}}{|x'|^p} dx \\ &\leq \left| \frac{p}{k-p} \right| \int_{\mathbb{R}^n} \frac{|f(x)|^{p-1}}{|x'|^p} |x' \cdot \nabla_k f| dx \\ &\leq \left| \frac{p}{k-p} \right| \int_{\mathbb{R}^n} \frac{|f(x)|^{p-1}}{|x'|^{p-1}} \frac{|x' \cdot \nabla_k f|}{|x'|} dx.\end{aligned}$$

By the Hölder inequality, it follows that

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x'|^p} dx \leq \left| \frac{p}{k-p} \right| \left(\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x'|^p} dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} \frac{|x' \cdot \nabla_k f|^p}{|x'|^p} dx \right)^{\frac{1}{p}},$$

which implies (4.1.1).

4.2 L^p identity on $\mathbb{R}^k \times \mathbb{R}^{n-k}$

Theorem 8. *Let $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ with $1 < p < \infty$ and $k > p$. Then for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$, we have*

$$\left(\frac{k-p}{p} \right)^p \left\| \frac{f}{|x'|_k} \right\|_{L^p(\mathbb{R}^n)}^p = \left\| \frac{x' \cdot \nabla_k}{|x'|_k} f \right\|_{L^p(\mathbb{R}^n)}^p - p \left(\frac{k-p}{p} \right)^p \int_{\mathbb{R}^n} I_p \left(\frac{f}{|x'|_k}, -\frac{p}{k-p} \frac{x' \cdot \nabla_k}{|x'|_k} f \right) \left| \frac{f}{|x'|_k} + \frac{p}{k-p} \frac{x' \cdot \nabla_k}{|x'|_k} f \right|^2 dx, \quad (4.2.1)$$

where

$$I_p(h, g) = (p-1) \int_0^1 |\xi h + (1-\xi) g|^{p-2} \xi d\xi.$$

Remark 11. *By dropping the last term of (4.2.1), we obtain the following inequality for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$:*

$$\left(\frac{k-p}{p} \right)^p \left\| \frac{f}{|x'|_k} \right\|_{L^p(\mathbb{R}^n)}^p \leq \left\| \frac{x' \cdot \nabla_k}{|x'|_k} f \right\|_{L^p(\mathbb{R}^n)}^p. \quad (4.2.2)$$

Here, using the Schwartz inequality on the right hand side of (4.2.2), we obtain the following Hardy inequality:

$$\left(\frac{k-p}{p} \right)^p \left\| \frac{f}{|x'|_k} \right\|_{L^p(\mathbb{R}^n)}^p \leq \|\nabla_k f\|_{L^p(\mathbb{R}^n)}^p$$

for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$.

Proof of Theorem 8. By using the identity (2.1.3) and the divergence theorem one calculates

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x'|_k^p} dx = \frac{1}{k-p} \int_{\mathbb{R}^n} |f(x)|^p \operatorname{div}_k \left(\frac{x'}{|x'|_k^p} \right) dx$$
$$= -\frac{p}{k-p} \int_{\mathbb{R}^n} f|f|^{p-2} \frac{x' \cdot \nabla_k f}{|x'|_k^p} dx.$$

For every real-valued function $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$ it becomes

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x'|_k^p} dx = -\frac{p}{k-p} \int_{\mathbb{R}^n} \frac{f|f|^{p-2}}{|x'|_k} \cdot \frac{x' \cdot \nabla_k f}{|x'|_k^{\frac{p}{p'}}} dx. \tag{4.2.3}$$

Using notations

$$u := u(x) = -\frac{p}{k-p} \frac{x' \cdot \nabla_k}{|x'|_k} f$$

and

$$v := v(x) = \frac{f}{|x'|_k}$$

formula (4.2.3) can be presented as

$$\|v\|^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |v|^{p-2} vu dx. \tag{4.2.4}$$

By Lemma 2 for all L^p-integrable real-valued functions u and v we have

$$\|u\|^p_{L^p(\mathbb{R}^n)} - \|v\|^p_{L^p(\mathbb{R}^n)} = p \int_{\mathbb{R}^n} I_p(v, u) |v - u|^2 dx,$$

where

$$I_p(h, g) = (p-1) \int_0^1 |\xi h + (1-\xi) g|^{p-2} \xi d\xi.$$

That is,

$$\left\| \frac{f}{|x'|_k} \right\|^p_{L^p(\mathbb{R}^n)} = \left(\frac{p}{k-p} \right)^p \left\| \frac{x' \cdot \nabla_k}{|x'|_k} f \right\|^p_{L^p(\mathbb{R}^n)} -$$
$$p \int_{\mathbb{R}^n} I_p \left(\frac{f}{|x'|_k}, -\frac{p}{k-p} \frac{x' \cdot \nabla_k}{|x'|_k} f \right) \left| \frac{f}{|x'|_k} + \frac{p}{k-p} \frac{x' \cdot \nabla_k}{|x'|_k} f \right|^2 dx.$$

The identity (4.2.1) is proved.

The proof of Theorem 8 is completed.

4.3 L^p identity on stratified Lie groups

In this section we establish L^p-Hardy identity in the setting of stratified Lie groups $\mathbb{G}$.

Theorem 9. *Let $\mathbb{G}$ be a stratified Lie group with N being the dimension of the first stratum. Let $N > p$ and $1 < p < \infty$. We denote by x' the variables from the first stratum of $\mathbb{G}$. Then for all real-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$, we have*

$$\left(\frac{N-p}{p}\right)^p \left\|\frac{f}{|x'|}\right\|^p_{L^p(\mathbb{G})} = \left\|\frac{x'\cdot\nabla_H}{|x'|}f\right\|^p_{L^p(\mathbb{G})} -$$
$$p\left(\frac{N-p}{p}\right)^p \int_{\mathbb{G}} I_p\left(\frac{f}{|x'|}, -\frac{p}{N-p}\frac{x'\cdot\nabla_H}{|x'|}f\right)\left|\frac{f}{|x'|}+\frac{p}{N-p}\frac{x'\cdot\nabla_H}{|x'|}f\right|^2 dx, \quad (4.3.1)$$

where

$$I_p(h,g) = (p-1)\int_0^1 |\xi h + (1-\xi)g|^{p-2}\xi d\xi.$$

Remark 12. *By dropping the last term on the right-hand side of* (4.3.1), *we obtain the following inequality for all real-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{x'=0\})$:*

$$\left(\frac{N-p}{p}\right)^p \left\|\frac{f}{|x'|}\right\|^p_{L^p(\mathbb{G})} \leq \left\|\frac{x'\cdot\nabla_H}{|x'|}f\right\|^p_{L^p(\mathbb{G})}. \quad (4.3.2)$$

Remark 13. *In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, ..., \partial_{x_n})$, so* (4.3.1) *implies the following L^p-Hardy identity:*

$$\left(\frac{n-p}{p}\right)^p \left\|\frac{f}{|x|_E}\right\|^p_{L^p(\mathbb{R}^n)} = \left\|\frac{x\cdot\nabla}{|x|_E}f\right\|^p_{L^p(\mathbb{R}^n)} -$$
$$p\left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} I_p\left(\frac{f}{|x|_E}, -\frac{p}{n-p}\frac{x\cdot\nabla}{|x|_E}f\right)\left|\frac{f}{|x|_E}+\frac{p}{n-p}\frac{x\cdot\nabla}{|x|_E}f\right|^2 dx, \quad (4.3.3)$$

for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$.

Remark 14. *By dropping the last term on the right-hand side of* (4.3.3) *we get*

$$\left(\frac{n-p}{p}\right)^p \left\|\frac{f}{|x|_E}\right\|^p_{L^p(\mathbb{R}^n)} \leq \left\|\frac{x\cdot\nabla}{|x|_E}f\right\|^p_{L^p(\mathbb{R}^n)} \leq \|\nabla f\|^p_{L^p(\mathbb{R}^n)}, \quad (4.3.4)$$

for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$.

Proof of Theorem 9. By using the identity (2.1.3) we get

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^p}dx = \frac{1}{N-p}\int_{\mathbb{G}} |f(x)|^p \operatorname{div}_{\mathrm{H}}\left(\frac{x'}{|x'|^p}\right) dx.$$

Theorem 9. *Let $\mathbb{G}$ be a stratified Lie group with N being the dimension of the first stratum. Let $N > p$ and $1 < p < \infty$. We denote by x' the variables from the first stratum of $\mathbb{G}$. Then for all real-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$, we have*

$$\left(\frac{N-p}{p}\right)^p \left\|\frac{f}{|x'|}\right\|^p_{L^p(\mathbb{G})} = \left\|\frac{x'\cdot\nabla_H}{|x'|}f\right\|^p_{L^p(\mathbb{G})} -$$
$$p\left(\frac{N-p}{p}\right)^p \int_{\mathbb{G}} I_p\left(\frac{f}{|x'|}, -\frac{p}{N-p}\frac{x'\cdot\nabla_H}{|x'|}f\right)\left|\frac{f}{|x'|}+\frac{p}{N-p}\frac{x'\cdot\nabla_H}{|x'|}f\right|^2 dx, \quad (4.3.1)$$

where

$$I_p(h,g) = (p-1)\int_0^1 |\xi h + (1-\xi)g|^{p-2}\xi d\xi.$$

Remark 12. *By dropping the last term on the right-hand side of (4.3.1), we obtain the following inequality for all real-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$:*

$$\left(\frac{N-p}{p}\right)^p \left\|\frac{f}{|x'|}\right\|^p_{L^p(\mathbb{G})} \le \left\|\frac{x'\cdot\nabla_H}{|x'|}f\right\|^p_{L^p(\mathbb{G})}. \quad (4.3.2)$$

Remark 13. *In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, ..., \partial_{x_n})$, so (4.3.1) implies the following L^p-Hardy identity:*

$$\left(\frac{n-p}{p}\right)^p \left\|\frac{f}{|x|_E}\right\|^p_{L^p(\mathbb{R}^n)} = \left\|\frac{x\cdot\nabla}{|x|_E}f\right\|^p_{L^p(\mathbb{R}^n)} -$$
$$p\left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} I_p\left(\frac{f}{|x|_E}, -\frac{p}{n-p}\frac{x\cdot\nabla}{|x|_E}f\right)\left|\frac{f}{|x|_E}+\frac{p}{n-p}\frac{x\cdot\nabla}{|x|_E}f\right|^2 dx, \quad (4.3.3)$$

for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$.

Remark 14. *By dropping the last term on the right-hand side of (4.3.3) we get*

$$\left(\frac{n-p}{p}\right)^p \left\|\frac{f}{|x|_E}\right\|^p_{L^p(\mathbb{R}^n)} \le \left\|\frac{x\cdot\nabla}{|x|_E}f\right\|^p_{L^p(\mathbb{R}^n)} \le \|\nabla f\|^p_{L^p(\mathbb{R}^n)}, \quad (4.3.4)$$

for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$.

Proof of Theorem 9. By using the identity (2.1.3) we get

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^p}dx = \frac{1}{N-p}\int_{\mathbb{G}} |f(x)|^p \operatorname{div}_H\left(\frac{x'}{|x'|^p}\right) dx.$$

Integrating by parts

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^p} dx = \frac{1}{N-p} \int_{\mathbb{G}} |f(x)|^p \operatorname{div}_{\mathrm{H}} \left(\frac{x'}{|x'|^p} \right) dx$$

$$= -\frac{p}{N-p} \int_{\mathbb{G}} f|f|^{p-2} \frac{x' \cdot \nabla_H f}{|x'|^p} dx.$$

For all real-valued functions $f \in C_0^\infty(\mathbb{G} \backslash \{x' = 0\})$ it becomes

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^p} dx = -\frac{p}{N-p} \int_{\mathbb{G}} \frac{f|f|^{p-2}}{|x'|} \cdot \frac{x' \cdot \nabla_H f}{|x'|^{\frac{p}{p'}}} dx. \tag{4.3.5}$$

Using notations

$$u := u(x) = -\frac{p}{N-p} \frac{x' \cdot \nabla_H}{|x'|} f$$

and

$$v := v(x) = \frac{f}{|x'|}$$

formula (4.3.5) can be rewritten as

$$\|v\|_{L^p(\mathbb{G})}^p = \int_{\mathbb{G}} |v|^{p-2} v u dx. \tag{4.3.6}$$

By Lemma 2 for all L^p-integrable real-valued functions u and v we have

$$\|u\|_{L^p(\mathbb{G})}^p - \|v\|_{L^p(\mathbb{G})}^p = p \int_{\mathbb{G}} I_p(v, u) |v - u|^2 dx,$$

where

$$I_p(h, g) = (p-1) \int_0^1 |\xi h + (1-\xi) g|^{p-2} \xi d\xi.$$

Summing up all above we arrive at

$$\left\| \frac{f}{|x'|} \right\|_{L^p(\mathbb{G})}^p = \left(\frac{p}{N-p} \right)^p \left\| \frac{x' \cdot \nabla_H}{|x'|} f \right\|_{L^p(\mathbb{G})}^p -$$
$$p \int_{\mathbb{G}} I_p \left(\frac{f}{|x'|}, -\frac{p}{N-p} \frac{x' \cdot \nabla_H}{|x'|} f \right) \left| \frac{f}{|x'|} + \frac{p}{N-p} \frac{x' \cdot \nabla_H}{|x'|} f \right|^2 dx,$$

which gives (4.3.1).

The proof of Theorem 9 is finished.

4.4 Logarithmic Hardy identity on the quasi-ball $B(0, R)$

In this section, we present L^p-Hardy identity of logarithmic type on the quasi-ball $B(0, R)$ in the setting of a homogeneous Lie group $\mathbb{G}$.

Theorem 10. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q and let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$. Let $1 < p < Q$ and $B(0, R) \subset \mathbb{G}$ be a quasi-ball with radius $R > 0$. Then for all real-valued functions $f \in C_0^\infty(B(0, R)\backslash\{0\})$ we have*

$$\left(\frac{p}{p-1}\right)^p \|\mathcal{R}_{|x|} f\|^p_{L^p(B(0,R))} - \left\| \frac{f}{|x| \left(log \frac{R}{|x|}\right)} \right\|^p_{L^p(B(0,R))}$$

$$-p\left(\frac{Q-p}{p-1}\right) \int_{B(0,R)} \frac{|f|^p}{|x|^p \left(log\frac{R}{|x|}\right)^{p-1}} dx = p \int_{B(0,R)} I_p \left(\frac{f}{|x|\left(log\frac{R}{|x|}\right)}, -\frac{p}{p-1}\mathcal{R}_{|x|}f \right)$$

$$\times \left| \frac{f}{|x|\left(log\frac{R}{|x|}\right)} + \frac{p}{p-1}\mathcal{R}_{|x|}f \right|^2 dx, \quad (4.4.1)$$

where $\mathcal{R}_{|x|}$ is the radial derivative defined by (2.2.1).

Proof of Theorem 10. Note that

$$\mathcal{R}_r \left(\frac{r^{Q-p}}{(log\frac{R}{r})^{p-1}} \right) = \frac{(Q-p)r^{Q-1-p}}{(log\frac{R}{r})^{p-1}} + (p-1)\frac{r^{Q-p-1}}{(log\frac{R}{r})^p},$$

then introducing polar coordinates $(r, y) = \left(|x|, \frac{x}{|x|}\right) \in (0, \infty) \times \wp$ on $\mathbb{G}$, where the quasi-sphere $\wp$, we have

$$\int_{B(0,R)} \frac{|f(x)|^p}{|x|^p \left(log\frac{R}{|x|}\right)^p} dx = \int_0^\infty \int_{\mathfrak{S}} \frac{|f(ry)|^p r^{Q-1-p}}{(log\frac{R}{r})^p} d\sigma(y) dr$$

$$= \frac{1}{p-1} \int_0^\infty \int_{\mathfrak{S}} \left(\mathcal{R}_r \left(\frac{r^{Q-p}}{(log\frac{R}{r})^{p-1}} \right) - \frac{(Q-p)r^{Q-1-p}}{(log\frac{R}{r})^{p-1}} \right) |f(ry)|^p d\sigma(y) dr.$$

Then the integrating by parts implies that

$$\int_0^\infty \int_{\mathfrak{S}} \frac{|f(ry)|^p r^{Q-1-p}}{(log\frac{R}{r})^p} d\sigma(y)dr$$

$$= \frac{1}{p-1}\int_0^\infty \int_{\mathfrak{S}} \left(\mathcal{R}_r \left(\frac{r^{Q-p}}{(log\frac{R}{r})^{p-1}} \right) - \frac{(Q-p)r^{Q-1-p}}{(log\frac{R}{r})^{p-1}} \right) |f(ry)|^p d\sigma(y)dr$$

$$= -\frac{p}{p-1}\int_0^\infty \int_{\mathfrak{S}} f(ry)|f(ry)|^{p-2}\frac{df(ry)}{dr} \frac{r^{Q-p}}{(log\frac{R}{r})^{p-1}} d\sigma(y)dr$$

$$- \frac{(Q-p)}{p-1}\int_0^\infty \int_{\mathfrak{S}} \frac{r^{Q-1-p}}{(log\frac{R}{r})^{p-1}} |f(ry)|^p d\sigma(y)dr$$

$$= -\frac{p}{p-1}\int_{B(0,R)} \frac{f(x)|f(x)|^{p-2}\mathcal{R}_{|x|}f}{|x|^{p-1}\left(log\frac{R}{|x|}\right)^{p-1}}dx - \frac{(Q-p)}{p-1}\int_{B(0,R)} \frac{|f(x)|^p}{|x|^p\left(log\frac{R}{|x|}\right)^{p-1}}dx. \tag{4.4.2}$$

Applying notations

$$u := u(x) = -\frac{p}{p-1}\mathcal{R}_{|x|}f$$

and

$$v := v(x) = \frac{f}{|x|\left(log\frac{R}{|x|}\right)}$$

formula (4.4.2) can be rewritten as

$$\int_{B(0,R)} |v|^p dx = \int_{B(0,R)} |v|^{p-2}vu dx - \frac{Q-p}{p-1}\int_{B(0,R)} \frac{|f(x)|^p}{|x|^p\left(log\frac{R}{|x|}\right)^{p-1}}dx. \tag{4.4.3}$$

By Lemma 2.2.3 and Proposition 1 for all L^p-integrable real-valued functions u and v we have

$$\|u\|^p_{L^p(B(0,R))} - \|v\|^p_{L^p(B(0,R))} + p\int_{B(0,R)} (|v|^p - |v|^{p-2}vu)dx$$

$$= \int_{B(0,R)} (|u|^p + (p-1)|v|^p - p|v|^{p-2}vu)dx$$

$$= p\int_{B(0,R)} \frac{\frac{|u|^p}{p} + \frac{p-1}{p}|v|^p - |v|^{p-2}vu}{|v-u|^2}|v-u|^2 dx = p\int_{B(0,R)} I_p(v,u)|v-u|^2 dx,$$

where

$$I_p(h,g) = (p-1)\int_0^1 |\xi h + (1-\xi)g|^{p-2}\xi d\xi.$$

Combining this with (4.4.3) we arrive at

$$\left(\frac{p}{p-1}\right)^p \|\mathcal{R}_{|x|}\|^p_{L^p(B(0,R))} - \left\|\frac{f}{|x|\left(log\frac{R}{|x|}\right)}\right\|^p_{L^p(B(0,R))} - p\left(\frac{Q-p}{p-1}\right)\int_{B(0,R)} \frac{|f|^p}{|x|^p\left(log\frac{R}{|x|}\right)^{p-1}}dx$$
$$= p\int_{B(0,R)} I_p\left(\frac{f}{|x|\left(log\frac{R}{|x|}\right)}, -\frac{p}{p-1}\mathcal{R}_{|x|}f\right)\left|\frac{f}{|x|\left(log\frac{R}{|x|}\right)} + \frac{p}{p-1}\mathcal{R}_{|x|}f\right|^2 dx. \tag{4.4.4}$$

The equality (4.4.1) is proved.

4.5 Applications

Here, in this section we give applications of the cylindrical improved Hardy identities and inequalities from the previous sections in Heisenberg-Paul-Weyl type uncertainty principle and Caffarelli-Kohn-Nirenberg type inequalities.

4.5.1 Heisenberg-Paul-Weyl type uncertainty principle

As a by-product of (4.1.1), we obtain the Heisenberg-Paul-Weyl type uncertainty principle.

Corollary 1. *Let $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $2 \le k \le n$, $1 < p, q < \infty$. Let $1 < p \le k$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$ we have*

$$\left(\int_{\mathbb{R}^n}\left|\frac{x'\cdot\nabla_k}{|x'|}f\right|^p dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}|x'|^q|f|^q dx\right)^{\frac{1}{q}} \ge \frac{k-p}{p}\int_{\mathbb{R}^n}|f|^2 dx. \tag{4.5.1}$$

Proof of Corollary 1. From the inequality (4.1.1) we get

$$\left(\int_{\mathbb{R}^n}\left|\frac{x'\cdot\nabla_k}{|x'|_k}f\right|^p dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}|x'|_k^q|f|^q dx\right)^{\frac{1}{q}} \ge \frac{k-p}{p}\left(\int_{\mathbb{R}^n}\frac{|f|^p}{|x'|_k^p}dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}|x'|_k^q|f|^q dx\right)^{\frac{1}{q}}$$
$$\ge \frac{k-p}{p}\int_{\mathbb{R}^n}|f|^2 dx,$$

where we have used Hölder's inequality in the last line. This shows (4.5.1).

4.5.2 Caffarelli-Kohn-Nirenberg type inequalities on $\mathbb{R}^k \times \mathbb{R}^{n-k}$

Now we discuss the cylindrical Caffarelli-Kohn-Nirenberg type inequality. The proof is relying on Hardy identities and the Hölder inequality.

Theorem 11. *Let $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $2 \le k \le n$, $k > p$, $1 < p, q < \infty$, $0 < r < \infty$, with $p + q \ge r$, $\delta \in [0,1] \cap \left[\frac{r-q}{r}, \frac{p}{r}\right]$ and $b, c \in \mathbb{R}$. Assume that*

$$\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$$

and

$$c = -\delta + b(1-\delta).$$

Then for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$, we have

$$\left(\frac{k-p}{p}\right)^{\delta} \left\| |x'|_k^c f \right\|_{L^r(\mathbb{R}^n)} \le \left[\left\| \frac{x' \cdot \nabla_k}{|x'|_k} f \right\|^p_{L^p(\mathbb{R}^n)} - p \left(\frac{k-p}{p}\right)^p \right.$$
$$\left. \times \int_{\mathbb{R}^n} I_p \left(\frac{f}{|x'|_k}, -\frac{p}{k-p} \frac{x' \cdot \nabla_k}{|x'|_k} f \right) \left| \frac{f}{|x'|_k} + \frac{p}{k-p} \frac{x' \cdot \nabla_k}{|x'|_k} f \right|^2 dx \right]^{\frac{\delta}{p}} \left\| |x'|_k^p f \right\|^{1-\delta}_{L^q(\mathbb{R}^n)}. \tag{4.5.2}$$

Remark 15. *By dropping the last term on the right-hand side of* (4.5.2), *we obtain*

$$\left(\frac{k-p}{p}\right)^{\delta} \left\| |x'|_k^c f \right\|_{L^r(\mathbb{R}^n)} \le \left\| \frac{x' \cdot \nabla_k}{|x'|_k} f \right\|^{\delta}_{L^p(\mathbb{R}^n)} \left\| |x'|_k^b f \right\|^{1-\delta}_{L^q(\mathbb{R}^n)} \tag{4.5.3}$$

for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$. Then, using the Schwartz inequality on the right-hand side of (4.5.3), *we obtain*

$$\left(\frac{k-p}{p}\right)^{\delta} \left\| |x'|_k^c f \right\|_{L^r(\mathbb{R}^n)} \le \left\| \nabla_k f \right\|^{\delta}_{L^p(\mathbb{R}^n)} \left\| |x'|_k^b f \right\|^{1-\delta}_{L^q(\mathbb{R}^n)}$$

for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$.

Proof of Theorem 11. Case $\delta = 0$. In this case, we have $q = r$ and $b = c$ by $\frac{\delta r}{2} + \frac{(1-\delta)r}{q} = 1$ and $c = -\delta + b(1-\delta)$, respectively. Then, the inequality (4.5.2) reduces to the trivial estimate

$$\left\| |x'|_k^c f \right\|_{L^r(\mathbb{R}^n)} \le \left\| |x'|_k^b f \right\|_{L^q(\mathbb{R}^n)}.$$

Case $\delta = 1$. Notice that in this case, $p = r$ and $c = -1$. We have the inequality

$$\||x'|_k^c f\|_{L^r(\mathbb{R}^n)} \leq \left(\frac{p}{k-p}\right)\left[\left\|\frac{x'\cdot\nabla_k}{|x'|_k}f\right\|^p_{L^p(\mathbb{R}^n)} - p\left(\frac{k-p}{p}\right)^p\right.$$
$$\left.\times\int_{\mathbb{R}^n} I_p\left(\frac{f}{|x'|_k}, -\frac{p}{k-p}\frac{x'\cdot\nabla_k}{|x'|_k}f\right)\left|\frac{f}{|x'|_k}+\frac{p}{k-p}\frac{x'\cdot\nabla_k}{|x'|_k}f\right|^2 dx\right]^{\frac{1}{p}}. \quad (4.5.4)$$

which is the inequality (4.2.1) that we already have proved in the previous Section 4.2.
Let us prove the case $\delta \in (0,1)\cap\left[\frac{r-q}{r}, \frac{p}{r}\right]$. Taking into account $c = -\delta + b(1-\delta)$, a direct calculation gives

$$\||x'|_k^c f\|_{L^r(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |x'|_k^{cr}|f(x)|^r dx\right)^{\frac{1}{r}} = \left(\int_{\mathbb{R}^n} \frac{|f(x)|^{\delta r}}{|x'|_k^{\delta r}}\cdot\frac{|f(x)|^{(1-\delta)r}}{|x'|_k^{-br(1-\delta)}}dx\right)^{\frac{1}{r}}.$$

Since we have $\delta \in (0,1)\cap\left[\frac{r-q}{r}, \frac{p}{r}\right]$ and $p+q \leq r$, then by using Hölder's inequality for $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$, we obtain

$$\begin{aligned}\||x'|_k^p f\|_{L^r(\mathbb{R}^n)} &\leq \left(\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x'|_k^p}dx\right)^{\frac{\delta}{p}}\left(\int_{\mathbb{R}^n} \frac{|f(x)|^q}{|x'|_k^{-bq}}dx\right)^{\frac{(1-\delta)}{q}} \\ &= \left\|\frac{f}{|x'|_k}\right\|^{\delta}_{L^p(\mathbb{R}^n)}\cdot\left\|\frac{f}{|x'|_k^{-b}}\right\|^{1-\delta}_{L^q(\mathbb{R}^n)}.\end{aligned} \quad (4.5.5)$$

By Theorem 8 we obtain

$$\||x'|_k^c f\|_{L^r(\mathbb{R}^n)} \leq \left(\frac{p}{k-p}\right)^{\delta}\left[\left\|\frac{x'\cdot\nabla_k}{|x'|_k}f\right\|^p_{L^p(\mathbb{R}^n)} - p\left(\frac{k-p}{p}\right)^p\right.$$
$$\left.\times\int_{\mathbb{R}^n} I_p\left(\frac{f}{|x'|_k}, -\frac{p}{k-p}\frac{x'\cdot\nabla_k}{|x'|_k}f\right)\left|\frac{f}{|x'|_k}+\frac{p}{k-p}\frac{x'\cdot\nabla_k}{|x'|_{|k|}}f\right|^2 dx\right]^{\frac{\delta}{p}} \||x'|_k^b f\|^{1-\delta}_{L^q(\mathbb{R}^n)}.$$

The proof of Theorem 11 is completed.

4.5.3 Caffarelli-Kohn-Nirenberg type inequalities on stratified Lie groups

Theorem 12. *Let $\mathbb{G}$ be a stratified Lie group and let $N > p$. Let $1 < p, q < \infty$, $0 < r < \infty$ with $p+q \leq r$ and $\delta \in [0,1]\cap\left[\frac{r-q}{r}, \frac{p}{r}\right]$ and $b, c \in \mathbb{R}$. Assume that $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ and $c = -\delta + b(1-\delta)$. Then for all real-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{x' = 0\})$ we have the following Caffarelli-Kohn-Nirenberg type inequalities* *with remainder terms:*

$$\left(\frac{N-p}{p}\right)^{\delta}\||x'|^{c}f\|_{L^{r}(\mathbb{G})}\leq\left[\left\|\frac{x'\cdot\nabla_{H}}{|x'|}f\right\|^{p}_{L^{p}(\mathbb{G})}-p\left(\frac{N-p}{p}\right)^{p}\right.$$
$$\left.\times\int_{\mathbb{G}}I_{p}\left(\frac{f}{|x'|},-\frac{p}{N-p}\frac{x'\cdot\nabla_{H}}{|x'|}f\right)\left|\frac{f}{|x'|}+\frac{p}{N-p}\frac{x'\cdot\nabla_{H}}{|x'|}f\right|^{2}dx\right]^{\frac{\delta}{p}}\||x'|^{b}f\|^{1-\delta}_{L^{q}(\mathbb{G})}. \tag{4.5.6}$$

Remark 16. *By dropping the non-negative term from* (4.5.6), *we obtain*

$$\left(\frac{N-p}{p}\right)^{\delta}\||x'|^{c}f\|_{L^{r}(\mathbb{G})}\leq\left\|\frac{x'\cdot\nabla_{H}}{|x'|}f\right\|^{\delta}_{L^{p}(\mathbb{G})}\||x'|^{b}f\|^{1-\delta}_{L^{q}(\mathbb{G})} \tag{4.5.7}$$

for all real-valued functions $f\in C_0^{\infty}(\mathbb{G}\backslash\{x'=0\})$.

Proof of Theorem 12. Case $\delta=0$. In this case, we have $q=r$ and $b=c$ by $\frac{\delta r}{p}+\frac{(1-\delta)r}{q}=1$ and $c=-\delta+b(1-\delta)$, respectively. Then, the inequality (4.5.6) is reduces the trivial estimate

$$\||x'|^{c}f\|_{L^{r}(\mathbb{G})}\leq\||x'|^{b}f\|_{L^{q}(\mathbb{G})}.$$

Case $\delta=1$. Notice that in this case, $p=r$ and $c=-1$. We have the inequality

$$\left(\frac{N-p}{p}\right)\||x'|^{c}f\|_{L^{r}(\mathbb{G})}\leq\left[\left\|\frac{x'\cdot\nabla_{H}}{|x'|}f\right\|^{p}_{L^{p}(\mathbb{G})}-p\left(\frac{N-p}{p}\right)^{p}\right.$$
$$\left.\times\int_{\mathbb{G}}I_{p}\left(\frac{f}{|x'|},-\frac{p}{N-p}\frac{x'\cdot\nabla_{H}}{|x'|}f\right)\left|\frac{f}{|x'|}+\frac{p}{N-p}\frac{x'\cdot\nabla_{H}}{|x'|}f\right|^{2}dx\right]^{\frac{1}{p}}, \tag{4.5.8}$$

which is the inequality (4.3.1) that we already have proved in the previous Section 4.3.
Case $\delta\in(0,1)\cap\left[\frac{r-q}{r},\frac{p}{r}\right]$. Taking into account $c=-\delta+b(1-\delta)$, a direct calculation gives

$$\||x'|^{c}f\|_{L^{r}(\mathbb{G})}=\left(\int_{\mathbb{G}}|x'|^{cr}|f(x)|^{r}dx\right)^{\frac{1}{r}}=\left(\int_{\mathbb{G}}\frac{|f(x)|^{\delta r}}{|x'|^{\delta r}}\cdot\frac{|f(x)|^{(1-\delta)r}}{|x'|^{-br(1-\delta)}}dx\right)^{\frac{1}{r}}.$$

Since we have $\delta\in(0,1)\cap\left[\frac{r-q}{r},\frac{p}{r}\right]$ and $p+q\leq r$, then by using Hölder's inequality for $\frac{\delta r}{p}+\frac{(1-\delta)r}{q}=1$, we obtain

$$\begin{aligned}\||x'|^c f\|_{L^r(\mathbb{G})} &\leq \left(\int_{\mathbb{G}}\frac{|f(x)|^p}{|x'|^p}dx\right)^{\frac{\delta}{p}}\left(\int_{\mathbb{G}}\frac{|f(x)|^q}{|x'|^{-bq}}dx\right)^{\frac{(1-\delta)}{q}} \\ &= \left\|\frac{f}{|x'|}\right\|^{\delta}_{L^p(\mathbb{G})}\cdot\left\|\frac{f}{|x'|^{-b}}\right\|^{1-\delta}_{L^q(\mathbb{G})}.\end{aligned} \tag{4.5.9}$$

By Theorem 9 we obtain

$$\||x'|^c f\|_{L^r(\mathbb{G})} \leq \left(\frac{p}{N-\alpha}\right)^{\delta}\left[\left\|\frac{x'\cdot\nabla_H}{|x'|}f\right\|^p_{L^p(\mathbb{G})} - p\left(\frac{N-p}{p}\right)^p\right.$$
$$\left.\times\int_{\mathbb{G}} I_p\left(\frac{f}{|x'|}, -\frac{p}{N-p}\frac{x'\cdot\nabla_H}{|x'|}f\right)\left|\frac{f}{|x'|}+\frac{p}{N-p}\frac{x'\cdot\nabla_H}{|x'|}f\right|^2 dx\right]^{\frac{\delta}{p}} \||x'|^b f\|^{1-\delta}_{L^q(\mathbb{G})}.$$

4.5.4 Caffarelli-Kohn-Nirenberg type inequality on the ball B_R

Let us recall the following Theorem 13 from [41, Theorem 2], which will be used in our proof:

Theorem 13. *Let $1<p\leq N$. Then the following inequality*

$$\left(\frac{p-1}{p}\right)^p\int_{B_R}\frac{|f|^p}{|x|^p\left(\log\frac{R}{|x|}\right)^p}dx \leq \int_{B_R}\left|\nabla f\cdot\frac{x}{|x|}\right|^p dx \tag{4.5.10}$$

holds for any $f\in W_0^{1,p}(B_R)$. And the constant $\left(\frac{p-1}{p}\right)^p$ is optimal and is not attained. Furthermore the following improved Hardy inequality

$$\left(\frac{p-1}{p}\right)^p\int_{B_R}\frac{|f|^p}{|x|^p\left(\log\frac{R}{|x|}\right)^p}dx + \phi_{N,p}(f) \leq \int_{B_R}\left|\nabla f\cdot\frac{x}{|x|}\right|^p dx$$

holds for any $f\in W_0^{1,p}(B_R)$, where

$$\phi_{N,p}(f) = (N-p)\left(\frac{p-1}{p}\right)^{p-1}\int_{B_R}\frac{|f|^p}{|x|^p\left(\log\frac{R}{|x|}\right)^{p-1}}dx.$$

Theorem 14. *Let B_R be the ball with radius $R>1$ and let $1<p<N$, then the*

following inequality holds for all real-valued functions $f \in C_0^\infty(B_R\backslash\{0\})$:

$$\left\| |x|^c \left(log\frac{R}{|x|}\right)^c f \right\|_{L^r(B_R)} \leq \left(\frac{p}{p-1}\right)^\delta \times \left\| \nabla f \cdot \frac{x}{|x|} \right\|_{L^p(B_R)}^\delta \left\| |x|^b \left(log\frac{R}{|x|}\right)^b f \right\|_{L^q(B_R)}^{1-\delta}. \quad (4.5.11)$$

Proof of Theorem 14. Case $\delta = 0$. In this case, we have $q = r$ and $b = c$ by $\frac{\delta r}{2} + \frac{(1-\delta)r}{q} = 1$ and $c = -\delta + b(1-\delta)$, respectively. Then, the inequality (4.5.11) becomes the trivial estimate

$$\left\| |x|^c \left(log\frac{R}{|x|}\right)^c f \right\|_{L^r(B_R)} \leq \left\| |x|^b \left(log\frac{R}{|x|}\right)^b f \right\|_{L^q(B_R)}.$$

Case $\delta = 1$. Notice that in this case, $r = p$ and $c = -1$. It means that

$$\left\| |x|^c \left(log\frac{R}{|x|}\right)^c f \right\|_{L^r(B_R)} \leq \left(\frac{p}{p-1}\right) \left\| \nabla f \cdot \frac{x}{|x|} \right\|_{L^p(B_R)},$$

which was proved by Sano in [41].

Case $\delta \in (0,1) \cap \left[\frac{r-q}{r}, \frac{p}{r}\right]$. Taking into account $c = -\delta + b(1-\delta)$, a direct calculation gives

$$\left\| |x|^c \left(log\frac{R}{|x|}\right)^c f \right\|_{L^r(B_R)} = \left(\int_{B_R} |x|^{cr} \left(log\frac{R}{|x|}\right)^{cr} |f(x)|^r dx\right)^{\frac{1}{r}}$$

$$= \left(\int_{B_R} \frac{|f(x)|^{\delta r}}{|x|^{\delta r}\left(log\frac{R}{|x|}\right)^{\delta r}} \cdot \frac{|f(x)|^{(1-\delta)r}}{|x|^{br(1-\delta)}\left(log\frac{R}{|x|}\right)^{br(1-\delta)}} dx\right)^{\frac{1}{r}}.$$

Since we have $\delta \in (0,1) \cap \left[\frac{r-q}{r}, \frac{p}{r}\right]$ and $p+q \leq r$, then by using Hölder's inequality for $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$, we get

$$\left\| |x|^c \left(log\frac{R}{|x|}\right)^c f \right\|_{L^r(B_R)} \leq \left(\int_{B_R} \frac{|f(x)|^p}{|x|^p\left(log\frac{R}{|x|}\right)^p} dx\right)^{\frac{\delta}{p}} \left(\int_{B_R} \frac{|f(x)|^q}{|x|^{-bq}\left(log\frac{R}{|x|}\right)^{-bq}} dx\right)^{\frac{(1-\delta)}{q}}$$

$$= \left\| \frac{f}{|x|\left(log\frac{R}{|x|}\right)} \right\|_{L^p(B_R)}^\delta \cdot \left\| \frac{f}{|x|^{-b}\left(log\frac{R}{|x|}\right)^{-b}} \right\|_{L^q(B_R)}^{1-\delta}.$$

By Theorem [41, Theorem2] we get the following Caffarelli-Kohn-Nirenberg type inequality:

$$\left\| |x|^c \left(log\frac{R}{|x|}\right)^c f \right\|_{L^r(B_R)} \le \left(\frac{p}{p-1}\right)^\delta \times \left\| \nabla f \cdot \frac{x}{|x|} \right\|^\delta_{L^p(B_R)} \left\| |x|^b \left(log\frac{R}{|x|}\right)^b f \right\|^{1-\delta}_{L^q(B_R)},$$

which is (4.5.11).

Chapter 5

Conclusions and future work

The Hardy inequality has numerous applications in the analysis of linear and nonlinear partial differential equations. However, the situation with cylindrical versions of functional inequalities changes the form of differential equations.

It is worthy to mention that importance of cylindrical extensions of the classical Hardy inequalities due to applications in partial differential equations arising in astrophysics. First, in this direction Badiale and Tarantello [19] obtained analogue of the Hardy inequality and used it in investigating cylindrical solution to a model describing the dynamics of galaxies. We also refer to [20], [21], [42] and [43] for cylindrical versions of Hardy, Hardy-Sobolev and Hardy-Rellich inequalities.

In this thesis, we obtained sharp remainder formulae for the cylindrical Hardy inequalities, namely cylindrical Hardy identities. For more general p we obtained cylindrical L^p Hardy inequalities for complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$. As an application of the cylindrical Hardy inequality, we established Heisenberg-Paul-Weyl type uncertainty principle. Moreover, for real-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$ we established L^p Hardy identities. The thesis also contains L^2 and L^p identities involving logarithmic functions on the ball $B(0, R)$ with a radius $R > 0$.

Furthermore, we discussed more general cylindrical Caffarelli-Kohn-Nirenberg inequalities with an explicit constant and remainder terms. In particular cases, these inequalities imply new functional inequalities, which are not covered by the classical Caffarelli-Kohn-Nirenberg inequalities.

In addition, we established L^p inequalities, L^2 and L^p identities on Folland and Stein's stratified Lie groups. Also, we obtained L^2 and L^p identities with logarithmic functions on homogeneous Lie groups. Namely, on stratified Lie groups we obtained horizontal versions of the results, while on general homogeneous Lie groups we discussed them for the radial derivative operator with any homogeneous quasi-norm.

One possible future directions can be to investigate cylindrical extensions of other classical functional inequalities. Another possible direction is to study optimality of the constants in cylindrical functional inequalities. Finally, it would be also interesting to apply these results in studying cylindrical solution of partial differential equations.

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