Ministry of Science and Higher Education of the Republic of Kazakhstan
SDU University





Tekebay Alpamys

3-nil alternative, pre-Lie and assosymmetric operads

THESIS

Presented in Partial Fulfillment for the
Degree of Master of Science in Mathematics
(degree code: 7M05401)
Department of Mathematics and Natural Sciences
School of Information Technologies and Applied Mathematics

Supervisor: **Bauyrzhan Sartayev, PhD.**

Kaskelen, 2026

SDU University
School of Information Technologies and Applied Mathematics

Dean of the School
Assistant Professor
Ph.D. Ramis Akhmedov
"____" _____ 2026

DIPLOMA PROJECT

3-nil alternative, pre-Lie and assosymmetric operads

7M05401 - "Mathematics"

Supervisor: Bauyrzhan Sartayev, Phd.

Student: Alpamys Tekebay

Kaskelen, 2026

Abstract

An algebra is called alternative if it satisfies the following identities:

$$(ab)c - a(bc) = -(ac)b + a(cb),$$

$$(ab)c - a(bc) = -(ba)c + b(ac).$$

In this dissertation, we consider free alternative algebra with the additional identity $x^3 = 0$. For motivation, we refer to the dual operad of the alternative operad. Also, we obtain pre-Lie algebra with the identity $x^3 = 0$ from binary perm algebra. Finally, we consider assosymmetric algebra with identity $x^3 = 0$.

Аңдатпа

Алгебра альтернативті деп аталады, егер ол келесі тепе-теңдіктерді қанағаттандырса:

$$(ab)c - a(bc) = -(ac)b + a(cb),$$
$$(ab)c - a(bc) = -(ba)c + b(ac).$$

Осы диссертацияда біз $x^3 = 0$ қосымша тепе-теңдігі бар еркін альтернативті алгебраны қарастырамыз. Мотивация ретінде біз альтернативті операданың қос операдасын қарастырамыз. Сондай-ақ, біз $x^3 = 0$ теңдігі бар пред-лиев алгебрасын бинарлы пермутативті алгебрадан аламыз. Соңында, біз $x^3 = 0$ теңдігі бар ассосиметриялық алгебраны қарастырамыз.

Аннотация

Алгебра называется альтернативной, если она удовлетворяет следующим тождествам:

$$(ab)c - a(bc) = -(ac)b + a(cb),$$
$$(ab)c - a(bc) = -(ba)c + b(ac).$$

В данной диссертации мы рассматриваем свободную альтернативную алгебру с дополнительным тождеством $x^3 = 0$. Для мотивации мы ссылаемся на двойственную операду альтернативной операды. Также мы получаем предлиеву алгебру с тождеством $x^3 = 0$ из бинарной пермутативной алгебры. Наконец, мы рассматриваем ассосиметричную алгебру с тождеством $x^3 = 0$.

Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Alpamys Tekebay

June 2026

Acknowledgements

I would like to thank my supervisor, Bauyrzhan Sartaev, very much. He helped me learn to think in a more scientific and abstract way and to feel comfortable in this field. I am grateful for his patience and for always answering my questions.

During my master's studies, he became not only a teacher, but also like an older brother to me.

Dedication

This thesis is dedicated to:
My parents, SDU and many other for their support, help, sense of humour and useful comments for improving this project.

Contents

Abstract i

Аңдатпа ii

Аннотация iii

Declaration iv

Acknowledgements v

Dedication v

1 Introduction 2

1.1 Historical Context and Previous Work 2

1.2 Research Objectives 2

1.3 Literature Review 3

1.3.1 The variety of alternative algebras 3

1.3.2 Koszul dual operad 4

1.3.3 Assosymmetric algebras 4

1.3.4 Some properties of algebras with identity $x^3 = 0$ 4

2 Main Results 7

2.1 Some identities under commutator 15

2.2 Some identities under anti-commutator 23

2.3 Anti-Leibniz algebras 25

3 Conclusion 33

4 References 34

Chapter 1

Introduction

An algebra is called alternative if it satisfies the following identities:

$$(ab)c - a(bc) = -(ac)b + a(cb), \tag{1.1}$$

$$(ab)c - a(bc) = -(ba)c + b(ac). \tag{1.2}$$

1.1 Historical Context and Previous Work

A natural source of alternative algebras is Artin's theorem, which states that its every two-generated subalgebra of alternative algebra is associative [15]. Let us demonstrate some works related to the subvarieties of the variety of alternative algebras. In [10], the authors constructed a basis of the free alternative algebra with identity $[a, b][c, d] = 0$ and proved that every metabelian Malcev algebra can be embedded into appropriate alternative algebra under commutator. In [5], the authors considered a variety of alternative algebras with the identity

$$(ab)c + (cb)a = (ac)b + (ca)b, \tag{1.3}$$

which coincides with a variety of binary perm algebras.

1.2 Research Objectives

The overall aim of this research is to systematically study the algebraic and operadic properties of varieties of algebras defined by the nilpotency condition $x^3 = 0$ within the framework of alternative, pre-Lie, and assosymmetric algebras. The specific research objectives are as follows:

(i) Analyze the polynomial identities that define the variety Alt_3, derive their multilinear consequences, and prove that the corresponding operad is self-dual.

(ii) Compute the dual operad $\mathcal{P}_2^!$ for binary perm algebras with the additional identity $x^3 = 0$, and prove whether this operad is Koszul by computing.

(iii) Obtain explicit algebraic and operadic inclusions between the varieties Alt_3, $\mathcal{P}_2^!$, and 3-nil assosymmetric algebras, and clarify their mutual connections via commutator and anti-commutator operations.

(iv) Compute the Hilbert series and Koszul duals for assosymmetric algebras defined by additional identities of degree 3, and determine whether the corresponding operads are Koszul.

(v) Obtain explicit polynomial identities of degrees 3 and 5 for the commutator algebras $Alt_3^{(-)}$ and $\mathcal{P}_2^{!(-)}$, as well as for the anti-commutator algebra $Alt_3^{(+)}$, and prove that some higher-degree identities are consequences of lower-degree ones.

(vi) Use computational algebraic methods (e.g., the Dotsenko–Heijltjes package) to compute the dimensions of multilinear components of the operads, and verify theoretical results on Koszulness and self-duality.

1.3 Literature Review

1.3.1 The variety of alternative algebras

There is given a basis of the free alternative algebra with identity [1.1] and described a complete list of identities of algebras that appear under commutator and anti-commutator.

The variety of alternative algebras is a natural generalization of the variety of associative algebras. On the other side, the dual operad of the alternative operad is an associative operad with additional identity $x^3 = 0$. So, we obtain

$$\mathcal{A}lt^! = \mathcal{A}s + \{x^3 = 0\}.$$

Also, we obtain the following trivial result which immediately follows from the definitions given above:

Theorem 1.1. *Let $\mathcal{A}lt_3$ be a variety of alternative algebras defined by identity $x^3 = 0$. Then every two-generated algebra from $\mathcal{A}lt_3$ lies in $\mathcal{A}lt^!$, i.e., $\mathcal{A}lt + \{x^3 = 0\} = \mathcal{A}lt^!{}_2$.*

All described motivations can be illustrated as inclusions of the varieties as follows:

$$\begin{array}{lll} \mathcal{A}s & \supset & \mathcal{A}s + \{x^3 = 0\} = \mathcal{A}lt^! \\ \cap & & \cap \qquad\qquad \cap \\ \mathcal{A}lt & \supset & \mathcal{A}lt + \{x^3 = 0\} = \mathcal{A}lt^!{}_2 \end{array}$$

1.3.2 Koszul dual operad

We consider Koszul dual operad ${\mathcal{P}_2}^!$, where $\mathcal{P}_2$ is a variety of binary perm algebra, i.e., it is an alternative operad with additional identity (1.3). It turns out that ${\mathcal{P}_2}^!$ is a variety of pre-Lie algebras with two additional independent identities, where one of them is $x^3 = 0$. In addition, it is observed the fact that an algebra from ${\mathcal{P}_2}^!$ is a Lie algebra with an additional independent identity of degree 5. The situation looks like for the Novikov algebras under commutator [3]. For Novikov algebras, there occurs a standard identity of degree 5

$$\sum_{\sigma\in S_4}(-1)^\sigma[x_{\sigma(1)},[x_{\sigma(2)},[x_{\sigma(3)},[x_{\sigma(4)},x_5]]]]=0.$$

1.3.3 Assosymmetric algebras

Finally, we consider assosymmetric algebra with identities generated by 1-dimensional invariant basis vectors which are described in [8]. These identities are

$$\sum_{\sigma\in S_3}(-1)^\sigma x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}) \quad \text{and} \quad \sum_{\sigma\in S_3} x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}).$$

Indeed, considered alternative and pre-Lie algebras with identity $x^3 = 0$ is equivalent to the alternative and pre-Lie algebras with identity generated by 1-dimensional invariant basis vectors of identities space of degree 3, see [8]. For more details on alternative, assosymmetric and pre-Lie algebras, see [1, 6, 7, 9, 11, 12, 13, 14].

We consider all algebras over a field $\mathbb{K}$ of characteristic 0. .

1.3.4 Some properties of algebras with identity $x^3 = 0$

Definition 1.2. An alternative algebra with additional identity $x^3 = 0$ is called a 3-nil alternative algebra. We denote by Alt_3 and $Alt_3\langle X\rangle$ the variety of 3-nil alternative algebras and free algebra if the variety Alt_3, respectively.

In characteristic 0, the identity $x^3 = 0$ is equivalent to its polarization:

$$\sum_{\sigma\in S_3}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}=0,$$

where x_1, x_2, x_3 are distinct variables. Substituting $(x_1, x_2, x_3) = (x, y, z)$ yields exactly equation (4). Monomials with repeated elements (e.g., $(x_1x_1)x_2$) do not appear in this sum, since polarization yields a sum only over distinct variables. So, we obtain

$$(xy)z + (yx)z + (xz)y + (zx)y + (yz)x + (zy)x = 0. \tag{1.4}$$

By using modified identity from (1.1) and (1.2)

$$(ab)c + (ba)c = b(ac) + a(bc)$$

the identity (1.4) can be rewritten as

$$x(yz) + x(zy) + y(xz) + y(zx) + z(xy) + z(yx) = 0.$$

Both identities that are obtained from $x^3 = 0$ give

$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0. \tag{1.5}$$

Proposition 1.3. *The polarization of 3-nil alternative algebra gives*

$$[x, \{y, z\}] = \{[x, y], z\} + \{[x, z], y\}$$

and

$$\{\{x, y\}, z\} = 1/3([x, [y, z]] - [[x, z], y]).$$

Proof We use the standard polarization

$$xy = \frac{1}{2}\big([x, y] + \{x, y\}\big).$$

The defining identities of a 3-nil alternative algebra are

$$(xy)z - x(yz) = -(xz)y + x(zy),$$

$$(xy)z - x(yz) = -(yx)z + y(xz),$$

and

$$(xy)z + (yx)z + (xz)y + (zx)y + (yz)x + (zy)x = 0.$$

We rewrite these identities in terms of the commutator and anti-commutator. For example,

$$(xy)z = \frac{1}{4}\Big([[x, y], z] + [\{x, y\}, z] + \{[x, y], z\} + \{\{x, y\}, z\}\Big),$$

and

$$x(yz) = \frac{1}{4}\Big([x, [y, z]] + [x, \{y, z\}] + \{x, [y, z]\} + \{x, \{y, z\}\}\Big).$$

We choose the following ordered list of monomials:

$$[x, \{y, z\}], \quad \{[x, y], z\}, \quad \{[x, z], y\}, \quad \{\{x, y\}, z\}, \quad [x, [y, z]], \quad [[x, z], y].$$

After collecting coefficients of these monomials from the polarized identities, we obtain the following matrix:

$$\begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 & 1 \end{pmatrix}.$$

Therefore, the corresponding system of identities is

$$[x,\{y,z\}]-\{[x,y],z\}-\{[x,z],y\}=0$$

and

$$3\{\{x,y\},z\}-[x,[y,z]]+[[x,z],y]=0.$$

Hence,

$$[x,\{y,z\}]=\{[x,y],z\}+\{[x,z],y\}$$

and

$$\{\{x,y\},z\}=\frac{1}{3}\big([x,[y,z]]-[[x,z],y]\big).$$

This completes the proof. □

Indeed, the identity (1.4) and its consequence in alternative algebra can be rewritten as

$$\sum_{\sigma\in S_3}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}=0$$

and

$$\sum_{\sigma\in S_3}x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)})=0.$$

These identities correspond to the 1-dimensional invariant basis vectors.

Main Results

Theorem 2.1. *An operad $\mathcal{A}lt_3$ is self-dual.*

Proof Firstly, let us fix a multilinear basis of algebra $\mathcal{A}lt_3$ of degree 3. That is

$$c(ba) = -(ac)b + (cb)a + a(cb),$$

$$c(ab) = (ca)b + (ac)b - a(cb),$$

$$b(ca) = (ac)b + (bc)a - a(cb),$$

$$(ab)c = -(ac)b + a(cb) + a(bc),$$

$$b(ac) = -(ca)b - (ac)b - (cb)a - (bc)a - a(bc)$$

and

$$(ba)c = -(ca)b - (cb)a - (bc)a - a(cb) - a(bc).$$

The Lie-admissibility condition for $S \otimes U$ gives the defining identities of the operad $\mathcal{A}lt_3^!$, where S is a 3-nil alternative algebra. Then

$$\begin{aligned}[[a \otimes u, b \otimes v], c \otimes w] &= (ab)c \otimes (uv)w \\ &- (ba)c \otimes (vu)w - c(ab) \otimes w(uv) + c(ba) \otimes w(vu) = \\ &(-(ac)b + a(cb) + a(bc)) \otimes (uv)w \\ &- (-(ca)b - (cb)a - (bc)a - a(cb) - a(bc)) \otimes (vu)w \\ &- ((ca)b + (ac)b - a(cb)) \otimes w(uv) + (-(ac)b + (cb)a + a(cb)) \otimes w(vu).\end{aligned}$$

$$\begin{aligned}[[b \otimes v, c \otimes w], a \otimes u] &= (bc)a \otimes (vw)u \\ &- (cb)a \otimes (wv)u - a(bc) \otimes u(vw) + a(cb) \otimes u(wv).\end{aligned}$$

$$\begin{aligned}[[c \otimes w, a \otimes u], b \otimes v] &= (ca)b \otimes (wu)v - (ac)b \otimes (uw)v \\ &- b(ca) \otimes v(wu) + b(ac) \otimes v(uw) = \\ &(ca)b \otimes (wu)v - (ac)b \otimes (uw)v \\ &- ((ac)b + (bc)a - a(cb)) \otimes v(wu) \\ &+ (-(ca)b - (ac)b - (cb)a - (bc)a - a(bc)) \otimes v(uw).\end{aligned}$$

Calculating the sum and collecting the same basis monomials, we obtain

$$\begin{aligned}[[a\otimes u, b\otimes v], c\otimes w] + [[b\otimes v, c\otimes w], a\otimes u] + [[c\otimes w, a\otimes u], b\otimes v] = \\ (ac)b\otimes(-(uv)w - w(uv) - w(vu) - (uw)v - v(wu) - v(uw)) \\ + a(cb)\otimes((uv)w + (vu)w + w(uv) + w(vu) + u(wv) + v(wu)) \\ + a(bc)\otimes((uv)w \\ + (vu)w - u(vw) - v(uw)) + (ca)b\otimes((vu)w - w(uv) + (wu)v - v(uw)) \\ + (cb)a\otimes((vu)w + w(vu) - (wv)u - v(uw)) \\ + (bc)a\otimes((vu)w + (vw)u - v(wu) - v(uw)) = 0.\end{aligned}$$

From the right sides of the tensors, we obtain the identities (1.1), (1.2) and (1.4) which means that the operad $\mathcal{A}lt_3$ is self-dual. □

We denote by $\mathcal{P}_2$ and $\mathcal{P}_2\langle X\rangle$ the variety of binary perm algebras and free binary perm algebra. Let us calculate the dual operad of binary perm algebra ${\mathcal{P}_2}^!$. As above, we first fix the multilinear basis of binary perm algebra of degree 3. That is

$$\begin{aligned}(bc)a &= (ba)c - (ac)b + (ab)c,\\ (cb)a &= (ca)b + (ac)b - (ab)c,\\ a(bc) &= c(ab) - (ca)b + (ab)c,\\ a(cb) &= -c(ab) + (ca)b + (ac)b,\\ b(ac) &= -c(ab) + (ca)b + (ba)c,\\ c(ba) &= -c(ab) + 2(ca)b + (ac)b - (ab)c\end{aligned}$$

and

$$b(ca) = c(ab) - (ca)b + (ba)c - (ac)b + (ab)c.$$

Performing similar calculations as above, we obtain

$$\begin{aligned}[[a\otimes u, b\otimes v], c\otimes w] + [[b\otimes v, c\otimes w], a\otimes u] + [[c\otimes w, a\otimes u], b\otimes v] = \\ (ab)c\otimes(uv)w - (ba)c\otimes(vu)w - c(ab)\otimes w(uv) + c(ba)\otimes w(vu) \\ + (bc)a\otimes(vw)u - (cb)a\otimes(wv)u - a(bc)\otimes u(vw) + a(cb)\otimes u(wv) \\ + (ca)b\otimes(wu)v - (ac)b\otimes(uw)v - b(ca)\otimes v(wu) + b(ac)\otimes v(uw) = \\ (ab)c\otimes(uv)w - (ba)c\otimes(vu)w - c(ab)\otimes w(uv) \\ + (-c(ab) + 2(ca)b + (ac)b - (ab)c)\otimes w(vu)\end{aligned}$$

$$
\begin{aligned}
&+((ba)c-(ac)b+(ab)c)\otimes(vw)u-((ca)b+(ac)b-(ab)c)\otimes(wv)u\\
&-(c(ab)-(ca)b+(ab)c)\otimes u(vw)\\
&+(-c(ab)+(ca)b+(ac)b)\otimes u(wv)\\
&+(ca)b\otimes(wu)v-(ac)b\otimes(uw)v\\
&-(c(ab)-(ca)b+(ba)c-(ac)b+(ab)c)\otimes v(wu)\\
&+(-c(ab)+(ca)b+(ba)c)\otimes v(uw)=\\
&(ab)c\otimes((uv)w-w(vu)+(vw)u+(wv)u-u(vw)-v(wu))\\
&+(ba)c\otimes(-(vu)w+(vw)u-v(wu)+v(uw))\\
&+c(ab)\otimes(-w(uv)-w(vu)-u(vw)-u(wv)-v(wu)-v(uw))\\
&+(ca)b\otimes(2w(vu)-(wv)u+u(vw)+u(wv)+(wu)v+v(wu)+v(uw))\\
&(ac)b\otimes(w(vu)-(vw)u-(wv)u+u(wv)-(uw)v+v(wu)+v(uw))=0.
\end{aligned}
$$

From all calculations, we obtain the following result:

Theorem 2.2. *The following identities define an operad which corresponds to* ${\mathcal{P}_2}^!$*:*

$$(v,w,u)=(v,u,w),$$

$$(u,w,v)+(w,v,u)+(v,w,u)=0$$

and

$$w(uv)+w(vu)+u(vw)+u(wv)+v(wu)+v(uw)=0,$$

where (v,w,u) *stands for associator.*

Theorem 2.3. *The operad* ${\mathcal{P}_2}^!$ *is not Koszul.*

Proof Calculating the dimension of the operad ${\mathcal{P}_2}^!$ by means of the package [2], we get the following result:

n	1	2	3	4	5
$\dim({\mathcal{P}_2}^!(n))$	1	2	7	26	67

According to the obtained table and [5], the first few terms of the Hilbert series of the operads ${\mathcal{P}_2}^!$ and $\mathcal{P}_2$ are

$$H(t)=-t+t^2-5t^3/6+6t^4/24-5t^5/120+O(t^6)$$

and

$$H^!(t)=-t+t^2-7t^3/6+26t^4/24-67t^5/120+O(t^6)$$

Thus,

$$H(H^!(t))=t+31t^5/60+O(t^6)\neq t.$$

By [4], the operad ${\mathcal{P}_2}^!$ is not Koszul. □

Proposition 2.4. *The polarization of* $\mathcal{P}_2{}^!$ *algebra gives*

$$[[a,b],c]+[[b,c],a]+[[c,a],b]=0,$$

$$\{\{a,b\},c\}+\{\{b,c\},a\}+\{\{c,a\},b\}=0,$$

$$[\{a,b\},c]+[\{b,c\},a]+[\{c,a\},b]=0$$

and

$$\{[b,c],a\}=\{\{a,b\},c\}+\{[c,a],b\}+[\{a,b\},c]+1/3[[b,c],a]-1/3[[c,a],b].$$

Proof Recall that the operad $\mathcal{P}_2{}^!$ is defined by the identities

$$(x,y,z)=(x,z,y),$$

$$(x,z,y)+(z,y,x)+(y,z,x)=0,$$

and

$$z(xy)+z(yx)+x(yz)+x(zy)+y(zx)+y(xz)=0,$$

where

$$(x,y,z)=(xy)z-x(yz)$$

is the associator.

We rewrite the product in terms of the commutator and anti-commutator:

$$xy=\frac{1}{2}\big([x,y]+\{x,y\}\big).$$

Then

$$(xy)z=\frac{1}{4}\Big([[x,y],z]+[\{x,y\},z]+\{[x,y],z\}+\{\{x,y\},z\}\Big),$$

and

$$x(yz)=\frac{1}{4}\Big([x,[y,z]]+[x,\{y,z\}]+\{x,[y,z]\}+\{x,\{y,z\}\}\Big).$$

First, polarizing the identity

$$(x,z,y)+(z,y,x)+(y,z,x)=0$$

and collecting all skew-symmetric terms, we obtain

$$[[a,b],c]+[[b,c],a]+[[c,a],b]=0.$$

This is exactly the Jacobi identity.

Next, collecting all purely symmetric terms in the polarization of

$$z(xy)+z(yx)+x(yz)+x(zy)+y(zx)+y(xz)=0,$$

we obtain

$$\{\{a,b\},c\}+\{\{b,c\},a\}+\{\{c,a\},b\}=0.$$

Now we collect the mixed terms containing one commutator and one anti-commutator. From the same polarized system we get

$$[\{a,b\},c]+[\{b,c\},a]+[\{c,a\},b]=0.$$

It remains to rewrite the remaining mixed relation. Using the identities already obtained, the mixed part may be reduced to

$$\{[b,c],a\}-\{\{a,b\},c\}-\{[c,a],b\}-[\{a,b\},c]-\frac{1}{3}[[b,c],a]+\frac{1}{3}[[c,a],b]=0.$$

Equivalently,

$$\{[b,c],a\}=\{\{a,b\},c\}+\{[c,a],b\}+[\{a,b\},c]+\frac{1}{3}[[b,c],a]-\frac{1}{3}[[c,a],b].$$

Therefore, the polarization of the defining identities of ${\mathcal{P}_2}^!$ gives precisely the required system of identities. □

The next natural operad that we have to consider is an assosymmetric operad with identity $x^3=0$.

Definition 2.5. An algebra is called a 3-nil assosymmetric if it satisfies the following identities:

$$(x,y,z)=(x,z,y),$$
$$(x,y,z)=(y,x,z)$$

and

$$x(yz)+x(zy)+y(xz)+y(zx)+z(xy)+z(yx)=0.$$

In other words, this is an assosymmetric algebra with identity generated by a 1-dimensional invariant basis vector

$$\sum_{\sigma\in S_3} x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}).$$

Remark. A 3-nil assosymmetric algebra does not satisfy the identity (1.4) under anti-commutator.

Let us calculate $S\otimes U$, where S is a 3-nil assosymmetric algebra.

$$\begin{aligned}[[a\otimes u,b\otimes v],c\otimes w]+[[b\otimes v,c\otimes w],a\otimes u]+[[c\otimes w,a\otimes u],b\otimes v]=\\ (ab)c\otimes(uv)w-(ba)c\otimes(vu)w-c(ab)\otimes w(uv)+c(ba)\otimes w(vu)\\ +(-c(ba)-5c(ab)+4(ca)b-(ba)c-(ac)b-(ab)c)\otimes(vw)u\\ -(c(ba)-c(ab)+(ca)b)\otimes(wv)u\\ -(c(ab)-(ca)b+(ab)c)\otimes u(vw)\\ +(c(ab)-(ca)b+(ac)b)\otimes u(wv)\end{aligned}$$

$$
\begin{aligned}
&+ (ca)b \otimes (wu)v - (ac)b \otimes (uw)v \\
&\quad - (-c(ba) - 4c(ab) + 3(ca)b - (ba)c - (ac)b - (ab)c) \otimes v(wu) \\
&\quad + (c(ab) - (ca)b + (ba)c) \otimes v(uw) = \\
&\quad (ab)c \otimes ((uv)w - (vw)u - u(vw) + v(wu)) \\
&\quad + (ba)c \otimes (-(vu)w - (vw)u + v(wu) + v(uw)) \\
&+ c(ab) \otimes (-w(uv) - 5(vw)u + (wv)u - u(vw) + u(wv) - 4v(wu) + v(uw))
\end{aligned}
$$

$$
\begin{aligned}
&+ c(ba) \otimes (w(vu) - (vw)u - (wv)u + v(wu)) \\
&+ (ca)b \otimes (4(vw)u - (wv)u + u(vw) - u(wv) + (wu)v - 3v(wu) - v(uw)) \\
&\quad + (ac)b \otimes (-(vw)u + u(wv) - (uw)v + v(wu)).
\end{aligned}
$$

The above calculations give the following result:

Theorem 2.6. *The dual operad of 3-nil assosymmetric operad is an alternative operad with additional identity:*

$$(uv)w - (vu)w - (uw)v + (wu)v + (vw)u - (wv)u = 0.$$

So, this is an alternative algebra with additional identity generated by a 1-dimensional invariant basis vector

$$\sum_{\sigma \in S_3} (-1)^{\sigma} (x_{\sigma(1)} x_{\sigma(2)}) x_{\sigma(3)}.$$

Theorem 2.7. *The operad governed by the variety of 3-nil assosymmetric algebras is not Koszul.*

Proof Calculating the dimension of these operads by means of the package [2], the first few terms of the Hilbert series of these operads are

$$H(t) = -t + t^2 - t^3 + 13t^4/24 - 15t^5/120 + O(t^6)$$

and

$$H^!(t) = -t + t^2 - t^3 + 13t^4/24 - 9t^5/120 + O(t^6).$$

Thus,

$$H(H^!(t)) = t + 19t^5/20 + O(t^6) \neq t.$$

By [4], such operad is not Koszul. □

Let us define an assosymmetric algebra with additional identity which is generated by the 1-dimensional invariant basis vector

$$\sum_{\sigma \in S_3} (-1)^{\sigma} (x_{\sigma(1)} x_{\sigma(2)}) x_{\sigma(3)}.$$

Such algebra also satisfies another identity

$$\sum_{\sigma \in S_3} (-1)^{\sigma} x_{\sigma(1)} (x_{\sigma(2)} x_{\sigma(3)}).$$

Theorem 2.8. *An assosymmetric operad with identity*

$$\sum_{\sigma\in S_3}(-1)^\sigma(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}=0$$

is self-dual.

Proof As before, we compute

$$\begin{gathered}
[[a\otimes u,b\otimes v],c\otimes w]+[[b\otimes v,c\otimes w],a\otimes u]+[[c\otimes w,a\otimes u],b\otimes v]=\\
(ab)c\otimes(uv)w-(ba)c\otimes(vu)w-c(ab)\otimes w(uv)+c(ba)\otimes w(vu)\\
+(c(ba)-c(ab)+(ba)c+(ac)b-(ab)c)\otimes(vw)u\\
-(c(ba)-c(ab)+(ca)b)\otimes(wv)u\\
-(c(ab)-(ca)b+(ab)c)\otimes u(vw)+(c(ab)-(ca)b+(ac)b)\otimes u(wv)\\
+(ca)b\otimes(wu)v-(ac)b\otimes(uw)v\\
-(c(ba)-(ca)b+(ba)c+(ac)b-(ab)c)\otimes v(wu)\\
+(c(ab)-(ca)b+(ba)c)\otimes v(uw)=\\
(ab)c\otimes((uv)w-(vw)u-u(vw)+v(wu))\\
+(ba)c\otimes(-(vu)w+(vw)u-v(wu)+v(uw))\\
+c(ab)\otimes(-w(uv)-(vw)u+(wv)u-u(vw)+u(wv)+v(uw))\\
+c(ba)\otimes(w(vu)+(vw)u-(wv)u-v(wu))\\
+(ac)b\otimes((vw)u+u(wv)-(uw)v-v(wu))\\
+(ca)b)\otimes(-(wv)u+u(vw)-u(wv)+(wu)v+v(wu)-v(uw)).
\end{gathered}$$

The right part of the tensors are equal to 0 if and only if the given operad is self-dual. □

Theorem 2.9. *The operad governed by the variety of assosymmetric algebras with identity*

$$\sum_{\sigma\in S_3}(-1)^\sigma(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}=0$$

is not Koszul.

Proof Calculating the dimension of this operad by means of the package [2], the first few terms of the Hilbert series of this operad is

$$H(t)=H^!(t)=-t+t^2-t^3+14t^4/24-12t^5/120+O(t^6)$$

Thus,

$$H(H^!(t))=t+7t^5/10+O(t^6)\neq t.$$

As before, such operad is not Koszul. □

The last remaining algebra is assosymmetric algebra which admits the identity

$$\{\{x,y\},z\}+\{\{y,z\},x\}+\{\{z,x\},y\}=0$$

under anti-commutator. For such an operad, let us calculate its Koszul dual operad:

$$\begin{aligned}
&[[a\otimes u,b\otimes v],c\otimes w]+[[b\otimes v,c\otimes w],a\otimes u]+[[c\otimes w,a\otimes u],b\otimes v]=\\
&(ab)c\otimes(uv)w-(ba)c\otimes(vu)w-c(ab)\otimes w(uv)+c(ba)\otimes w(vu)\\
&+(-c(ba)-2c(ab)+(ca)b-(ba)c-(ac)b-(ab)c)\otimes(vw)u\\
&-(c(ba)-c(ab)+(ca)b)\otimes(wv)u\\
&-(c(ab)-(ca)b+(ab)c)\otimes u(vw)\\
&+(c(ab)-(ca)b+(ac)b)\otimes u(wv)\\
&+(ca)b\otimes(wu)v-(ac)b\otimes(uw)v\\
&-(-c(ba)-c(ab)-(ba)c-(ac)b-(ab)c)\otimes v(wu)\\
&+(c(ab)-(ca)b+(ba)c)\otimes v(uw)=\\
&(ab)c\otimes((uv)w-(vw)u-u(vw)+v(wu))\\
&+(ba)c\otimes(-(vu)w-(vw)u+v(wu)+v(uw))\\
&c(ab)\otimes(-w(uv)-2(vw)u+(wv)u-u(vw)+u(wv)+v(wu)+v(uw))\\
&+c(ba)\otimes(w(vu)-(vw)u-(wv)u+v(wu))\\
&+(ac)b\otimes(-(vw)u+u(wv)-(uw)v+v(wu))\\
&+(ca)b\otimes((vw)u-(wv)u+u(vw)-u(wv)+(wu)v-v(uw)).
\end{aligned}$$

We obtain an alternative operad with the identity

$$(vw)u-(wv)u+(wu)v-u(wv)-v(uw)+u(vw)=0.$$

Let us check the Koszulness condition for the last considered operad:

Theorem 2.10. *An assosymmetric operad with identity*

$$\{\{x,y\},z\}+\{\{y,z\},x\}+\{\{z,x\},y\}=0$$

is not Koszul.

Proof Calculating the dimension of these operads by means of the package [2], the first few terms of the Hilbert series of these operads are

$$H(t)=-t+t^2-t^3+12t^4/24-15t^5/120+O(t^6)$$

and

$$H^!(t)=-t+t^2-t^3+12t^4/24-9t^5/120+O(t^6)$$

Thus,

$$H(H^!(t))=t+6t^5/5+O(t^6)\neq t.$$

So, such an operad is not Koszul. □

2.1 Some identities under commutator

For $\mathcal{A}\langle X\rangle$, we define commutator algebra $\mathcal{A}^{(-)}\langle X\rangle$ which is obtained from $\mathcal{A}\langle X\rangle$ under the operation

$$[x, y] = xy - yx.$$

Analogically, we define anti-commutator algebra $\mathcal{A}^{(+)}\langle X\rangle$ under the operation $\{x, y\} = xy + yx$.

Theorem 2.11. *An algebra ${\mathcal{P}_2}^{!\,(-)}\langle X\rangle$ satisfies the following identities:*

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0,$$

$$\begin{aligned}\frac{1}{3}([[[[a, b], d], c], e] + [[[[a, b], d], e], c] + [[[[a, d], b], c], e] + [[[[a, d], b], e], c])\\ + [[[[a, c], b], e], d] + [[[[a, c], d], e], b] + [[[[a, e], b], c], d] + [[[[a, e], d], c], b] = 0,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}([[[[a, b], c], d], e] + [[[[a, b], d], c], e] + [[[[a, c], b], d], e]\\ + [[[[a, c], d], b], e] + [[[[a, d], b], c], e] + [[[[a, d], c], b], e]) =\\ [[[[a, b], c], e], d] + [[[[a, b], d], e], c] + [[[[a, c], b], e], d] +\\ [[[[a, c], d], e], b] + [[[[a, d], b], e], c] + [[[[a, d], c], e], b].\end{aligned}$$

Proof The first identity is the Jacobi identity, since every algebra in ${\mathcal{P}_2}^{!}$ is pre-Lie, and every pre-Lie algebra is Lie-admissible.

Let us prove the two identities of degree 5. Recall that ${\mathcal{P}_2}^{!}$ is defined by

$$(x, y, z) = (x, z, y),$$

$$(x, z, y) + (z, y, x) + (y, z, x) = 0,$$

and

$$z(xy) + z(yx) + x(yz) + x(zy) + y(zx) + y(xz) = 0,$$

where

$$(x, y, z) = (xy)z - x(yz).$$

We use the commutator

$$[x, y] = xy - yx.$$

Consider the multilinear component of degree 5 of the free ${\mathcal{P}_2}^{!}$-algebra. We expand every commutator in the two degree 5 identities by the rule

$$[x, y] = xy - yx.$$

Thus each identity becomes a linear combination of nonassociative degree 5 monomials in the variables a, b, c, d, e.

Now we reduce these linear combinations modulo the defining identities of ${\mathcal{P}_2}^!$. More precisely, we take all multilinear consequences of the identities

$$(x, y, z) - (x, z, y) = 0,$$

$$(x, z, y) + (z, y, x) + (y, z, x) = 0,$$

and

$$z(xy) + z(yx) + x(yz) + x(zy) + y(zx) + y(xz) = 0$$

in degree 5. We write them as rows of a matrix with respect to the ordered basis of all degree 5 nonassociative multilinear monomials.

Let R_5 be this matrix. After elementary row operations, the reduced row echelon form of R_5 gives all relations in the degree 5 multilinear component of the operad ${\mathcal{P}_2}^!$.

Let F_1 denote the left-hand side of the second identity:

$$\begin{aligned} F_1 = \frac{1}{3}[[[[y_1, y_2], y_4], y_3], y_5] + [[[[y_1, y_2], y_4], y_5], y_3] + [[[[y_1, y_4], y_2], y_3], y_5] \\ +[[[[y_1, y_4], y_2], y_5], y_3]) + [[[[y_1, y_3], y_2], y_5], y_4] + [[[[y_1, y_3], y_4], y_5], y_2] \\ +[[[[y_1, y_5], y_2], y_3], y_4] + [[[[y_1, y_5], y_4], y_3], y_2]. \end{aligned}$$

Then

$$\begin{aligned} F_1 = \frac{1}{3}\Big(& (((((y_1y_2)y_4)y_3)y_5) - (y_5(((y_1y_2)y_4)y_3)) - ((y_3((y_1y_2)y_4))y_5) \\ & + (y_5(y_3((y_1y_2)y_4))) - (((y_4(y_1y_2))y_3)y_5) + (y_5((y_4(y_1y_2))y_3)) + ((y_3(y_4(y_1y_2)))y_5) \\ & - (y_5(y_3(y_4(y_1y_2)))) - ((((y_2y_1)y_4)y_3)y_5) + (y_5(((y_2y_1)y_4)y_3)) + ((y_3((y_2y_1)y_4))y_5) \\ & - (y_5(y_3((y_2y_1)y_4))) + (((y_4(y_2y_1))y_3)y_5) - (y_5((y_4(y_2y_1))y_3)) - ((y_3(y_4(y_2y_1)))y_5) \\ & + (y_5(y_3(y_4(y_2y_1)))) + ((((y_1y_2)y_4)y_5)y_3) - (y_3(((y_1y_2)y_4)y_5)) - ((y_5((y_1y_2)y_4))y_3) \\ & + (y_3(y_5((y_1y_2)y_4))) - (((y_4(y_1y_2))y_5)y_3) + (y_3((y_4(y_1y_2))y_5)) + ((y_5(y_4(y_1y_2)))y_3) \\ & - (y_3(y_5(y_4(y_1y_2)))) - ((((y_2y_1)y_4)y_5)y_3) + (y_3(((y_2y_1)y_4)y_5)) + ((y_5((y_2y_1)y_4))y_3) \\ & - (y_3(y_5((y_2y_1)y_4))) + (((y_4(y_2y_1))y_5)y_3) - (y_3((y_4(y_2y_1))y_5)) - ((y_5(y_4(y_2y_1)))y_3) \\ & + (y_3(y_5(y_4(y_2y_1)))) + ((((y_1y_4)y_2)y_3)y_5) - (y_5(((y_1y_4)y_2)y_3)) - ((y_3((y_1y_4)y_2))y_5) \\ & + (y_5(y_3((y_1y_4)y_2))) - (((y_2(y_1y_4))y_3)y_5) + (y_5((y_2(y_1y_4))y_3)) + ((y_3(y_2(y_1y_4)))y_5) \\ & - (y_5(y_3(y_2(y_1y_4)))) - ((((y_4y_1)y_2)y_3)y_5) + (y_5(((y_4y_1)y_2)y_3)) + ((y_3((y_4y_1)y_2))y_5) \\ & - (y_5(y_3((y_4y_1)y_2))) + (((y_2(y_4y_1))y_3)y_5) - (y_5((y_2(y_4y_1))y_3)) - ((y_3(y_2(y_4y_1)))y_5) \\ & + (y_5(y_3(y_2(y_4y_1)))) + ((((y_1y_4)y_2)y_5)y_3) - (y_3(((y_1y_4)y_2)y_5)) - ((y_5((y_1y_4)y_2))y_3) \\ & + (y_3(y_5((y_1y_4)y_2))) - (((y_2(y_1y_4))y_5)y_3) + (y_3((y_2(y_1y_4))y_5)) + ((y_5(y_2(y_1y_4)))y_3) \\ & - (y_3(y_5(y_2(y_1y_4)))) - ((((y_4y_1)y_2)y_5)y_3) + (y_3(((y_4y_1)y_2)y_5)) + ((y_5((y_4y_1)y_2))y_3) \\ & - (y_3(y_5((y_4y_1)y_2))) + (((y_2(y_4y_1))y_5)y_3) - (y_3((y_2(y_4y_1))y_5)) - ((y_5(y_2(y_4y_1)))y_3) \\ & + (y_3(y_5(y_2(y_4y_1))))\Big) \end{aligned}$$

$$+((((y_1y_3)y_2)y_5)y_4)-(y_4(((y_1y_3)y_2)y_5))-((y_5((y_1y_3)y_2))y_4)+(y_4(y_5((y_1y_3)y_2)))$$
$$-(((y_2(y_1y_3))y_5)y_4)+(y_4((y_2(y_1y_3))y_5))+((y_5(y_2(y_1y_3)))y_4)-(y_4(y_5(y_2(y_1y_3))))$$
$$-((((y_3y_1)y_2)y_5)y_4)+(y_4(((y_3y_1)y_2)y_5))+((y_5((y_3y_1)y_2))y_4)-(y_4(y_5((y_3y_1)y_2)))$$
$$+(((y_2(y_3y_1))y_5)y_4)-(y_4((y_2(y_3y_1))y_5))-((y_5(y_2(y_3y_1)))y_4)+(y_4(y_5(y_2(y_3y_1))))$$
$$+((((y_1y_3)y_4)y_5)y_2)-(y_2(((y_1y_3)y_4)y_5))-((y_5((y_1y_3)y_4))y_2)+(y_2(y_5((y_1y_3)y_4)))$$
$$-(((y_4(y_1y_3))y_5)y_2)+(y_2((y_4(y_1y_3))y_5))+((y_5(y_4(y_1y_3)))y_2)-(y_2(y_5(y_4(y_1y_3))))$$
$$-((((y_3y_1)y_4)y_5)y_2)+(y_2(((y_3y_1)y_4)y_5))+((y_5((y_3y_1)y_4))y_2)-(y_2(y_5((y_3y_1)y_4)))$$
$$+(((y_4(y_3y_1))y_5)y_2)-(y_2((y_4(y_3y_1))y_5))-((y_5(y_4(y_3y_1)))y_2)+(y_2(y_5(y_4(y_3y_1))))$$
$$+((((y_1y_5)y_2)y_3)y_4)-(y_4(((y_1y_5)y_2)y_3))-((y_3((y_1y_5)y_2))y_4)+(y_4(y_3((y_1y_5)y_2)))$$
$$-(((y_2(y_1y_5))y_3)y_4)+(y_4((y_2(y_1y_5))y_3))+((y_3(y_2(y_1y_5)))y_4)-(y_4(y_3(y_2(y_1y_5))))$$
$$-((((y_5y_1)y_2)y_3)y_4)+(y_4(((y_5y_1)y_2)y_3))+((y_3((y_5y_1)y_2))y_4)-(y_4(y_3((y_5y_1)y_2)))$$
$$+(((y_2(y_5y_1))y_3)y_4)-(y_4((y_2(y_5y_1))y_3))-((y_3(y_2(y_5y_1)))y_4)+(y_4(y_3(y_2(y_5y_1))))$$
$$+((((y_1y_5)y_4)y_3)y_2)-(y_2(((y_1y_5)y_4)y_3))-((y_3((y_1y_5)y_4))y_2)+(y_2(y_3((y_1y_5)y_4)))$$
$$-(((y_4(y_1y_5))y_3)y_2)+(y_2((y_4(y_1y_5))y_3))+((y_3(y_4(y_1y_5)))y_2)-(y_2(y_3(y_4(y_1y_5))))$$
$$-((((y_5y_1)y_4)y_3)y_2)+(y_2(((y_5y_1)y_4)y_3))+((y_3((y_5y_1)y_4))y_2)-(y_2(y_3((y_5y_1)y_4)))$$
$$+(((y_4(y_5y_1))y_3)y_2)-(y_2((y_4(y_5y_1))y_3))-((y_3(y_4(y_5y_1)))y_2)+(y_2(y_3(y_4(y_5y_1)))).$$

After expanding all commutators and reducing modulo the rows of R_5 by Algebra Software as [2], we obtain

$$F_1 \equiv 0.$$

Hence

$$\frac{1}{3}\Big([[[[y_1,y_2],y_4],y_3],y_5]+[[[[y_1,y_2],y_4],y_5],y_3]+[[[[y_1,y_4],y_2],y_3],y_5]$$
$$+[[[[y_1,y_4],y_2],y_5],y_3]+[[[[y_1,y_3],y_2],y_5],y_4]+[[[[y_1,y_3],y_4],y_5],y_2]$$
$$+[[[[y_1,y_5],y_2],y_3],y_4]+[[[[y_1,y_5],y_4],y_3],y_2]=0.$$

Similarly, let F_2 denote the difference between the two sides of the third identity:

$$F_2=\frac{1}{2}\Big([[[[y_1,y_2],y_3],y_4],y_5]+[[[[y_1,y_2],y_4],y_3],y_5]+[[[[y_1,y_3],y_2],y_4],y_5]$$
$$+[[[[y_1,y_3],y_4],y_2],y_5]+[[[[y_1,y_4],y_2],y_3],y_5]+[[[[y_1,y_4],y_3],y_2],y_5]\Big)$$
$$-\Big([[[[y_1,y_2],y_3],y_5],y_4]+[[[[y_1,y_2],y_4],y_5],y_3]+[[[[y_1,y_3],y_2],y_5],y_4]$$
$$+[[[[y_1,y_3],y_4],y_5],y_2]+[[[[y_1,y_4],y_2],y_5],y_3]+[[[[y_1,y_4],y_3],y_5],y_2]\Big).$$

Then

$$
\begin{aligned}
F_2 = \frac{1}{2}\Big(& (((((y_1y_2)y_3)y_4)y_5) - (y_5(((y_1y_2)y_3)y_4)) - ((y_4((y_1y_2)y_3))y_5) \\
& + (y_5(y_4((y_1y_2)y_3))) - (((y_3(y_1y_2))y_4)y_5) + (y_5((y_3(y_1y_2))y_4)) \\
& + ((y_4(y_3(y_1y_2)))y_5) - (y_5(y_4(y_3(y_1y_2)))) - ((((y_2y_1)y_3)y_4)y_5) \\
& + (y_5(((y_2y_1)y_3)y_4)) + ((y_4((y_2y_1)y_3))y_5) - (y_5(y_4((y_2y_1)y_3))) \\
& + (((y_3(y_2y_1))y_4)y_5) - (y_5((y_3(y_2y_1))y_4)) - ((y_4(y_3(y_2y_1)))y_5) \\
& + (y_5(y_4(y_3(y_2y_1)))) + ((((y_1y_2)y_4)y_3)y_5) - (y_5(((y_1y_2)y_4)y_3)) \\
& - ((y_3((y_1y_2)y_4))y_5) + (y_5(y_3((y_1y_2)y_4))) - (((y_4(y_1y_2))y_3)y_5) \\
& + (y_5((y_4(y_1y_2))y_3)) + ((y_3(y_4(y_1y_2)))y_5) - (y_5(y_3(y_4(y_1y_2)))) \\
& - ((((y_2y_1)y_4)y_3)y_5) + (y_5(((y_2y_1)y_4)y_3)) + ((y_3((y_2y_1)y_4))y_5) \\
& - (y_5(y_3((y_2y_1)y_4))) + (((y_4(y_2y_1))y_3)y_5) - (y_5((y_4(y_2y_1))y_3)) \\
& \quad - ((y_3(y_4(y_2y_1)))y_5) + (y_5(y_3(y_4(y_2y_1)))) + ((((y_1y_3)y_2)y_4)y_5)
\end{aligned}
$$

$$
\begin{aligned}
& - (y_5(((y_1y_3)y_2)y_4)) - ((y_4((y_1y_3)y_2))y_5) + (y_5(y_4((y_1y_3)y_2))) \\
& \quad - (((y_2(y_1y_3))y_4)y_5) + (y_5((y_2(y_1y_3))y_4)) + ((y_4(y_2(y_1y_3)))y_5) \\
& \quad - (y_5(y_4(y_2(y_1y_3)))) - ((((y_3y_1)y_2)y_4)y_5) + (y_5(((y_3y_1)y_2)y_4)) \\
& \quad + ((y_4((y_3y_1)y_2))y_5) - (y_5(y_4((y_3y_1)y_2))) + (((y_2(y_3y_1))y_4)y_5) \\
& \quad - (y_5((y_2(y_3y_1))y_4)) - ((y_4(y_2(y_3y_1)))y_5) + (y_5(y_4(y_2(y_3y_1)))) \\
& \quad + ((((y_1y_3)y_4)y_2)y_5) - (y_5(((y_1y_3)y_4)y_2)) - ((y_2((y_1y_3)y_4))y_5) \\
& \quad + (y_5(y_2((y_1y_3)y_4))) - (((y_4(y_1y_3))y_2)y_5) + (y_5((y_4(y_1y_3))y_2)) \\
& \quad + ((y_2(y_4(y_1y_3)))y_5) - (y_5(y_2(y_4(y_1y_3)))) - ((((y_3y_1)y_4)y_2)y_5) \\
& \quad\quad + (y_5(((y_3y_1)y_4)y_2)) + ((y_2((y_3y_1)y_4))y_5) - (y_5(y_2((y_3y_1)y_4)))
\end{aligned}
$$

$$
\begin{aligned}
& + (((y_4(y_3y_1))y_2)y_5) - (y_5((y_4(y_3y_1))y_2)) - ((y_2(y_4(y_3y_1)))y_5) \\
& \quad + (y_5(y_2(y_4(y_3y_1)))) + ((((y_1y_4)y_2)y_3)y_5) - (y_5(((y_1y_4)y_2)y_3)) \\
& \quad - ((y_3((y_1y_4)y_2))y_5) + (y_5(y_3((y_1y_4)y_2))) - (((y_2(y_1y_4))y_3)y_5) \\
& \quad + (y_5((y_2(y_1y_4))y_3)) + ((y_3(y_2(y_1y_4)))y_5) - (y_5(y_3(y_2(y_1y_4)))) \\
& \quad - ((((y_4y_1)y_2)y_3)y_5) + (y_5(((y_4y_1)y_2)y_3)) + ((y_3((y_4y_1)y_2))y_5) \\
& \quad - (y_5(y_3((y_4y_1)y_2))) + (((y_2(y_4y_1))y_3)y_5) - (y_5((y_2(y_4y_1))y_3)) \\
& \quad - ((y_3(y_2(y_4y_1)))y_5) + (y_5(y_3(y_2(y_4y_1)))) + ((((y_1y_4)y_3)y_2)y_5) \\
& \quad - (y_5(((y_1y_4)y_3)y_2)) - ((y_2((y_1y_4)y_3))y_5) + (y_5(y_2((y_1y_4)y_3))) \\
& \quad - (((y_3(y_1y_4))y_2)y_5) + (y_5((y_3(y_1y_4))y_2)) + ((y_2(y_3(y_1y_4)))y_5) \\
& \quad - (y_5(y_2(y_3(y_1y_4)))) - ((((y_4y_1)y_3)y_2)y_5) + (y_5(((y_4y_1)y_3)y_2)) \\
& \quad\quad + ((y_2((y_4y_1)y_3))y_5) - (y_5(y_2((y_4y_1)y_3))) + (((y_3(y_4y_1))y_2)y_5)
\end{aligned}
$$

$$
\begin{aligned}
&- (y_5((y_3(y_4y_1))y_2)) - ((y_2(y_3(y_4y_1)))y_5) + (y_5(y_2(y_3(y_4y_1))))\Big) \\
&\qquad - ((((y_1y_2)y_3)y_5)y_4) + (y_4(((y_1y_2)y_3)y_5)) + ((y_5((y_1y_2)y_3))y_4) \\
&\qquad - (y_4(y_5((y_1y_2)y_3))) + (((y_3(y_1y_2))y_5)y_4) - (y_4((y_3(y_1y_2))y_5)) \\
&\qquad - ((y_5(y_3(y_1y_2)))y_4) + (y_4(y_5(y_3(y_1y_2)))) + ((((y_2y_1)y_3)y_5)y_4) \\
&\qquad - (y_4(((y_2y_1)y_3)y_5)) - ((y_5((y_2y_1)y_3))y_4) + (y_4(y_5((y_2y_1)y_3))) \\
&\qquad - (((y_3(y_2y_1))y_5)y_4) + (y_4((y_3(y_2y_1))y_5)) + ((y_5(y_3(y_2y_1)))y_4) \\
&\qquad - (y_4(y_5(y_3(y_2y_1)))) - ((((y_1y_2)y_4)y_5)y_3) + (y_3(((y_1y_2)y_4)y_5)) \\
&\qquad + ((y_5((y_1y_2)y_4))y_3) - (y_3(y_5((y_1y_2)y_4))) + (((y_4(y_1y_2))y_5)y_3) \\
&\qquad - (y_3((y_4(y_1y_2))y_5)) - ((y_5(y_4(y_1y_2)))y_3) + (y_3(y_5(y_4(y_1y_2)))) \\
&\qquad + ((((y_2y_1)y_4)y_5)y_3) - (y_3(((y_2y_1)y_4)y_5)) - ((y_5((y_2y_1)y_4))y_3) \\
&\qquad + (y_3(y_5((y_2y_1)y_4))) - (((y_4(y_2y_1))y_5)y_3) + (y_3((y_4(y_2y_1))y_5)) \\
&\qquad + ((y_5(y_4(y_2y_1)))y_3) - (y_3(y_5(y_4(y_2y_1)))) - ((((y_1y_3)y_2)y_5)y_4) \\
&\qquad + (y_4(((y_1y_3)y_2)y_5)) + ((y_5((y_1y_3)y_2))y_4) - (y_4(y_5((y_1y_3)y_2))) \\
&\qquad + (((y_2(y_1y_3))y_5)y_4) - (y_4((y_2(y_1y_3))y_5)) - ((y_5(y_2(y_1y_3)))y_4) \\
&\qquad + (y_4(y_5(y_2(y_1y_3)))) + ((((y_3y_1)y_2)y_5)y_4) - (y_4(((y_3y_1)y_2)y_5)) \\
&\qquad - ((y_5((y_3y_1)y_2))y_4) + (y_4(y_5((y_3y_1)y_2))) - (((y_2(y_3y_1))y_5)y_4) \\
&\qquad + (y_4((y_2(y_3y_1))y_5)) + ((y_5(y_2(y_3y_1)))y_4) - (y_4(y_5(y_2(y_3y_1)))) \\
&\qquad\qquad - ((((y_1y_3)y_4)y_5)y_2) + (y_2(((y_1y_3)y_4)y_5)) + ((y_5((y_1y_3)y_4))y_2)
\end{aligned}
$$

$$
\begin{aligned}
&- (y_2(y_5((y_1y_3)y_4))) + (((y_4(y_1y_3))y_5)y_2) - (y_2((y_4(y_1y_3))y_5)) \\
&\qquad - ((y_5(y_4(y_1y_3)))y_2) + (y_2(y_5(y_4(y_1y_3)))) + ((((y_3y_1)y_4)y_5)y_2) \\
&\qquad - (y_2(((y_3y_1)y_4)y_5)) - ((y_5((y_3y_1)y_4))y_2) + (y_2(y_5((y_3y_1)y_4))) \\
&\qquad - (((y_4(y_3y_1))y_5)y_2) + (y_2((y_4(y_3y_1))y_5)) + ((y_5(y_4(y_3y_1)))y_2) \\
&\qquad - (y_2(y_5(y_4(y_3y_1)))) - ((((y_1y_4)y_2)y_5)y_3) + (y_3(((y_1y_4)y_2)y_5)) \\
&\qquad + ((y_5((y_1y_4)y_2))y_3) - (y_3(y_5((y_1y_4)y_2))) + (((y_2(y_1y_4))y_5)y_3) \\
&\qquad - (y_3((y_2(y_1y_4))y_5)) - ((y_5(y_2(y_1y_4)))y_3) + (y_3(y_5(y_2(y_1y_4)))) \\
&\qquad + ((((y_4y_1)y_2)y_5)y_3) - (y_3(((y_4y_1)y_2)y_5)) - ((y_5((y_4y_1)y_2))y_3) \\
&\qquad + (y_3(y_5((y_4y_1)y_2))) - (((y_2(y_4y_1))y_5)y_3) + (y_3((y_2(y_4y_1))y_5)) \\
&\qquad + ((y_5(y_2(y_4y_1)))y_3) - (y_3(y_5(y_2(y_4y_1)))) - ((((y_1y_4)y_3)y_5)y_2) \\
&\qquad + (y_2(((y_1y_4)y_3)y_5)) + ((y_5((y_1y_4)y_3))y_2) - (y_2(y_5((y_1y_4)y_3))) \\
&\qquad + (((y_3(y_1y_4))y_5)y_2) - (y_2((y_3(y_1y_4))y_5)) - ((y_5(y_3(y_1y_4)))y_2) \\
&\qquad + (y_2(y_5(y_3(y_1y_4)))) + ((((y_4y_1)y_3)y_5)y_2) - (y_2(((y_4y_1)y_3)y_5)) \\
&\qquad - ((y_5((y_4y_1)y_3))y_2) + (y_2(y_5((y_4y_1)y_3))) - (((y_3(y_4y_1))y_5)y_2) \\
&\qquad\qquad + (y_2((y_3(y_4y_1))y_5)) + ((y_5(y_3(y_4y_1)))y_2) - (y_2(y_5(y_3(y_4y_1)))).
\end{aligned}
$$

Again, after expanding all commutators and reducing modulo the defining

relations of ${\mathcal{P}_2}^!$ in degree 5 by Algebra Software package such as [2], we get

$$F_2 \equiv 0.$$

Therefore,

$$\begin{aligned}\frac{1}{2}\Big([[[[a,b],c],d],e] + [[[[a,b],d],c],e] + [[[[a,c],b],d],e] \\ + [[[[a,c],d],b],e] + [[[[a,d],b],c],e] + [[[[a,d],c],b],e]\Big) \\ = [[[[a,b],c],e],d] + [[[[a,b],d],e],c] + [[[[a,c],b],e],d] \\ + [[[[a,c],d],e],b] + [[[[a,d],b],e],c] + [[[[a,d],c],e],b].\end{aligned}$$

Thus, both degree 5 identities hold in ${\mathcal{P}_2}^{!\,(-)}\langle X\rangle$. □

Theorem 2.12. *An algebra $Alt_3^{(-)}\langle X\rangle$ satisfies the following identities:*

$$[[a,c],[b,d]] = [[[a,b],c],d] + [[[b,c],d],a] + [[[c,d],a],b] + [[[d,a],b],c]$$

and

$$\begin{aligned}&[[[[a,b],d],e],c] + [[[[a,b],e],d],c] + [[[[a,c],d],e],b] + [[[[a,c],e],d],b] \\ &- [[[[a,d],b],c],e] - [[[[a,d],c],b],e] - [[[[a,e],b],c],d] - [[[[a,e],c],b],d] = 0.\end{aligned}$$

Proof The first identity is the Malcev identity. Indeed, every alternative algebra is Malcev-admissible under the commutator

$$[x,y] = xy - yx.$$

Hence the algebra $Alt_3^{(-)}\langle X\rangle$ satisfies

$$[[a,c],[b,d]] = [[[a,b],c],d] + [[[b,c],d],a] + [[[c,d],a],b] + [[[d,a],b],c].$$

Let us prove the second identity. Recall that Alt_3 is the variety of alternative algebras satisfying the additional identity

$$x^3 = 0.$$

In characteristic 0, this identity is equivalent to its polarization

$$(xy)z + (yx)z + (xz)y + (zx)y + (yz)x + (zy)x = 0.$$

Together with the alternative identities

$$(xy)z - x(yz) = -(xz)y + x(zy),$$

$$(xy)z - x(yz) = -(yx)z + y(xz),$$

these identities define the operad Alt_3.

Now consider the multilinear component of degree 5 of the free Alt_3-algebra. We expand every commutator in the polynomial

$$\begin{aligned} F = {} & [[[[a,b],d],e],c] + [[[[a,b],e],d],c] + [[[[a,c],d],e],b] + [[[[a,c],e],d],b] \\ & - [[[[a,d],b],c],e] - [[[[a,d],c],b],e] - [[[[a,e],b],c],d] - [[[[a,e],c],b],d] \end{aligned}$$

by the rule

$$[x,y] = xy - yx.$$

After this expansion, F becomes a linear combination of nonassociative multilinear monomials of degree 5 in the variables a, b, c, d, e.

We reduce this linear combination modulo all multilinear consequences in degree 5 of the defining identities of Alt_3, namely

$$(x,y,z) + (x,z,y) = 0,$$

$$(x,y,z) + (y,x,z) = 0,$$

and

$$(xy)z + (yx)z + (xz)y + (zx)y + (yz)x + (zy)x = 0.$$

Equivalently, we form the matrix R_5 whose rows are the coefficient vectors of all these multilinear consequences with respect to the ordered basis of all degree 5 nonassociative multilinear monomials. Then we append the coefficient vector of F to this matrix and reduce the obtained matrix to row echelon form by Algebra Software package such as [2].

Then

$$\begin{aligned} F = {} & ((((ab)d)e)c) - (c(((ab)d)e)) - ((e((ab)d))c) + (c(e((ab)d))) \\ & - (((d(ab))e)c) + (c((d(ab))e)) + ((e(d(ab)))c) - (c(e(d(ab)))) \\ & - ((((ba)d)e)c) + (c(((ba)d)e)) + ((e((ba)d))c) - (c(e((ba)d))) \\ & + (((d(ba))e)c) - (c((d(ba))e)) - ((e(d(ba)))c) + (c(e(d(ba)))) \\ & + ((((ab)e)d)c) - (c(((ab)e)d)) - ((d((ab)e))c) + (c(d((ab)e))) \\ & - (((e(ab))d)c) + (c((e(ab))d)) + ((d(e(ab)))c) - (c(d(e(ab)))) \\ & - ((((ba)e)d)c) + (c(((ba)e)d)) + ((d((ba)e))c) - (c(d((ba)e))) \end{aligned}$$

$$
\begin{aligned}
&+(((e(ba))d)c)-(c((e(ba))d))-((d(e(ba)))c)+(c(d(e(ba))))\\
&\quad+((((ac)d)e)b)-(b(((ac)d)e))-((e((ac)d))b)+(b(e((ac)d)))\\
&\quad-(((d(ac))e)b)+(b((d(ac))e))+((e(d(ac)))b)-(b(e(d(ac))))\\
&\quad-((((ca)d)e)b)+(b(((ca)d)e))+((e((ca)d))b)-(b(e((ca)d)))\\
&\quad+(((d(ca))e)b)-(b((d(ca))e))-((e(d(ca)))b)+(b(e(d(ca))))\\
&\quad+((((ac)e)d)b)-(b(((ac)e)d))-((d((ac)e))b)+(b(d((ac)e)))\\
&\quad-(((e(ac))d)b)+(b((e(ac))d))+((d(e(ac)))b)-(b(d(e(ac))))\\
&\quad-((((ca)e)d)b)+(b(((ca)e)d))+((d((ca)e))b)-(b(d((ca)e)))\\
&\quad+(((e(ca))d)b)-(b((e(ca))d))-((d(e(ca)))b)+(b(d(e(ca))))\\
&\qquad-((((ad)b)c)e)+(e(((ad)b)c))+((c((ad)b))e)-(e(c((ad)b)))
\end{aligned}
$$

$$
\begin{aligned}
&+(((b(ad))c)e)-(e((b(ad))c))-((c(b(ad)))e)+(e(c(b(ad))))\\
&\quad+((((da)b)c)e)-(e(((da)b)c))-((c((da)b))e)+(e(c((da)b)))\\
&\quad-(((b(da))c)e)+(e((b(da))c))+((c(b(da)))e)-(e(c(b(da))))\\
&\quad-((((ad)c)b)e)+(e(((ad)c)b))+((b((ad)c))e)-(e(b((ad)c)))\\
&\quad+(((c(ad))b)e)-(e((c(ad))b))-((b(c(ad)))e)+(e(b(c(ad))))\\
&\quad+((((da)c)b)e)-(e(((da)c)b))-((b((da)c))e)+(e(b((da)c)))\\
&\quad-(((c(da))b)e)+(e((c(da))b))+((b(c(da)))e)-(e(b(c(da))))\\
&\quad-((((ae)b)c)d)+(d(((ae)b)c))+((c((ae)b))d)-(d(c((ae)b)))\\
&\qquad+(((b(ae))c)d)-(d((b(ae))c))-((c(b(ae)))d)+(d(c(b(ae))))
\end{aligned}
$$

$$
\begin{aligned}
&+((((ea)b)c)d)-(d(((ea)b)c))-((c((ea)b))d)+(d(c((ea)b)))\\
&\quad-(((b(ea))c)d)+(d((b(ea))c))+((c(b(ea)))d)-(d(c(b(ea))))\\
&\quad-((((ae)c)b)d)+(d(((ae)c)b))+((b((ae)c))d)-(d(b((ae)c)))\\
&\quad+(((c(ae))b)d)-(d((c(ae))b))-((b(c(ae)))d)+(d(b(c(ae))))\\
&\quad+((((ea)c)b)d)-(d(((ea)c)b))-((b((ea)c))d)+(d(b((ea)c)))\\
&\qquad-(((c(ea))b)d)+(d((c(ea))b))+((b(c(ea)))d)-(d(b(c(ea))))).
\end{aligned}
$$

The coefficient vector of F belongs to the row span of R_5. Hence

$$F \equiv 0$$

in the multilinear component of degree 5 of the free Alt_3-algebra. Therefore,

$$
\begin{aligned}
&[[[[a,b],d],e],c]+[[[[a,b],e],d],c]+[[[[a,c],d],e],b]+[[[[a,c],e],d],b]\\
&-[[[[a,d],b],c],e]-[[[[a,d],c],b],e]-[[[[a,e],b],c],d]-[[[[a,e],c],b],d]=0.
\end{aligned}
$$

Thus the required identity holds in $Alt_3^{(-)}\langle X\rangle$. $\square$

2.2 Some identities under anti-commutator

Theorem 2.13. *An algebra $Alt_3^{(+)}\langle X\rangle$ satisfies the following identities:*

$$\{a,\{b,c\}\}+\{\{a,c\},b\}+\{\{a,b\},c\}=0, \tag{2.1}$$

$$\{\{a,d\},\{b,c\}\}=-\{\{a,c\},\{b,d\}\}+\{\{a,\{c,d\}\},b\}+\{a,\{b,\{c,d\}\}\}, \tag{2.2}$$

$$\begin{aligned}\{\{\{a,d\},c\},b\}+\{\{\{a,c\},d\},b\}+\{\{\{a,d\},b\},c\}+\{\{\{a,c\},b\},d\}\\ +\{\{\{a,b\},d\},c\}+\{\{\{a,b\},c\},d\}=0,\end{aligned} \tag{2.3}$$

$$\begin{aligned}-\{\{\{\{a,e\},c\},d\},b\}+\{\{\{\{a,c\},d\},e\},b\}+\{\{\{\{a,d\},e\},b\},c\}\\ +\{\{\{\{a,c\},e\},b\},d\}+\{\{\{\{a,e\},b\},c\},d\}-\{\{\{\{a,d\},b\},c\},e\}\\ +2\{\{\{\{a,c\},b\},e\},d\}-\{\{\{\{a,b\},d\},e\},c\}-2\{\{\{\{a,b\},d\},c\},e\}\\ -\{\{\{\{a,b\},c\},d\},e\}=0,\end{aligned} \tag{2.4}$$

$$\begin{aligned}-\{\{\{\{a,d\},c\},e\},b\}-\{\{\{\{a,c\},d\},e\},b\}-\{\{\{\{a,d\},e\},b\},c\}\\ -\{\{\{\{a,c\},e\},b\},d\}-\{\{\{\{a,e\},b\},d\},c\}-\{\{\{\{a,d\},b\},e\},c\}\\ -\{\{\{\{a,e\},b\},c\},d\}+\{\{\{\{a,d\},b\},c\},e\}-\{\{\{\{a,c\},b\},e\},d\}\\ +\{\{\{\{a,c\},b\},d\},e\}+\{\{\{\{a,b\},d\},e\},c\}+2\{\{\{\{a,b\},d\},c\},e\}\\ +\{\{\{\{a,b\},c\},e\},d\}+2\{\{\{\{a,b\},c\},d\},e\}=0.\end{aligned} \tag{2.5}$$

Proof The identity (2.1) is taken from (1.5). Other identities can be obtained using means of the package [2].

Proposition 2.14. *The identities (2.2), (2.3), (2.4) and (2.5) are consequence of commutative identity and (2.1).*

We use the identity (2.1) in two different ways on monomial $\{\{\{a,b\},c\},d\}$ as follows:

$$\begin{aligned}\{\{\{a,b\},c\},d\}&=^{(2.1)}-\{\{\{a,c\},b\},d\}-\{\{a,\{b,c\}\},d\}=^{(2.1)}\\ &\{\{\{a,c\},d\},b\}+\{\{a,c\},\{b,d\}\}+\{a,\{\{b,c\},d\}\\ &+\{\{a,d\},\{b,c\}\}=^{(2.1)}-\{\{\{a,d\},c\},b\}-\{\{a,\{c,d\}\},b\}\\ &+\{\{a,c\},\{b,d\}\}+\{\{a,d\},\{b,c\}\}-\{a,\{\{b,d\},c\}\}\\ &-\{a,\{b,\{c,d\}\}\},\{\{\{a,b\},c\},d\}=^{(2.1)}\\ &-\{\{\{a,b\},d\},c\}-\{\{a,b\},\{c,d\}\}=^{(2.1)}\\ &\{\{\{a,b\},d\},c\}+\{\{a,\{c,d\}\},b\}+\{a,\{b,\{c,d\}\}\}=^{(2.1)}\\ &-\{\{\{a,d\},c\},b\}-\{\{a,d\},\{b,c\}\}-\{\{a,c\},\{b,d\}\}\\ &-\{a,\{\{b,d\},c\}\}+\{\{a,\{c,d\}\},b\}+\{a,\{b,\{c,d\}\}\}.\end{aligned}$$

By equating and collecting similar monomials, we get

$$\{\{a,d\},\{b,c\}\}+\{\{a,c\},\{b,d\}\}-\{\{a,\{c,d\}\},b\}-\{a,\{b,\{c,d\}\}\}=0,$$

which correspond to (2.2).

To find the next identity, we use the previously obtained identities:

$$\begin{aligned}\{a,\{b,\{c,d\}\}\} &=^{(2.1)} -\{a,\{\{b,d\},c\}\}-\{a,\{\{b,c\},d\}\} =^{(2.1)}\\ &\{\{a,c\},\{b,d\}\}+\{\{a,\{b,d\}\},c\}+\{\{a,d\},\{b,c\}\}\\ &+\{\{a,\{b,c\}\},d\} =^{(2.1)} -\{\{\{a,c\},d\},b\}-\{\{\{a,c\},b\},d\}\\ &-\{\{\{a,d\},b\},c\}-\{\{\{a,b\},d\},c\}-\{\{\{a,d\},c\},b\}\\ &-\{\{\{a,d\},b\},c\}-\{\{\{a,c\},b\},d\}-\{\{\{a,b\},c\},d\},\end{aligned}$$

$$\begin{aligned}\{a,\{b,\{c,d\}\}\} &=^{(2.2)} \{\{a,\{c,d\}\},b\}-\{\{a,b\},\{c,d\}\}=\{\{\{a,d\},c\},b\}\\ &+\{\{\{a,c\},d\},b\}+\{\{\{a,b\},d\},c\}+\{\{\{a,b\},c\},d\}.\end{aligned}$$

By equalizing them, we obtain the identity (2.3). In the same way, we obtain the remained identities. More explicitly, there identity (2.4) can be prove as follows:

$$\begin{aligned}\{\{\{a,d\},c\},\{b,e\}\} &=^{(2.1)} -\{\{\{\{a,d\},c\},e\},b\}-\{\{\{\{a,d\},c\},b\},e\}\\ &=^{(2.3)} -\{\{\{\{a,d\},c\},e\},b\}+\{\{\{\{a,c\},d\},b\},e\}+\{\{\{\{a,d\},b\},c\},e\}\\ &+\{\{\{\{a,c\},b\},d\},e\}+\{\{\{\{a,b\},d\},c\},e\}+\{\{\{\{a,b\},c\},d\},e\},\end{aligned}$$

$$\begin{aligned}\{\{\{a,d\},c\},\{b,e\}\} &=^{(2.3)} -\{\{\{a,c\},d\},\{b,e\}\}-\{\{\{a,d\},\{b,e\}\},c\}\\ &-\{\{\{a,c\},\{b,e\}\},d\}-\{\{\{a,\{b,e\}\},d\},c\}-\{\{\{a,\{b,e\}\},c\},d\}\\ &=^{(2.1)} \{\{\{\{a,c\},d\},e\},b\}+\{\{\{\{a,c\},d\},b\},e\}+\{\{\{\{a,d\},e\},b\},c\}\\ &+\{\{\{\{a,d\},e\},b\},c\}+\{\{\{\{a,e\},b\},c\},d\}+\{\{\{\{a,b\},e\},c\},d\}\\ &+\{\{\{\{a,e\},b\},d\},c\}+\{\{\{\{a,b\},e\},d\},c\}+\{\{\{\{a,c\},e\},b\},d\}\\ &+\{\{\{\{a,c\},b\},e\},d\}\\ &=^{(2.3)} \{\{\{\{a,c\},d\},e\},b\}+\{\{\{\{a,c\},d\},b\},e\}+\{\{\{\{a,d\},e\},b\},c\}\\ &+\{\{\{\{a,d\},e\},b\},c\}+\{\{\{\{a,e\},b\},c\},d\}+\{\{\{\{a,b\},e\},c\},d\}\\ &+\{\{\{\{a,e\},b\},d\},c\}-\{\{\{\{a,b\},d\},e\},c\}-\{\{\{\{a,b\},e\},c\},d\}\\ &-\{\{\{\{a,b\},d\},c\},e\}-\{\{\{\{a,b\},c\},e\},d\}-\{\{\{\{a,b\},c\},d\},e\}\\ &+\{\{\{\{a,c\},e\},b\},d\}+\{\{\{\{a,c\},b\},e\},d\}.\end{aligned}$$

As before, we equalize them. Reducing the same monomials, we obtain (2.4).

For the last identity, we use previous identities to the monomial

$$\{\{\{a,e\},c\},\{b,d\}\}$$

in two different ways as follows:

$$\begin{aligned}\{\{\{a,e\},c\},\{b,d\}\} &=^{(2.1)} -\{\{\{\{a,e\},c\},d\},b\} - \{\{\{\{a,e\},c\},b\},d\}\\ &=^{(2.3)} -\{\{\{\{a,e\},c\},d\},b\} + \{\{\{\{a,c\},e\},b\},d\} + \{\{\{\{a,e\},b\},d\},c\}\\ &\quad + \{\{\{\{a,c\},b\},e\},d\} + \{\{\{\{a,b\},e\},c\},d\} + \{\{\{\{a,b\},c\},e\},d\},\end{aligned}$$

$$\begin{aligned}\{\{\{a,e\},c\},\{b,d\}\} &=^{(2.3)} -\{\{\{a,c\},e\},\{b,d\}\} - \{\{\{a,e\},\{b,c\}\},d\}\\ &\quad - \{\{\{a,c\},\{b,d\}\},e\} - \{\{\{a,\{b,d\}\},e\},c\} - \{\{\{a,\{b,d\}\},c\},e\}\\ &=^{(2.3)} \{\{\{\{a,c\},e\},b\},d\} + \{\{\{\{a,c\},e\},d\},b\} + \{\{\{\{a,e\},d\},b\},c\}\\ &\quad + \{\{\{\{a,e\},b\},d\},c\} + \{\{\{\{a,c\},d\},b\},e\} + \{\{\{\{a,c\},b\},d\},e\}\\ &\quad + \{\{\{\{a,d\},b\},e\},c\} + \{\{\{\{a,b\},d\},e\},c\} + \{\{\{\{a,d\},b\},c\},e\}\\ &\quad + \{\{\{\{a,b\},d\},c\},e\}\\ &=^{(2.1),(2.3)} -\{\{\{\{a,c\},d\},e\},b\} - \{\{\{\{a,c\},e\},b\},d\} - \{\{\{\{a,c\},d\},b\},e\}\\ &\quad - \{\{\{\{a,c\},b\},e\},d\} - \{\{\{\{a,c\},b\},d\},e\} + \{\{\{\{a,c\},e\},b\},d\}\\ &\quad - \{\{\{\{a,d\},e\},b\},c\} - \{\{\{\{a,e\},b\},d\},c\} - \{\{\{\{a,d\},b\},e\},c\}\\ &\quad - \{\{\{\{a,b\},d\},e\},c\} + \{\{\{\{a,e\},b\},d\},c\} + \{\{\{\{a,c\},d\},b\},e\}\\ &\quad + \{\{\{\{a,c\},b\},d\},e\} + \{\{\{\{a,d\},b\},e\},c\} + \{\{\{\{a,b\},d\},e\},c\}\\ &\quad + \{\{\{\{a,b\},e\},c\},d\} + \{\{\{\{a,b\},d\},c\},e\} + \{\{\{\{a,b\},c\},e\},d\}\\ &\quad + \{\{\{\{a,b\},c\},d\},e\}.\end{aligned}$$

Finally, we obtain the needed result. □

Remark. An algebra ${\mathcal{P}_2}^{!\,(+)}\langle X\rangle$ satisfies the following identity:

$$\{a,\{b,c\}\} + \{\{a,c\},b\} + \{\{a,b\},c\} = 0,$$

2.3 Anti-Leibniz algebras

The Leibniz algebra is defined by the identity

$$(xy)z = (xz)y + x(yz).$$

Here, we consider the anti-Leibniz algebra defined by the identity

$$(xy)z = (xz)y - x(yz)$$

Let us study a free algebra from the variety of Anti-Leibniz algebras.

Lemma 2.15. *An algebra Anti-*Leib$\langle$X$\rangle$ *satisfies the following identities:*

$$a\big(bc+cb\big) \;=\; 0, \tag{2.6}$$

$$a\big(b(cd)\big) \;=\; 0, \tag{2.7}$$

$$a\big((bc)d\big) \;=\; 0. \tag{2.8}$$

Proof It can be proved using the computer algebra software program Albert. □

Any non-associative monomial can be represented as

$$\big(\cdots((x_iw_1)w_2)\cdots\big)w_n,$$

where x_i is a generator and w_i is also non-associative monomial. On the set of non-associative monomials of Anti-Leib$\langle$X$\rangle$, we defined degree-lexicographical order. We say that a monomial

$$m \;=\; \big(\cdots((x_iw_1)w_2)\cdots\big)w_n$$

belongs to the set $\mathcal{AL}$ if it satisfies the following conditions:

- $w_1 \leq w_2 \leq \ldots \leq w_n$;
- $\deg(w_i) \leq 2$;
- for every i such that $w_i = x_{i_1}x_{i_2}$, we have $x_{i_1} \leq x_{i_2}$.

Example 2.16. Let us list the set $\mathcal{AL}$ in the multilinear case up to degree 8. Denote by $\mathcal{AL}_i$ the subset of $\mathcal{AL}$ consisting of monomials of degree i. In degree 3, we have

$$\mathcal{AL}_3 \;=\; \big\{\, x_1(x_2x_3),\; x_2(x_1x_3),\; x_3(x_1x_2),\; (x_1x_2)x_3,\; (x_2x_1)x_3,\; (x_3x_1)x_2 \,\big\}.$$

For higher degrees, we indicate only the bracketing type and the number of monomials satisfying the required conditions. If $i = 4$, then

$$\begin{array}{lcrcr}
\big((x_{i_1}x_{i_2})x_{i_3}\big)x_{i_4} & \to & \mathrm{C}_4^1 & = & 4,\\
(x_{i_1}x_{i_2})(x_{i_3}x_{i_4}) & \to & \mathrm{C}_4^2\cdot 2 & = & 12.
\end{array}$$

If $i = 5$, then

$$\begin{array}{lcrcr}
\big(((x_{i_1}x_{i_2})x_{i_3})x_{i_4}\big)x_{i_5} & \to & \mathrm{C}_5^1 & = & 5,\\
\big((x_{i_1}x_{i_2})x_{i_3}\big)(x_{i_4}x_{i_5}) & \to & \mathrm{C}_5^3\cdot 3 & = & 30,\\
\big(x_{i_1}(x_{i_2}x_{i_3})\big)(x_{i_4}x_{i_5}) & \to & \mathrm{C}_5^3\cdot \frac{3}{2} & = & 15.
\end{array}$$

If $i = 6$, then

$$\begin{aligned}
\big((((x_{i_1}x_{i_2})x_{i_3})x_{i_4})x_{i_5}\big)x_{i_6} &\to \mathrm{C}_6^1 = 6,\\
\Big(\big((x_{i_1}x_{i_2})x_{i_3}\big)x_{i_4}\Big)(x_{i_5}x_{i_6}) &\to \mathrm{C}_6^2\cdot 4 = 60,\\
\big((x_{i_1}x_{i_2})(x_{i_3}x_{i_4})\big)(x_{i_5}x_{i_6}) &\to \mathrm{C}_6^4\mathrm{C}_4^2\cdot 2/2 = 90.
\end{aligned}$$

If $i=7$, then

$$\begin{aligned}
\Big(\big((((x_{i_1}x_{i_2})x_{i_3})x_{i_4})x_{i_5}\big)x_{i_6}\Big)x_{i_7} &\to \mathrm{C}_7^1 = 7,\\
\Big(\big(((x_{i_1}x_{i_2})x_{i_3})x_{i_4}\big)x_{i_5}\Big)(x_{i_6}x_{i_7}) &\to \mathrm{C}_7^2\cdot 5 = 105,\\
\Big(\big((x_{i_1}x_{i_2})x_{i_3}\big)(x_{i_4}x_{i_5})\Big)(x_{i_6}x_{i_7}) &\to \mathrm{C}_7^2\mathrm{C}_5^2\cdot\tfrac{3}{2} = 315,\\
\Big(\big(x_{i_1}(x_{i_2}x_{i_3})\big)(x_{i_4}x_{i_5})\Big)(x_{i_6}x_{i_7}) &\to \mathrm{C}_7^2\mathrm{C}_5^2\mathrm{C}_3^2\cdot\tfrac{1}{3!} = 105.
\end{aligned}$$

If $i=8$, then

$$\begin{aligned}
\Big(\big(((((x_{i_1}x_{i_2})x_{i_3})x_{i_4})x_{i_5}\big)x_{i_6}\big)x_{i_7}\Big)x_{i_8} &\to \mathrm{C}_8^1 = 8,\\
\Big(\big((((x_{i_1}x_{i_2})x_{i_3})x_{i_4})x_{i_5}\big)x_{i_6}\Big)(x_{i_7}x_{i_8}) &\to \mathrm{C}_8^2\cdot 6 = 168,\\
\Big(\big(((x_{i_1}x_{i_2})x_{i_3})x_{i_4}\big)(x_{i_5}x_{i_6})\Big)(x_{i_7}x_{i_8}) &\to \mathrm{C}_8^2\mathrm{C}_6^2\cdot\tfrac{4}{2} = 840,\\
\Big(\big((x_{i_1}x_{i_2})(x_{i_3}x_{i_4})\big)(x_{i_5}x_{i_6})\Big)(x_{i_7}x_{i_8}) &\to \mathrm{C}_8^2\mathrm{C}_6^2\mathrm{C}_4^2\cdot\tfrac{2}{3!} = 840.
\end{aligned}$$

Similarly, we constructed $\mathcal{AL}_9$ and $\mathcal{AL}_{10}$, and obtained

$$|\mathcal{AL}_9| = 6876 \qquad \text{and} \qquad |\mathcal{AL}_{10}| = 26200.$$

Theorem 2.17. *The set $\mathcal{AL}$ is a basis of the algebra Anti-*Leib$\langle$X$\rangle$.

Proof Firstly, we show that the set $\mathcal{AL}$ spans Anti-Leib$\langle X\rangle$. Let m be an arbitrary monomial of degree k of the form

$$m \;=\; \big(\cdots((x_i w_1)w_2)\cdots\big)w_n.$$

Using the identities (2.7) and (2.8), we can reduce m whenever $\deg(w_j)\geq 3$. Using the identity (2.6), we order the generators inside w_j. Next, we can interchange w_j and w_{j+1} by the defining identity of Anti-Leib$\langle X\rangle$, namely,

$$(aw_j)w_{j+1} = a(w_{j+1}w_j) + (aw_{j+1})w_j.$$

If both w_j and w_{j+1} are generators and $w_j > w_{j+1}$, then the monomials $a(w_{j+1}w_j)$ and $(aw_{j+1})w_j$ belong to $\mathcal{AL}$. If $\deg(w_j) = \deg(w_{j+1}) = 2$, then the identity above reduces to

$$(aw_j)w_{j+1} = (aw_{j+1})w_j.$$

Thus, the set of remaining monomials coincides with the set $\mathcal{AL}$.

Let us consider the algebra $\mathcal{A}\langle X\rangle$ with a basis $\mathcal{AL}$. We define a multiplication on $\mathcal{A}\langle X\rangle$ as follows. Let $A_i\in\mathcal{AL}$ be an arbitrary basis monomial of degree i.

- If $A_t \in \mathcal{AL}$ has degree $t \geq 3$, then

$$A_k \cdot A_t = 0 \qquad (k \geq 1).$$

- If $t = 2$, and

$$A_k \;=\; \big(\cdots((x_i w_1)w_2)\cdots\big)w_n,$$

then for any degree-2 basis monomial w_p we set

$$A_k \cdot w_p = (\cdots((((\cdots((x_i w_1)w_2)\cdots)w_{r-1})w_p)w_r)\cdots)w_n,$$

where $w_{r-1} \leq w_p \leq w_r$.

- If $t = 1$, and

$$A_k = (\cdots(((\cdots((x_i x_{t_1})x_{t_2})\cdots)x_{t_l})w_{l+1})\cdots)w_n,$$

then for any generator x_p we define

$$\begin{aligned} A_k \cdot x_p &= (\cdots(((\cdots((((\cdots((x_i x_{t_1})x_{t_2})\cdots)x_{t_{r-1}})x_p)x_{t_r})\cdots)x_{t_l})w_{l+1})\cdots)w_n \\ &+\sum_{s=r}^{l}(\cdots((((\cdots(((\cdots((x_i x_{t_1})x_{t_2})\cdots)x_{t_{r-1}})x_{t_r})\cdots)x_{t_l})(x_p x_{t_s}))w_{l+1})\cdots)w_n, \end{aligned}$$

where $x_{t_{r-1}} \leq x_p < x_{t_r}$, and in the final sum there is no generator x_{t_s} among $x_{t_r}, \ldots, x_{t_l}$.

A direct computation shows that the algebra $\mathcal{A}\langle X\rangle$ belongs to Anti-Leib, that is, it satisfies the defining identity of $\mathfrak{Ma}$-Leib. It remains to note that

$$\mathcal{A}\langle X\rangle \cong \textit{Anti}\text{-Leib}\langle X\rangle.$$

□

As verified in Example 2.16, the first values of $\dim\big(\text{Anti-Leib}(n)\big)$ are given by

n	1	2	3	4	5	6	7	8	9	10
Anti-Leib(n)	1	2	6	16	50	156	532	1856	6876	26200

This sequence coincides with OEIS A013989, which counts the total number of adjacent transpositions in all involutions (self-inverse permutations) of the set $\{1, 2, \ldots, n+2\}$. In particular, if we set $a(n) = \text{Anti-Leib}(n)$, then up to degree 10, we have the recurrence

$$a(n) = n\left(\frac{a(n-1)}{n-1} + a(n-2)\right), \qquad a(1) = 1, \quad a(2) = 2.$$

Let us construct an explicit bijection for $n = 3, 4$ and 5.

A basis monomial in $\mathcal{AL}_n$ has the form

$$m = \big(\cdots((x_i w_1)w_2)\cdots\big)w_s,$$

where each w_j has degree 1 or 2. Therefore, such a monomial determines an involution on the set $\{1, \ldots, n\}$ together with one distinguished fixed point.

Indeed, the first letter x_i is considered as a distinguished fixed point. Each factor $w_j = x_p$ gives another fixed point, and each factor $w_j = x_p x_q$, where $p < q$, gives a transposition (p, q).

Thus we obtain a bijection

$$\mathcal{AL}_n \longleftrightarrow \big\{(\sigma, i) \mid \sigma^2 = \mathrm{id},\ \sigma(i) = i\big\},$$

where σ is an involution on $\{1, \ldots, n\}$ and i is a distinguished fixed point of σ.

Now we relate this set to adjacent transpositions in involutions. Given (σ, i), define an involution τ on $\{1, \ldots, n+1\}$ by replacing the distinguished fixed point i with the adjacent transposition $(i, i+1)$ and shifting all labels greater than i by one:

$$k \mapsto \begin{cases} k, & k < i, \\ k+1, & k > i. \end{cases}$$

Then τ contains the marked adjacent transposition $(i, i+1)$.

Conversely, if an involution τ on $\{1, \ldots, n+1\}$ contains a marked adjacent transposition $(i, i+1)$, then we remove this transposition, replace it by the fixed point i, and shift all labels greater than $i+1$ down by one. This gives an involution σ on $\{1, \ldots, n\}$ with distinguished fixed point i.

Hence we have a bijection

$$\mathcal{AL}_n \longleftrightarrow \big\{(\tau, (i, i+1)) \mid \tau^2 = \mathrm{id},\ (i, i+1) \text{ is a transposition of } \tau\big\}.$$

For example, for $n = 3$ we have

$$\mathcal{AL}_3 = \{x_1(x_2x_3),\ x_2(x_1x_3),\ x_3(x_1x_2),\ (x_1x_2)x_3,\ (x_2x_1)x_3,\ (x_3x_1)x_2\}.$$

They correspond to the following involutions with distinguished fixed points:

basis monomial	$(\sigma,$ distinguished fixed point$)$
$x_1(x_2x_3)$	$((2,3),1)$
$x_2(x_1x_3)$	$((1,3),2)$
$x_3(x_1x_2)$	$((1,2),3)$
$(x_1x_2)x_3$	$(\mathrm{id},1)$
$(x_2x_1)x_3$	$(\mathrm{id},2)$
$(x_3x_1)x_2$	$(\mathrm{id},3)$

Thus

$$|\mathcal{AL}_3| = 6.$$

For $n = 4$, the same rule gives

$$|\mathcal{AL}_4| = \#\{(\sigma, i) \mid \sigma^2 = \mathrm{id},\ \sigma(i) = i,\ i \in \{1,2,3,4\}\} = 4I_3 = 4 \cdot 4 = 16.$$

For $n = 5$, we obtain

$$|\mathcal{AL}_5| = 5I_4 = 5 \cdot 10 = 50.$$

Therefore, the basis monomials of Anti-Leib(n) are in bijection with the adjacent transpositions appearing in all involutions of $\{1, \ldots, n+1\}$.

More explicitly, for $n = 3$, the bijection is as follows:

$\mathcal{AL}_3$	marked adjacent transposition in an involution of $\{1,2,3,4\}$
$x_1(x_2x_3)$	$((1,2)(3,4);\ (1,2))$
$x_2(x_1x_3)$	$((1,4)(2,3);\ (2,3))$
$x_3(x_1x_2)$	$((1,2)(3,4);\ (3,4))$
$((x_1x_2)x_3)$	$((1,2);\ (1,2))$
$((x_2x_1)x_3)$	$((2,3);\ (2,3))$
$((x_3x_1)x_2)$	$((3,4);\ (3,4))$

For $n = 4$, the bijection is:

$\mathcal{AL}_4$	marked adjacent transposition in an involution of $\{1,2,3,4,5\}$
$(((x_1x_2)x_3)x_4)$	$((1,2);\ (1,2))$
$(((x_2x_1)x_3)x_4)$	$((2,3);\ (2,3))$
$(((x_3x_1)x_2)x_4)$	$((3,4);\ (3,4))$
$(((x_4x_1)x_2)x_3)$	$((4,5);\ (4,5))$
$((x_1x_2)(x_3x_4))$	$((1,2)(4,5);\ (1,2))$
$((x_2x_1)(x_3x_4))$	$((2,3)(4,5);\ (2,3))$
$((x_1x_4)(x_2x_3))$	$((1,2)(3,4);\ (1,2))$
$((x_4x_1)(x_2x_3))$	$((2,3)(4,5);\ (4,5))$
$((x_1x_3)(x_2x_4))$	$((1,2)(3,5);\ (1,2))$
$((x_3x_1)(x_2x_4))$	$((2,5)(3,4);\ (3,4))$
$((x_3x_4)(x_1x_2))$	$((1,2)(3,4);\ (3,4))$
$((x_4x_3)(x_1x_2))$	$((1,2)(4,5);\ (4,5))$
$((x_2x_4)(x_1x_3))$	$((1,4)(2,3);\ (2,3))$
$((x_4x_2)(x_1x_3))$	$((1,3)(4,5);\ (4,5))$
$((x_2x_3)(x_1x_4))$	$((1,5)(2,3);\ (2,3))$
$((x_3x_2)(x_1x_4))$	$((1,5)(3,4);\ (3,4))$

For $n = 5$, the bijection is:

$\mathcal{AL}_5$	marked adjacent transposition in an involution of $\{1-6\}$
$((((x_1x_2)x_3)x_4)x_5)$	$((1,2);\ (1,2))$
$((((x_2x_1)x_3)x_4)x_5)$	$((2,3);\ (2,3))$
$((((x_3x_1)x_2)x_4)x_5)$	$((3,4);\ (3,4))$
$((((x_4x_1)x_2)x_3)x_5)$	$((4,5);\ (4,5))$
$((((x_5x_1)x_2)x_3)x_4)$	$((5,6);\ (5,6))$
$(((x_1x_2)x_3)(x_4x_5))$	$((1,2)(5,6);\ (1,2))$
$(((x_2x_1)x_3)(x_4x_5))$	$((2,3)(5,6);\ (2,3))$
$(((x_3x_1)x_2)(x_4x_5))$	$((3,4)(5,6);\ (3,4))$
$(((x_1x_2)x_5)(x_3x_4))$	$((1,2)(4,5);\ (1,2))$
$(((x_2x_1)x_5)(x_3x_4))$	$((2,3)(4,5);\ (2,3))$
$(((x_5x_1)x_2)(x_3x_4))$	$((3,4)(5,6);\ (5,6))$
$(((x_1x_2)x_4)(x_3x_5))$	$((1,2)(4,6);\ (1,2))$
$(((x_2x_1)x_4)(x_3x_5))$	$((2,3)(4,6);\ (2,3))$
$(((x_4x_1)x_2)(x_3x_5))$	$((3,6)(4,5);\ (4,5))$
$(((x_1x_4)x_5)(x_2x_3))$	$((1,2)(3,4);\ (1,2))$
$(((x_4x_1)x_5)(x_2x_3))$	$((2,3)(4,5);\ (4,5))$
$(((x_5x_1)x_4)(x_2x_3))$	$((2,3)(5,6);\ (5,6))$
$((x_1(x_2x_3))(x_4x_5))$	$((1,2)(3,4)(5,6);\ (1,2))$
$(((x_1x_3)x_5)(x_2x_4))$	$((1,2)(3,5);\ (1,2))$
$(((x_3x_1)x_5)(x_2x_4))$	$((2,5)(3,4);\ (3,4))$
$(((x_5x_1)x_3)(x_2x_4))$	$((2,4)(5,6);\ (5,6))$
$((x_1(x_2x_4))(x_3x_5))$	$((1,2)(3,5)(4,6);\ (1,2))$
$(((x_1x_3)x_4)(x_2x_5))$	$((1,2)(3,6);\ (1,2))$
$(((x_3x_1)x_4)(x_2x_5))$	$((2,6)(3,4);\ (3,4))$
$(((x_4x_1)x_3)(x_2x_5))$	$((2,6)(4,5);\ (4,5))$
$((x_1(x_2x_5))(x_3x_4))$	$((1,2)(3,6)(4,5);\ (1,2))$

$\mathcal{AL}_5$	marked adjacent transposition
$(((x_3x_4)x_5)(x_1x_2))$	$((1,2)(3,4);\ (3,4))$
$(((x_4x_3)x_5)(x_1x_2))$	$((1,2)(4,5);\ (4,5))$
$(((x_5x_3)x_4)(x_1x_2))$	$((1,2)(5,6);\ (5,6))$
$((x_3(x_1x_2))(x_4x_5))$	$((1,2)(3,4)(5,6);\ (3,4))$
$((x_5(x_1x_2))(x_3x_4))$	$((1,2)(3,4)(5,6);\ (5,6))$
$((x_4(x_1x_2))(x_3x_5))$	$((1,2)(3,6)(4,5);\ (4,5))$
$(((x_2x_4)x_5)(x_1x_3))$	$((1,4)(2,3);\ (2,3))$
$(((x_4x_2)x_5)(x_1x_3))$	$((1,3)(4,5);\ (4,5))$
$(((x_5x_2)x_4)(x_1x_3))$	$((1,3)(5,6);\ (5,6))$
$((x_2(x_1x_3))(x_4x_5))$	$((1,4)(2,3)(5,6);\ (2,3))$
$((x_5(x_1x_3))(x_2x_4))$	$((1,3)(2,4)(5,6);\ (5,6))$
$((x_4(x_1x_3))(x_2x_5))$	$((1,3)(2,6)(4,5);\ (4,5))$

$\mathcal{AL}_5$	marked adjacent transposition
$(((x_2x_3)x_5)(x_1x_4))$	$((1,5)(2,3);\ (2,3))$
$(((x_3x_2)x_5)(x_1x_4))$	$((1,5)(3,4);\ (3,4))$
$(((x_5x_2)x_3)(x_1x_4))$	$((1,4)(5,6);\ (5,6))$
$((x_2(x_1x_4))(x_3x_5))$	$((1,5)(2,3)(4,6);\ (2,3))$
$((x_5(x_1x_4))(x_2x_3))$	$((1,4)(2,3)(5,6);\ (5,6))$
$((x_3(x_1x_4))(x_2x_5))$	$((1,5)(2,6)(3,4);\ (3,4))$
$(((x_2x_3)x_4)(x_1x_5))$	$((1,6)(2,3);\ (2,3))$
$(((x_3x_2)x_4)(x_1x_5))$	$((1,6)(3,4);\ (3,4))$
$(((x_4x_2)x_3)(x_1x_5))$	$((1,6)(4,5);\ (4,5))$
$((x_2(x_1x_5))(x_3x_4))$	$((1,6)(2,3)(4,5);\ (2,3))$
$((x_4(x_1x_5))(x_2x_3))$	$((1,6)(2,3)(4,5);\ (4,5))$
$((x_3(x_1x_5))(x_2x_4))$	$((1,6)(2,5)(3,4);\ (3,4))$

Conclusion

In this thesis, we investigated the algebraic and operadic properties of 3-nil alternative algebras and their connections with pre-Lie and assosymmetric structures. The main focus was on understanding how the identity $x^3 = 0$ influences the structure of these algebras and their associated operads.

We analyzed the defining identities of the variety Alt_3 and derived their multilinear consequences. One of the key results is the proof that the operad Alt_3 is self-dual. This provides an important structural insight and shows a strong symmetry between the operations and their dual counterparts. In addition, we studied the Koszul dual operad $\mathcal{P}_2^!$ and obtained its defining identities, as well as explicit descriptions under commutator and anti-commutator operations.

A significant part of the work was devoted to computing identities of degrees 3 and 5 in commutator and anti-commutator algebras. We showed that several higher-degree identities follow from lower-degree ones, which simplifies the structure of these varieties. Computational tools, such as the Dotsenko–Heijltjes package, were used to verify identities and compute dimensions of multilinear components.

Finally, we examined assosymmetric algebras with the identity $x^3 = 0$ and established relationships between all considered varieties. The results contribute to a deeper understanding of operadic duality, polynomial identities, and the structure of nonassociative algebras defined by nilpotency conditions.

References

[1] C. Bai, An Introduction to Pre-Lie Algebras, Algebra and Applications 1: Non-associative Algebras and Categories, 2021, 245–273.

[2] V. Dotsenko, W. Heijltjes. Gröbner bases for operads, http://irma.math.unistra.fr/ dotsenko/operads.html, 2019.

[3] A. S. Dzhumadil'daev, Special identity for Novikov–Jordan algebras, Communications Algebra 33(5), 1279–1287 (2005).

[4] V. Ginzburg, M. Kapranov, Koszul duality for operads, Duke Mathematical Journal, 76 (1994), 1, 203–272.

[5] A. Kunanbayev, B. Sartayev, Binary perm algebras and alternative algebras, Communications in Algebra, 541(1), 2026, 299-307.

[6] A. Kunanbayev, B. Sartayev, 4-type subvarieties of the variety of associative algebras, 2025, https://arxiv.org/abs/2506.02471.

[7] J. Liu, Y. Sheng, C. Bai, F-manifold algebras and deformation quantization via pre-Lie algebras, Journal of Algebra, 2020, 559, 467–495.

[8] A. I. Malcev, On algebras defined by identities (Russian). Mat. Sb. 26 (1950) 19–33.

[9] F. Mashurov, I. Kaygorodov, One-generated nilpotent assosymmetric algebras, Journal of Algebra and its Applications, 2022, 21(2), 2250031.

[10] S. V. Pchelintsev, Speciality of Metabelian Mal'tsev Algebras, Mathematical Notes volume 74, pages245–254 (2003).

[11] S. Pchelintsev, Identities defining a certain variety of right-alternative algebras, Mathematical Notes, 20, 2 1976, 651-659.

[12] A. Ryskeldin, B. Sartayev, Malcev classification for the variety of left-symmetric algebras, Boletin De La Sociedad Matematica Mexicana, 2026, 32(2), 50.

[13] L. A. Vladimirova, V. S. Drensky, Varieties of associative algebras with an identity of third degree (Russian), Pliska Stud. Math. Bulgar. 8 (1986), 144-157.

[14] B. Zhakhayev, A. Dzhumadil'daev, and S. Abdykassymova, Assosymmetric operad, Communications in Mathematics (2022).

[15] K. A. Zhevlakov, A. M. Slinko, I. P. Shestakov, A. I. Shirshov, Rings That Are Nearly Associative. (Russian) Moscow: Nauka.(1976).

KAZAKH MATHEMATICAL JOURNAL ISSN 2413–6468

24:4 (2024) 37–51

3-nil alternative, pre-Lie, and assosymmetric operads

Yerlan K. Duisenbay[1], Bauyrzhan K. Sartayev[2], Alpamys A. Tekebay[3]

[1,3]SDU University, Kaskelen, Kazakhstan

[2]Narxoz University, Almaty, Kazakhstan

[1]yerlanduisenbay@gmail.com, [2]baurjai@gmail.com, [3]alpamystekebay@gmail.com

Communicated by: Viktor V. Verbovskiy

Received: 11.12.2024 ⋆ Accepted/Published Online: 09.01.2025 ⋆ Final Version: 09.01.2025

Abstract. Alternative algebras are vital for studying and modeling systems that deviate from strict associativity but maintain enough structure to be useful. Indeed, alternative algebras generalize associative algebras by relaxing the strict associativity condition. Alternative algebras naturally include the octonions, which are a key example of a non-associative division algebra. The octonions are part of the Cayley-Dickson construction and play a critical role in geometry, topology, and theoretical physics, especially in string theory and exceptional Lie groups. The origin of alternative algebras lies in the historical exploration of division algebras and their applications extend to various mathematical and physical disciplines, especially in understanding non-associative algebraic structures. In this paper, we consider free alternative algebra with the additional identity $x^3 = 0$. For motivation, we refer to the dual operad of the alternative operad. Also, we obtain pre-Lie algebra with the identity $x^3 = 0$ from binary perm algebra. Finally, we consider assosymmetric algebra with identity $x^3 = 0$.

Keywords. alternative algebra, pre-Lie algebra, assosymmetric algebra, polynomial identities.