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**Süleyman Demirel University**  
**Faculty of Economics**

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**H.N.Aliyev**

**STATISTICS**

**Solved problems and exercises**  
**Part 1**

Almaty – 2006

**Süleyman Demirel University**  
**Faculty of Economics**

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## To the students

Nowadays, all companies use statistical methods in making decisions. Consequently, the study of statistical methods has taken on a prominent role in the education of student majoring in management and economics. Here is some advice that will help you to succeed in statistics (and in other subjects too).

**Tip1:** Understanding the process of solving a particular type of problem is emphasized on memorizing formulas. In most cases, if you understand the concepts, memorizing a formula becomes completely unnecessary, because you construct the necessary tools when needed.

**Tip2:** Classes are held for your benefit. If attending class was not important, all university courses would be by correspondence, and your tuition would be much lower. During class your instructor will go over examples, which are important, and most likely not in the book. Statistics courses are sequential, so the stuff you see in, for example lecture 6, will enable you to make sense of a much material you will see in lecture 7. As instructors, we note a definite correlation between grades and class attendance. **Go to class!!**

**Tip3:** Statistics books are not meant to be read like novels (even though they are often exciting). It is better generally to read the sections of the book to be covered in lecture through quickly to get some idea of what there is before going to lecture. After the lecture read it through carefully, with pencil and paper in hand, working through examples.

**Tip4:** Just as you must play a lot of football to be good at it, you must do a lot of statistics problems in order to be successful as well. At minimum, work on every problem your instructor suggests. If you are having trouble or want more practice, work on other problems in that section or get another book and work problems out of it. If you are having trouble getting a correct answer to a problem, think about what is going wrong. By doing

these you can learn something new and prevent yourself from making the same error in the future. Work on problems more than once. Work on problems until you can do them quickly. Remember, the process is usually more important than the result.

**Tip5:** The fastest way to get into trouble in statistics is to not do homework. Remember, similar problems will probably show up on quizzes and exams, where you will be expected to work them quickly and accurately, probably without the book in front of you.

**Tip6:** Contrary to many students' opinions, your instructor wants you to succeed. Extremely rare is the instructor who will intentionally put completely different material on an exam that was covered in class. For this reason, pay attention to your instructor and take notes. Then read your notes, and be sure you understand them, filling any missing details. Review your notes regularly.

The goal of this book is to present statistics in a clear and interesting way. Students, who will use this book, are not required to have a strong background in mathematics. The chapters are divided into sections, and each section contains necessary theoretical background and solved problems. A set of exercises appears at the end of each section. Answers are right after exercises.

In the end, any suggestion from readers would be greatly appreciated.

Dr. Humbet Aliyev

## Chapter 1 Organization and description of data

### 1.1. Introduction

Statistics is a group of methods that are used to collect, analyse, present, interpret data and make decisions.

Statistics is sometimes divided into two main areas:

1. Descriptive statistics
2. Inferential statistics.

**Descriptive statistics** consists of the collection, organization, summation, and presentation of data.

**A population** is a complete set of units (usually people, objects, events) that we are interested in studying.

A subset of the population selected for study is called **a sample**.

**Inferential statistics** is an estimate or prediction about a population based on information contained in a sample.

### 1.2. The mean

The mean for ungrouped data, also known as the arithmetic average, is found by adding the values of the data and dividing by the total number of values. Thus,

Mean for population data:

$$\mu = \frac{\sum_{i=1}^N x_i}{N}$$

Mean for sample data:

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$$

where  $N$  – is the population size,  $n$  – is sample size,  $\mu$  – (Greek letter mu) is the population mean, and  $\bar{x}$  – (read as “x-bar”) is the sample mean.

**Example:**

Calculate the mean of the following six sample observations:  
5, 2, 6, 8, 7, 8

**Solution:**

Using the definition of sample mean, we find

$$\bar{x} = \frac{\sum_{i=1}^6 x_i}{6} = \frac{5+2+6+8+7+8}{6} = \frac{36}{6} = 6.$$

Thus, the mean of this sample is 6.

**Example:**

The salaries of all 7 employees of a small company are:

\$ 320, 410, 310, 480, 530, 370, 240

Find the mean salary.

**Solution:**

Since the given data set includes all 7 employees of the company, it represents the population. Hence,  $N = 7$ . The population mean is

$$\mu = \frac{\sum_{i=1}^N x_i}{N} = \frac{320 + 410 + 310 + 480 + 530 + 370 + 240}{7} = \frac{2660}{7} = \$380.$$

Thus, the mean salary of the employees of this company is \$380.

### 1.3. The median

The median is the middle term in a data set. Before one can find this point the data must be arranged in increasing (or decreasing) order. The calculation of the median for ungrouped data consists of the following two steps:

1. Rank the given data set in increasing (or decreasing) order.
2. Find  $\left(\frac{n+1}{2}\right)^{th}$  term in a ranked data set.

The value of  $\left(\frac{n+1}{2}\right)^{th}$  term is the median.

There are two possibilities

- 1) If  $n$  is odd, then the median is given by the value of the middle term in ranked data.
- 2) If  $n$  is even, then the median is given by the average of the values of two middle terms.

**Remark:** If the given data set represents a population, replace  $n$  by  $N$ .

**Example:**

Consider again the seven salaries of employees of a small company  
\$ 320, 410, 310, 480, 530, 370, 240

Calculate the median of this population.

**Solution:**

First of all, let us rank salaries in ascending order:

\$ 240, 310, 320, 370, 410, 480, 530

$$N=7 \text{ and } \left[\frac{N+1}{2}\right]^{th} = \left[\frac{7+1}{2}\right]^{th} = 4^{th}$$

Therefore, the median is the value of the fourth term in the ranked data

\$ 240, 310, 320, 370, 410, 480, 530

↑  
Median

Thus, the median value for this population is \$370.

**Example:**

The ages of a sample of 10 university students are

18, 22, 19, 20, 21, 18, 22, 19, 23, 17

Calculate the median of this sample.

**Solution:**

First we order the data in increasing order. The ordered values are

17, 18, 18, 19, 19, 20, 21, 22, 22, 23

There are 10 values in the data set. Hence,

$$n = 10 \text{ and } \left[\frac{n+1}{2}\right]^{th} = \left[\frac{10+1}{2}\right]^{th} = 5.5^{th}$$

Therefore, the median is given by the mean of fifth and sixth values in the ranked data.

$5^{th}$  value = 19,  $6^{th}$  value = 20

$$\text{Median} = \frac{19 + 20}{2} = 19.5.$$

Hence, the median age is 19.5.

## 1.4. The Mode

The value that occurs most often in a data set is called the mode.

**Example:**

The following data give the GPA of 7 students

3.6; 3.2; 2.8; 3.6; 3.8; 3.6; 2.9

Find the mode.

**Solution:**

It is helpful to arrange the data in order, although it is not necessary

2.8; 2.9; 3.2; 3.6; 3.6; 3.6; 3.8

Since 3.6 occurs three times and 3.6 has a frequency larger than any other number- the mode for the data set is 3.6.

A data set can have more than one mode or no mode at all, whereas it will have only one mean and only one median.

A data set with each value occurring only once has no mode.

A data set with two (or more) values occurring with the same (highest) frequency has two (or more) modes.

**Example:**

Find the mode of the set data set:

2, 3, 3, 1, 2, 7, 7, 3, 1, 2

**Solution:** Since 2 and 3 both occur three times, the modes are 2 and 3. This data set is said to be bimodal.

**Example:**

Last year's income of six randomly selected families were

210,000, 300,000, 325,000, 280,000, 315,000, 410,000

Find the mode.

**Solution:**

Since each value occurs only once, there is no mode.

**Remark:** One advantage of the mode is that it can be calculated for both kinds of data, quantitative and qualitative, whereas the mean and median can be calculated only for quantitative data.

**Example:**

6 students are selected at random. Their statuses are:

Senior, sophomore, senior, junior, senior, sophomore.

Find the mode.

**Solution:**

Since senior occurs more frequently than the other categories, it is the mode for this data set. However, we can not calculate the mean and

median for this data set.

For a data set, the mean, median, and mode can be quite different. Consider the following example.

**Example:**

The number of days the first six heart transplant patients survived after their operations were

15; 3; 46; 623; 126; 64

Find the mean, median, and mode.

**Solution:**

The sample mean is

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n} = \frac{15 + 3 + 46 + 623 + 126 + 64}{6} = \frac{877}{6} = 146.2 \text{ days}$$

To find median, first we rank the data. The ordered values are

3, 15, 46, 64, 126, 623                       $n = 6$

$$\text{Median} = \left[ \frac{n+1}{2} \right]^{\text{th}} = \left[ \frac{6+1}{2} \right]^{\text{th}} = 3.5^{\text{th}}$$

$$\text{Median} = \frac{3^{\text{rd}} \text{ value} + 4^{\text{th}} \text{ value}}{2} = \frac{46 + 64}{2} = 55 \text{ days}$$

Since each value occurs only once, there is no mode.

In this example, the mean is much higher than the median. Only 1 out of 6

patients survived longer than  $\bar{x} = 146.2$  days. It is because of only large survival time that greatly inflates the mean. In this and similar situations, the median should be used as the measure of central tendency.

To sum up, we can not conclude which of the three measures of central tendency is a better measure overall. Each of them may be better under different situations. But the mean is the most used measure of central tendency. The advantage of the mean is that its calculation includes each value of the data set.

### Exercises

1. Calculate the mean and median for each of the following data set.

a) 6, 10, 7, 14, 8

b) 2, 1, 4, 2, 1

2. The following data set belongs to a population

5, -7, 2, 0, -9, 12, 10, 7

Calculate the mean, median and mode.

3. Find the mode of each of the following samples

a) 5, 8, 11, 9, 8, 6, 8

b) 7, 12, 8, 7, 10, 11, 8, 6, 10, 13, 7, 8

4. Twelve secretaries were given a typing test, and the time (in minutes) to complete it were as follows:

8, 12, 15, 9, 6, 8, 10, 9, 8, 6, 7, 8

Find the mean, median, and mode.

5. The grade point average (GPA) of 10 students who applied for financial aid are shown below

3.15, 3.62, 2.54, 2.81, 3.97, 1.85, 1.95, 2.63, 2.50, 2.80

Find the mean, median, and mode.

6. During a year, the major earthquakes had Richter magnitudes as shown below

7.0, 6.2, 7.7, 8.0, 6.4, 6.2, 7.2, 5.4, 6.4, 6.5, 7.2, 5.4

Find the mean, median, and mode.

7. Eight participants in a bike race had following finishing times in minutes

28, 22, 26, 33, 21, 23, 37, 24

Find the mean and median for the finishing times.

8. The monthly income in dollars for seven families are

950, 775, 925, 2500, 1150, 850, 975

a) Calculate the mean and median income

b) Which of the two is preferable as a measure of center,

and why?

9. The numbers of defective parts observed on 16 different days are shown below:

11, 14, 18, 14, 21, 17, 13, 21, 25, 19, 17, 13, 28, 13, 17, 18

What statistical measures of central tendency would help summarize this data set?

What information would you report to the production manager?

10. Consider the following two data sets:

Data set I: 10, 12, 17, 8, 15

Data set II: 13, 15, 20, 11, 18

Notice that each value of the second data set is obtained by adding 3 to the corresponding value of the first data set. Calculate the mean for each of these data sets. Comment on the relationship between the two means.

11. Consider the following two data sets:

Data set I: 2, 5, 7, 9, 8

Data set II: 6, 15, 21, 27, 24

Notice that each value of the second data set is obtained by multiplying the corresponding value of the first data set by 3. Calculate the mean for each of these data sets. Comment on the relationship between the two means.

### Answers

1. a)  $\bar{x} = 9$ ; Median = 8; b)  $\bar{x} = 2$ ; Median = 2; 2.  $\bar{x} = 2.5$ ; Median = 3.5; no mode; 3. a) 8; b) 7, 8; 4.  $\bar{x} = 8.83$ ; Median = 8; mode = 8; 5.  $\bar{x} = 2.78$ ; Median = 2.715; no mode; 6.  $\bar{x} = 6.6$ ; Median = 6.45; no mode; 7.  $\bar{x} = 26.75$ ; Median = 25; 8. a)  $\bar{x} = 1160.7$ ; Median = 950.

### 1.5. Measures of dispersion for ungrouped data

In statistics, in order to describe the data set accurately statisticians must know more than measures of central tendency. Two data sets with the same mean may have completely different spreads. The variation among values of observations for one data set may be much larger or smaller than for the other data set.

#### Remark:

The words dispersion, spread, and variation have the same meaning.

#### Example:

Consider the following two samples:

Sample 1: 66, 66, 66, 67, 67, 67, 68, 69

Sample 2: 43, 44, 50, 54, 67, 90, 91, 97

The mean of sample 1 is

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n} = \frac{536}{8} = 67$$

The mean of sample 2 is

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n} = \frac{536}{8} = 67$$

Each of these samples has a mean equal to 67. However, the dispersion of the observations in the two samples differs greatly. In the first sample all observations are grouped within 2 units of the mean. Only one observation (67) is closer than 13 units to the mean of the second sample, and some as far away as 30 units. Thus, the mean, median, or mode is usually not itself a sufficient measure to reveal the shape of the distribution of a data set. We also need a measure that can provide some information about the variation among data values. The measures that help us to know about the spread of data set are called the measures of dispersion. The measures of central tendency and dispersion taken together give a better picture of a data set than measure of central tendency alone. Several quantities that used as measures of dispersion are the *range*, the *mean absolute deviation*, the *variance*, and the *standard deviation*.

### 1.5.1. Range

The simplest measure of variability for a set of data is the range.

#### Definition:

The range for a set of data is the difference between the largest and smallest values in the set.

$$\text{Range} = \text{Largest value} - \text{Smallest value}$$

#### Example:

Find the range for the sample observations

13, 23, 11, 17, 25, 18, 14, 24

#### Solution:

We see that the largest observation is 25 and the smallest observation is 11. The range is  $25 - 11 = 14$ .

#### Example:

A sample is composed of the observations

67, 79, 87, 97, 93, 57, 44, 80, 47, 78, 81, 90, 88, 91

Find the range.

#### Solution:

The largest observation is 97; the smallest observation is 44. The range is  $97 - 44 = 53$ .

### 1.5.2. The mean absolute deviation

The mean absolute deviation is defined exactly as the words indicate. The word "deviation" refers to the deviation of each member from the mean of the population. The term "absolute deviation" means the numerical (i.e. positive) value of the deviation, and the "mean absolute deviation" is simply the arithmetic mean of the absolute deviations.

Let  $x_1, x_2, x_3, \dots, x_N$  denote the  $N$  members of a population, whose mean is  $\mu$ . Their mean absolute deviation, denoted by *M.A.D.*, is

$$M.A.D. = \frac{\sum_{i=1}^N |x_i - \mu|}{N}$$

For the sample of  $n$  observations, with mean  $\bar{x}$ , mean absolute deviation is defined analogously

$$M.A.D. = \frac{\sum_{i=1}^n |x_i - \bar{x}|}{n}$$

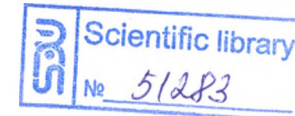
To calculate mean absolute deviation it is necessary to take following steps:

1. Find  $\bar{x}$  (or  $\mu$ )
2. Find and record the signed differences
3. Find and record the absolute differences
4. Find  $\sum_{i=1}^n |x_i - \bar{x}|$  (or  $\sum_{i=1}^N |x_i - \mu|$ )
5. Find the mean absolute deviation.

#### Example:

Suppose that sample consists of the observations

21, 17, 13, 25, 9, 19, 6, and 10



Find the mean absolute deviation.

**Solution:**

Perhaps the best manner to display the computations in steps 1, 2, 3, and is to make use of a table 1.1 composed of three columns

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n} = \frac{120}{8} = 15$$

$$M.A.D. = \frac{\sum_{i=1}^n |x_i - \bar{x}|}{n} = \frac{44}{8} = 5.5$$

Table 1.

$x_i$	$x_i - \bar{x}$	$ x_i - \bar{x} $
21	21-15=6	6
17	17-15=2	2
13	13-15=-2	2
25	25-15=10	10
9	9-15=-6	6
19	19-15=4	4
6	6-15=-9	9
10	10-15=-5	5
120		44

On the average, each observation is 5.5 units from the sample.

**1.5.3. The variance and the standard deviation**

A key step in developing a measure of variability that includes all the items involves the computations of the differences between the data value and the mean for the data set. The difference between  $x_i$  and the mean

( $\bar{x}$  for a sample,  $\mu$  for a population) is called a deviation about the mean. Since we are seeking a descriptive statistical measure that summarizes variability or dispersion in the entire data set, we want to consider the deviation of each data value about the mean. Thus for a sample size  $n$ , data values  $x_1, x_2, \dots, x_n$ , we will need to compute the deviations

$$(x_1 - \bar{x}), (x_2 - \bar{x}), \dots, (x_n - \bar{x}).$$

We might think of summarizing the dispersion in a data set by computing the average deviation about the mean. The only trouble with such an

attempted definition is that it would not give us much information about

the variation present in the data; the mean  $\frac{\sum_{i=1}^n (x_i - \bar{x})}{n}$  would be zero for

every sample, because the sum  $\sum_{i=1}^n (x_i - \bar{x})$  equals zero for every sample.

The positive and negative deviations cancel each other out. Hence if we were to use the deviations from the mean as a measure of dispersion we must find another approach. As we already know, one way is computing the average absolute deviation as a measure of variability. While this measure is sometimes used, the one most often used is based on squaring the deviations to eliminate the negative values. The average of the squared deviations for a data set representing a population or sample is given a special name in statistics. It is called the **variance**.

The **population variance** is denoted by the Greek symbol  $\sigma^2$  (pronounced "sigma squared"). The formula for population variance is

$$(1) \quad \sigma^2 = \frac{\sum_{i=1}^N (x_i - \mu)^2}{N}$$

where  $x_i$  - population data

$\mu$  - population mean

$N$  - population size

is frequently desirable to have a measure of dispersion whose units are the same as those of the observations. Since the variance is given in squared units, the square root of the variance would be given in units that need.

Thus, if we take the square root of the variance, we have the measure of dispersion that is known as the population standard deviation and denoted  $\sigma$ . By definition we have

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum_{i=1}^N (x_i - \mu)^2}{N}}$$

In many statistical applications, the data set we are working with is a sample. When we compute a measure of variability for the sample, we often are interested in using the sample statistic obtained as an estimate of the population parameter,  $\sigma^2$ . At this point it might seem that the average of the squared deviations in the sample would provide a good estimate of the population variance. However, statisticians have found that the average squared deviation for the sample has the undesirable feature that it tends to underestimate the population variance  $\sigma^2$ . Because of this tendency toward underestimation we say it provides a **biased estimate**.

Fortunately, it can be shown that if the sum of the squared deviations in a sample is divided by  $(n-1)$ , and not  $n$ , then the resulting sample statistic will provide an unbiased estimate of the population variance. For this reason the **sample variance** is not defined to be the average squared deviation in the sample. Sample variance is denoted by  $s^2$  and is defined as follows:

$$(2) \quad s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}$$

To find the sample standard deviation (denoted by  $s$ ), one must take the square root of the sample variance:

$$\text{Sample standard deviation} = s = \sqrt{s^2} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}}$$

**Example:**

Find the variance and the standard deviation for the sample data 21, 17, 13, 25, 9, 19, 6, and 10

**Solution:**

When we compute  $s^2$  by applying formula (2), the computation can most conveniently be shown in a table. The table will be composed of three columns: a column for the observations  $x_i$ , a column for the deviations of the observations from the sample mean  $(x_i - \bar{x})$ , and a column for the squared deviations  $(x_i - \bar{x})^2$ . (Table 1.2)

(Table 1.2)

$x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
21	21-15=6	36
17	17-15=2	4
13	13-15=-2	4
25	25-15=10	100
9	9-15=-6	36
19	19-15=4	16
6	6-15=-9	81
10	10-15=-5	25
120		302

$$\sum_{i=1}^8 x_i = 120; \quad \bar{x} = \frac{\sum_{i=1}^8 x_i}{n} = \frac{120}{8} = 15$$

$$s^2 = \frac{\sum_{i=1}^8 (x_i - \bar{x})^2}{n-1} = \frac{302}{7} = 43.14;$$

$$\text{and } s = \sqrt{s^2} = \sqrt{43.14} = 6.57.$$

From the computational point of view, it is easier and more efficient to use **short-cut** formulas to calculate the variance. By using the short-cut formula, we reduce the computation time and round off errors. The short-cut formulas for calculating variance are as follows:

$$\sigma^2 = \frac{1}{N} \left[ \sum x_i^2 - \frac{(\sum x_i)^2}{N} \right] = \frac{1}{N} \left[ \sum x_i^2 - N \cdot \mu^2 \right]$$

$$s^2 = \frac{1}{n-1} \left[ \sum x_i^2 - \frac{(\sum x_i)^2}{n} \right] = \frac{1}{n-1} \left[ \sum x_i^2 - n \cdot (\bar{x})^2 \right]$$

**Example:**

Find the variance and the standard deviation for the sample of 16, 19, 15, 15, and 14

**Solution:**

Let us apply

$$s^2 = \frac{1}{n-1} \left[ \sum x_i^2 - \frac{(\sum x_i)^2}{n} \right]$$

**Step1:** Find the sum of values,

$$\sum x = 16 + 19 + 15 + 15 + 14 = 79$$

**Step2:** Square each value and find the sum

$$\sum x^2 = 16^2 + 19^2 + 15^2 + 15^2 + 14^2 = 1263$$

**Step3:** Substitute in the formula and calculate

$$s^2 = \frac{1}{n-1} \left[ \sum x_i^2 - \frac{(\sum x_i)^2}{n} \right] = \frac{1}{4} \left[ 1263 - \frac{79^2}{5} \right] = 3.7$$

$$s = \sqrt{3.7} = 1.9$$

Hence the sample variance is 3.7 and sample standard deviation is 1.9.

#### 1.5.4. Interpretation of the population standard deviation

Often in statistical studies we are interested in specifying the percentage items in a data set that lie within some specified interval when only the mean and standard deviation for the data set are known. Two rules are commonly used for forming such estimates.

The first is true for any data set.

#### **Chebyshev's theorem:**

For any set of data and any  $k \geq 1$ , at least  $100 \cdot \left(1 - \frac{1}{k^2}\right)\%$  of the values

in the data set must be within plus or minus  $k$  standard deviations of the mean.

#### **Remark:**

In applying Chebyshev's theorem we treat every data set as if it were a population, and the formula for a population standard deviation is used.

$k$	1.5	2	2.5
$100 \cdot \left(1 - \frac{1}{k^2}\right)\%$	55.6%	75%	84%

According to Chebyshev's rule, at least 55.6% of the population data lie within 1.5 standard deviations around the mean, at least 75% of the

population data lie within 2 standard deviations around the mean and so on.

#### **Example:**

Let  $\mu = 70$ ,  $\sigma = 1.5$

we let  $k = 3$  from  $100 \cdot \left(1 - \frac{1}{k^2}\right)\%$  we obtain that

$$100 \cdot \left(1 - \frac{1}{k^2}\right)\% = 100 \cdot \left(1 - \frac{1}{9}\right)\% = \frac{8}{9} \cdot 100\% = 88.89\%$$

The theorem states that at least 88.89% of data values will fall within 3 standard deviations of the mean. 88.89% of data falls within  $(\mu \pm 3\sigma)$  or

$$70 + 3 \cdot 1.5 = 74.5 \text{ and}$$

$$70 - 3 \cdot 1.5 = 65.5$$

for  $\mu = 70$ ,  $\sigma = 1.5$ , at 88.89% of the data values fall between 74.5, 65.5.

#### **Rule of Thumb:**

When a distribution is bell-shaped the following statements, which are called Thumb rule, are true:

Approximately 68% of the population members lie within one standard deviation of the mean.

Approximately 95% of the population members lie within two standard deviations of the mean.

Approximately 99.7% of the population members lie within three standard deviations of the mean.

**Example:** suppose that scores on entrance exam have a mean of 480 and standard deviation of 90. If these scores are normally distributed, then

Approximately 68% will fall between 390 and 570 ;

$$(480 - 1 \cdot 90 = 390 \text{ and } 480 + 1 \cdot 90 = 570)$$

Approximately 95% of the scores will fall between 300 and 660

$$(480 - 2 \cdot 90 = 300 \text{ and } 480 + 2 \cdot 90 = 660)$$

Approximately 99.7% of the scores will fall between 210 and 750

$$(480 - 3 \cdot 90 = 210 \text{ and } 480 + 3 \cdot 90 = 750)$$

### 1.5.5. The interquartile range

Quartiles are the summary measures that divide a ranked data set into four equal parts. Three measures will divide any data set into four equal parts. These three measures are the first quartile (denoted by  $Q_1$ ), the second quartile (denoted by  $Q_2$ ), and the third quartile (denoted by  $Q_3$ ). The data should be ranked in increasing order before the quartiles are determined. The quartiles are defined as follows:

$$Q_1 = \left[ \frac{(n+1)}{4} \right]^{th} \quad \text{- ordered observation}$$

$$Q_3 = \left[ \frac{3 \cdot (n+1)}{4} \right]^{th} \quad \text{- ordered observation.}$$

The difference between the third and the first quartiles gives the interquartile range. That is

$$IQR = \text{Interquartile range} = Q_3 - Q_1.$$

#### Example:

A teacher gives a 20-point test to 10 students. The scores are shown below: 18, 15, 12, 6, 8, 2, 3, 5, 20, 10

Find the interquartile range.

#### Solution:

First, we rank the given data in increasing order:

$$2, 3, 5, 6, 8, 10, 12, 15, 18, 20$$

$$n = 10$$

$$Q_1 = \left[ \frac{(n+1)}{4} \right]^{th} \quad \text{- ordered observation.}$$

$$Q_1 = \left[ \frac{10+1}{4} \right]^{th} = \left[ 2 \frac{3}{4} \right]^{th}$$

Hence, the first quartile is three-quarter way from the 2<sup>nd</sup> data (3) to the 3<sup>rd</sup> third (5). Therefore,

$$\text{First quartile} = Q_1 = 3 + \frac{3}{4}(5-3) = 3 + \frac{3}{2} = \frac{9}{2}$$

$$\text{Similarly, since } Q_3 = \left[ \frac{3(n+1)}{4} \right]^{th} = \left[ \frac{3 \cdot 11}{4} \right]^{th} = \left[ \frac{33}{4} \right]^{th} = \left[ 8 \frac{1}{4} \right]^{th}$$

the third quartile is one-quarter of the way from the 8<sup>th</sup> observation (15) to the 9<sup>th</sup> observation (18). Thus we have

$$\text{Third quartile} = Q_3 = 15 + \frac{1}{4}(18-15) = 15 + \frac{3}{4} = 15 \frac{3}{4}$$

Therefore, the interquartile range is the difference between the third and first quartiles:

$$\begin{aligned} \text{Interquartile range} = IQR &= Q_3 - Q_1 = \\ &= 15 \frac{3}{4} - 3 \frac{3}{2} = \frac{63}{4} - \frac{9}{2} = \frac{45}{4} = 11.25 \end{aligned}$$

#### Example:

The following are the ages of nine employees of an insurance company: 47, 28, 39, 51, 33, 37, 59, 24, 33. Find the interquartile range.

#### Solution:

First, we arrange the data in order from smallest to largest: 24, 28, 33, 33, 37, 39, 47, 51, 59

$$\begin{aligned} Q_1 &= \left[ \frac{(n+1)}{4} \right]^{th} = \left[ \frac{9+1}{4} \right]^{th} = (2.5)^{th} = 2^{nd} + \frac{1}{2}(3^{rd} - 2^{nd}) = \\ &= 28 + \frac{1}{2}(33-28) = 28 + 2.5 = 30.5 \end{aligned}$$

$$\begin{aligned} Q_3 &= \left[ \frac{3(n+1)}{4} \right]^{th} = \left[ \frac{3 \cdot 10}{4} \right]^{th} = \left[ 7 \frac{3}{4} \right]^{th} = 7^{th} + \frac{3}{4}(8^{th} - 7^{th}) = \\ &= 47 + \frac{3}{4}(51-47) = 47 + 9 = 56 \end{aligned}$$

Therefore, the interquartile range is

$$IQR = Q_3 - Q_1 = 56 - 30.5 = 25.5$$

### Exercises

1. Fifteen students were selected randomly and asked how many hours each studied for the final exam in statistics. Their answers are recorded here

8, 6, 3, 0, 0, 5, 9, 2, 1, 3, 7, 10, 0, 3, 6

- Find the range
- Find the mean absolute deviation
- Find the sample variance and sample standard deviation
- Find the interquartile range.

2. The following data give the hourly wage rate of all 12 employees of a small company

21, 22, 27, 36, 22, 29, 22, 23, 22, 28, 36, 33

- Find the population variance and standard deviation
- Find the mean absolute deviation
- Find the range
- Find the interquartile range.

3. The number of words printed in each of 12 randomly selected storybooks for children is listed below

502, 213, 335, 197, 414, 469, 497, 367, 409, 297, 309, 414

- Find the sample variance and sample standard deviation
- Find the range
- Find the mean absolute deviation
- Find the interquartile range.

4. The weights of sample of nine football players are recorded as follows

78, 72, 68, 73, 75, 69, 74, 73, 72

- Find the range
- Find the variance
- Find the standard deviation

5. The following data give the number of cars that stopped at a service station during each of the 10 hours observed

29, 35, 42, 31, 24, 18, 16, 27, 39, 34

Find the range, variance, and standard deviation.

6. The following data give the number of new cars sold at a dealership during a 12-day period

13, 5, 9, 6, 8, 11, 9, 15, 4, 11, 7, 5

Find the range, variance, standard deviation, and interquartile range.

Consider the following two data sets:

Data set I: 12, 25, 37, 8, 41

Data set II: 19, 32, 44, 15, 48

Notice that each value of the second data set is obtained by adding 7 to the corresponding value of the first data set. Calculate the standard deviation for each of these two data sets using the formula for sample data. Comment on the relationship between the standard deviations.

Consider the following two data sets:

Data set I: 4, 8, 15, 9, 11

Data set II: 8, 16, 30, 18, 22

Notice that each value of the second data set is obtained by multiplying the corresponding value of the first data set by 2. Calculate the standard deviation for each of these data sets using the formula for the sample data. Comment on the relationship between the standard deviations.

The number of patients treated at the hospital per day are shown below. The data are from a random sample of 12 days:

45, 50, 36, 59, 28, 42, 55, 67, 33, 35, 40, 50

Compute the mean, median, mode, range, variance, and standard deviation for these data.

Light bulbs manufactured by a well-known electrical equipment firm are known to have a mean life of 800 hours with a standard deviation of 100 hours.

Find a range in which it can be guaranteed that 84% of lifetimes of light bulbs lie.

Using the rule of thumb, find a range in which it can be estimated that approximately 68% of these light bulbs lie.

Tires of a particular brand have lifetimes with mean of 29,000 km and standard deviation of 3,000 km.

Find a range in which it can be guaranteed that 75% of the lifetimes of tires of this brand lie.

Using the rule of thumb, find a range in which it can be estimated that approximately 95% of the lifetimes of tires of this brand lie.

The mean of a distribution is 20 and the standard deviation is 2.

Use Chebyshev's theorem to answer:

At least what percentage of the values will fall between 10 and 30?

At least what percentage of the values will fall between 12 and 28?

A sample of hourly wages of employees who work in restaurants in a city has a mean of \$5.02 and a standard deviation of \$0.09. Using

Chebyshev's theorem, find the range in which at least 75% of the data lie.

**14.** The average score on a special test of knowledge has a mean of 95 and a standard deviation of 2. Using Chebyshev's theorem, find the range in which at least 88.89% of the data will fall.

**15.** During a recent football season, it was reported that the average attendance for games was 45,000. The standard deviation in the attendance figure was  $\sigma = 4,000$ . Use Chebyshev's theorem to answer the following:

- a) Develop an interval that contains the attendance figure for at least 75% of the games.  
 b) The commissioner claims that at least 90% of the games had attendances between 29,000 and 61,000. Is this statement warranted given the information we have?

**Answers**

- 1.** a) 10; b) 2.8; c) 11.3; 3.4; d) 6; **2.** a) 29.52; 5.43; b) 4.75; c) 15; d) 10; **3.** a)  $s^2 = 10325.9$ ;  $s = 101.62$ ; b) 305; c) 82.25; d) 156.5;  
**4.** a) 10; b) 9; c) 3; **5.** range = 26; b)  $s^2 = 72.25$ ; c)  $s = 8.50$ ;  
**6.** range = 11;  $s^2 = 11.720$ ;  $s = 3.42$ ; **7.**  $s = 14.64$  for both data sets  
**8.**  $s_1 = 4.04$  and  $s_2 = 8.08$ ; **9.** 45; 43.5; 50; range = 39;  $s^2 = 134.36$ ;  
 $s = 11.59$ ; **10.** a) 550-1050; b) 700-900; **11.** a) 23,000-35,000; b) 23,000-35,000; **12.** a)  $1 - \frac{1}{5^2} = 0.96$  or 96%; b) 0.9375 or 93.75%;  
**13.** \$4.84-\$5.20; **14.** 89-101; **15.** a) 37,000-53,000.

**1.6. Numerical summary of grouped data**

**1.6.1. Mean for data with multiple-observation values**

Suppose that a data set contains values  $m_1, m_2, \dots, m_k$  occurring with frequencies,  $f_1, f_2, \dots, f_k$  respectively.

or a population of  $N$  observations, so that

$$N = \sum_{i=1}^k f_i$$

the mean is

$$\mu = \frac{\sum_{i=1}^k f_i \cdot m_i}{N}$$

or a sample of  $n$  observations, so that

$$n = \sum_{i=1}^k f_i$$

the mean is

$$\bar{x} = \frac{\sum_{i=1}^k f_i \cdot m_i}{n}$$

The arithmetic mean is most conveniently set out in tabular form.

**Example:**

The score for the sample of 25 students on a 5-point quiz are shown below. Find the mean.

**Solution:**

must find  $\bar{x} = \frac{\sum_{i=1}^6 f_i \cdot m_i}{n}$ . We need a

table to display the computation of the mean. The table below shows the frequency  $f_i \cdot x_i$  (Table 1.3):

Score ( $m_i$ )	Frequency ( $f_i$ )
0	1
1	2
2	6
3	12
4	3
5	1
$n = 25$	

In the end,

$$\bar{x} = \frac{\sum_{i=1}^n f_i \cdot m_i}{n} = \frac{67}{25} = 2.68 \approx 2.7.$$

Hence the mean of the scores is approximately 2.7.

Table 1.3

Score ( $m_i$ )	Frequency ( $f_i$ )	$f_i \cdot m_i$
0	1	0 · 1 = 0
1	2	1 · 2 = 2
2	6	2 · 6 = 12
3	12	3 · 12 = 36
4	3	4 · 3 = 12
5	1	5 · 1 = 5
$n = 25$		$\sum_{i=1}^n f_i \cdot m_i = 67$

Table 1.4

Class	Number of sets sold	Frequency (month)	Cumulative frequency
1	1	3	3
2	2	8	11
3	3	5	16
4	4	4	20
5	5	2	22
6	6	1	23
7	7	1	24
		$n = 24$	

∴ 12<sup>th</sup> and 13<sup>th</sup> values fall in class 3.

∴ 12<sup>th</sup> value = 3 ; 13<sup>th</sup> value = 3.

∴ Therefore, Median =  $\frac{3+3}{2} = 3$ .

### 1.6.2. Median for data with multiple-observation values

For an ungrouped frequency distribution, find the median by examining the cumulative frequency to locate the middle value, as shown in the example.

**Example:**

The number of videocassette recorders sold per month over a two-year period is recorded below. Find the median.

**Solution:**

As we know the median is  $\left[ \frac{n+1}{2} \right]^{th}$  observation.

Since  $n = 24$  then median =  $\left[ \frac{24+1}{2} \right]^{th} = \frac{12^{th} + 13^{th}}{2}$ .

To find 12<sup>th</sup> and 13<sup>th</sup> observations we write corresponding cumulative frequency distribution (Table 1.4).

### 1.6.3. Mode for data with multiple-observation values

As we already know, the mode is the most frequently occurring value. A similar concept can be used when the data are available in multiple-observation form.

**Example:**

The following data were collected on the number of blood tests a hospital conducted for a random sample of 50 days. Find the mode.

Number of tests per day	Frequency (days)
26	5
27	9
28	12
29	18
30	5
31	0
32	1

**Solution:**

Since 29 days were given on 18 days (the number of tests that occurs n often), the mode is 29.

**1.6.4. Variance for data with multiple-observation values**

Suppose that a data set contains values  $m_1, m_2, \dots, m_k$  occurring with frequencies  $f_1, f_2, \dots, f_k$  respectively.

1. For a population of  $N$  observations, so that

$$N = \sum_{i=1}^k f_i$$

The variance is

$$\sigma^2 = \frac{\sum_{i=1}^k f_i (m_i - \mu)^2}{N} = \frac{\sum_{i=1}^k f_i \cdot m_i^2}{N} - \mu^2$$

The standard deviation is  $\sigma = \sqrt{\sigma^2}$ .

2. For a sample of  $n$  observations, so that

$$n = \sum_{i=1}^k f_i$$

The variance is

$$s^2 = \frac{\sum_{i=1}^k f_i (m_i - \bar{x})^2}{n-1} = \frac{1}{n-1} \left[ \sum_{i=1}^k f_i \cdot m_i^2 - n \cdot \bar{x}^2 \right]$$

The standard deviation is  $s = \sqrt{s^2}$ .

The arithmetic is most conveniently set out in tabular form.

**Example:**

The score for the sample of 25 students on a 5-point quiz are shown below. Find a sample variance and standard deviation.

Score ( $m_i$ )	Frequency ( $f_i$ )
0	1
1	2
2	6
3	12
4	3
5	1

**Caution:**

**Remark:** The denominator in the formula

$$\frac{\sum_{i=1}^k f_i \cdot m_i}{n}$$

is obtained by summing the frequencies ( $\sum f = n$ ). It is not the number of classes.

To calculate variance we need three columns to display the computation of

quantities  $(m_i - \bar{x})^2$  a column for the  $m_i$ , a column for the

$(m_i - \bar{x})$  and a column for the  $(m_i - \bar{x})^2$ . We also need a column for

and a final column for the products  $f_i \cdot (m_i - \bar{x})^2$ . (Table 1.5)

The necessary computations for finding  $\sum_{i=1}^k f_i (m_i - \bar{x})^2$  are shown below.

Table 1.5

Score ( $m_i$ )	Frequency ( $f_i$ )	$(m_i - \bar{x})$	$(m_i - \bar{x})^2$	$f_i \cdot (m_i - \bar{x})^2$
0	1	0-2.7=-2.7	7.29	0·7.29=0
1	2	1-2.7=-1.7	2.89	1·2.89=2.89
2	6	2-2.7=-0.7	0.49	2·0.49=0.98
3	12	3-2.7=0.3	0.09	3·0.09=0.27
4	3	4-2.7=1.3	1.69	4·1.69=6.76
5	1	5-2.7=2.3	5.29	5·5.29=26.45
	25			$\sum_{i=1}^k f_i (m_i - \bar{x})^2 = 37.35$

Thus we have

$$s^2 = \frac{\sum_{i=1}^6 f_i (m_i - \bar{x})^2}{n-1} = \frac{37.35}{24} = 1.56$$

$$s = \sqrt{1.56} = 1.25$$

**Example:**

The number of television sets sold per month over a two year period is reported below. Find the variance and standard deviation for the data.

Number of sets sold ( $m_i$ )	Frequency ( $f_i$ )
5	2
6	3
7	8
8	1
9	6
10	4

**Solution:**

Let us apply

$$s^2 = \frac{1}{n-1} \left[ \sum_{i=1}^6 f_i \cdot m_i^2 - n \cdot \bar{x}^2 \right]$$

Make a table as shown below

Sets ( $m_i$ )	Frequency ( $f_i$ )	$m_i \cdot f_i$	$m_i^2$	$f_i \cdot m_i^2$
5	2	5 · 2 = 10	25	50
6	3	6 · 3 = 18	36	108
7	8	7 · 8 = 56	49	392
8	1	8 · 1 = 8	64	64
9	6	9 · 6 = 54	81	486
10	4	10 · 4 = 40	100	400
	$n = 24$	$\sum_{i=1}^6 m_i \cdot f_i = 186$		$\sum_{i=1}^6 f_i \cdot m_i^2 = 1500$

$$\bar{x} = \frac{\sum_{i=1}^6 f_i \cdot m_i}{n} = \frac{186}{24} = 7.75$$

$$s^2 = \frac{1}{23} [1500 - 24 \cdot (7.75)^2] = 2.5$$

To find standard deviation we take the square root of variance

$$s = \sqrt{s^2} = \sqrt{2.5} = 1.6$$

**Exercises**

The following numbers of books were read by each of the 28 students in literature class.

- Find the mean
- Find the median
- Find the mode
- Find the variance and standard deviation.

Number of books	Frequency (students)
0	2
1	6
2	12
3	5
4	3

The all forty students in a class found the following figures for number of hours spent studying in the week before final exam.

- Find the mean time for study
- Find the median
- Find the mode
- Find the variance and standard deviation for population.

Time (hours)	Number of students
1	1
2	7
3	15
4	10
5	7

A sample of fifty personal property insurance policies found the following numbers of claims over the past 2 years

Number of claims	0	1	2	3	4	5	6
Number of policies	21	13	5	4	2	3	2

- a) Find the mean number of claims per day policy
- b) Find the sample median of claims
- c) Find the modal number of claims for this sample
- d) Find the sample variance and standard deviation.

4. For sample of 50 antique car owners, the following numbers of cars ages was obtained

- a) Find the mean age of cars
- b) Find the median
- c) Find the modal number
- d) Find the sample variance and standard deviation.

Ages (in years)	Freqs	510	520	880	820	780	810	580	555
17	2	505	610	620	650	680	350	530	495
18	1	695	610	710	810	525	530	680	705
19	1	760	590	705	300	590	390	460	590
20	1	540	690	480	420	410	595	750	620
		585	690	570	560				

5. The following data represents the net worth (in million of dollars) of persons do not like to examine a mass of numbers, and many others national corporations

- a) Find the sample mean net worth
- b) Find the median
- c) Find the mode
- d) Find the sample variance and standard deviation

Net worth (in million of dollars)	Freq
15	1
20	1
25	1
30	1
35	1
40	1

**Answers.**

- 1. a) 2.04; b) 2; c) 2; d) 1.09; 1.04; 2. a) 3.375; b) 3; c) 3; d) 1.08; 1.04
- 3. a) 1.4; b) 1; c) 0; d) 3.061; 1.75; 4. a) 17.92; b) 18; c) 17; d) 0.89; 0.89
- 5. a) 27.7; b) 25; c) 25; d) 41.98; 6.48.

**Frequency distribution. Grouped data and histograms**

Suppose a researcher wished to do study on the monthly earnings of a sample of 50 employees of a large company. The researcher would first to collect the data by asking each of 50 employees. When data are collected in original form, they are called **raw data**. In this case, the data

follows:

Persons do not like to examine a mass of numbers, and many others do not have the time to do so. Therefore, it would be advantageous if the information could somehow be "compressed" so that the distribution of observations could be seen at a glance. We find, after some searching, the smallest observation is 300 and the largest observation is 880. Let us group the observations. We could subdivide the range of data and count the number of values in each subinterval. If the lowest and highest values in the data set are known, the following expression often is helpful in determining both the width of the class interval and the number of classes required:

$$\text{Approximate number of classes} = \frac{\text{Highest value} - \text{Lowest value}}{\text{Width of class}}$$

Using this formula with a trial class width of 100 shows that

$$\frac{880 - 300}{100} = 5.8$$

Rounding up, we find that 6 classes would be required for the data.

Table 1.6

Monthly earnings (in dollars)	Number of employees
301 - 400	4
401 - 500	8
501 - 600	16
601 - 700	10
701 - 800	7
801 - 900	5

The numbers 301, 400, 401, 500 are known as class limits. To find the midpoint of the upper limit of the first class and the lower limit of the second class in table 1.6 we divide the sum of these two limits by 2. Thus, midpoint is

$$\frac{400 + 401}{2} = 400.5$$

The value 400.5 is called the upper boundary of the first class and the lower boundary of the second class. By using this technique, we can convert the class limits of table 1.7 to class boundaries, which are also called real class limits.

Table 1.7

Class limits	Class boundaries	Class width	Class midpoint
301 to 400	300.5-400.5	100	350.5
401 to 500	400.5-500.5	100	450.5
501 to 600	500.5-600.5	100	550.5
601 to 700	600.5-700.5	100	650.5
701 to 800	700.5-800.5	100	750.5
801 to 900	800.5-900.5	100	850.5

**Definition:**

The class boundary is given by the midpoint of the upper limit of one class and the lower limit of the next class.

**Definition:**

The difference between the two boundaries of a class is called the class width.

$$\text{Class width} = \text{Upper boundary} - \text{Lower boundary}$$

**Definition:**

Class midpoint (or mark) is the average of the two limits (or two class boundaries)

$$\text{Class midpoint (or mark)} = \frac{\text{Lower limit} + \text{Upper limit}}{2}$$

**Definition:**

Class widths may be considered in (1); the decision on the class width and the number of classes is up to the user.

**Definition:**

Frequency distribution is a table used to organize data. The left column (or classes or groups) included numerical intervals on a scale being studied. The right column is a list of the frequencies, or number of observations, for each class. Data presented in the form of frequency distributions are called **grouped data**.

Subintervals into which the data are broken down are called classes. In a frequency distribution the values 300 and 400 of the first class are called class boundaries. For any particular class, the **cumulative frequency** is the total number of observations in that and previous classes. (Table 1.8)

Table 1.8

Monthly earnings (in dollars)	Number of employees	Cumulative frequencies
301 to 400	4	4
401 to 500	8	12
501 to 600	16	28
601 to 700	10	38
701 to 800	7	45
801 to 900	5	50

**Definition:**

Relative frequency is the proportion of observations in each class. It is calculated as:

$$\text{Relative frequency of a class} = \frac{\text{frequency of that class}}{\text{sum of all frequencies}} = \frac{f_i}{\sum_{i=1}^n f_i}$$

In addition, we often want to consider the proportion of observations are either in that or one of the earlier classes. These proportions are called **cumulative relative frequencies**.

Example in the table 1.9 illustrates how to construct relative frequency cumulative relative frequency distributions.

Table 1.9

Monthly earnings (in dollars)	Number of employees	Cumulative frequencies	Relative frequencies	Cumulative relative frequency
301 but less than 400	4	4	4/50	4/50
401 but less than 500	8	12	8/50	12/50
501 but less than 600	16	28	16/50	28/50
601 but less than 700	10	38	10/50	38/50
701 but less than 800	7	45	7/50	45/50
800 but less than 900	5	50	5/50	50/50 = 1
	50		50/50 = 1	50/50

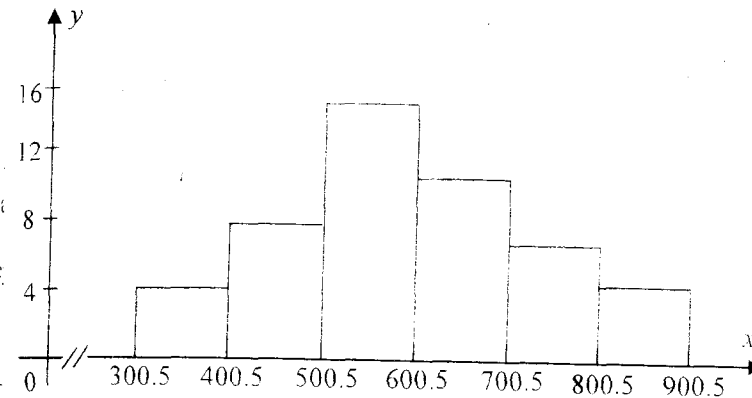


Fig. 1.1 Monthly earnings salaries frequency histogram

mark:

symbol “-//” used in the horizontal axis represents a break, called truncation, in the horizontal axis. It indicates that entire horizontal axis is shown in this figure. As can be noticed, the zero to 300.5 portion of horizontal axis has been omitted in the figure 1.1.

**Definition:**

A histogram is a graph in which classes are marked on a horizontal axis and either the frequencies, relative frequencies, or cumulative relative frequencies are marked on the vertical axis. The frequencies, relative frequencies, or cumulative relative frequencies are represented by the heights of the bars. In a histogram, the bars are drawn adjacent to each other.

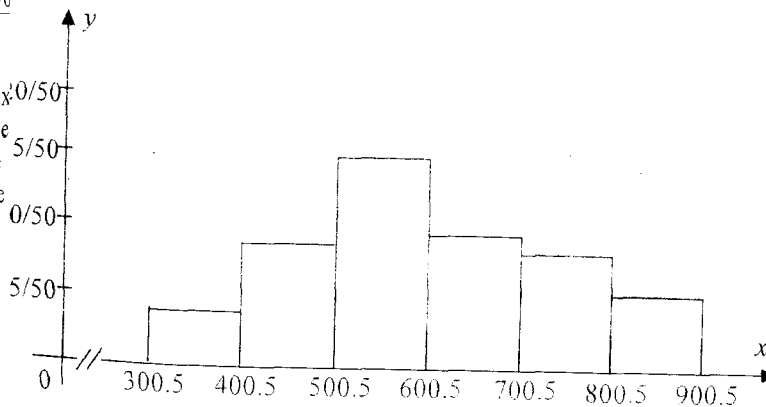


Fig. 1.2 Relative frequency for monthly earnings salaries

From the figure 1.2., we see, for example, that 16/50 of all employees monthly earnings are between 500.5 and 600.5.

The cumulative relative frequencies are the cumulated sums of the relative frequencies. For the first class, the cumulative relative frequency is the same as the relative frequency. For subsequent classes, the cumulative relative frequency for the class to the cumulative relative frequency is obtained by adding the relative frequency for the class to the cumulative relative frequency of the previous class.

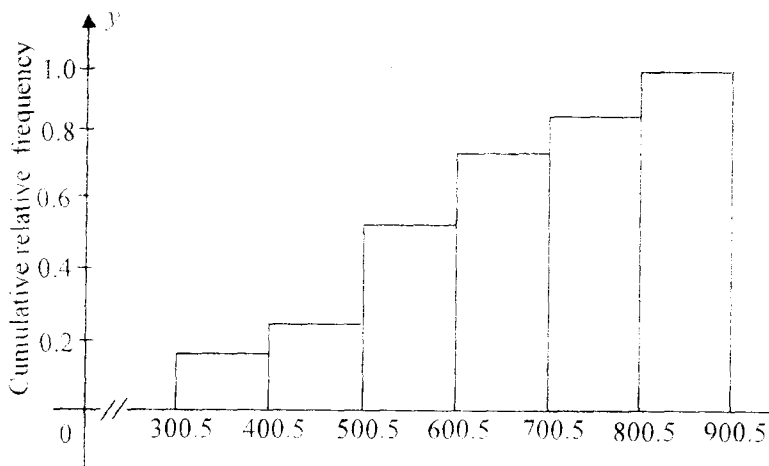


Fig. 1.3 Cumulative relative frequency for monthly-earning salaries

The interpretation of these quantities is very valuable. For example, 38/50 of all employees' monthly earnings are less than 700. The information contained in the cumulative relative frequency can also be presented pictorially, as in Fig. 1.3

### 1.7.1. Less than method for writing classes

The classes in frequency distribution given in table 1.9 for the data on monthly-earning salaries for 50 employees were written as 301-400, 401-500, etc. Alternatively, we can write the classes in a frequency distribution table using the *less than* method. The technique for writing classes in previous topic is more commonly used for data sets that do not contain fractional values. The *less than* method is more appropriate when the data set contains fractional values.

### Example:

The following data give the hourly wage rates for a sample of 30 employees selected from a population.

12.25 9.20 13.90 8.10 7.30 7.25 8.75  
 5.20 15.85 11.20 10.20 14.50 10.50 8.25  
 7.45 10.20 12.20 10.80 9.25 14.35 16.50  
 6.40 15.20 10.30 11.75 12.45 13.25 10.80  
 10.35 9.75

Construct a frequency distribution table. Find the relative frequency distribution table. Find the relative frequency and cumulative frequencies.

### Solution:

The minimum value in the data set is 5.20 and the maximum value is 16.50. If we decide to group these data using six classes of equal width,

$$\text{Approximate width of class} = \frac{16.50 - 5.20}{6} = 1.883$$

Round this number to a more convenient number, say 2. If we take 2 as the width of each class, the classes will be written as 5 to less than 7, 7 to less than 9, and so on as shown in table 1.10.

Hourly wage rate (dollars)	<i>f</i>	Relative frequencies	Cumulative relative frequencies
less than 7	2	2/30=0.067	2/30=0.067
less than 9	6	6/30=0.2	8/30=0.267
less than 11	10	10/30=0.333	18/30=0.6
less than 13	5	5/30=0.167	23/30=0.767
less than 15	4	4/30=0.133	27/30=0.9
less than 17	3	3/30=0.1	30/30=1
	30	Sum=1.00	Sum=1

A histogram for frequencies can be drawn in the same way as for the data in table 1.10. (Fig. 1.4; Fig. 1.5; Fig. 1.6)

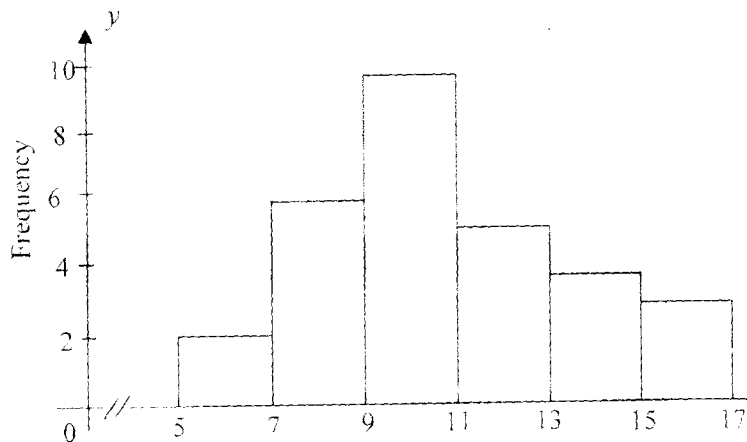


Fig. 1.4 Hourly wage rate

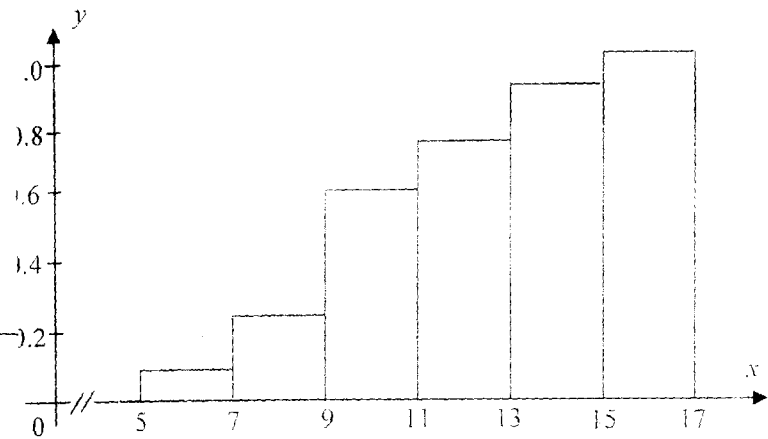


Fig. 1.5 Cumulative relative frequency for hourly wage rate

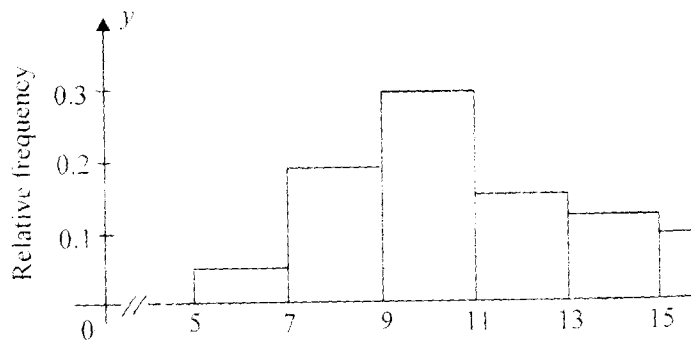


Fig. 1.5 Relative frequency for hourly wage rate

Exercises

survey of 20 patients who smoked, the following data were obtained. value represents the number of cigarettes the patient smoked per day. Construct a frequency distribution, using six classes.

8 6 14 22 13 17 19 11  
 18 14 13 12 15 15 5 11  
 11

Construct histograms for frequency, relative frequency and cumulative relative frequency.

For 75 employees of a large department store, the following distribution of years of service was obtained. Construct histograms for frequency, relative frequency, and cumulative relative frequency.

Class limits	Frequency
1-5	21
6-10	25
11-15	15
16-20	0
21-25	8
26-30	6

3. In a study of 32 student grade point averages (GPA), the following were obtained.

3.2 2.0 3.3 2.7 2.1 3.9 1.1 3.5  
 1.7 0.8 2.6 0.6 4.0 3.5 2.3 1.6  
 2.6 1.6 1.6 2.4 2.6 2.3 3.8 2.1  
 3.0 1.7 4.0 1.2 3.1

Construct histograms for frequency, relative frequency and cumulative relative frequency.

4. To determine their lifetimes, 80 randomly selected batteries were tested. The following frequency distribution was obtained. The data values are in hours given in table 1.11

Construct histograms for frequency, relative frequency and cumulative relative frequency.

Table 1.11

Class boundaries	Frequency
63.5-74.5	10
74.5-85.5	15
85.5-96.5	22
96.5-107.5	17
107.5-118.5	11
118.5-129.5	5

### 1.8. Mean for grouped data

1.9. In statistical measures we have presented for the central location and dispersion of data sets are computed using the individual data values. The computational procedures we have discussed provide the most common methods for computing measures of central location and dispersion. However, in some situations the data available only in grouped or frequency distribution form. In these cases special procedures are used in order to obtain approximations to the common measures of central location and dispersion.

2.8. The formulas used to calculate the mean for grouped data are as follows:

1 for the population data:

$$\mu = \frac{\sum_{i=1}^k m_i \cdot f_i}{N}$$

1 for the sample data:

$$\bar{x} = \frac{\sum_{i=1}^k m_i \cdot f_i}{n}$$

where  $m_i$  is the midpoint of  $i^{th}$  class,  $f_i$  is the frequency of  $i^{th}$  class,  $N$  is the total number of classes.

To calculate the mean for grouped data, first find the midpoint of each class and then multiply by the frequencies of the corresponding classes.

The sum of these products, denoted by  $\sum_{i=1}^k f_i \cdot m_i$ , gives an approximation

of the sum of all values. To find the value of the mean, divide this sum by the total number of observations in the data.

**Example:**

The following table gives the frequency distribution of daily commuting time (in minutes) from home to work for all 25 employees of a company

Daily commuting time (minutes)	Number of employees
0 to less than 10	4
10 to less than 20	9
20 to less than 30	6
30 to less than 40	4
40 to less than 50	2

Calculate the mean of daily commuting time.

**Solution:**

Note that because the data set includes all 25 employees of the company,

it represents the population. Table 1.12 shows the calculation of  $\sum_{i=1}^5 m_i f_i$ .

In table 1.12-  $m_i$  denotes the midpoint of the classes.

Daily commuting time (minutes)	$f_i$	$m_i$
0 to less than 10	4	5
10 to less than 20	9	15
20 to less than 30	6	25
30 to less than 40	4	35
40 to less than 50	2	45
	$N = 25$	

To calculate the mean, we first find the midpoint of each class. The midpoints are recorded in the third column of Table 1.12. The products of the midpoints and the corresponding frequencies are listed in the fourth column of that table. The sum of column, denoted by  $\sum m_i \cdot f_i$ , gives the approximate total daily commuting time (in minutes) for all 25 employees. The mean is obtained by dividing this sum by the total frequency.

Therefore 
$$\mu = \frac{\sum_{i=1}^5 m_i f_i}{N} = \frac{535}{25} = 21.40 \text{ minutes}$$

Thus, the employees of this company spend an average of 21.40 minutes per day commuting from home to work.

**1.9. The Median for grouped data**

The median is different for odd and for even numbers of observations when the data are not in the grouped form. However, if the  $n$  observations are written in grouped form, then median is simply defined as the  $(\frac{n}{2})^{\text{th}}$  observation.

Table 1.12

if we have the frequency distribution of 100 observations, then the 50<sup>th</sup> observation in order of size would be the median; if we have 101 observations then the "50.5<sup>th</sup>" observation would be the median. To find the median, first, we need to find the class which contains the middle observation. Let  $M$  denotes the number of this class, where  $M$  is the number of classes from 1 to  $k$ . If the median occurs in the fifth class then  $M=5$ ; if it occurs in the seventh class, then  $M=7$ ; and so on.

The frequency of the  $M^{\text{th}}$  class be denoted by  $f_M$ . Next, note how many observations are in  $(M - 1)$  classes preceding the median class: denote this cumulative frequency by  $F_{M-1}$ .

The general formula for median is

$$\text{Median} = L_M + \frac{\frac{n}{2} - F_{M-1}}{f_M} \cdot C$$

- $L_M$  - lower boundary of the median class
- $n$  - number of observations
- $f_M$  - the number of observations in the median class
- $F_{M-1}$  - the number of observations in the  $(M - 1)$  classes preceding the median class
- $C$  - width of the median class

**Example:** Find the median of the frequency distribution

Starting monthly salary(in dollars)	Frequency
900-1000	2
1000-1100	4
1100-1200	3
1200-1300	1
1300-1400	1
1400-1500	0
1500-1600	1
	$n=12$

**Solution:**

First of all, let us divide  $n$  (the number of all observations) to find the halfway point.

$$\text{Median} = \left[ \frac{n}{2} \right]^{th} \text{ observation} = \left[ \frac{12}{2} \right]^{th} = 6^{th} \text{ observation}$$

To find the class that contains  $6^{th}$  observation it is necessary to form cumulative frequency distribution. This class is called the median class contains the median:

Starting monthly salary(in dollars)	Frequency	Cumulative frequency
900-1000	2	2
1000-1100	4	6
1100-1200	3	9
1200-1300	1	10
1300-1400	1	11
1400-1500	0	11
1500-1600	1	12

$6^{th}$  observation is in  $2^{nd}$  class. So, the median class is 1000-1100. Now let us apply

$$\text{Median} = L_M + \frac{\frac{n}{2} - F_{M-1}}{f_M} \cdot C$$

in case

$$L_M = 1000; \quad n = 12; \quad F_{M-1} = 2; \quad f_M = 4; \quad C = 100$$

substituting we get

$$\text{Median} = 1000 + \frac{6 - 2}{4} \cdot 100 = 1100$$

Median is 1100. In other words, median as a measure of center states that average value of monthly salaries of 12 employees is 1100\$.

### 1.10. Modal class

The mode for grouped data is the modal class. The modal class is the class with the largest frequency.

**Example:**

Find the mode of the frequency distribution

The modal class is 20-25, since it has the largest frequency. Sometimes the midpoint of the class is used rather than the boundaries; hence the mode will be given as 22.5.

Class	Frequency
5-10	1
10-15	2
15-20	3
20-25	7
25-30	4
30-35	3
	$n=20$

**Exercises**

**1.** Dinner check amounts at the restaurant have the following frequency distribution:  
Compute the mean, median, and mode for the above data.

Dinner check (dollars)	Frequency
4-8	10
8-12	15
12-16	12
16-20	8
20-24	5
24-28	3

**2.** The following table gives the frequency distribution of entertainment expenditures (in dollars) incurred by 50 families during the past week. Find the mean, median and mode.

Entertainment expenditure (dollars)	Number of families
0-10	5
10-20	10
20-30	15
30-40	12
40-50	5
50-60	3

**3.** The following table gives the frequency distribution of total hours studying during the semester for sample of 40 university students enrolled in an introductory business statistics course. Find the mean, median, and mode.

Hours of study	Number of students
24-40	3
40-56	5
56-72	10
72-88	12
88-104	5
104-120	5

This frequency distribution represents the data obtained from sample of copying machine service technicians. The values represent the days between service calls for various copying machines. Find the mean, median, and mode.

Class boundaries	Frequency
15.5-18.5	14
18.5-21.5	12
21.5-24.5	18
24.5-27.5	10
27.5-30.5	15
30.5-33.5	6

For 35 antique car owners, the following distribution of cars' age was obtained. Find the mean, median, and mode.

Class limits	Frequency
13-19	2
20-26	7
27-33	12
34-40	5
41-47	6
48-54	1
55-61	0
62-68	2
$n=35$	

**Answers**

1. 12.57; 14; 2. 27.20; 26.7; 25; 3. 74.40; 74.67; 80; 4. 23.72; 23.417; 23.8; 31.5; modal class = 27-33.

**1.11. Variance and standard deviation for grouped data** (late the variance and standard deviation.)

**ion:**

Suppose that we have data grouped into  $K$  classes, with frequencies  $f_1, f_2, \dots, f_k$ . If the midpoints of these classes are  $m_1, m_2, \dots$

the mean and variance of the grouped data are estimated by using following formulas

$$\sigma^2 = \frac{\sum_{i=1}^k f_i (m_i - \mu)^2}{N}$$

1. For a population of  $N$  observations, so that

$$N = \sum_{i=1}^k f_i$$

The variance is

$$\sigma^2 = \frac{\sum_{i=1}^k f_i (m_i - \mu)^2}{N} = \frac{\sum_{i=1}^k f_i \cdot m_i^2}{N} - \mu^2$$

The standard deviation is  $\sigma = \sqrt{\sigma^2}$ .

2. For a sample of  $n$  observations, so that  $n = \sum_{i=1}^k f_i$

The variance is

$$s^2 = \frac{\sum_{i=1}^k f_i (m_i - \bar{x})^2}{n-1} = \frac{1}{n-1} \left[ \sum_{i=1}^k f_i \cdot m_i^2 - n \cdot (\bar{x})^2 \right]$$

The standard deviation is  $s = \sqrt{s^2}$ .

**Example:**

The following table gives the distribution of the number of days for all 40 employees of a company were absent during the last year

Number of days absent	Number of employees
0-2	13
3-5	14
6-8	6
9-11	4
12-14	3

we need to find  $m_i$  and  $\mu$

Number of days absent	Number of employees ( $f_i$ )	Class mark ( $m_i$ )	$m_i \cdot f_i$
0-2	13	1	13
3-5	14	4	56
6-8	6	7	42
9-11	4	10	40
12-14	3	13	39
	40		$\sum_{i=1}^5 m_i f_i = 190$

$$\mu = \frac{\sum_{i=1}^5 m_i \cdot f_i}{N} = \frac{190}{40} = 4.75$$

we need to find  $\sum_{i=1}^k f_i \cdot (m_i - \mu)^2$ .

er to find products  $f_i \cdot (m_i - \mu)^2$  we must first find the square ities  $(m_i - \mu)^2$ . We need three columns to display the computation quantities  $(m_i - \mu)^2$  - a column for  $m_i$ , a column for  $(m_i - \mu)$ , column for the  $(m_i - \mu)^2$ . We also need a column for  $f_i$  and a final m for the products  $f_i \cdot (m_i - \mu)^2$ . The necessary computations are 1 below in the table 1.13.

Table 1.13. Number of orders

Class $i$	Class mark ( $m_i$ )	$f_i$	$(m_i - \mu)$	$(m_i - \mu)^2$	$f_i(m_i - \mu)$	Number of orders	$f_i$	$m_i$	$m_i f_i$	$m_i^2$	$f_i \cdot m_i^2$
1	1	13	1-4.75=-3.75	14.06	182.1	10-12	4	11	44	121	484
2	4	14	4-4.75=-0.75	0.56	7.8	13-15	12	14	168	196	2352
3	7	6	7-4.75=2.25	5.06	30.3	16-18	20	17	340	289	5780
4	10	4	10-4.75=5.25	27.56	110.2	19-21	14	20	280	400	5600
5	13	3	13-4.75=8.25	68.06	204.1		$N=50$		832		14216
					535.1						

$$\bar{x} = \frac{832}{50} = 16.64$$

In the end,

$$\sigma^2 = \frac{535.51}{40} = 13.39 \quad \text{and} \quad \sigma = \sqrt{\sigma^2} = 3.66$$

**Example:**

The following table gives the frequency distribution of the number of orders received each day during the past 50 days at the office of a mail-order company. Calculate the variance and standard deviation.

Number of orders	$f$
10-12	4
13-15	12
16-18	20
19-21	14

**Solution:**

Because the data includes only 50 days, it represents a sample. Hence, we will use sample formulas to calculate variance and standard deviation.

Let us apply

$$s^2 = \frac{1}{n-1} \left[ \sum_{i=1}^k f_i \cdot m_i^2 - n \cdot (\bar{x})^2 \right]$$

All the information required for the calculation of the variance and standard deviation appears in the following table

Substituting the values in the formula for the sample variance, we

$$\frac{1}{n-1} \left[ \sum_{i=1}^k f_i \cdot m_i^2 - n \cdot \bar{x}^2 \right] = \frac{1}{50-1} [14216 - 50 \cdot (16.64)^2] = 7.582$$

Therefore, the standard deviation is  $s = \sqrt{s^2} = \sqrt{7.580} = 2.75$ .

Therefore, the standard deviation of the number of orders received at office of mail-order company during the past 50 days is 2.75.

**1.12. Interquartile range for grouped data**

Assume that a class, with lower boundary  $L$  and upper boundary  $U$ , contains  $f$  observations. If these observations were to be arranged in ascending order, the  $j^{\text{th}}$  observation is estimated by

$$L + (j - \frac{1}{2}) \cdot \frac{(U - L)}{f} \quad \text{for } j = 1, 2, 3, \dots, f.$$

- $L$  is the lower limit of class containing  $j^{\text{th}}$  observation
  - $U$  is the upper limit of class containing  $j^{\text{th}}$  observation
  - $f$  is the frequency of class containing  $j^{\text{th}}$  observation
  - $j$  is the location of  $j^{\text{th}}$  observation in that class.
- Therefore, the interquartile range we need to find

$$Q_1 = \left[ \frac{N+1}{4} \right]^{th} \text{ and } Q_3 = \left[ \frac{3 \cdot (N+1)}{4} \right]^{th}$$

As we know I.Q.R. = Interquartile range =  $Q_3 - Q_1$ .

**Example:** The following table gives the frequency distribution of the number of orders received each day during the past 50 days at the office of a mail-order company. Calculate the interquartile range.

Number of orders
10-12
13-15
16-18
19-21

**Solution:**

First of all, let us write cumulative frequency distribution

Number of orders	$f$	Cumulative frequency
10-12	4	4
13-15	12	16
16-18	20	36
19-21	14	50

Since there are  $N=50$  observations, we have

$$Q_1 = \left[ \frac{N+1}{4} \right]^{th} = \left[ \frac{51}{4} \right]^{th} = \left[ 12 \frac{3}{4} \right]^{th} = 12^{th} + \frac{3}{4}(13^{th} - 12^{th}), \text{ so with } j=2, \text{ we estimate } 38^{th} \text{ observation by}$$

Hence the first quartile is the three-quarters way from the 12<sup>th</sup> observation to 13<sup>th</sup>. From cumulative distribution we see that the 12<sup>th</sup> value is the 8<sup>th</sup> value in the class 13-15. In our notation then

$$j=8; \quad f=12; \quad L=13; \quad U=15$$

The 12<sup>th</sup> observation is estimated by

$$L + \left( j - \frac{1}{2} \right) \cdot \frac{(U-L)}{f} = 13 + \left( 8 - \frac{1}{2} \right) \cdot \frac{15-13}{12} = 14 \frac{1}{4} = 14.25$$

Similarly, the 13<sup>th</sup> observation is the 9<sup>th</sup> value in the same class, so with  $j=9$ , we have

$$L + \left( j - \frac{1}{2} \right) \cdot \frac{(U-L)}{f} = 13 + \left( 9 - \frac{1}{2} \right) \cdot \frac{15-13}{12} = 14 \frac{5}{12} = 14.41$$

∴ the first quartile is three-quarters of the way from the twelfth observation to the thirteenth observation, we have

$$\begin{aligned} \text{first quartile} = Q_1 &= 12^{th} + \frac{3}{4}(13^{th} - 12^{th}) = \\ &= 14 \frac{1}{4} + \frac{3}{4} \left( 14 \frac{5}{12} - 14 \frac{1}{4} \right) = 14.375. \end{aligned}$$

∴ the third quartile, we have

$$\frac{3 \cdot (N+1)}{4} = \left[ \frac{150}{4} \right]^{th} = \left[ 37 \frac{1}{2} \right]^{th} = 37^{th} + \frac{1}{2}(38^{th} - 37^{th})$$

∴ before, when the observations are arranged in ascending order, the third quartile is half of the way from thirty-seventh to thirty-eighth.

Looking at table, we see that the thirty-seventh observation is the first observation in class the 19-21, which contains 14 observations. We have then  $f=14; \quad L=19; \quad U=21$

∴ the thirty-seventh observation is estimated by

$$j - \frac{1}{2} \cdot \frac{(U-L)}{f} = 19 + \left( 1 - \frac{1}{2} \right) \cdot \frac{21-19}{14} = 19 \frac{1}{14} = 19.07$$

∴ the thirty-eighth observation is the second value in the same class.

∴ the thirty-eighth observation is estimated by

$$j - \frac{1}{2} \cdot \frac{(U-L)}{f} = 19 + \left( 2 - \frac{1}{2} \right) \cdot \frac{21-19}{14} = 19 \frac{3}{14} = 19.21$$

∴ since the third quartile is half of the way from the 37<sup>th</sup> to 38<sup>th</sup> observation, we have

$$\begin{aligned} \text{third quartile} = Q_3 &= 37^{th} + \frac{1}{2}(38^{th} - 37^{th}) = \\ &= 19 \frac{1}{14} + \frac{1}{2} \left( 19 \frac{3}{14} - 19 \frac{1}{14} \right) = 19 \frac{2}{14} = 19.14 \end{aligned}$$

∴ the interquartile range is the difference between the third and first quartiles, so

$$\text{interquartile range} = 19 \frac{2}{14} - 14 \frac{1}{32} = 19.07 - 14.375 = 4.695$$

Thus, if the interquartile range is to be used as a measure of dispersion, estimate it by 4.695.

### Exercises

1. For 50 airplanes that arrived late at an airport during a week, the time by which they were late was observed. In the following table,  $x$  denotes the time (in minutes) by which an airplane was late and  $f$  denotes the number of airplanes.

$x$
0 to less than 20
20 to less than 40
40 to less than 60
60 to less than 80
80 to less than 100

- Find the mean
- Find the median
- Find the mode
- Find the variance and standard deviation
- Find the interquartile range.

2. The following table gives information on the amount (in dollars) of the electric bills for a sample of 40 families.

Amount of electric bill (dollars)	Number of families
4 to less than 8	10
8 to less than 12	12
12 to less than 16	8
16 to less than 20	5
20 to less than 24	5

- Estimate the sample mean
- Estimate the median
- Estimate the mode
- Estimate the variance and standard deviation.
- Estimate the interquartile range.

population of all twenty financial analysts was asked to give forecasts of earnings per share of a corporation for next year. The results are summarized in the following table.

Forecast \$ per share	Number of analysts
0.5-10.5	2
10.5-20.5	4
20.5-30.5	9
30.5-40.5	5

- Find the relative frequencies.
- Find the cumulative frequencies.
- Find the cumulative relative frequencies.
- Estimate the population mean.
- Estimate the population variance.
- Estimate the population standard deviation.
- Estimate the population mode.
- Estimate the population median.
- Estimate the interquartile range.
- Which class is modal class?

A sample was taken of flights arriving at a major airport to study the problem of air traffic delays. The table shows numbers of minutes late for a sample of 100 flights.

Minutes late	0-10	10-20	20-30	30-40	40-50	50-60
Number of flights	29	23	17	14	11	6

- Draw the histogram
- Find the sample relative frequencies
- Find and interpret the sample cumulative relative frequencies
- Estimate the sample mean number of minutes
- Estimate the sample variance and standard deviation
- Estimate the sample median number of minutes late
- Estimate the interquartile range
- Which is the modal class for this sample?

5. The following table gives the frequency distribution of the number of computers sold during the past 25 weeks at a computer store.

Computers sold
4 to 9
10 to 15
16 to 21
22 to 27
28 to 33

Calculate the mean, variance, and standard deviation.

6. Eighty randomly selected light bulbs were tested to determine their lifetimes (in hours). The following frequency distribution was obtained

Class limits
52.5-63.5
63.5-74.5
74.5-85.5
85.5-96.5
96.5-107.5
107.5-118.5

Find the variance and standard deviation.

7. The following data represent the scores (in words per minute) of 25 typists on a speed test.

Class limits
54-58
59-63
64-68
69-73
74-78
79-83
84-88

Find the variance and standard deviation.

Frequency a sample of fifty new full-size cars, fuel consumption figures were recorded and summarized in the accompanying table

Fuel consumption	14-16	16-18	18-20	20-22	22-24
Number of cars	3	6	13	20	8

Draw the histogram.

Find the sample relative frequencies.

Find and interpret the sample cumulative relative frequencies

Estimate the sample mean fuel consumption.

Estimate the sample standard deviation of fuel consumption.

Frequency Estimate the sample median fuel consumption.

Estimate the sample interquartile range.

Which is the modal class for this sample?

The fuel capacity in gallons of 30 randomly selected cars is shown

	Class	Frequency
Mean	12.5-27.5	6
Median	27.5-42.5	3
Modal class	42.5-57.5	5
Variance	57.5-72.5	8
Standard deviation	72.5-87.5	6
	87.5-102.5	2

Twelve batteries were tested after being used for one hour. The output (in volts) is

shown below.

Find each of the following

Volts	Frequency
2	1
3	4
4	5
5	1
6	1

Mean

Median

Mode

Range

Variance

Standard deviation.

11. For a sample of twenty-five students from a large class, the accompanying table shows the amount of time students spent studying a test

Study time (hours)	0-2	2-4	4-6	6-8	8-10
Number of students	3	4	8	7	3

- Draw the histogram.
- Find and interpret the cumulative relative frequencies.
- Estimate the sample mean study time.
- Estimate the sample median.
- Find the modal class.
- Estimate the sample variance.
- Estimate the sample standard deviation study time.
- Estimate the sample interquartile range.

### Answers

1. a)  $\bar{x} = 36.80$ ; b) 32.22; c) 30; d)  $s^2 = 597.714$ ;  $s = 24.45$ ;  
 e)  $I.R. = 35.28$ ; 2. a)  $\bar{x} = 14.50$ ; b) 14.25; c) 14; d)  $s^2 = 18.205$ ;  
 $s = 4.27$ ; e)  $I.R. = 6.556$ ; 3. a) 2/20; 4/20; 9/20; 5/20; b) 2; 6; 15;  
 c) 2/20; 6/20; 15/20; 20/20; d)  $\mu = 480/20 = 24$ ;  $\sigma^2 = 82.75$ ;  
 e)  $\sigma = 9.097$ ; f) median=24.944; i) I.Q.R. =13.735; j) modal class:  
 20.5-30.5; 4. b) 0.29; 0.23; 0.17; 0.14; 0.11; 0.06; c) 0.29; 0.52; 0.69;  
 0.94; 1.0; d)  $\bar{x} = 2230/100 = 22.3$ ; e)  $s^2 = 246.176$ ;  $s = 15.69$ ; f)  
 median=19.13; g) 25.931; h) modal class 0-10; 5.  $\bar{x} = 18.98$ ;  
 $s^2 = 44.760$ ;  $s = 6.69$ ; 6.  $s^2 = 211.2$ ;  $s = 14.5$ ; 7.  $s^2 = 80.3$ ;  $s = 9$ ;  
 8. b) 3/50; 6/50; 13/50; 20/50; 8/50; c) 3/50; 9/50; 22/50; 42/50; 50/50;  
 $\bar{x} = 19.96$ ; e)  $s = 2.185$ ; f) median=20.3; g) I.Q.R. = 3.075; h) modal  
 20-22; 9. a)  $\bar{x} = 55.5$ ; b) median=59.4; c) modal class;  
 57.5-72.5; d)  $s^2 = 566.1$ ; e)  $s = 23.8$ ; 10. a)  $\bar{x} = 3.8$ ; b) median=4;  
 c) mode = 4; d) range = 4; e)  $s^2 = 1.11$ ; f)  $s = 1.05$ ; 11. b) 3/25; 7/25;  
 15/25; 22/25; 25/25; c) 5.24; d) 5.375; e) modal class = 4-6; f) 5.773;  
 2.403; h) 3.64.

## Chapter 2. Probability

### 2.1. Introduction

Concepts of probability is a familiar one to everyone. Statements such as the following are heard frequently: "You had better take an umbrella because it is likely to rain", "It is not likely to snow today", "He will probably read at least three books during the next two weeks", "I am almost certain that we will be home for the holidays".

Examples illustrate that most of us use the concept of probability in our everyday speech. They also illustrate that there is a great deal of imprecision involved in such statements. For instance, a family has been arguing about whether they will go to the wife's home for holidays, and have not reached a decision. Even though the wife may say, "It is almost certain that we will go for the holidays", the husband might not even think that such an assurance is likely. This sort of imprecision is intolerable in mathematics. Difficulties can be avoided if we restrict our discussion of probability to situations which are outcomes of experiments that can be repeated, and if we deal only with idealized situations.

We think of probability as the language in which we discuss uncertainty. If we can communicate with one another in this language, we need to develop a common vocabulary. Moreover, as in any other language, rules of grammar are needed so that a clear statement can be made with our vocabulary.

### 2.2. Random experiment, outcomes, and sample space

**Definition:** A random experiment is a process that, when performed, results in one and only one of at least two observations.

**Definition:** Possible outcomes of a random experiment are called the basic events.

**Definition:** The set of all basic outcomes is called the sample space. The sample space will be denoted by S.

1 illustrates a list of a few examples of random experiments, their sample spaces, and their sample spaces.

Table 2.1

Experiment	Outcomes	Sample space
Toss a coin once	Head, Tail	$S = \{\text{Head, Tail}\}$
Roll a die once	1,2,3,4,5,6	$S = \{1,2,3,4,5,6\}$
Toss a coin twice	HH,HT,TH,TT	$S = \{HH,HT,TH,TT\}$
Take a test	Pass, Fail	$S = \{\text{Pass, Fail}\}$
Select a student	Male, Female	$S = \{\text{Male, Female}\}$

**Event:** An event is a collection of one or more of the outcomes from the sample space.

An event is denoted by  $E_1, E_2, E_3, \dots$ , and so forth. We can denote it by letters too, that is, by  $A, B, C$  and so forth.

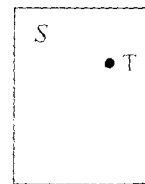
**Simple Event:** An event that includes one and only one of the outcomes for an experiment is called a simple event.

The sample space for an experiment can also be described by drawing a Venn diagram. A Venn diagram is a picture (a closed geometric shape like a rectangle, a square, or circle) that depicts all the possible outcomes of an experiment.

**Example:** Draw the Venn diagram for the experiment of tossing a coin.

**Solution:** This experiment has two possible outcomes: head and tail. Sample space given by  $S = \{H, T\}$ ; where  $H = \text{head}$ ,  $T = \text{Tail}$ .

To draw Venn diagram for this example, we draw a rectangle and mark two points inside this rectangle that represent two outcomes, head and tail. The rectangle is labelled by  $S$  because it represents the sample space. (Fig. 2.1)



**Example:** Suppose we randomly select two employees from a company and observe whether the employee selected each time is a male or female. Write all the possible outcomes for this experiment and draw the Venn diagram.

**Solution:** Let us denote selection of male by  $M$  and that of female by  $F$ . There are four final possible outcomes:  $MM, MF, FM$ , and  $FF$ . We can write sample space as  $S = \{MM, MF, FM, FF\}$ . The Venn diagram is given in Fig.2.2

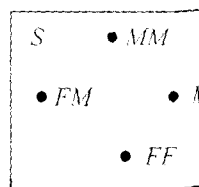


Fig.2.2.

**Intersection:** Let  $A$  and  $B$  be two events defined in sample space  $S$ . The intersection of  $A$  and  $B$  represents the collection of all outcomes that are common to both  $A$  and  $B$  and is denoted by either  $A \cap B$  or  $AB$ .

$A \cap B$  or  $AB$  occurs if and only if both  $A$  and  $B$  occur. Generally, given  $N$  events  $E_1, E_2, \dots, E_N$ , their intersection,

$E_1 \cap E_2 \cap \dots \cap E_N$ , is the set of all basic outcomes that belong to every event  $E_i$  for  $i = 1, 2, \dots, N$ .

**Example:** Let  $A$  be the event that a family owns washing machine and  $B$  be the event that family owns a VCR. Then intersection of these events is all the families who own both washing machine and VCR.

**Union:** Let  $A$  and  $B$  be two events defined in sample space  $S$ . Their union, denoted by  $A \cup B$ , is the set of all basic outcomes in  $S$  that belong to at least one of these events. The union  $A \cup B$  occurs if and only if either  $A$  or  $B$  (or both) occurs.

Generally, given  $N$  events  $E_1, E_2, \dots, E_N$ , their union,  $E_1 \cup E_2 \cup \dots \cup E_N$ , is the set of all basic outcomes that belong to at least one of these  $N$  events.

**Mutually Exclusive Events:** Two events  $A$  and  $B$  are called mutually exclusive events if they have no common basic outcomes and their intersection  $A \cap B$  is said to be empty set.

**Definition:**

Let  $E_1, E_2, \dots, E_N$  be  $N$  events in the sample space  $S$ .

If  $E_1 \cup E_2 \cup \dots \cup E_N = S$ , then these  $N$  events are said to be collectively exhaustive events.

**Definition:**

Let  $A$  be an event in the sample space. The complement of event  $A$ , denoted by  $\bar{A}$  and read as "A bar" or "A complement" is the event that includes all outcomes for an experiment that are not in  $A$ .

**Example:**

A statistical experiment has eight equally likely outcomes that are denoted by 1, 2, 3, 4, 5, 6, 7, and 8. Let  $A = \{2, 5, 7\}$  and  $B = \{2, 4, 8\}$ .

- a) Find  $A \cap B$
- b) Find  $A \cup B$
- c) Are events  $A$  and  $B$  mutually exclusive events?
- d) Are events  $A$  and  $B$  collectively exhaustive events?
- e) Find  $\bar{A}$  and  $\bar{B}$

**Solution:**

- a) These two events have only one common element  $A \cap B = \{2\}$
- b)  $A \cup B = \{2, 4, 5, 7, 8\}$
- c) Events  $A$  and  $B$  are not mutually exclusive events, because they have a common element, and their intersection is not empty set.
- d) Events  $A$  and  $B$  are not collectively exhaustive events, because their union does not equal to sample space.
- e)  $\bar{A} = \{1, 3, 4, 6, 8\}$ ;  $\bar{B} = \{1, 3, 5, 6, 7\}$

**Example:**

At a busy international airport arriving planes land a first come, first served basis. Let  $A$ ,  $B$ , and  $C$  be the events that there are at least five, at most five, and exactly two planes waiting to land, respectively. Then

- 1.  $\bar{A}$  is the event that at most four planes are waiting to land.
- 2.  $\bar{B}$  is the event that at least four planes are waiting to land.
- 3.  $A$  is a subset of  $\bar{B}$ ; that is if  $A$  occurs, then  $\bar{B}$  occurs. Therefore,  $A \cap \bar{B} = A$ .
- 4.  $C$  is a subset of  $B$ ; that is if  $C$  occurs, then  $B$  occurs. Therefore,  $B \cap C = C$ .
- 5.  $A$  and  $B$  are mutually exclusive; that is,  $A \cap B = \emptyset$ .

$A$  and  $C$  are also mutually exclusive since  $A \cap C = \emptyset$ .

$\bar{C}$  is the event that number of planes waiting to land is zero, one or more.

### 2.3. Three conceptual approaches to probability

There are three conceptual approaches to probability:

- 1. Classical probability
- 2. Relative frequency concept of probability
- 3. Subjective probability.

#### 2.3.1. Classical probability

Classical probability assumes that all outcomes in the sample space are equally likely to occur. It was developed originally in the analysis of gambling problems, where the assumption of equally likely outcomes often is reasonable. When the assumption of equally likely outcomes is used as a basis for assigning probabilities, the approach is referred to as the classical method.

In the classical method, the probability of a single event is equal to the number of outcomes favorable to the event divided by the total number of outcomes for the experiment. On the other hand, the probability of a compound event  $A$  is equal to the number of outcomes favorable to event  $A$ , divided by the total number of outcomes for the experiment.

$$P(A) = \frac{\text{Number of outcomes favorable to } A}{\text{Total number of outcomes for the experiment}}$$

#### 2.3.2. Relative frequency concept of probability

The difference between classical probability and relative frequency probability is that classical method assumes that certain outcomes are equally likely (e.g., the outcomes when a die is rolled) while relative frequency method uses actual experience to determine the likelihood of outcomes. In relative frequency method, one might actually roll a given die 1000 times and observe the relative frequencies and use these frequencies to determine the probability of each outcome. This method of assigning a probability to an event is called the relative frequency concept of probability.

**Definition:**

If an experiment is repeated  $n$  times and an event  $A$  is observed  $f$  times according to the relative frequency concept of probability:

$$P(A) = \frac{f}{n}$$

Because relative frequencies are determined by performing an experiment, probabilities calculated using relative frequencies may change almost any time an experiment is repeated. But the variation in probabilities will be small if the sample size is large.

**2.3.3. Subjective probability**

**Subjective probability** uses a probability value based on an educated guess or estimate, employing opinions.

In subjective probability, a person or group makes an educated guess as to the chance that an event will occur. This guess is based on person's experience and evaluation of solution. For example, a sportswriter may say that there is a 65% probability that the Milan will win championship cup next year. A doctor might say that on the basis of his diagnosis, there is a 40% chance that a patient will need an operation. A seismologist might say there is a 60% probability that an earthquake will occur in a certain area.

**Definition:**

Subjective probability is the probability assigned to an event based on subjective judgment, experience, information, and belief.

All three types of probability (classical, relative frequency, and subjective) are used to solve variety of problems in business, economics, engineering and other fields.

**2.4. Probability and its postulates**

Probability, which gives the likelihood of occurrence of an event, is denoted by  $P$ .

Let  $S$  denote the sample space of a random experiment,  $E_i$  the basic outcomes, and  $A$  an event. The probability that an event  $A$  will occur is denoted by  $P(A)$ .

**The probability has the following important properties:**

For any event in the sample space  $S$ , then  $0 \leq P(A) \leq 1$ .

An event that can not occur has zero probability; such event is called an impossible event. An event that is certain to occur has a probability equal to 1 called sure event.

For impossible event  $M: P(M) = 0$

For a sure event  $C: P(C) = 1$

Let  $A$  be an event in sample space  $S$ , and let  $E_i$  denote the basic outcomes.

$$P(A) = \sum_A P(E_i),$$

The notation implies that the summation extends over all the basic outcomes in  $A$ .

The sum of the probabilities for all the basic outcomes in the sample space is 1. Thus

$$P(S) = \sum_{i=1}^n P(E_i) = P(E_1) + P(E_2) + \dots + P(E_n) = 1.$$

**2.5. Formula for classical probability**

Classical probability uses sample spaces to determine the numerical probability that an event will happen.

Classical probability assumes that all outcomes in the sample space are equally likely to occur. For example, when a single die is rolled, each face has the same probability of occurring. Since there are six outcomes, each face has a probability of 1/6. When a card is selected from an ordinary deck of 52 cards, we assume that the deck has been shuffled, and each card has the same probability of being selected. In this case, it is 1/52.

**Definition:**

If a sample space  $S$  contains  $n$  equally likely basic outcomes and the event  $A$  consists of  $m$  of these outcomes ( $m \leq n$ ), then

$$P(A) = \frac{m}{n}$$

The probability of event  $A$  equals to number of basic outcomes included by the total number of outcomes in the sample space".

The definition above as  $P(A) = \frac{n(A)}{n(S)}$ .

**Example:**

For a card drawn from an ordinary deck find the probability of getting

**Solution:**

Let  $A$  be an event getting a queen. Since there are four queens then

$$n(A) = 4. \text{ Hence, } P(A) = \frac{4}{52} = \frac{1}{13}$$

**Example:**

Find the probability of obtaining an even number in one roll of a die.

**Solution:**

In this experiment  $S = \{1, 2, 3, 4, 5, 6\}$ . Let  $A$  be an event that an even number is observed on the die. Event  $A$  has three outcomes: 2, 4, and 6.

If any one of these three numbers is obtained, event  $A$  is said to occur.

$$P(A) = \frac{\text{number of outcomes included in } A}{\text{total number of outcomes}} = \frac{3}{6} = 0.5$$

**2.6. Consequences of the postulates**

1. Let  $A$  and  $B$  be mutually exclusive events. Then the probability of their union is the sum of their individual probabilities; that is  $P(A \cup B) = P(A) + P(B)$

More generally, if  $E_1, E_2, \dots, E_N$  are mutually exclusive events, then the probability of their union is  $P(E_1 \cup E_2 \cup \dots \cup E_N) = P(E_1) + P(E_2) + \dots + P(E_N)$

2. If  $E_1, E_2, \dots, E_N$  are collectively exhaustive events, then the probability of their union is  $P(E_1 \cup E_2 \cup \dots \cup E_N) = 1$ .

Since the events are collectively exhaustive, their union is the whole sample space  $S$  and  $P(S) = 1$ .

**Example:**

A drawer contains three pairs of red socks, two pairs of black socks and one pair of brown socks. If a person in a dark room selects a pair of socks, find the probability that the pair will be either black or brown. (Note: The socks are folded together in matching pairs).

**Solution:**

Let us define the following events

$A$  = the selected socks are black

$B$  = the selected socks are brown.

There are nine pairs of socks,

$$P(\text{black}) = P(A) = \frac{2}{9};$$

$$P(\text{brown}) = P(B) = \frac{4}{9}$$

$$P(\text{black or brown}) = P(A \cup B) = P(A) + P(B) = \frac{2}{9} + \frac{4}{9} = \frac{2}{3}$$

**Example:**

If a day of the week is selected at random. Find the probability that it is a weekend day.

**Solution:**

$A$  = the selected day is Saturday

$B$  = the selected day is Sunday

$$P(A) = \frac{1}{7}; \quad P(B) = \frac{1}{7} \quad \text{and} \quad P(A \cup B) = P(A) + P(B) = \frac{2}{7}$$

**Exercises**

1. A box contains three red and five blue balls. Define the sample space for the experiment of recording the colours of three balls that are drawn from the box one, with replacement.

2. A sample space for the experiment of putting three different books in random order. If two of these three books are a two-volume set, describe the event that these volumes stand in increasing order side by side (i.e., volume I precedes volume II).

3. A single card is drawn from an ordinary pack of playing cards. What is the probability that the card is

- a) An ace
- b) A five
- c) A red card
- d) A club

4. There are 15 slips of paper in a hat, numbered from 1 to 15. If one slip is drawn at random, find the probability that

- a) The number drawn is 5
- b) The number drawn is even
- c) The number drawn is odd

d) The number drawn is divisible by 3

5. Two events,  $A$  and  $B$ , are mutually exclusive:  $P(A) = \frac{1}{5}$  and  $P(B) = \frac{1}{5}$

Find the probability that

- a) Either  $A$  or  $B$  will occur
- b) Both  $A$  and  $B$  will occur
- c) Neither  $A$  nor  $B$  will occur

6. The manager of a furniture store sells from zero to four sofas each week. Based on past experience, the following probabilities are assigned to the number of sofas sold:

- $P(0) = 0.08$
- $P(1) = 0.18$
- $P(2) = 0.32$
- $P(3) = 0.30$
- $P(4) = 0.12$

- a) Are these valid probability assignments? Why or why not?
- b) Let  $A$  be the event that two or fewer are sold in one week. Find  $P(A)$ .
- c) Let  $B$  be the event that four or more are sold in one week. Find  $P(B)$ .
- d) Are  $A$  and  $B$  mutually exclusive? Find  $P(A \cap B)$  and  $P(A \cup B)$ .

7. Bektur, Janat, and Linar are the finalists in the spelling contest of the school. The winner and the first runner-up will be sent to a city-wide competition.

- a) List the sample space of concerning the out comes of the local contest.
- b) Give the composition of each of the following events
  - $A$  = Linar wins the local contest
  - $B$  = Bektur does not go to the city-wide contest.

8. In a large department store, there are two managers, four department clerks, and four stokers. If a person selected at random, find the probability that the person is either a clerk or a manager.

9. On a small college campus, there are five English professors, four mathematics professors, two science professors, three psychology professors, and three history professors. If a professor is selected at random, find the probability that the professor is the following

- a) An English or psychology professor.
- b) A mathematics or science professor.
- c) A history, science, or mathematics professor.

An English, mathematics, or history professor.

A hospital has monitored the length of time a patient spends in a hospital. The probability for numbers of days a patient spends in the hospital, are given in following table:

number of days	0	1-3	4-6	7-9	10-12	More than 12
probability	0.14	0.39	0.23	0.15	0.06	0.03

Let  $A$  be the event "The patient spends at least one days in the hospital", and let  $B$  be the event "The patient spends less than 10 days in the hospital".

- a) Find the probability of event  $A$ .
- b) Find the probability of event  $B$ .
- c) Find the probability of the complement of  $A$ .
- d) Find the probability of the union of  $A$  and  $B$ .
- e) Find the probability of the intersection of  $A$  and  $B$ .
- f) Are  $A$  and  $B$  mutually exclusive events?
- g) Are  $A$  and  $B$  collectively exhaustive events?

### Answers

- 1.  $\{RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB\}$ ; 2.  $\{d_1d_2a, d_1ad_2, d_2ad_1, ad_2d_1\}$ ;  $\{d_1d_2a, ad_1d_2\}$ ; 3. a)  $\frac{1}{13}$ ; b)  $\frac{1}{13}$ ; c)  $\frac{1}{2}$ ; d)  $\frac{1}{15}$ ; e)  $\frac{7}{15}$ ; f)  $\frac{8}{15}$ ; g)  $\frac{1}{3}$ ; 4. a)  $\frac{8}{15}$ ; b) 0; c)  $\frac{7}{15}$ ; 5. a) Yes; b) No; c) 0.12; d) Yes;  $P(A \cap B) = 0$ ;  $P(A \cup B) = 0.70$ ; 6. a)  $\{BJ, BL, LJ, LB, LJ\}$ ; b)  $A = \{LB, LJ\}$ ;  $B = \{JL, LJ\}$ ; 7. a)  $\frac{9}{13}$ ; b)  $\frac{8}{17}$ ; c)  $\frac{6}{17}$ ; d)  $\frac{12}{17}$ ; 8. a) 0.86; b) 0.91; c) 0.14; d) 1; e) 0.77; f) no, because  $P(A \cap B) \neq \emptyset$ ; g) Yes, because  $P(A \cup B) = 1$ .

## 2.7. Counting principle. Permutation and combination

### Counting principle:

If the set  $E$  contains  $n$  elements and the set  $F$  contains  $m$  elements, there are  $n \times m$  ways in which we can choose first an element of  $E$  and then an element of  $F$ .

### Example:

We toss a coin two times. This experiment has two steps: the first step is the second toss. Each step has two outcomes: a head and a tail. Thus, there are  $2 \times 2 = 4$  outcomes for two tosses of a coin.

The four outcomes for this experiment are:  $HH, HT, TH, TT$

### Generalized counting principle:

Let  $E_1, E_2, E_3, \dots, E_k$  be sets with  $n_1, n_2, n_3, \dots, n_k$  elements, respectively. Then there are  $n_1 \times n_2 \times n_3 \times \dots \times n_k$  ways in which we can first choose an element of  $E_1$ , then an element of  $E_2, \dots$ , and finally an element of  $E_k$ .

### Example:

How many outcomes does the experiment of throwing five dice have?

### Solution:

Let  $E_i, 1 \leq i \leq 5$  be set of all possible outcomes of  $i^{\text{th}}$  die.

Then  $E_i = \{1, 2, 3, 4, 5, 6\}$ . The number of the outcomes of the experiment of throwing five dice equals the number of ways we can first choose an element of  $E_1$ , then an element of  $E_2, \dots$ , and finally an element of  $E_5$ .

That is  $6 \times 6 \times 6 \times 6 \times 6 = 6^5$

### 2.7.1. Permutation

#### Definition:

An  $n$ -element permutation of a set with  $n$  objects is simply called a **permutation**, denoted by  $P_n^n$ . The number of permutations of a set of  $n$  elements is  $P_n^n = n(n-1)(n-2)\dots 2 \cdot 1 = n!$

#### Example:

The 3 digits 1, 2, 3 can be arranged in  $3! = 6$  different orders:

123, 132, 213, 231, 312, 321

#### Definition:

The number of permutations,  $P_r^n$  of  $r$  objects chosen from  $n$  is the number of possible arrangements when  $r$  objects are to be selected from a total of  $n$  and arranged in order. This number is

$$P_r^n = \frac{n!}{(n-r)!}$$

#### Remark:

In the definition of  $P_r^n$ , the symbols  ${}_n P_r$  and  $P(n, r)$  are frequently used to denote the number of permutations of  $n$  objects taken  $r$  at a time. Different authors frequently use different symbols.

#### Example:

Three students, Kanat, Askhat, and Marat must be scheduled for a job interview. In how many different orders can this be done?

#### Solution:

The number of different orders is equal to the number of permutations of the set  $\{\text{Kanat, Askhat, Marat}\}$ . So there are  $3! = 6$  possible orders for the interview.

#### Example:

Five persons are to pose for a photograph by standing in a row. how many different arrangements are possible?

#### Solution:

The number of different arrangements of 5 "objects" 5 at a time, the number of different permutations is given by  $P_5^5 = 5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$ .

#### Example:

Five boys and five girls sit in a row in a random order, what is the probability that no two children of the same sex sit together?

#### Solution:

There are  $10!$  ways for 10 persons to sit in a row. In order that no two of the same sex sit together, boys must occupy positions 1, 3, 5, 7, 9 and girls must occupy positions 2, 4, 6, 8, 10, or vice versa. In each case there are  $5! \times 5!$  possibilities.

The desired possibility is  $\frac{2 \cdot 5! \cdot 5!}{10!} \approx 0.008$ .

**Theorem:** The number of distinguishable permutations of  $n$  objects of  $k$  different types, where  $n_1$  are alike,  $n_2$  are alike, ...,  $n_k$  are alike and

$$n = n_1 + n_2 + \dots + n_k \text{ is}$$

$$\frac{n!}{n_1! \times n_2! \times \dots \times n_k!}$$

**Example:**

How many different 10-letter codes can be made using three  $a$ 's, four  $b$ 's and three  $c$ 's?

**Solution:**

By theorem, the number of such codes is  $\frac{10!}{3! \cdot 4! \cdot 3!} = 4200$ .

### 2.7.2. Combination

If order is of no importance, then we have a combination rather a permutation. A combination of  $n$  objects taken  $r$  at a time is a selection of  $r$  objects taken from the  $n$ , without regard to the order in which they are selected or arranged. Order is irrelevant.

**Definition:**

The number of combinations,  $C_r^n$ , of  $r$  objects chosen from  $n$  is the number of possible selections that can be made. This number is

$$C_r^n = \frac{n!}{r!(n-r)!}$$

**Remark:**

Some other symbols which are used to denote the number of combinations of

$n$  objects taken  $r$  at a time are  ${}_n C_r$ ,  $C(n, r)$  and  $\binom{n}{r}$ .

**Example:**

If a club has a membership of ten, then how many three-man committees are possible?

**Solution:**

Order is not important, so this is combination problem. The number of possible committees is equal to the number of ways three persons can be selected from ten persons, namely  $C_3^{10}$ .

We have:

$$C_3^{10} = \frac{10!}{3!7!} = 120.$$

**Example:**

In how many ways can two math and three biology books be selected from eight math and six biology books?

**Solution:**

There are  $C_2^8$  possible ways to select two math books and  $C_3^6$  possible ways to select three biology books. Therefore, by counting principle,

$C_2^8 \cdot C_3^6 = \frac{8!}{2!6!} \cdot \frac{6!}{3!3!} = 560$  is the total number of ways in which two math and three biology books can be selected.

**Example:**

A box contains 24 transistors, four of which are defective. If four are sold at random, find the following probabilities:

- a) Exactly two are defective
- b) None is defective
- c) All are defective
- d) At least one is defective

**Solution:**

There are  $C_4^{24}$  ways to sell four transistors, so the denominator in each case will be  $C_4^{24} = 10626$

a) Two defective transistors can be selected as  $C_2^4$  and two nondefective ones as  $C_2^{20}$ . Hence

$$P(2 \text{ defective}) = \frac{C_2^4 \cdot C_2^{20}}{C_4^{24}} = \frac{1140}{10626} = \frac{190}{1771}$$

b) The number of ways to choose no defective is  $C_4^{20}$ . Hence

$$P(\text{no defective}) = \frac{C_4^{20}}{C_4^{24}} = \frac{4845}{10626} = \frac{1615}{3542}$$

c) The number of ways to choose four defectives from four is  $C_4^4$ , or 1. Hence

$$P(\text{all defective}) = \frac{C_4^4}{C_4^{24}} = \frac{1}{10626}$$

d) To find the probability of at least one defective transistor, find the probability that there are no defective transistors, and then subtract that probability from 1.

$$P(\text{at least 1 defective}) = 1 - P(\text{no defective}) = 1 - \frac{C_4^{20}}{C_4^{24}} = \frac{1927}{3542}$$

### Exercises

1. How many permutations of the set  $\{a, b, c, d, e\}$  begin with  $a$  and end with  $e$ ?
2. How many different messages can be sent by five dashes and three dots?
3. Roman has eight guests, two of whom are Jane and John. If the guests arrive in a random order, what is the probability that John will not arrive after Jane?
4. Find the number of distinguishable permutations of the letters MISSISSIPPI.
5. There are 20 chairs in a room numbered 1 through 20. If eight girls and boys sit on these chairs at random, what is the probability that the thirteenth chair is occupied by a boy?
6. If we put five math, six biology, eight history, and three literature books on a bookshelf at random, what is the probability that all the math books are together?
7. Five boys and five girls sit in a row at random. What is the probability the boys are together and the girls are together?
8. A man has 20 friends. If he decides to invite six of them to his birthday party, how many choices does he have?
9. A panel consists of 20 men and 20 women. How many choices do we have for a jury of six men and six women from this panel?
10. In a company there are seven executives: four women and three men. Three are selected to attend a management seminar. Find the following probabilities:
  - a) All three selected will be women.
  - b) All three selected will be men.
  - c) Two men and one woman will be selected.
  - d) One man and two women will be selected.
11. In a class of 18 students, there are 11 men and seven women. Four students are selected to present a demonstration on the use of the calculator. Find the probability that the group consists of the following:
  - a) All men
  - b) All women
  - c) Three men and one woman

- d) One man and three women
- e) Two men and two women.
2. A committee of four people is to be formed from six doctors and eight dentists. Find the probability that the committee will consist of the following:
  - a) All dentists
  - b) Two dentists and two doctors
  - c) All doctors
  - d) Three doctors and dentist
  - e) One doctor and three dentists
3. From a faculty of six professors, six associate professors, 10 assistant professors, and 12 instructors, a committee of size 6 is formed randomly. What is the probability that
  - a) There are exactly two professors on the committee?
  - b) All committee members are of the same rank?
4. Almas has three sets of classics in literature, each set having four volumes. How many ways can he put them in a bookshelf so that books of each set are not separated?

### Answers

1. 24; 2. 40320; 3. 0.875; 4. 34 650; 5. 0.6; 6. 0.00068; 7. 0.0079; 8. 38 760; 9. 1502337600; 10. a) 4/35; b) 1/35; c) 12/35; d) 18/35; 11. a) 11/102; b) 7/612; c) 77/204; d) 77/612; e) 77/204; 12. a) 10/143; b) 60/143; c) 15/1001; d) 160/1001; e) 48/143; 13. a) 0.228; b) 0.00084; 14. 82944.

## 2.8. Probability rules

### 1. Complement rule:

Let  $A$  be an event and  $\bar{A}$  its complement. Then the **complement rule** is:

$$P(\bar{A}) = 1 - P(A)$$

In words, the probability of the occurrence of any event equals one minus the probability of the occurrence of its complementary event.

#### Example:

A club has a membership of six men and four women. A three-person committee is chosen at random. What is the probability that at least one woman will be selected?

#### Solution:

Let  $A$  be the event "at least one woman will be selected". We will start solution by computing the probability of the complement:  $\bar{A}$  - "no woman selected", and then using the complement rule will compute the probability of  $A$ .

$$P(\bar{A}) = \frac{C_3^6}{C_3^{10}} = \frac{1}{6}$$

And therefore the required probability is  $P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{6} = \frac{5}{6}$ .

### 2. The addition rule of probabilities:

Let  $A$  and  $B$  be two events. The probability of their union is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

#### Example:

The probability that a randomly selected student from a university is a senior is 0.18, a business major is 0.14, and a senior and a business major is 0.04. Find the probability that a student selected at random from this university is a senior or a business major.

#### Solution:

Let  $A$  be the event "Chosen student is a senior student" and  $B$  the event "Chosen student is a business major student". Thus we have

$$P(A) = 0.18, \quad P(B) = 0.14 \quad \text{and} \quad P(A \cap B) = 0.04$$

The required probability is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.018 + 0.014 - 0.04 = 0.28.$$

#### Example:

Suppose that in a community of 400 adults, 300 bike or swim or do both, 160 swim and 120 swim and bike. What is the probability that an adult selected at random from this community bikes?

#### Solution:

Let  $A$  be the event that person swims and  $B$  be the event that he or she bikes, then  $P(A \cup B) = 300/400$ ;  $P(A) = 160/400$ ;  $P(A \cap B) = 120/400$ . Hence the relation  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$  implies that

$$P(B) = P(A \cup B) + P(A \cap B) - P(A) = \frac{300}{400} + \frac{120}{400} - \frac{160}{400} = 0.65.$$

## Exercises

The probability that a randomly selected elementary school teacher from a city is a female is 0.68, holds a second job is 0.42, and is a female and holds a second job is 0.29. Find the probability that an elementary school teacher selected at random from this city is a female or holds a second job?

It was estimated that 35% of all students were seriously concerned about employment prospects, 20% were seriously concerned about grades, and 15% were seriously concerned about both. What is the probability that a randomly chosen student is seriously concerned about at least one of these two things?

It was found that 45% of students think that professors must be "more tolerant" to the students. If a student is selected randomly what is the probability that he or she will disagree or have no opinion on the issue.

In a statistics class there are 18 juniors and 10 seniors; 6 of the seniors are males, and 12 of the juniors are males. If a student is selected at random, find the probability of the following:

- A junior or a female
- A senior or a female
- A junior or a senior

If a die is rolled three times, find the probability of getting at least one 6.

If a die is rolled three times, find the probability of getting at least one even number.

7. A number is selected at random from the set of natural numbers  $\{1, 2, 3, \dots, 1000\}$ . What is the probability that it is divisible by 4 but not by 5 nor by 7?

8. There are four tickets numbered 1, 2, 3, and 4. Suppose a two-digit number will be formed by first drawing one ticket at random and then drawing a second ticket at random from remaining three. (For instance, if the first ticket drawn shows 4 and the second shows 1, the number recorded is 41.) List the sample space and determine the following probabilities

- a) What is the probability of getting an even number?
  - b) What is the probability of getting a number larger than 20?
  - c) What is the probability that obtained number is between 22 and 30?
9. A three-digit number is formed by arranging the digits 1, 5, and 6 in random order.
- a) List the sample space.
  - b) Find the probability of getting a number larger than 400.
  - c) What is the probability that an even number is obtained?

**Answer**

1. 0.81; 2. 0.40; 3. 0.55; 4. a) 6/7; b) 4/7; c) 1; 5. 91/216; 6. 7/8; 7. 0.17; 8. a) 0.5; b) 0.75; c) 0.167; 9. b) 2/3; c) 1/3;

**2.8.1. Conditional probability**

**Definition:**

Conditional probability is the probability that an event will occur given another event has already occurred. If  $A$  and  $B$  are two events, then the conditional probability of  $A$  is written as  $P(A/B)$  and read as "the probability of  $A$  given that  $B$  has already occurred".

If  $A$  and  $B$  are two events, then

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{and}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

given that  $P(A) > 0$  and  $P(B) > 0$ .

**Example:**

A box contains black chips and white chips. A person selects two chips without replacement. If the probability of selecting a black chip and a white chip is  $15/56$ , and the probability of selecting a black chip on the first draw is  $3/8$ , find the probability of selecting the white chip on the second draw, given that the first chip selected was a black chip.

**Solution:**

Let  $B$  = selecting a black chip  
 $W$  = selecting a white chip.

$$\text{Then } P(W/B) = \frac{P(W \cap B)}{P(B)} = \frac{15/56}{3/8} = \frac{5}{7}$$

Hence, the probability of selecting a white chip on the second draw given that the first chip selected was black is  $5/7$ .

**Example:**

In a certain region of Kazakhstan, the probability that a person lives at least 80 years is 0.75 and the probability that he or she lives at least 90 years is 0.63. What is the probability that randomly selected 80-year old person from this region will survive to become 90?

**Solution:**

Let  $A$  and  $B$  be the events that the person selected survives to become 90 and 80 years old, respectively. We are interested in  $P(A/B)$ . By definition,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{0.63}{0.75} = 0.84$$

Note that in this case  $P(A \cap B) = P(A)$ .

**2.8.2. The multiplication rule of probability**

If  $A$  and  $B$  be two events. The probability of their intersection is

$$P(A \cap B) = P(A/B) \cdot P(B)$$

$$\text{so } P(A \cap B) = P(B/A) \cdot P(A)$$

**Example:**

Suppose that seven nondefective and three defective goods have been mixed. To find defective goods, we test them one by one, at random, and without replacement. What is the probability that we are lucky and find both of the defective goods in the first two tests?

**Solution:**

Let  $D_1$  and  $D_2$  be the events of finding defective goods in the first and second tests respectively. We are interested in

$$P(D_1 \cap D_2) = P(D_2 / D_1) \cdot P(D_1)$$

As we know, there are three defective goods in total 10 goods. Consequently, the probability of selecting a defective good at the first selection

is  $P(D_1) = \frac{3}{10}$ . To calculate the probability  $P(D_2 / D_1)$ , we know that the

first good is defective because  $D_1$  has already occurred. Because the selections are made without replacement, there are 9 total goods and 2 of them are defective at the time of the second selection. Therefore  $P(D_2 / D_1) = \frac{2}{9}$ .

Hence the required probability is

$$P(D_1 \cap D_2) = P(D_2 / D_1) \cdot P(D_1) = \frac{3}{10} \cdot \frac{2}{9} = \frac{1}{15}$$

**Remark:** Multiplication rule can be generalized for calculating the probability of the joint occurrence of several events.

For example, if  $P(A \cap B) > 0$ , then

$$P(A \cap B \cap C) = P(A) \cdot P(B / A) \cdot P(C / (A \cap B))$$

### 2.8.3. Multiplication rule for independent events

Two events  $A$  and  $B$  are **independent** if  $P(A / B) = P(A)$

Equivalent conditions are  $P(B / A) = P(B)$  or  $P(A \cap B) = P(A) \cdot P(B)$ .

**Example:**

An urn contains three red balls, two blue balls, and five white balls. A ball is selected and its colour is noted. Then it is replaced. A second ball is selected and its colour is noted. Find the probability of

- a) Selecting two blue balls
- b) Selecting a blue and then white ball
- c) Selecting a red ball and then a blue ball

**Solution:**

a)  $P(\text{blue and blue}) = P(\text{blue}) \cdot P(\text{blue}) = \frac{2}{10} \cdot \frac{2}{10} = \frac{1}{25}$

b)  $P(\text{blue and white}) = P(\text{blue}) \cdot P(\text{white}) = \frac{2}{10} \cdot \frac{5}{10} = \frac{1}{10}$

c)  $P(\text{red and blue}) = P(\text{red}) \cdot P(\text{blue}) = \frac{3}{10} \cdot \frac{2}{10} = \frac{3}{50}$

**Example:**

An urn contains five red and seven blue balls. Suppose that two balls are selected at random with replacement. Let  $A$  and  $B$  be the events that the first

and the second balls are red, respectively. Then we get  $P(A \cap B) = \frac{5}{12} \cdot \frac{5}{12}$ .

Now  $P(A \cap B) = P(A)P(B)$  since  $P(A) = \frac{5}{12}$  and  $P(B) = \frac{5}{12}$ .

Thus  $A$  and  $B$  are independent.

If we do the same experiment without replacement, then  $P(B / A) = \frac{4}{11}$

while  $P(B) = P(B / A) \cdot P(A) + P(B / \bar{A}) \cdot P(\bar{A}) = \frac{4}{11} \cdot \frac{5}{12} + \frac{5}{11} \cdot \frac{7}{12} = \frac{5}{12}$

as expected. Thus  $P(B / A) \neq P(B)$ , implying that  $A$  and  $B$  are dependent.

**Remark:**

Multiplication rule for independent events can also be extended to three or more independent events by using the formula

$$P(A \cap B \cap C \cap \dots \cap K) = P(A) \cdot P(B) \cdot P(C) \cdot \dots \cdot P(K)$$

**Example:**

The probability that a specific medical test will show positive is 0.32. If four people are tested, find the probability that all four will show positive.

**Solution:**

Let  $T_i$  ( $i=1, 2, 3, 4$ ) be the symbol for a positive test result.

$$P(T_1 \cap T_2 \cap T_3 \cap T_4) = P(T_1) \cdot P(T_2) \cdot P(T_3) \cdot P(T_4) = 0.32^4 = 0.010$$

### Exercises

1. Suppose that  $P(A) = 0.30$ ,  $P(B) = 0.25$ , and  $P(A \cap B) = 0.20$

a) Find  $P(A \cup B)$ ,  $P(A|B)$ ,  $P(B|A)$ .

b) Are the events  $A$  and  $B$  independent? Why or why not?

2. Suppose that  $P(A) = 0.68$ ,  $P(B) = 0.55$ , and  $P(A \cap B) = 0.32$ . Find

a) The conditional probability that  $B$  occurs, given that  $A$  occurs.

b) The conditional probability that  $B$  does not occur given that  $A$  occurs.

c) The conditional probability that  $B$  occurs given that  $A$  does not occur.

3. Concerning the events  $A$  and  $B$ , the following probabilities are given

$$P(B) = \frac{1}{3}; P(A|B) = \frac{2}{3}; P(A|\bar{B}) = \frac{3}{7}.$$

Determine

a)  $P(A \cap \bar{B})$ ;      b)  $P(A)$ ;      c)  $P(\bar{B}|A)$

4. In a study of television viewing habits among married couples, a researcher found that for a popular Saturday night program 25% of the husbands view the program regularly and 30% of the wives viewed the program regularly. The study found that for couples where the husband watches the program regularly 80% of the wives also watch regularly.

a) What is the probability that both husband and wife watch the program regularly?

b) What is the probability that at least one-husband or wife-watches the program regularly?

c) What percentage of married couples do not have at least one regular viewer of the program?

5. Of 20 rats in a laboratory, 12 are males and 9 are infected with a virus. Of the 12 male rats, 7 infected with the virus. One rat is randomly selected from the laboratory.

a) If the selected rat is found to be infected, what is the probability that it is female?

b) If the selected rat is found to be a male, what is the probability that it is infected?

c) Are the events "the selected rat is infected" and "the selected rat is male" independent? Why or why not?

6. Suppose  $P(A) = 0.50$ ,  $P(B) = 0.22$ .

a) Determine  $P(A \cup B)$  if  $A$  and  $B$  are independent.

b) Determine  $P(A \cup B)$  if  $A$  and  $B$  are mutually exclusive.

c) Find  $P(A|\bar{B})$  if  $A$  and  $B$  are mutually exclusive.

7. An urn has three red and five blue balls. Suppose that 8 balls are selected at random and with replacement. What is the probability that the first three are red and the rest are blue balls?

### Answer

1. a)  $P(A \cup B) = 0.35$ ;  $P(A|B) = 0.8$ ;  $P(B|A) = 0.67$ ; b) No;

2. a) 0.471; b) 0.529; c) 0.719; 3. a) 2/7; b) 32/63; c) 9/16; 4. a) 0.2;

b) 0.35; c) 65%; 5. a) 2/9; b) 7/12; c) No; 6. a) 0.61; b) 0.72; c) 0.641;

7. 0.00503;

### 2.8.4. The law of total probability

#### Theorem:

Let  $B$  be an event with  $P(B) > 0$  and  $P(\bar{B}) > 0$ . Then for any event  $A$ ,

$$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B}).$$

#### Example:

An urn contains 10 white and 6 red balls. Two balls are selected at random without replacement. What is the probability that second selected ball is red?

#### Solution:

Let  $A$  be the event that second selected ball is red,  $B$  be event that the first ball is white. Then  $P(B) = \frac{10}{16}$ ,  $P(\bar{B}) = \frac{6}{16}$ ,  $P(A|B) = \frac{6}{15}$ ,  $P(A|\bar{B}) = \frac{5}{15}$ . Then by

the law of total probability:

$$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B}) = \frac{6}{15} \cdot \frac{10}{16} + \frac{5}{15} \cdot \frac{6}{16} = \frac{3}{8}.$$

#### Theorem:

Let  $\{B_1, B_2, \dots, B_n\}$  be a set of nonempty, mutually exclusive subsets of the sample space  $S$  and  $P(B_i) > 0$  for  $i = 1, 2, \dots, n$ , then for any event  $A$  of  $S$ ,

$$P(A) = P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + \dots + P(A/B_n)P(B_n) = \sum_{i=1}^n P(A/B_i)P(B_i).$$

**Example:**

Suppose that 70% of seniors, 60% of juniors, 55% of the sophomores, and 40% of the freshmen of a university use the library frequently. If 35% of all students are freshmen, 30% are sophomores, 20% are juniors, and 15% are seniors, what percent of all students use the library frequently?

**Solution:**

Let  $A$  be the event that a randomly selected student is using library frequently. Let  $F, O, J,$  and  $E$  be the events that he or she is a freshmen, sophomore, junior, or senior respectively. Thus

$$P(A) = P(A/F)P(F) + P(A/O)P(O) + P(A/J)P(J) + P(A/E)P(E) = 0.4 \cdot 0.35 + 0.55 \cdot 0.3 + 0.6 \cdot 0.2 + 0.7 \cdot 0.15 = 0.53.$$

Therefore, 53% of these students use the library frequently.

**Exercises**

1. In a country men constitute 58% of the labour force. The rates of unemployment are 6.2% and 4.3% among males and females respectively.
  - a) What is the overall rate of unemployed in the country?
  - b) If a worker selected at random is found to be unemployed, what is the probability that the worker is a woman?
2. In a shipment of 15 air conditioners, there are 4 with defective thermostats. Two air conditioners will be selected at random and inspected one after another. Find the probability that
  - a) The first is defective.
  - b) The first is defective and the second good.
  - c) Both are defective.
  - d) The second air conditioner is defective.
  - e) Exactly one is defective.
3. Suppose that 40% of the students are girls. If 25% of the girls and 15% of the boys of this university are A students, what is the probability that randomly selected student is A student?

4. A factory produces all its products by three machines. Machines I; II; and III produces 40%; 40% and 20% of the output, where 5%, 4%, and 2% of their outputs are defective, respectively. What percentage of the total product is defective?

5. A box contains 18 tennis balls. of which eight are new. Suppose that three balls are selected randomly, played with, and after play are returned to the box. If another three balls are selected for a second play, what is the probability that they are all new?

6. In an economical college all students are required to take calculus and economics course. Statistics shows that 37% of the students of this college get A's in calculus and 25% of them get A's in both economics and calculus. If randomly selected student of this college has passed calculus with an A, what is the probability that he or she got A in economics?

7. Suppose that 12% of the population of a country are unemployed women and 17% of population are unemployed. What percentage of the unemployed are women?

**Answer**

1. a) 5.4%; b) 0.334; 2. a) 4/15; b) 22/105; c) 2/35; d) 4/15; e) 44/105;  
 3. 0.19; 4. 4%; 5. 0.148; 6. 0.676; 7. 70.6%.

**2.9. Bayes' theorem**

Often, we begin our analysis with initial or *prior* probability estimates for specific events or interest. Then, from sources such as a sample, a special report, a product test, etc., we obtain some additional information about the events. Given this new information, we want to revise and update the prior probability values. The new or revised probabilities for the events are referred to as *posterior* probabilities. **Bayes' theorem**, which will be presented shortly, provides a means of computing these revised probabilities. To introduce Bayes' formula, let us consider the following example:

**Example 1:**

In the factory 40%, 30%, and 30% of the goods is produced by machines I, II, and III, respectively. If 5%, 4%, and 3% of the outputs of these machines is defective, what is the probability that a randomly selected good that is found to be defective is produced by machine III?

**Solution:**

Let  $A$  be the event that a randomly selected good is defective and  $B_3$  be the event that it is produced by machine III. We are asked to find  $P(B_3 / A)$ . We know that

$$P(B_3 / A) = \frac{P(B_3 \cap A)}{P(A)}$$

To find  $P(B_3 \cap A)$ , note that since  $P(A / B_3)$  and  $P(B_3)$  are known, we can use relation  $P(B_3 \cap A) = P(A / B_3) \cdot P(B_3)$ .

To calculate  $P(A)$ , we use the law of total probability. Let  $B_1$  and  $B_2$  be the events that the good is produced by machines I and II, respectively. Hence,

$$P(A) = P(A / B_1)P(B_1) + P(A / B_2)P(B_2) + P(A / B_3)P(B_3)$$

By substituting we obtain

$$\begin{aligned} P(B_3 / A) &= \frac{P(B_3 \cap A)}{P(A)} = \\ &= \frac{P(A / B_3) \cdot P(B_3)}{P(A / B_1)P(B_1) + P(A / B_2)P(B_2) + P(A / B_3)P(B_3)} = \\ &= \frac{0.03 \cdot 0.3}{0.05 \cdot 0.4 + 0.04 \cdot 0.3 + 0.03 \cdot 0.3} \approx 0.22. \end{aligned}$$

The formula for  $P(B_3 / A)$  is a particular case of Bayes' formula.

To write formula for  $P(B_3 / A)$  we can use tree diagram. (Fig. 2.1). Letter  $D$  stands for "defective" and  $N$  for "not defective".

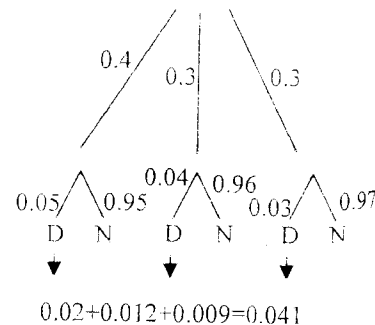


Fig.2.1. For example 1.

**Theorem: (Bayes' theorem)**

Let  $A$  and  $B$  be two events. Then Bayes' theorem states that

$$P(B / A) = \frac{P(A / B)P(B)}{P(A)} \quad \text{and}$$

$$P(A / B) = \frac{P(B / A)P(A)}{P(B)}$$

**Theorem (Bayes' theorem, general form):**

Let  $B_1, B_2, B_3, \dots, B_n$  be  $n$  mutually exclusive and collectively exhaustive events of the sample space  $S$ . Then for any other event  $A$  of  $S$  with  $P(A) > 0$

$$P(B_n / A) = \frac{P(A / B_n) \cdot P(B_n)}{P(A / B_1)P(B_1) + P(A / B_2)P(B_2) + \dots + P(A / B_n)P(B_n)}$$

**Example 2:**

A box contains 8 red and 11 blue balls. Two balls are selected at random without replacement and without their colour being seen. If the third ball is drawn randomly and observed to be red, what is the probability that both of previous selected balls were blue?

**Solution:** Let  $BB, BR,$  and  $RR$  be the events that first two selected balls are blue and blue, blue and red, and red and red. Let  $R$  be the event that the third ball drawn is red. We need to find  $P(BB / R)$ . Using Bayes' formula:

$$P(BB / R) = \frac{P(R / BB) \cdot P(BB)}{P(R / BB)P(BB) + P(R / BR)P(BR) + \dots + P(R / RR)P(RR)}$$

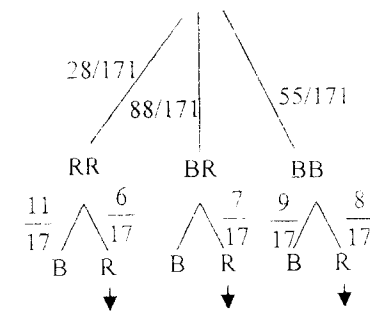
$$\text{Now } P(BB) = \frac{11}{19} \cdot \frac{10}{18} = \frac{55}{171}$$

$$P(RR) = \frac{8}{19} \cdot \frac{7}{18} = \frac{28}{171}$$

and

$$P(BR) = \frac{11}{19} \cdot \frac{8}{18} + \frac{8}{19} \cdot \frac{11}{18} = \frac{88}{171}$$

where  $BR$  is the union of two events:



$$\frac{6}{17} \cdot \frac{28}{171} + \frac{7}{17} \cdot \frac{88}{171} + \frac{8}{17} \cdot \frac{55}{171} = 0.42$$

Fig.2.2. for example 2.

namely, the first ball was blue, the second was red, and vice versa. Thus,

$$P(BB/R) = \frac{\frac{8}{17} \cdot \frac{55}{171}}{\frac{8}{17} \cdot \frac{55}{171} + \frac{7}{17} \cdot \frac{88}{171} + \frac{6}{17} \cdot \frac{28}{171}} \approx 0.36.$$

This can be found easily from tree diagram on Fig. 2.2. as well.

### Exercises

1. Given that  $P(A_1) = 0.66$ ,  $P(A_2) = 0.34$ ,  $P(B/A_1) = 0.57$ ,  $P(C/A_1) = 0.43$ ,  $P(B/A_2) = 0.61$ ,  $P(C/A_2) = 0.39$ , find the following probabilities:  $P(A_1/B)$ ,  $P(A_2/B)$ ,  $P(A_1/C)$ , and  $P(A_2/C)$ .
2. A store purchases electric irons from two companies. From company *A*, 500 irons are purchased, and 2% are defective. From company *B*, 850 irons are purchased, and 2% are defective. Given that an iron is defective, find the probability that it came from company *B*.
3. A store owner purchases telephones from two companies. From company *A*, 350 telephones are purchased, and 2% are defective. From company *B* 1050 telephones are purchased, and 4% are defective. Given that a phone is defective, find the probability that it came from company *B*.
4. A certain cancer is found in 1 person in 5000. If a person does have the disease, in 92% of the cases the diagnostic procedure will show that he or she actually has it. If a person does not have the disease, the diagnostic procedure in 1 out of 500 cases gives a false positive result. Determine the probability that a person with a positive test result has the cancer.
5. Suppose that 5% of the men and 2% of the women working for a corporation make over 4000\$ a year. If 30% of the employees of the corporation are women, what percent of those who make over 4000\$ a year are women?
6. Company purchases a certain part from three suppliers *A*, *B*, and *C*. Supplier *A* supplies 60% of the parts, *B* supplies 30% and *C* supplies 10%. The quality of parts is known to vary among suppliers, with *A*, *B*, and *C* parts having 0.25%, 1%, and 2% defective rates, respectively.
  - a) What percent of product of the company has a defect?
  - b) When a defective part is found, which supplier is the likely source?

7. At a department store, 20% of all customers spend \$50 or less and 80% spend more than \$50 per visit. Of those who spend \$50 or less, 75% pay by cash or check and 25% pay by credit card. Of those who spend more than \$50, 30% pay by cash or check and 70% pay by credit card. One randomly selected customer, who made a purchase at this store, paid by credit card. What is the probability that this customer spent more than \$50 at his store?
8. Each day companies introduce thousands of new products in the market. Usually new products are test marketed before they are introduced for sale. The probability is 0.65 that a new product introduced by a company will be successful. For an eventually successful product, probability is 0.95 that 50% or more of the people included in the test like it. However, for an eventually unsuccessful product, probability is 0.20 that 50% or more of the people included in the test like it. A company recently introduced a new product. What is the probability that this product will be successful if less than 50% of the people included in the test like it?
9. Professor classifies students as most accurate (in calculations and writing), moderately accurate, or poorly accurate, and finds 50%, 40%, and 10% respectively of all students fall into these categories. Professor found that A grade was got by 70% of the most accurate, by 50% of the moderately accurate, and by 30% of the poorly accurate students.
- What is the probability that a randomly chosen student is A grade student?
  - If the randomly selected student is A grade student, what is the probability that the student is from group of most accurate student?
  - If randomly selected student is A grade student, what is the probability that the student is not from the group of most accurate students?

### Answers

1.  $P(A_1 / B) = 0.645$ ;  $P(A_2 / B) = 0.355$ ;  $P(A_1 / C) = 0.682$ ;  
 $P(A_2 / C) = 0.318$ ; **2.** 0.063; **3.** 0.857; **4.** 0.084; **5.** 14.63%; **6.** a) 0.65%;  
 b) **7.** 0.91; **8.** 0.104; **9.** a) 0.58; b) 0.6034; c) 0.3966.

**2.10. Bivariate probabilities**  
**Joint and marginal probabilities**

Suppose that a problem involves two distinct sets of events that label  $A_1, A_2, A_3, \dots, A_n$ ,  $B_1, B_2, B_3, \dots, B_k$ . The events  $A_i$  and  $B_j$  are mutually exclusive and collectively exhaustive within their sets, but intersections  $A_i \cap B_j$  can occur between all events from the two sets. These intersections can be considered as a basic outcome of a random experiment. Two sets of events considered jointly in this way, are called **bivariate**, and their probabilities are called **bivariate probabilities**.

Table 2.1.

	$B_1$	$B_2$	$B_3$	$B_k$
$A_1$	$P(A_1 \cap B_1)$	$P(A_1 \cap B_2)$	$P(A_1 \cap B_3)$	$P(A_1 \cap B_k)$
$A_2$	$P(A_2 \cap B_1)$	$P(A_2 \cap B_2)$	$P(A_2 \cap B_3)$	$P(A_2 \cap B_k)$
.....	.....	.....	.....	.....
$A_n$	$P(A_n \cap B_1)$	$P(A_n \cap B_2)$	$P(A_n \cap B_3)$	$P(A_n \cap B_k)$

**Definition:**

In the table 2.1., the intersection probabilities  $P(A_i \cap B_j)$  are called joint probabilities. Marginal probability is the probability of a single event without consideration of any other event. Marginal probability can be computed by summing the corresponding row or column.

**Example:**

All the 420 employees of a company were asked if they smoke or not and whether they are university graduates or not. Based on this information, the following two-way classification table was prepared.

Table 2.2.

	University graduate	Not a university graduate
Smoker	35	80
Nonsmoker	130	175

In table 2.2. each box that contains a number is called a cell. There are four cells in table 2.2. Each cell gives the frequency for two characteristics. For example, 35 employees in this group possess two characteristics: They are university graduates and smoke. We can interpret the number in other cells the same way.

By adding the row of totals and the column of totals to table 2.2. we write table 2.3.

Table 2.3.

	University graduate	Not a university graduate	Total
Smoker	35	80	115
Nonsmoker	130	175	305
Total	165	255	420

Suppose one employee is selected at random from these 420 employees. This employee may be classified either on the basis of smoker or non-smoker alone or on the basis of university graduate or not. If only one characteristic is considered at a time, the employee selected can be a smoker, non-smoker, a university graduate, or not a university graduate. The probability of each of these four characteristics or events is called marginal probabilities because they are calculated by dividing the corresponding row margins (totals for rows) or column margins (totals for the columns) by the grand total. For table 2.3., the marginal probabilities are calculated as follows:

$$P(\text{smoker}) = \frac{\text{Number of smoker}}{\text{Total number of employees}} = \frac{115}{420} = 0.274$$

$P(\text{smoker}) = 0.274$  can be interpreted as "The probability that randomly selected employee is a smoker is 0.274". Similarly

$$P(\text{nonsmoker}) = \frac{305}{420} = 0.726$$

$$P(\text{university graduate}) = \frac{165}{420} = 0.393$$

$$P(\text{not a university graduate}) = \frac{255}{420} = 0.607$$

Now, suppose that one employee is selected at random from these 420 employees. Furthermore, assume that it is known that this (selected) employee is a smoker. In other words, the event that the employee selected is a smoker

has already occurred. What is the probability that the employee selected is a university graduate?

This probability,  $P$  (university graduate/smoker), as we know, is called a conditional probability, and it is read as "the probability that the employee selected is a university graduate given that this employee is a smoker". The required conditional probability is calculated as follows:

$$P(\text{university graduate/smoker}) = \frac{\text{Number of smokers who are university graduate}}{\text{total number of smokers}} = \frac{35}{115} = 0.304.$$

**Example:**

For the data of table 2.3, calculate the conditional probability that a randomly selected employee is a non-smoker given that this employee is not a university graduate.

**Solution:**

We are to compute the probability  $P(\text{nonsmoker / not a university graduate})$ .

$$P(\text{nonsmoker/ not a university graduate}) = \frac{\text{Number of not a university graduates who do not smoke}}{\text{total number of not university graduates}} = \frac{175}{255} = 0.686.$$

The probability that randomly selected employee who is not a university graduate, does not smoke is 0.686.

**Example:**

Refer to the information on 420 employees given in table 2.3., are the events "smoker (S)" and "university graduate (U)" independent?

**Solution:**

If the occurrence of one event affects the probability of the occurrence of the other event then the two events are said to be dependent events. Using probability notation, the two events will be dependent if either

$$P(A/B) \neq P(A) \text{ or } P(B/A) \neq P(B).$$

Events S and U will be independent if  $P(S) = P(S/U)$ , otherwise they will be dependent.

Using the information given in table 2.3., we compute the following two probabilities

$$P(S) = \frac{115}{420} = 0.274; \text{ and } P(S/U) = \frac{35}{165} = 0.212.$$

Because these two probabilities are not equal, the two events are dependent. Here, dependence of events means that percentage of smokers is different from percentage between a university graduates.

**Example:**

A recent survey asked 100 people if they thought women should be permitted to participate in weightlifting competition. The results of the survey are shown in the table.

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

Find the probabilities

- That a randomly selected person is a male.
- The respondent answered "yes", given that the respondent was a female.
- The respondent was a male, given that the respondent answered "no".

**Solution:**

Let  $M$  = respondent was a male;  $Y$  = respondent answered "Yes"

$F$  = respondent was a female;  $N$  = respondent answered "No".

a) We need to compute the probability  $P$  (male). The probability that randomly selected respondent is a male is obtained by dividing total number of row labelled "Male" (50) by the total number of respondents (100).

$$P(\text{male}) = \frac{50}{100} = \frac{1}{2}$$

b) The problem is to find  $P(Y/F)$ . The rule states

$$P(Y/F) = \frac{P(F \cap Y)}{P(F)}$$

The probability  $P(F \cap Y)$  is the number of females who responded "yes" divided by the total number of respondents  $P(F \cap Y) = \frac{8}{100}$ .

The probability  $P(F)$  is the probability of selecting a female:

$$P(F) = \frac{50}{100}.$$

Then

$$P(Y/F) = \frac{P(F \cap Y)}{P(F)} = \frac{8/100}{50/100} = \frac{4}{25}$$

c) The problem is to find  $P(M/N)$

$$P(M | N) = \frac{P(M \cap N)}{P(N)} = \frac{18/100}{60/100} = \frac{3}{10}$$

### Exercises

**1.** The following table shows the probabilities concerning two events  $A$  and  $B$ .

- Determine the missing entries.
- What is the probability that  $A$  occurs and  $B$  does not occur?
- Find the probability that either  $A$  or  $B$  occurs.
- Find the probability that one of these events occurs and other does not.

	$B$	$\bar{B}$
$A$	0.25	0.12
$\bar{A}$		
	0.40	

**2.** A woman's clothing store owner buys from three companies: A, B, and C. The purchases are shown below:

Product	Company A	Company B	Company C
Dresses	24	18	12
Blouses	13	36	15

If one item is selected at random, find the following probabilities:

- It is purchased from company A or it is a dress.
  - It was purchased from company B or company C.
  - It is a blouse or was purchased from company A.
- 3.** In a statistics class there are 18 local and 10 foreign students: 6 of the foreign students are females, and 12 of the local students are males. If a student is selected at random, what is the probability that
- randomly selected student is a local or a female?
  - randomly selected student is foreign or a female student?
  - randomly selected student is a local or a foreign student?
- 4.** Two thousand randomly selected adults were asked if they think they are financially better off than their parents. The following table gives the two-way classification of the responses based on the education levels of the persons included in the survey and whether they are financially better off, the same, or worse off than their parents.

	Education level		
	Less than High school	Hight school	More than high school
Better off	140	450	420
Same	60	250	110
Worse off	200	300	70

Suppose one adult is selected at random from these 2000 adults, find the probability that this adult is

- a) financially better off his (her) parents or high school student;
- b) more than high school student or financially worse off than his (her) parents;
- c) financially better off his (her) parents or financially worse off his (her) parents;
- d) financially better off than his (her) parents
- e) financially better off than his (her) parents given that he (she) has less than high school education;
- f) financially worse off than his (her) parents given that he (she) has high school education
- g) financially the same as his (her) parents given that he (she) has more than high school eduction.
- h) Are the events "better off " and "hight school education" mutually exclusive? What about the events "less than high school" and "more than high school"? Why or why not?
- i) Are the events "worse off "and ""more than high school" independent? Why or why not?

5. Eighty students in a university cafeteria were asked if they favoured a ban on smoking in the cafeteria. The results of the survey are shown in the table.

Class	Favour	Oppose	No opinion
Freshmen	15	27	8
Sophomore	23	5	2

- If a student is selected at random, find these probabilities:
- a) He or she opposes the ban, given that the student is a freshman.
  - b) Given that the student favours the ban, the student is a sophomore.

6. The following table gives a two-way classification of 200 randomly selected purchases made at department store.

	Paid by cash/check	Paid by credit card
Male	24	46
Female	77	53

If one of these 200 purchases is selected at random, find the probability that it is

- made by a female
- paid by cash/check
- paid by credit card given that the purchase is made by a male
- made by a female given that it is paid by cash/check
- made by a female and paid by a credit card
- paid by cash/check or made by a male
- Are the events "female" and "paid by credit card" independent? Are they mutually exclusive? Explain why or why not.

7. Three cable channels (6, 8, and 10) have quiz shows, comedies, and dramas. The table gives proportions in the nine joint classifications

Type of show \ Channels	Channel 6	Channel 8	Channel 10
	Quiz show	0.21	0.11
Comedy	0.08	0.21	0.08
Drama	0.01	0.07	0.17

- What proportion of shows is quiz show?
- What proportion of shows does Channel 6 have?
- If randomly selected show is quiz show, what is the probability that it was shown on channel 6?
- If the show was shown on channel 10, what is the probability that it was comedy?
- What is the probability that randomly chosen show is drama, or shown on channel 8, or both?

8. A supermarket manager classified customers according to whether their visits to the store as frequent or infrequent and whether they often,

sometimes, or never make a purchase. The accompanying table gives the proportions of people surveyed in each of six joint classifications:

Making purchase \ Frequency of visit		Often	Sometimes	Never
Frequent	0.12	0.48	0.19	
Infrequent	0.07	0.06	0.08	

- What is the probability that a customer is both a frequent shopper and often purchases?
  - What is the probability that a customer who never makes purchase visits the store frequently?
  - Are the events "Never makes a purchase" and "Visits the store frequently" independent?
  - What is the probability that a customer who infrequently visits the store often makes a purchase?
  - Are the events "Often makes a purchase" and "Visits the store infrequently" independent?
  - What is the probability that a customer frequently visits the store?
  - What is the probability that customer never makes a purchase in this store?
  - What is the probability that a customer either frequently visits the store or never makes a purchase, or both?
9. The accompanying table shows proportions of salespeople classified according to marital status and whether or not they own stocks.

Marital status \ Own stocks	Yes	No
	0.64	0.13
Single	0.17	0.06

- What is the probability that a randomly chosen salesperson was married?
- What is the probability that a randomly chosen salesperson does not own stocks?

c) What is the probability that randomly chosen single salesperson does not own stocks?

d) What is the probability that a randomly chosen salesperson who owns stocks was married?

**10.** Forty-two percent of employees in a large corporation were in favour of a modified health care plan, and 22% of the corporation employees favoured a proposal to change the work schedule. Thirty-four percent of those favouring the health plan modification favoured the work schedule change.

a) What is the probability that a randomly selected employee is in favour of both modified health care plan and the changed work schedule?

b) What is the probability that a randomly chosen employee is in favour of at least one of these two changes?

c) What is the probability that a randomly selected employee favouring the work schedule change also favours the modified health plan?

#### Answers

**1.** b)  $P(A \cap B) = 0.12$ ; c)  $P(A \cup B) = 0.52$ ; d)  $P(A \cap B \cup A \cap B) = 0.27$ ;  
**2.** a) 67/118; b) 81/118; c) 88/118; **3.** a) 6/7; b) 4/7; c) 1; **4.** a) 0.78; b) 0.55; c) 0.79; d) 0.505; e) 0.350; f) 0.300; g) 0.183; h) "Better off" and "high school education" are not mutually exclusive. "Less than high school" and "more than high school" are mutually exclusive events; i) "Worse off" and "more than high school" are not independent events; **5.** a) 0.54; b) 0.61; **6.** a) 0.65; b) 0.51; c) 0.66; d) 0.76; e) 0.27; f) 0.74; g) no and no; **7.** a) 0.38; b) 0.3; c) 0.55; d) 0.26; e) 0.57; **8.** a) 0.12; b) 0.704; c) No; d) 0.333; e) No; f) 0.79; g) 0.27; h) 0.87; **9.** a) 0.77; b) 0.19; c) 0.2609; d) 0.7901; **10.** a) 0.1428; b) 0.4972; c) 0.6491.

## Chapter 3

### Discrete random variables and probability distributions

#### 3.1. Random variables

Suppose that experiment of rolling two fair dice to be carried out. Let  $X$  be the sum of outcomes, then  $X$  can only assume the values 2, 3, 4, ..., 12 with the following probabilities:

$$P(X = 2) = P\{(1,1)\} = \frac{1}{36}$$

$$P(X = 3) = P\{(1,2); (2,1)\} = \frac{2}{36}$$

$$P(X = 4) = P\{(1,3); (3,1); (2,2)\} = \frac{3}{36} \text{ and so on.}$$

The numerical value of random variable depends on the outcomes of the experiment. In this example, for instance, if it is (3, 2), then  $X$  is 5, and if it is (6, 6) then  $X$  is 12. In this example  $X$  is called a random variable.

#### Definition:

A random variable is a variable whose value is determined by the outcome of a random experiment.

Conventionally, we use capital letters, such as  $X$ , to denote the random variable and corresponding lowercase  $x$  to denote a possible value.

The set of possible values of a random variable might be finite, infinite and countable, or uncountable.

#### Definition:

A random variable  $X$  is called a **discrete random variable** if it can take on more than a countable number of values.

Some examples of discrete random variable:

1. The number of employees working at a company.

2. The number of heads obtained in three tosses of a coin.

3. The number of customers visiting a bank during any given day.

A random variable whose values are not countable is called a **continuous random variable**.

#### Definition:

A random variable  $X$  is called a **continuous** if it can take any value in an interval.

Here are some examples of continuous random variables:

1. Prices of houses
2. The amount of oil imported.
3. Time taken by workers to learn a job.

### 3.2. Probability distributions for Discrete Random Variables

Let  $X$  be a discrete random, and  $x$  be one of its possible values. The probability that the random variable  $X$  takes the value  $x$  is denoted by  $P(X = x)$ .

**Definition:**

The probability distribution function,  $P(x)$ , of a discrete random variable  $X$  indicates that this variable takes the value  $x$ , as a function of  $x$ . That is

$$P(x) = P(X = x), \text{ for all values of } x.$$

**Example 1:**

In the experiment of tossing a fair coin three times, let  $X$  be number of heads obtained. Determine and sketch the probability function of  $X$ .

**Solution:**

Table 3.1

First,  $X$  is a variable and the number of heads in three tosses of a coin can have any of the values 0, 1, 2, or 3.

We can make a list of the outcomes and the associated values of  $X$ . (Table 3.1).

Note that, for each basic outcome there is only one value of  $X$ . However, several basic outcomes may yield the same value. We identify the events

(i.e., the collections of the distinct values of  $X$ ). (Table 3.2)

Outcome	Value of $X$
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

Table 3.2.

Numerical value of $X$ as an event	Composition of the event
$[X=0]$	{TTT}
$[X=1]$	{HTT, THT, TTH}
$[X=2]$	{HHT, HTH, THH}
$[X=3]$	{HHH}

The model of a fair coin entails that 8 basic outcomes are equally likely, so each is assigned the probability  $1/8$ .

The event  $[X=0]$  has a single outcome TTT, so its probability is  $1/8$ .

Similarly, the probabilities of  $[X=1]$ ,  $[X=2]$ , and  $[X=3]$  are found to be  $3/8$ ,  $3/8$ , and  $1/8$ , respectively. Collecting these results, we obtain the probability distribution of  $X$  shown in table 3.3.

Table 3.3. The probability distribution of  $X$ , the number of heads in 3 tosses of a coin.

Value of $X$	Probability
0	$1/8$
1	$3/8$
2	$3/8$
3	$1/8$
Total	1

**Remark:** When summed over all possible values of  $X$ , these probabilities must add up to 1.

The graphical representation of the probability of  $X$ , the number of heads in three tosses of a coin is shown in Fig. 3.1.

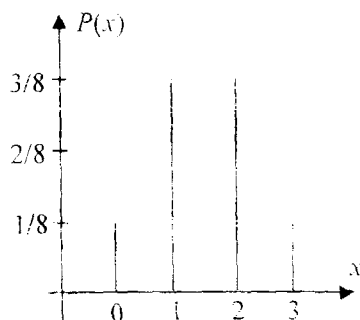


Fig. 3.1 Probability function for example 1.

In the development of the probability distribution for a discrete random variable, the following two conditions must always be satisfied:

**Properties of probability function of discrete random variables:**

Let  $X$  be a discrete random variable with probability  $P(x)$ . Then

1.  $P(x) \geq 0$  for any value  $x$ .
2. The individual probabilities sum to 1; that is  $\sum_x P(x) = 1$ , where the notation  $\sum_x$  indicates summation over all possible values of  $x$ .

Another representation of discrete probability distribution is also useful.

**Cumulative probability function  $F(x_0)$ :**

The cumulative probability function,  $F(x_0)$  of a random variable  $X$  expresses the probability that  $X$  does not exceed the value  $x_0$  as a function of  $x_0$ . That is

$$F(x_0) = P(X \leq x_0),$$

where the function is evaluated of all values  $x_0$ .

$$F(x_0) = \sum_{x \leq x_0} P(x)$$

**Properties of cumulative probability functions for discrete random variables:**

Let  $X$  be a discrete random variable with cumulative probability function  $F(x_0)$ . Then we can show that

1.  $0 \leq F(x_0) \leq 1$  for every number  $x_0$ .
2. If  $x_1$  and  $x_2$  are two numbers with  $x_1 < x_2$ , then  $F(x_1) < F(x_2)$ .

**Example 2:**

In the experiment of rolling a balanced die twice, let  $X$  be the minimum of the two numbers obtained. Determine and sketch the probability function and cumulative probability function of  $X$ .

**Solution:**

The possible values of  $X$  are 1, 2, 3, 4, 5, and 6. The sample space of this experiment consists of 36 basic outcomes. Hence the probability of any of them is  $1/36$ . In our experiment

$$P(X=1) = P\{(1,1)(1,2)(2,1)(1,3)(3,1)(1,4)(4,1)(1,5)(5,1)(1,6)(6,1)\} = 11/36$$

$$P(X=2) = P\{(2,2)(2,3)(3,2)(2,4)(4,2)(2,5)(5,2)(2,6)(6,2)\} = 9/36$$

$$P(X=3) = P\{(3,3)(3,4)(4,3)(3,5)(5,3)(3,6)(6,3)\} = 7/36$$

$$P(X=4) = P\{(4,4)(4,5)(5,4)(4,6)(6,4)\} = 5/36$$

$$P(X=5) = P\{(5,5)(5,6)(6,5)\} = 3/36$$

$$P(X=6) = P\{(6,6)\} = 1/36$$

The graphical representation of  $P(x)$  is shown in Fig.3.2.

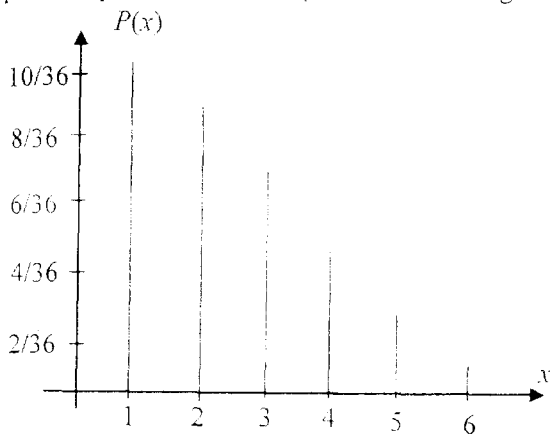


Fig. 3.2. Probability function for example 2.

Now let us form cumulative probability function.

If  $x_0$  is some number less than 1,  $X$  can not be less than  $x_0$ , so

$$F(x_0) = P(X \leq x_0) = 0 \text{ for all } x_0 < 1$$

If  $x_0$  is greater than or equal to 1 but strictly less than 2, the only one

number less than 2, the only way for  $X$  to be less than or equal to  $x_0$  is if

$X=1$ . Hence

$$F(x_0) = P(X \leq x_0) = P(1) = 11/36 \text{ for all } 1 \leq x_0 < 2$$

If  $x_0$  is greater than or equal to 2 but strictly less than 3,  $X$  is less than or

equal to the  $x_0$  if and only if either  $X=1$  or  $X=2$ , so

$$F(x_0) = P(X \leq x_0) = P(1) + P(2) = 20/36 \text{ for } 2 \leq x_0 < 3.$$

Continuing in this way we can write cumulative probability function as

$$F(x_0) = \begin{cases} 0 & \text{if } x_0 < 1 \\ 11/36 & \text{if } 1 \leq x_0 < 2 \\ 20/36 & \text{if } 2 \leq x_0 < 3 \\ 27/36 & \text{if } 3 \leq x_0 < 4 \\ 32/36 & \text{if } 4 \leq x_0 < 5 \\ 35/36 & \text{if } 5 \leq x_0 < 6 \\ 1 & \text{if } x_0 \geq 6 \end{cases}$$

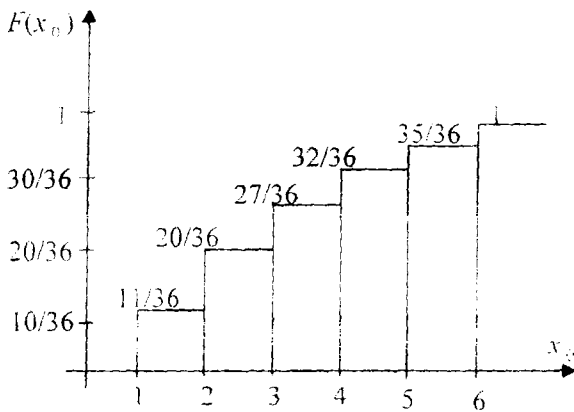


Fig. 3.3. Cumulative probability function for example 2.

The cumulative distribution function of  $X$ ,  $F(x_0)$ , is plotted in Fig. 3.3. It can be seen that the cumulative probability function increases in steps until the sum is 1.

**Example 3:**

A consumer agency surveyed all 2500 families living in a small town to collect data on the number of TV sets owned by them. The following table lists the frequency distribution of the data collected by this agency

Number of TV sets owned	0	1	2	3	4
Number of families	120	970	730	410	270

- Construct a probability distribution table. Draw a graph of the probability distribution.
- Calculate and draw the cumulative probability function.

c) Find the probabilities:  $P(X=1)$ ,  $P(X>1)$ ,  $P(X \leq 1)$ ,  $P(1 \leq X \leq 3)$ .

**Solution:**

a) In a chapter 2 we learned that the relative frequencies obtained from an experiment or a sample can be used as approximate probabilities. Using the relative frequencies, we can write the probability distribution on the discrete random variable  $X$  in the following table.

Number of TV sets owned, $x$	Probability $P(x)$
0	$120/2500=0.048$
1	$970/2500=0.388$
2	$730/2500=0.292$
3	$410/2500=0.164$
4	$270/2500=0.108$

Figure 3.4 shows the graphical presentation of the probability distribution.

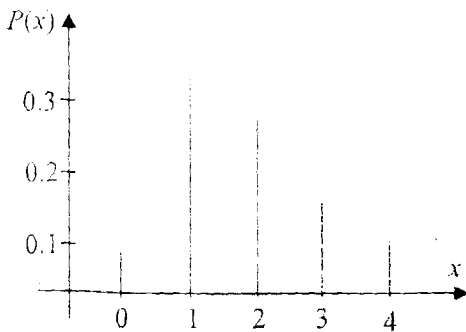


Figure 3.4

b) Let us form cumulative probability distribution function.

If  $x_0$  is less than 0, then

$$F(x_0) = P(X \leq x_0) = 0 \text{ for } x_0 < 0$$

If  $x_0$  is less than 1, then

$$F(x_0) = P(X \leq x_0) = P(0) = 0.048 \text{ for } 0 \leq x_0 < 1$$

Continuing in this way, we obtain

$$F(x_0) = \begin{cases} 0 & \text{if } x_0 < 0 \\ 0.048 & \text{if } 0 \leq x_0 < 1 \\ 0.436 & \text{if } 1 \leq x_0 < 2 \\ 0.728 & \text{if } 2 \leq x_0 < 3 \\ 0.892 & \text{if } 3 \leq x_0 < 4 \\ 1 & \text{if } x_0 \geq 4 \end{cases}$$

This function is plotted in Fig. 3.5.

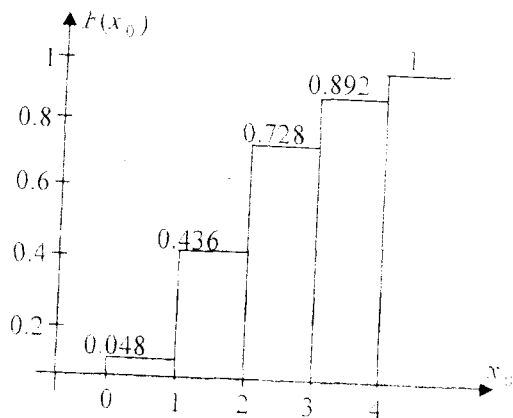


Fig. 3.5. Cumulative probability function for example 3.

- e)  $P(X = 1) = 0.388$   
 $P(X > 1) = 1 - F(1) = 1 - 0.436 = 0.564$   
 $P(X \leq 1) = F(1) = 0.436$   
 $P(1 \leq X \leq 3) = F(3) - F(0) = 0.892 - 0.048 = 0.844.$

### Exercises

1. Each of the following tables lists certain values of  $X$  and their probabilities. Determine if each of them satisfies the two conditions required for a valid probability distribution.

a)

$X$	$P(x)$
5	-0.39
6	0.67
7	0.31
8	0.28

c)

$X$	$P(x)$
0	0.16
1	0.00
2	0.43
3	0.41

b)

$X$	$P(x)$
2	0.22
3	0.23
5	0.65

For each case, list the values of  $x$  and  $P(x)$  and examine if the specification represents a probability distribution. If does not, state what properties are violated

a)  $P(x) = \frac{1}{10}(x-2)$  for  $x = 3, 4, 5, 6$

b)  $P(x) = \frac{1}{2}(x-2)$  for  $x = 1, 2, 3, 4$

c)  $P(x) = \frac{1}{20}(2x+4)$  for  $x = -2, -1, 0, 1, 2$

d)  $P(x) = 3/2^x$  for  $x = 2, 3, 4, 5$

3. The following table gives the probability distribution of a discrete random variable  $X$ .

$X$	0	1	2	3	4	5
$P(x)$	0.03	0.13	0.22	0.31	0.19	0.12

a) Draw the probability function.

b) Calculate and draw the cumulative probability function.

c) Find  $P(X=1)$ ;  $P(X \leq 1)$ ;  $P(X > 3)$ ;  $P(0 < X \leq 2)$

d) Find the probability that  $x$  assumes a value less than 3

e) Find the probability that  $x$  assumes a value in the interval 2 to 4.

4. Despite all safety measures, accidents do happen at the factory. Let  $X$  denote the number of accidents that occur during a month at this factory. The following table lists the probability distribution of  $X$ .

$X$	0	1	2	3	4
$P(x)$	0.25	0.30	0.20	0.15	0.10

- Draw the probability function.
- Calculate and draw the cumulative probability function.
- Determine the probability that the number of accidents that will occur during a given month at this company is exactly 4.
- What is the probability that number of accidents will be at least 2?
- What is the probability that number of accidents will be less than 3?
- What is the probability that number of accidents will be between 2 to 4?
- Two month are chosen at random. What is the probability that on both of these months there will be fewer than two accidents?

5. Let  $X$  be the number of shopping trips made by family during a month. The following table lists the frequency distribution of  $X$  of 1000 families.

$X$	4	5	6	7	8	9	10
$f$	70	180	240	210	170	90	40

- Draw the probability function.
- Calculate and draw the cumulative probability function.
- Find the following probabilities  $P(X = 5)$ ;  $P(X > 6)$ ;  $P(4 \leq X \leq 7)$ ;  $P(X \leq 6)$ .

6. The following table lists the probability distribution of the number of phone calls received per 10-minute period at an office.

Number of phone calls	0	1	2	3	4
Probability	0.12	0.26	0.34	0.18	0.10

Let  $X$  denote the number of phone calls received during a certain 10-minute period at this office.

- Find the probabilities:  $P(X = 1)$ ;  $P(X < 2)$ ;  $P(X > 2)$ ;  $P(1 \leq X \leq 3)$ .

b) Two 10-minute period are chosen at random. What is the probability that at least one of them there will be at least one received call?

7. In successive rolls of a fair die, let  $X$  be the number of rolls until the first 6. Determine the probability function.

8. In a tennis championship, player A competes against player B in consecutive sets and the game continues until one player wins three sets. Assume that, for each set  $P(A \text{ wins}) = 0.4$ ,  $P(B \text{ wins}) = 0.6$ , and the outcomes of different sets are independent. Let  $X$  stand for the number of sets played.

- List the possible values of  $X$  and identify the basic outcomes associated with each value.
- Obtain the probability distribution of  $X$ .

### Answers

1. a) this is not a valid probability distribution; b) this is not a valid probability distribution; c) this is a valid probability distribution;

2. a) this is a valid probability distribution; b) this is not a valid probability distribution; c) this is a valid probability distribution;

3. this is not a valid probability distribution; 3. c) 0.13; 0.16; 0.62; 0.38; d) 0.38; e) 0.72; 4. c) 0.10; d) 0.45; e) 0.75; f) 0.45; g) 0.3025; 5. c) 0.18; 0.51; 0.70; 0.49; 6. a) 0.26; 0.38; 0.28; 0.78; b) 0.9856;

7.  $P(x) = \left(\frac{5}{6}\right)^{x-1} \cdot \frac{1}{6}; x \geq 1$ ; 8. b)  $P(3) = 0.2800$ ;  $P(4) = 0.3744$ ;

$P(5) = 0.3456$ ;

### 3.3. Expected (mean) value and variance for discrete random variables

#### 3.3.1. Expected value

Once we have constructed the probability distribution for a random variable, we often want to compute the mean or expected value of the random variable. The mean of discrete random variable  $X$ , denoted either  $\mu_X$  or  $E(X)$ , is actually the mean of its probability distribution. The mean (or expected) value of a discrete random variable is the value that we expect to observe per repetition, on average, if we perform an experiment a large number of times. For example, we may expect a house salesperson to sell on average, 3.50 houses per month. It does not mean that every month the salesperson will sell exactly 3.50 houses. (Actually he (or she) can not sell exactly 3.50 houses). This simply means that if we observe for many months, this salesperson will sell a different number of houses different months. However, the average of all sold houses in these months will be 3.50.

#### **Definition:**

The mean (or expected value) of discrete random variable  $X$  is defined as

$$\mu_X = E(X) = \sum (\text{value} \cdot \text{probability}) = \sum x \cdot P(x)$$

Here the sum extends over all distinct values  $x$  of  $X$ .

In order to compute the expected value of a discrete random variable we must multiply each value of the random variable by the corresponding value of its probability function. We then add the resulting terms.

#### **Example:**

Sales show that five is the maximum number of cars sold on a given day at car selling company. Table 3.4 shows probability distribution of cars sold per day. Find the expected number of cars sold.

#### **Solution:**

To find the expected number (or mean) of cars sold, we multiply each value of  $x$  by its probability and add these results.

Table 3.4

$x$	$P(x)$
0	0.18
1	0.39
2	0.24
3	0.14
4	0.04
5	0.01

$x$	$P(x)$	$x \cdot P(x)$
0	0.18	0.00
1	0.39	0.39
2	0.24	0.48
3	0.14	0.42
4	0.04	0.16
5	0.01	0.05

$$\mu_x = E(X) = \sum_x x \cdot P(x) = 1.50$$

In fact, it is impossible for company to sell exactly 1.50 cars in any given day. But we examine selling cars at this company for many days into the future, and see that, the expected value of 1.50 cars provides a good estimate of the mean or average daily sales volume. The expected value can be important to the managers from both planning and decision making points of view.

For example, suppose that this company will be open 40 days during next 2-month. How many cars should the owner expect to be sold during this time?

While we can not specify the exact value of 1.50 cars, it provides an expected sale of  $40 \cdot 1.50 = 60$  cars for the next 2-month period.

### 3.3.2. Variance and standard deviation of discrete random variable

While the expected value gives us an idea of the average or central value for the random variable, often we would also like to measure the dispersion or variability of the possible values of the random variable. The variance of discrete random variable  $X$ , denoted by  $\sigma_x^2$ , measures the spread of its probability distribution. In defining the variance of a random variable, a weighted average of the squares of its possible discrepancies about the means is formed; the weight associated with  $(x - \mu_x)^2$  is the probability that the random variable takes the value  $x$ . The variance can be viewed as the average value that will be taken by the function  $(X - \mu_x)^2$  over a very large number of repeated trials.

The mathematical expression for the variance of a discrete random variable is

$$\sigma_X^2 = E[(X - \mu_X)^2] = \sum_x (x - \mu_X)^2 P(x).$$

The standard deviation,  $\sigma_X$ , is the positive square root of the variance. In some particular cases, an alternative but equivalent (sometimes called shortcut formula) formula for the variance can be used:

$$\sigma_X^2 = E(X^2) - \mu_X^2 = \sum_x x^2 P(x) - \mu_X^2$$

**Example:**

Find the variance for the example in previous topic, for the number of cars sold per day at a car selling company.

**Solution:**

Let us apply  $\sigma_X^2 = E[(X - \mu_X)^2] = \sum_x (x - \mu_X)^2 P(x)$ .

The calculations are shown in the table 3.5:

$x$	$(x - \mu)$	$(x - \mu)^2$	$P(x)$	$(x - \mu)^2 \cdot P(x)$
0	0-1.50=-1.50	2.25	0.18	2.25·0.18=0.4050
1	1-1.50=-0.50	0.25	0.39	0.25·0.39=0.0975
2	2-1.50=0.50	0.25	0.24	0.25·0.24=0.0600
3	3-1.50=1.50	2.25	0.14	2.25·0.14=0.3150
4	4-1.50=2.50	6.25	0.04	6.25·0.04=0.2500
5	5-1.50=3.50	12.25	0.01	12.25·0.01=0.1225
				$\sum (x - \mu)^2 \cdot P(x) = 1.25$

We see that the variance for the number of cars sold per day is 1.25. The standard deviation of the number of cars sold per day is

$$\sigma = \sqrt{1.25} = 1.118.$$

**Remark:**

For the purpose of easier managerial interpretation the standard deviation may be preferred over the variance because it is measured in the same units as the random variable.

**Example:**

The following table gives the probability distribution of  $X$ .

	0	1	2	3	4	5
$P(x)$	0.02	0.20	0.30	0.30	0.10	0.08

Compute the standard deviation of  $x$ .

**Solution:**

Let us apply equivalent (shortcut formula) formula for the variance

$$\sigma_x^2 = \sum x^2 P(x) - \mu_x^2$$

The following table shows all the calculations required for the computation of the standard deviation of  $x$ .

$x$	$P(x)$	$x P(x)$	$x^2$	$x^2 \cdot P(x)$
0	0.02	0.00	0	0.00
1	0.20	0.20	1	0.20
2	0.30	0.60	4	1.20
3	0.30	0.90	9	2.70
4	0.10	0.40	16	1.60
5	0.08	0.40	25	2.00
		$\sum xP(x) = 2.50$		$\sum x^2 P(x) = 7.70$

We perform the following steps to compute the standard deviation by shortcut formula:

**Step 1:** Compute the mean of discrete random variable:

$$\mu = \sum xP(x) = 2.50$$

**Step 2:** Compute the value of  $\sum x^2 P(x)$ .

**Step 3:** Substitute the values of  $\mu$  and  $\sum x^2 P(x)$  in the shortcut formula for the variance

$$\sigma_x^2 = \sum x^2 P(x) - \mu_x^2 = 7.70 - (2.50)^2 = 1.45$$

**Step 4:** Take positive square root of variance.

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{1.45} = 1.20.$$

**Example:**

A farmer will earn a profit of \$30 thousand in case of heavy rain next year, \$60 thousand in case of a moderate rain, and \$15 thousand in case of little rain. A meteorologist forecasts that the probability is 0.35 for heavy rain, 0.40 for moderate rain, and 0.25 for little rain next year. Let  $X$  be the random variable that represents next year's profit in thousands of dollars for this farmer. Write the probability distribution of  $x$ . Find the mean and standard deviation of  $x$ . Give a brief interpretation of the values of the mean and standard deviation.

**Solution:**

The table 3.6 lists the probability distribution of  $x$

Table 3.6

$x$	$P(x)$
30	0.35
60	0.40
15	0.25

The table 3.7 shows all calculations needed for the computation of the mean and standard deviation.

Table 3.7

$x$	$P(x)$	$xP(x)$	$x^2$	$x^2 \cdot P(x)$
30	0.35	10.5	900	315
60	0.40	24	3600	1440
15	0.25	3.75	225	56.25
		$\sum xP(x) = 38.25$		$\sum x^2 P(x) = 1811.25$

The mean of  $x$  is  $\mu_x = \sum xP(x) = \$38.25$  thousand. The standard deviation is  $\sigma_x = \sqrt{\sum x^2 P(x) - \mu_x^2} = \sqrt{1811.25 - (38.25)^2} = \$18.660$ .

Thus, it is expected that a farmer will earn an average of \$38.25 thousand profits in next year with a standard deviation of \$18.660 thousand.

**3.3.3. Mean and variance of linear function of a random variable**

Let  $X$  be a random variable that takes the value  $x$  with probability  $P(x)$  and consider a new random variable  $Y$ , defined by  $Y = a + bX$ .

Suppose that random variable  $X$  has mean  $\mu_X$  and variance  $\sigma_X^2$ .

Then mean and variance of  $Y$  are

$$\mu_Y = E(a + bX) = a + b \cdot \mu_X \quad \text{and}$$

$$\sigma_Y^2 = \text{Var}(a + bX) = b^2 \sigma_X^2 \quad \text{so that standard deviation of } Y \text{ is}$$

$$\sigma_Y = |b| \cdot \sigma_X.$$

**Example:**

A car salesman estimates the following probabilities for the number of cars that he will sell in next month.

Number cars	0	1	2	3	4
Probability	0.12	0.20	0.25	0.25	0.18

- Find the expected number of cars that will be sold in the next month.
- Find the standard deviation of the number of cars that will be sold in next month.
- The salesperson receives for the month a salary of \$300, plus an additional \$200 for each car sold. Find the mean and standard deviation of his total monthly salary.

**Solution:**

a) The random variable  $X$  has mean

$$\begin{aligned} \mu_X &= E(X) = \sum xP(x) = \\ &= 0 \cdot 0.12 + 1 \cdot 0.20 + 2 \cdot 0.25 + 3 \cdot 0.25 + 4 \cdot 0.18 = 2.17. \end{aligned}$$

b) Variance  $\sigma_X^2 = \sum (x - \mu_X)^2 \cdot P(x) =$

$$\begin{aligned} &= 0 - 2.17)^2 (0.12) + (1 - 2.17)^2 (0.20) + (2 - 2.17)^2 (0.25) + (3 - 2.17)^2 (0.25) \\ &+ (4 - 2.17)^2 (0.18) = 1.621 \end{aligned}$$

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{1.621} = 1.273.$$

c) Total monthly salary of salesperson can be written as  $Y = 300 + 200X$ .

Then

$$\mu_Y = E(Y) = E(300 + 200X) = 300 + 200 \cdot \mu_X = 300 + 200 \cdot 2.17 = \$734.$$

$$\text{Var}(\mu_Y) = \text{Var}(300 + 200X) = 200^2 \cdot 1.621 = 64840.$$

$$\sigma_Y = \$254.64.$$

**Summary results for the mean and variance of special linear functions:**

a) Let  $b=0$  in the linear function,  $Y = a + bX$ . Then  $Y = a$  for any constant  $a$ .  
 $E(a) = a$  and  $\text{Var}(a) = 0$

If a random variable always takes the value  $a$ , it will have a mean  $a$  and a variance 0.

b) Let  $a=0$  in the linear function,  $Y = a + bX$ . Then  $Y = bX$ .  
 $E(bx) = b \cdot \mu_x$  and  $\text{var}(bx) = b^2 \sigma_x^2$ .

**Exercises**

**1.** Find the mean and standard deviation for each of the following probability distributions.

a)

$X$	$P(x)$
0	0.12
1	0.27
2	0.43
3	0.18

b)

$X$	$P(x)$
6	0.36
7	0.26
8	0.21
9	0.17

**2.** Given the following probability distribution.

Find  $E(X), \sigma^2, \sigma$

$X$	$P(x)$
0	0.4
1	0.3
2	0.2
3	0.1

**3.**

Let  $x$  be the number of errors that a randomly selected page of a book contains. The following table lists the probability distribution of  $x$ .

$X$	0	1	2	3	4
$P(x)$	0.73	0.16	0.06	0.04	0.01

Find the mean and standard deviation.

4. Suppose the probability function of a random variable  $X$  is given by the

formula  $P(x) = \frac{60}{77} \cdot \frac{1}{x}$  for  $x=2, 3, 4, 5$

Calculate the mean and standard deviation of this distribution.

5. Given the two probability distributions

$X$	$P(x)$	$X$	$P(x)$
1	0.2	0	0.1
2	0.6	1	0.2
3	0.2	2	0.4
		3	0.2
		4	0.1

a) Verify that both distributions have the same mean.

b) Compare the two standard deviations.

6. An instant lottery ticket costs \$2. Out of a total of 10 000 tickets printed for this lottery, 1000 tickets contain a prize of \$5 each, 100 tickets have a prize of \$10 each, 5 tickets have a prize of \$1000 each, and 1 ticket has a prize of \$5000. Let  $x$  be the random variable that denotes the net amount a player wins by playing this lottery. Write the probability distribution of  $x$ . Determine the mean and standard deviation of  $x$ . How will you interpret the values of the mean and standard deviation of  $x$ ?

7. A TV repairer estimates the probabilities for the number of hours required to complete some job as follows:

Time taken (Hours)	1	2	3	4	5
Probability	0.05	0.2	0.35	0.3	0.1

a) Find the expected time to complete the job.

b) The TV repairer's service is made up of two parts- a fixed cost of \$20, plus \$2 for each hour taken to complete the job. Find the mean and standard deviation of total cost.

**8.** Consider the following probability distribution for the random variable  $X$ .

$X$	$P(x)$
10	0.20
20	0.40
30	0.25
40	0.15

- a) Find the expected value of  $X$ .  
 b) Find the variance and standard deviation.  
 c) If  $Y = 3X + 5$ , find the expected value, variance, and standard deviation for  $Y$ .

**Answers**

- 1.** a) 1.67; 0.906; b) 7.19; 1.102; **2.** 1; 1; 1; **3.** 0.44; 0.852; **4.** 3.12; 1.09;  
**5.** a)  $\mu_1 = \mu_2 = 2$ ; b)  $\sigma_1 = 0.63$ ;  $\sigma_2 = 1.1$ ; **6.** 1.6; 54.78; **7.** a) 3.2;  
 b) 26.4; 2.06; **8.** a) 23.5; b) 92.75 and 9.63; c) 75.5; 834.75; 28.89.

**3.4. Jointly distributed discrete random variable**

Although the probability distributions studied so far have involved only one random variable, many decisions are based upon an analysis of two or more random variables. In problem situations that involve two or more random variables, the resulting probability distribution is referred to as a **joint probability distribution**.

**Example:**

The number of between-meal snacks eaten by students in a day during final examinations week depends on the number of tests a student had to take on that day. The accompanying table shows joint probabilities, estimated from a survey.

Table 3.8

Number of snacks ( $Y$ )	Number of tests ( $X$ )			$P(y)$
	0	1	2	
0	0.05	0.08	0.09	0.22
1	0.07	0.09	0.11	0.27
2	0.11	0.04	0.10	0.25
3	0.08	0.07	0.11	0.26
$P(x)$	0.31	0.28	0.41	1.00

**Definition:**

Let  $X$  and  $Y$  be a pair of discrete random variables. Their joint probability function expresses the probability that simultaneously  $X$  takes the specific value  $x$  and  $Y$  takes the value  $y$ , as a function of  $x$  and  $y$ .

The notation used is  $P_{X,Y}(x,y)$  so,

$$P_{X,Y}(x,y) = P(X=x \cap Y=y)$$

For example,  $P_{X,Y}(2,3) = 0.11$ . It means that, the probability that randomly chosen student has 2 tests and eats 3 snacks is 0.11.

**Definition:**

Let  $X$  and  $Y$  be a pair of jointly distributed random variables. The probability function of the random variable  $X$  is called its **marginal** probability function, denoted by  $P_X(x)$ , and is obtained by summing the joint probabilities over all possible values; that is

$$P_X(x) = \sum_y P(x,y).$$

Similarly, the **marginal** probability function of the random variable  $Y$  is

$$P_Y(y) = \sum_x P(x,y).$$

Marginal probability functions  $P_X(x)$  and  $P_Y(y)$  are shown in the lower row and the right column of the table 3.8.

For example,  $P_X(x=0) = 0.31$  expresses the probability that, randomly chosen student has no tests is 0.31.

$P_Y(y=2) = 0.25$ , expresses the probability that randomly chosen student eats 2 snacks is 0.25.

**Properties of joint probability functions of discrete random variables**

Let  $X$  and  $Y$  be a discrete random variables with joint probability function  $P_{X,Y}(x,y)$ . Then

1.  $P_{X,Y}(x,y) \geq 0$  for any pairs of  $x$  and  $y$ .
2. The sum of the joint probabilities  $P_{X,Y}(x,y)$  over all possible pairs of values must be 1.

### 3.4.1. Conditional probability function

Let  $X$  and  $Y$  be a pair of jointly distributed discrete random variables. The **conditional** probability function of the random variable  $Y$ , given that the random variable  $X$  takes the value  $x$ , expresses the probability that  $Y$  takes the value  $y$ , as a function of  $y$ , when the value  $x$  is specified for  $X$ .

This is denoted by  $P_{Y/X}(y/x)$ , and defined as

$$P_{Y/X}(y/x) = \frac{P_{X,Y}(x,y)}{P_X(x)}$$

Similarly, the conditional probability function of  $X$ , given  $Y=y$  is

$$P_{X/Y}(x/y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

For example, using the table 3.8, we can compute the conditional probability of  $y=2$ , given that  $x=1$  as

$$P_{Y/X}(2/1) = \frac{P_{X,Y}(2,1)}{P_X(1)} = \frac{0.04}{0.28} = \frac{1}{7}$$

It means, the probability that randomly chosen student who has 1 test eats 2 snacks is  $1/7$ .

The probability of  $x=2$ , given that  $y=3$  is

$$P_{X/Y}(2/3) = \frac{P_{X,Y}(2,3)}{P_Y(3)} = \frac{0.11}{0.26} = \frac{11}{26}$$

It means, the probability that randomly chosen student who eats 3 snacks has 2 tests is  $11/26$ .

### 3.4.2. Independence of jointly distributed random variables

#### Definition:

Let  $X$  and  $Y$  be a pair of jointly distributed discrete random variables. They are said to be **independent** if and only if their joint probability function is the product of their marginal probability functions:

$$P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)$$

for all possible pairs of values  $x$  and  $y$ . Otherwise they are said to be dependent.

As an example, from table 3.8, let  $x=1$ ,  $y=2$ .

Then

$$P_{X,Y}(x,y) = P(1,2) = 0.04; \quad P_X(1) = 0.28; \quad P_Y(2) = 0.25.$$

$$0.04 \neq 0.28 \cdot 0.25.$$

so number of eaten snacks and number of tests are not independent.

### 3.3. Expected value of the function of jointly distributed random variables

Let  $X$  and  $Y$  be a pair of discrete random variables with probability function  $P_{X,Y}(x,y)$ .

The mean of random variable  $X$  is

$$\mu_X = E(X) = \sum_x x \cdot P(x)$$

The mean of random variable  $Y$  is

$$\mu_Y = E(Y) = \sum_y y \cdot P(y)$$

The mean, or expectation of any function  $g(X,Y)$  of the random variables  $X$  and  $Y$  is defined as:

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) \cdot P(x,y).$$

As an example let us calculate means of  $X$ ,  $Y$ , and  $g(X,Y)$  for the Example above.

The mean of  $X$  is:

$$E(X) = \sum_x x \cdot P(x) = 0 \cdot 0.31 + 1 \cdot 0.28 + 2 \cdot 0.41 = 0.28 + 0.82 = 1.1.$$

Means, on average we expect that each student eats 1.1 snacks per day during final examination week.

The mean of  $Y$  is:

$$\mu_Y = E(Y) = \sum_y y \cdot P(y) =$$

$$0.22 + 1 \cdot 0.27 + 2 \cdot 0.25 + 3 \cdot 0.26 = 0.27 + 0.5 + 0.78 = 1.55$$

Means, on average, we expect that each student has 1.55 tests per day during final examination week.

### 3.4.4. Covariance

Suppose that  $X$  and  $Y$  are pair of random variables and they are dependent. We use covariance to measure the nature and strength of the relationship between them.

**Definition:**

Let  $X$  be a random variable with mean  $\mu_X$ , and let  $Y$  be a random variable with mean  $\mu_Y$ . The expected value of  $(X - \mu_X)(Y - \mu_Y)$  is called the covariance between  $X$  and  $Y$ , denoted  $Cov(X, Y)$ , defined as

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = \sum_x \sum_y (x - \mu_X)(y - \mu_Y) \cdot P(x, y).$$

An equivalent expression for  $Cov(X, Y)$  is:

$$Cov(X, Y) = E(XY) - \mu_X \cdot \mu_Y = \sum_x \sum_y x \cdot y \cdot P(x, y) - \mu_X \cdot \mu_Y.$$

If  $Cov(X, Y)$  is a positive, then there is a positive linear association between  $X$  and  $Y$ , if  $Cov(X, Y)$  is a negative value, then there is a negative linear association between  $X$  and  $Y$ . An expectation of 0 for  $Cov(X, Y)$  would imply an absence of linear association between  $X$  and  $Y$ .

Let us calculate  $Cov(X, Y)$  for probability distribution shown in the table 3.8.

Using an equivalent expression for  $Cov(X, Y)$  yields:

$$\begin{aligned} Cov(X, Y) &= \sum_x \sum_y x \cdot y \cdot P(x, y) - \mu_X \cdot \mu_Y \\ &= \sum_x \sum_y x \cdot y \cdot P(x, y) = 0 \cdot 0 \cdot 0.05 + 0 \cdot 1 \cdot 0.08 + 0 \cdot 2 \cdot 0.09 + \\ &+ 1 \cdot 0 \cdot 0.07 + 1 \cdot 1 \cdot 0.09 + 1 \cdot 2 \cdot 0.11 + 2 \cdot 0 \cdot 0.11 + 2 \cdot 1 \cdot 0.04 + 2 \cdot 2 \cdot 0.10 + \\ &+ 3 \cdot 0 \cdot 0.08 + 3 \cdot 1 \cdot 0.07 + 3 \cdot 2 \cdot 0.11 = 0.09 + 0.22 + 0.08 + \\ &+ 0.40 + 0.21 + 0.66 = 1.66 \\ Cov(X, Y) &= \sum_x \sum_y x \cdot y \cdot P(x, y) - \mu_X \cdot \mu_Y = \\ &= 1.66 - 1.1 \cdot 1.55 = 1.66 - 1.705 = -0.045 \end{aligned}$$

It means that there is a weak negative association between number of tests taken a day during a final examination week and number of eaten snacks.

### Exercises

1. Shown below is the joint probability distribution for two random variables  $X$  and  $Y$ .

$X$	$Y$		$\Sigma$
	5	10	
10	0.12	0.08	0.20
20	0.30	0.20	0.50
30	0.18	0.12	0.30
$\Sigma$	0.60	0.40	1.00

- a) Find  $P_{X,Y}(10,10)$ ,  $P_{X,Y}(30,5)$ , and  $P_{X,Y}(20,5)$ .
- b) Specify the marginal probability distributions for  $X$  and  $Y$ .
- c) Compute the mean and variance for  $X$  and  $Y$ .
- d) Are  $X$  and  $Y$  independent random variables? Justify your answer.

2. There is a relationship between the number of lines in a newspaper advertisement for an apartment and the volume of interest from the potential renters. Let volume of interest be denoted by the random variable  $X$ , with the value 0 for little interest, 1 for moderate interest, and 2 for heavy interest. Let  $Y$  be the number of lines in a newspaper. Their joint probabilities are shown in the table

Number of lines ( $Y$ )	Volume of interest ( $X$ )		
	0	1	2
3	0.09	0.14	0.07
4	0.07	0.23	0.16
5	0.03	0.10	0.11

- a) Find and interpret  $P_{X,Y}(2,4)$ .
- b) Find the joint cumulative probability function at  $X=2$ ,  $Y=4$ , and interpret your result.
- c) Find and interpret the conditional probability function for  $Y$ , given  $X=0$ .
- d) Find and interpret the conditional probability function for  $X$ , given  $Y=4$ .

- e) If the randomly selected advertisement contains 5 lines, what is the probability that it has heavy interest from the potential renters?
- f) Find expected number of volume of interest.
- g) Find and interpret covariance between  $X$  and  $Y$ .
- h) Are the number of lines in the advertisement and volume of interest independent of one another?

**3.** Students at a university were classified according to the years at the university ( $X$ ) and number of visits to a museum in the last year. ( $Y=0$  for no visits, 1 for one visit, 2 for two visits, 3 for more than two visits). The accompanying table shows joint probabilities.

Number of visits ( $Y$ )	Years at the university ( $X$ )			
	1	2	3	4
0	0.06	0.08	0.07	0.02
1	0.08	0.07	0.06	0.01
2	0.05	0.05	0.12	0.02
3	0.03	0.06	0.18	0.04

- a) Find and interpret  $P_{X,Y}(4,3)$
- b) Find and interpret the mean number of  $X$ .
- c) Find and interpret the mean number of  $Y$ .
- d) If the randomly selected student is a 2<sup>nd</sup> year student, what is the probability that he or she visits museum at least 3 times?
- e) If the randomly selected student has 1 visit to a museum, what is the probability that he (or she) is a 3<sup>rd</sup> year student?
- f) Are number of visits to a museum and years at the university independent of each other?

**4.** It was found that 20% of all people both watched the show regularly and could correctly identify the advertised product. Also, 27% of all people regularly watched the show and 53% of all people could correctly identify the advertised product. Define a pair of random variables as follows:

$$\begin{aligned}
 X &= 1 \text{ if regularly watch the show;} & X &= 0 \text{ otherwise} \\
 Y &= 1 \text{ if product correctly identified;} & Y &= 0 \text{ otherwise.}
 \end{aligned}$$

- a) Find the joint probability function of  $X$  and  $Y$ .
- b) Find the conditional probability function of  $Y$ , given  $X=0$ .
- c) If randomly selected person could identify the product correctly, what is the probability that he (or she) regularly watch the show?
- d) Find and interpret the covariance between  $X$  and  $Y$ .

## Answers

1. a) 0.08; 0.18; 0.30; b)  $P_X(10) = 0.20$ ;  $P_X(20) = 0.50$ ;  $P_X(30) = 0.30$ ;  
 $P_Y(5) = 0.60$ ;  $P_Y(10) = 0.40$ ; c)  $\mu_X = 21$ ;  $\sigma_X^2 = 49$ ;  $\mu_Y = 7$ ;  $\sigma_Y^2 = 6$ ; d) Yes;
2. a) 0.16; b) 0.76; c)  $P(3/0) = 9/19$ ;  $P(4/0) = 7/19$ ;  $P(5/0) = 3/19$ ;  
d)  $P(0/4) = 7/46$ ;  $P(1/4) = 1/2$ ;  $P(2/4) = 8/23$ ; e)  $11/24$ ; f) 1.15;  
g) 0.109; h) No; 3. a) 0.04; b) 2.39; c) 1.63; d)  $3/13$ ; e)  $3/11$ ; f) No;
4. a)  $P_{X,Y}(0,0) = 0.40$ ;  $P_{X,Y}(0,1) = 0.33$ ;  $P_{X,Y}(1,0) = 0.07$ ;  $P_{X,Y}(1,1) = 0.20$ ;  
b)  $P_{Y|X}(0/0) = 40/73$ ;  $P_{Y|X}(1/0) = 33/73$ ; c)  $20/53$ ; d) 0.057.

### 3.5. The binomial distribution

An experiment that satisfies the following four conditions is called a binomial experiment:

1. There are  $n$  identical trials. In other words, the given experiment is repeated  $n$  times. All these repetitions are performed under identical conditions.
2. Each trial has two and only two outcomes. These outcomes are usually called a success and a failure.
3. The probability of success is denoted by  $p$  and that of failure by  $q$ , and  $p + q = 1$ . The probabilities  $p$  and  $q$  remain constant for each trial.
4. The trials are independent. In other words, the outcome of one trial does not affect the outcome of another trial.

It is important to note that one of the two outcomes of a trial is called a **success** and the other a **failure**. Note that a success does not mean that the corresponding outcome is considered favorable or desirable. Similarly, a failure does not necessarily refer to an unfavorable or undesirable outcome. Success and failure simply the names used to denote the two possible outcomes of a trial.

The random variable  $x$  that represents the number of successes in  $n$  trial for a binomial experiment is called a binomial random variable.

#### **Binomial formula:**

For a binomial experiment, the probability of exactly  $x$  successes in  $n$  trials is given by the binomial formula:

$$P(x) = C_n^x \cdot p^x \cdot q^{n-x}$$

where

$n$  = total number of trials

$p$  = probability of success

$q = 1 - p$  = probability of failure

$x$  = number of success in  $n$  trials

$n - x$  = number of failures in  $n$  trials.

To find the probability of  $x$  successes in  $n$  trials for a binomial experiment, the only values needed are those of  $n$  and  $q$ . These are called the parameters of the binomial distribution or simply the binomial parameters.

**Example:**

A certain drug is effective in 30 per cent of the cases in which it has been prescribed. If a doctor is now administering this drug to four patients, what is the probability that it will be effective for at least three of the patients?

**Solution:**

We can consider the administration of the drug to each patient as a trial. Thus, this experiment has four trials. There are only two outcomes for each trial: the drug is effective or the drug is not effective. The event "effective for at least three" can be broken down into two mutually exclusive events (outcomes), "effective for three or effective for four". If we use the term "success" instead of "effective" we can say that

$$P(\text{at least 3 successes}) = P(3 \text{ successes or } 4 \text{ successes}) = \\ = P(3 \text{ successes}) + P(4 \text{ successes}) = P(x=3) + P(x=4).$$

Now we can find  $P(x=3)$  and  $P(x=4)$  separately. Since the drug is effective in 30% of the cases, we say that the probability that the drug is effective in any single case is  $p=0.3$ .

Hence,  $q = 1 - p = 0.7$ , then the equation of the particular binomial distribution is

$$P(x) = C_n^x \cdot 0.3^x \cdot 0.7^{4-x}$$

Hence, we have

$$P(x=3) + P(x=4) = C_3^4 \cdot 0.3^3 \cdot 0.7^1 + C_4^4 \cdot 0.3^4 \cdot 0.7^0 = 0.0837.$$

Practically interpreted, this number means that if the drug is administrated to 10 000 sets of four patients, in about 837 of the 10 000 sets will be effective for at least three patients out of four.

**Example:**

It is known from past data that despite all efforts, 2% of the packages mailed through post office do not arrive at their destinations within the specified time. A corporation mailed 10 packages through post office.

- a) Find the probability that exactly one of these 10 packages will not arrive at this destination within the specified time.
- b) Find the probability that at most one of these 10 packages will not arrive at this destination within the specified time.

**Solution:**

Let us call it a success if a package does not arrive at its destination within the specified time and a failure if it does arrive within the specified time.

Then

$$n=10; \quad p=0.02; \quad q=0.98$$

- a) For this part,

$$x = \text{number of successes} = 1$$

$$n-x = \text{number of failures} = 10-1=9$$

Substituting all values in the binomial formula, we obtain:

$$P(x=1) = C_1^{10} \cdot (0.02)^1 \cdot (0.98)^9 = 0.1667.$$

Thus, there is a 0.1667 probability that exactly one of the 10 packages mailed will not arrive at its destination within the specified time.

- b) The probability that at most one of the 10 packages will not arrive at its destination within the specified time is given by the sum of the probabilities of  $x=0$  and  $x=1$ . Thus,

$$P(x \leq 1) = P(x=0) + P(x=1) = C_0^{10} \cdot (0.02)^0 \cdot (0.98)^{10} + C_1^{10} \cdot (0.02)^1 \cdot (0.98)^9 = 0.8171 + 0.1667 = 0.9838.$$

Thus, the probability that at most one of the 10 packages will not arrive at its destination within the specified time is 0.9838.

**3.5.1. Mean and standard deviation of the binomial distribution**

The mean and standard deviation for a binomial distribution are

$$\mu = n \cdot p \quad \text{and} \quad \sigma = \sqrt{n \cdot p \cdot q},$$

where

$n$ -is the total number of trials,

$p$ -is the probability of success, and

$q$ -is the probability of failure.

**Example:**

The probability that a certain rifleman will get a hit on any given shot at the rifle range is  $\frac{3}{10}$ . If he fires one hundred shots, find the theoretical mean and standard deviation of  $x$ , the number of hits.

**Solution:**

We have  $n = 100$ ;  $p = \frac{3}{10}$  and  $q = \frac{7}{10}$ .

Then, by formula, the mean is

$$\mu = n \cdot p = 100 \cdot \frac{3}{10} = 30$$

and standard deviation is

$$\sigma = \sqrt{n \cdot p \cdot q} = \sqrt{100 \cdot \frac{3}{10} \cdot \frac{7}{10}} = \sqrt{21} = 4.583.$$

**Exercises**

1. Let  $x$  be a discrete random variable that possesses a binomial distribution. Using binomial formula, find the following probabilities:

- $P(x = 5)$  for  $n=8$  and  $p=0.60$
- $P(x = 3)$  for  $n=4$  and  $p=0.30$
- $P(x = 2)$  for  $n=6$  and  $p=0.20$

2. Determine the probability of getting:

- exactly three heads in 6 tosses of a fair coin;
- at least 3 heads in 6 tosses of a fair coin.

3. A card is drawn from an ordinary pack of playing cards, and its suit (clubs, diamonds, hearts, spades) noted, then it is replaced, the pack is shuffled, and another card is drawn. This is done until four cards have been drawn.

- What is the probability that two spades will be drawn in four draws?
- What is the probability that at least two spades will be drawn in four draws?
- What is the probability that two red cards will be drawn in four draws?

What is the probability that at most two red cards will be drawn in four draws?

At a particular university it has been found that 20% of the students withdraw without completing the business statistics course. Assume that 100 students have registered for the course.

What is the probability that two or fewer will withdraw?

What is the probability that exactly four will withdraw?

What is the probability that more than three will withdraw?

What is the expected number and standard deviation of withdrawals?

For the binomial distribution with  $n=4$  and  $p=0.25$  find the probability of

- a) three or more successes
- b) at most three successes
- c) two or more failures.

Calculate the mean and standard deviation of the binomial distribution with

- a)  $n=16$ ;  $p=0.5$
- b)  $n=25$ ;  $p=0.1$
- c)  $n=25$ ;  $p=0.9$

19% of cars in the country were at least 12 years old. Find the probability that in a random sample of 10 cars

- a) exactly 4 are at least 12 years old;
- b) exactly 2 are at least 12 years old;
- c) none are at least 12 years old;
- d) exactly 5 are at least 12 years old.

Suppose that, for a particular type of a cancer, treatment provides a 5-year or more years survival rate of 80% if the disease could be detected at an early stage. Among 18 patients diagnosed to have this form of cancer at an early stage who are just starting this treatment, find the probability that

- a) fourteen will survive beyond 5-years;
- b) six will die within 5-years;
- c) the number of patients surviving beyond 5-years will be between 9 and 14 (inclusive);
- d) find the expectation and standard deviation of the number of 5-years survivors.

It is known that 3% of produced goods have some defects. Eight of these goods are selected randomly.

What is the probability that none of these goods are defective?

What is the probability that one of these goods is defective?

c) What is the probability that at least two of these goods are defective?

**10.** A certain type of infection is spread by contact with an infected person. Let the probability that a healthy person gets the infection, in one contact, be  $p=0.4$ .

a) An infected person has contact with five healthy persons. Specify the distribution of  $X$  = number of persons who contract the infection.

b) Find  $P[X < 3]$ ;  $P[X = 0]$ ; and  $E[X]$ .

**11.** A salesman of home computers will contact four customers during a week. Each contact can result in either a sale, with probability 0.20, or no sale with probability 0.80. Assume that customer contacts are independent. Let  $X$  denotes the number of computers sold during the week.

a) Obtain the probability distribution of  $X$ .

b) Calculate the expected value of  $X$ .

#### Answers

**1.** a) 0.279; b) 0.076; c) 0.246; **2.** a) 5/16; b) 21/32; **3.** a) 0.211; b) 0.262;

c) 0.375; d) 0.688; **4.** a) 0.2060; b) 0.2182; c) 0.5886; d) 4; 1.790;

**5.** a) 0.051; b) 0.996; c) 0.949; **6.** a) 8;2; b) 2.5; 1.5; c) 22.5; 1.5;

**7.** a) 0.0773; b) 0.3010; c) 0.1216; d) 0.0218; **8.** a) 0.215; b) 0.082; c) 0.283;

d) 14.4; 1.697; **9.** a) 0.784; b) 0.194; c) 0.022; **10.** a) binomial distribution

with  $n=5$ ;  $p=0.4$ ; b) 0.913; 0.078; 2; **11.** a)  $P(0)=0.4096$ ;  $P(1)=0.4096$ ;

$P(2)=0.1536$ ;  $P(3)=0.0256$ ;  $P(4)=0.0016$ ; b) 0.8;

### 3.6. The hypergeometric probability distribution

In previous section, we have learned that one of the conditions required to apply the binomial probability distribution is that the trials are independent so that the probabilities of the two outcomes (success and failure) remains constant. If the trials are not independent, we can not apply the binomial probability distribution to find probability of  $x$  successes in  $n$  trials. In such cases we replace the binomial distribution by the **hypergeometric probability distribution**. Such a case occurs when a sample is drawn without replacement from a finite population.

#### Definition:

Let

$N$  = total number of elements in the population

$S$  = number of successes in the population

$N - S$  = number of failures in the population

$n$  = number of trials (sample size)

$x$  = number of successes in  $n$  trials

$n - x$  = number of failures in  $n$  trials.

The probability of  $x$  successes in  $n$  trials is given by

$$P(x) = \frac{C_x^S \cdot C_{n-x}^{N-S}}{C_n^N} = \frac{S! (N-S)!}{x!(S-x)! (n-x)!(N-S-n+x)!} \cdot \frac{N!}{n!(N-n)!}$$

#### Example:

A company has 12 employees who hold managerial positions. Of them, 7 are females and 5 are males. The company is planning to send 3 of these 12 managers to a conference. If 3 managers are randomly selected out of 12,

- find the probability that all 3 of them are female
- find the probability that at most 1 of them is a female.

#### Solution:

Let the selection of a female be called a success and the selection of a male be called a failure.

- a) From the given information,

$N$  = total number of managers in the population = 12

$S$  = number of successes (females) in the population = 7

$N-S$ =number of failures (males) in the population=5

$n$ = number of selections (sample size) =3

$x$ = number of successes (females) in three selections =3

$n - x$  = number of failures (males) in three selections =0.

Using the hypergeometric formula, we find

$$P(x=3) = \frac{C_x^S \cdot C_{n-x}^{N-S}}{C_n^N} = \frac{C_3^7 \cdot C_0^5}{C_3^{12}} = \frac{35 \cdot 1}{220} = 0.1591$$

Thus, the probability that all three of the selected managers are female is 0.1591.

b) The probability that at most one of them is a female is given by the sum of the probabilities that either none or one of the selected managers is a female.

To find the probability that none of the selected managers is a female:

$N=12$

$S=7$

$N-S=5$

$n=3$

$x=0$

$n-x=3$

$$P(x=0) = \frac{C_0^7 \cdot C_3^5}{C_3^{12}} = \frac{1 \cdot 110}{220} = 0.0455$$

To find the probability that one of the selected managers is a female:

$N=12$

$S=7$

$N-S=5$

$n=3$

$x=1$

$n-x=2$

$$P(x=1) = \frac{C_1^7 \cdot C_2^5}{C_3^{12}} = \frac{7 \cdot 10}{220} = 0.3182$$

In the end,

$$P(x \leq 1) = P(x=0) + P(x=1) = 0.0455 + 0.3182 = 0.3637.$$

### Exercises

1. Let  $N=14$ ,  $S=6$ , and  $n=5$ . Using the hypergeometric probability distribution formula, find

- a)  $P(x = 4)$ ;                      b)  $P(x = 5)$ ;                      c)  $P(x \leq 1)$

2. Let  $N=16$ ,  $S=10$ , and  $n=5$ . Using the hypergeometric probability distribution formula, find

- a)  $P(x = 5)$ ;                      b)  $P(x = 0)$ ;                      c)  $P(x \leq 1)$

3. There are 25 goods, 5 of which are defective. We randomly select 4 goods. What is the probability that three of those four goods are nondefective and one is defective?

4. A committee of two members is to be formed from the list of 8 candidates. Of the 8 candidates, 5 are management and 3 are economics department students. Find the probability that

- a) both candidates are managers  
b) neither of the candidates are managers  
c) at most one of the candidates is a manager.

5. A company buys keyboards from another company. The keyboards are received in shipments of 100 boxes, each box containing 20 keyboards. The quality control department first randomly selects one box from each shipment, and then randomly selects five keyboards from that box. The shipment is accepted if not more than one of the five keyboards is defective. The quality control inspector selects a box from a recently received shipment of keyboards. Unknown to the inspector, this box contains six defective keyboards.

- a) What is the probability that this shipment will be accepted?  
b) What is the probability that this shipment will not be accepted?

### Answers

1. a) 0.0599; b) 0.0030; c) 0.2378; 2. a) 0.0577; b) 0.0014; c) 0.0357;  
3. 0.4506; 4. a) 0.3571; b) 0.1071; c) 0.6429; 5. a) 0.5165; b) 0.4835.

### 3.7. The Poisson probability distribution

The Poisson probability distribution, named after the French mathematician Siemon D. Poisson, is another important probability distribution of a discrete random variable that has a large number of applications.

A Poisson probability distribution is modeled according to certain assumptions:

1.  $x$  is a discrete random variable;
2. The occurrences are random.
3. The occurrences are independent.

In the Poisson probability distribution terminology, the average number of occurrences in an interval is denoted by  $\lambda$  (Greek letter lambda). The actual number of occurrences in that interval is denoted by  $x$ .

#### **Poisson probability distribution formula:**

According to the Poisson probability distribution, the probability of  $x$  occurrences in an interval is

$$P(x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

where

$P(x)$  = the probability of  $x$  successes over an interval;

$\lambda$  = is the mean number of occurrences in that interval

$e = 2.71828$  (the base of natural logarithms)

The **mean** and **variance** of the Poisson probability distribution are:

$$\mu_X = E(X) = \lambda \quad \text{and} \quad \sigma_X^2 = E[(X - \mu_X)^2] = \lambda.$$

**Remark:** As it is obvious from the Poisson probability distribution formula, we need to know only the value  $\lambda$  to compute the probability of any given value of  $x$ . We can read the value of  $e^{-\lambda}$  for a given  $\lambda$  from Table 1 of the Appendix.

#### **Example:**

A computer breaks down at an average of three times per month. Using the Poisson probability distribution formula, find the probability that during the next month this computer will have

- a) exactly three breakdowns;  
 b) at most one breakdown.

**Solution:**

Let  $\lambda$  be the mean number of breakdowns per month and  $x$  be the actual number of breakdowns observed during the next month for this computer.

Then  $\lambda = 3$ .

a) The probability that exactly three breakdowns will be observed during the next month is

$$P(x=3) = \frac{3^3 \cdot e^{-3}}{3!} = \frac{27 \cdot 0.049787}{6} = 0.224$$

b) The probability that at most one breakdown will be observed during the next month is given by the sum of the probabilities of zero and one breakdown. Then

$P(\text{at most one breakdown}) = P(0 \text{ or } 1 \text{ breakdown}) = P(x=0) + P(x=1) =$

$$= \frac{3^0 \cdot e^{-3}}{0!} + \frac{3^1 \cdot e^{-3}}{1!} = 0.049787 + 3 \cdot 0.049787 = 0.1991.$$

**Example:**

A car salesperson sells an average of 0.9 cars per day. Find the probability of selling

- a) exactly 2  
 b) at least 3 cars per day  
 c) find the mean, variance and standard deviation of selling cars per day.

**Solution:**

Let  $\lambda$  be the mean number of cars sold per day by this salesperson.

Let  $x$  be the number of cars sold by this salesperson. Hence,  $\lambda = 0.9$

a)  $P(x=2) = \frac{0.9^2 \cdot e^{-0.9}}{2!} = \frac{0.81 \cdot 0.406570}{2} = 0.1647.$

b)  $P(\text{at least 3 cars sold}) = P(x=3) + P(x=4) + \dots =$   
 $= 1 - P(x=0) - P(x=1) - P(x=2) = 1 - 0.0406570 - 0.9 \cdot 0.406570 -$   
 $- 0.1647 = 0.0628.$

c)  $\mu = \lambda = 0.9$

$$\sigma^2 = \lambda = 0.9 \quad \text{and} \quad \sigma = \sqrt{\lambda} = \sqrt{0.9} = 0.949.$$

### Exercises

- 1.** Using the Poisson formula, find the following probabilities
- $P(x < 2)$  for  $\lambda = 3$
  - $P(x = 8)$  for  $\lambda = 5.3$
- 2.** Let  $x$  be a Poisson random variable. Using the Poisson probabilities table, write the probability distribution of  $x$  for each of the following. Find the mean and standard deviation for each of these probability distributions.
- $\lambda = 0.6$ ;
  - $\lambda = 1.8$
- 3.** An average of 7.5 crimes are reported per day to police in a city. Use the Poisson formula to find the probability that
- exactly 3 crimes will be reported to a police on a certain day
  - at least 2 crimes will be reported to a police on a certain day.
- 4.** A mail-order company receives an average 1.3 complaints per day. Find the probability that it will receive
- exactly 3 complaints
  - 2 to 3 complaints
  - more than 3 complaints
  - less than 3 complaints on a certain day.
- 5.** An average of 4.5 customers come to the bank per half hour.
- Find the probability that exactly 2 customers will come to this bank during a given hour;
  - Find the probability that during a given hour, the number of customers who will come to the bank is at most 2.
- 6.** An average of 0.6 accidents occur per month at a large company.
- Find the probability that no accident will occur at this company during a given month.
  - Find the mean, variance, and standard deviation of the number of accidents that will occur at this company during a given month.

### Answers

- 1.** a) 0.1991; b) 0.0771; **2.** a)  $\mu = 0.6$ ;  $\sigma = 0.7746$ ; b)  $\mu = 1.8$ ;  $\sigma = 1.3416$ ;  
**3.** a) 0.03888; b) 0.9953; **4.** a) 0.0998; b) 0.3301; c) 0.0431; d) 0.8569;  
**5.** a) 0.0050; b) 0.0062; **6.** a) 0.5488; b)  $\mu = 0.6$ ;  $\sigma^2 = 0.6$ ;  $\sigma = 0.7746$ .

## Chapter 4

### Continuous random variables and their probability distributions

#### 4.1. Introduction

Up to this point, we have limited our discussion to probability distributions of discrete random variables. Recall that a discrete random variable takes on only some isolated values, usually integers representing a count. We now turn our attention to the probability distribution of a continuous random variable- one that can ideally assume any value in an interval. Variables measured on an underlying continuous scale, such as weight, strength, life length, and temperature, have this feature.

Figure 4.1 displays the histogram and polygon for some continuous data set. The smoothed polygon is an approximation of the probability distribution curve of the continuous random variable  $X$ . The probability distribution curve of a continuous random variable is also called its probability density function.

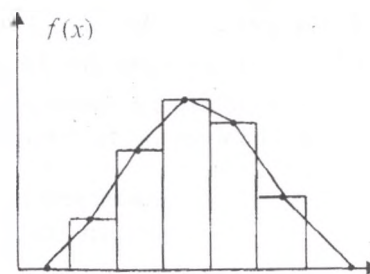


Fig.4.1. Histogram and polygon

The probability density function, denoted by  $f(x)$  possesses the following characteristics:

1.  $f(x) > 0$  for all  $x$ .
2. The area under the probability density function  $f(x)$  over all possible values of the random variable  $X$  is equal to 1.
3. Let  $a$  and  $b$  be two possible values of the random variable  $X$ , with  $a < b$ . Then the probability that  $X$  lies between  $a$  and  $b$  is the area under the density function between  $a$  and  $b$ . (Fig.4.2)
4. The cumulative distribution function  $F(x_0)$  is the area under

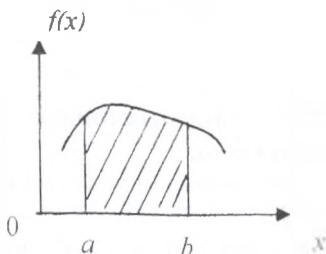


Fig.4.2. Shaded area is the probability that  $X$  lies between  $a$  and  $b$ .

the probability density function  $f(x)$  up to  $x_0$

$$F(x_0) = \int_{x_0}^{x_0} f(x) dx$$

where  $x_0$  is the minimum value of the random variable  $X$ .

#### 4.2. Areas under continuous probability density functions

Let  $X$  be a continuous random variable with probability density function  $f(x)$  and cumulative distribution function  $F(x)$ . Then:

1. The total area under the curve  $f(x)$  is 1.
2. The area under the curve  $f(x)$  to the left of  $x_0$  is  $F(x_0)$ , where  $x_0$  is any value that the random variable  $X$  can take.

The area under the probability distribution curve of a continuous random variable between any two points is between 0 and 1, as shown in Figure 4.3.

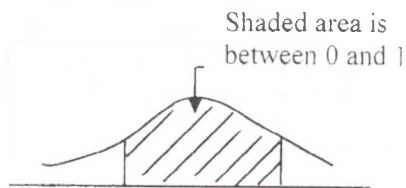


Fig.4.3. Area under a curve between two points

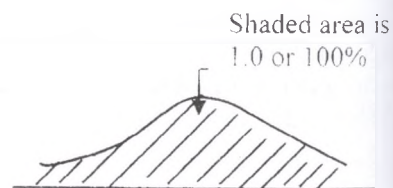


Fig.4.4. Total area under a probability distribution curve

The total area under the probability distribution curve of a continuous random variable is always 1.0 or 100% as shown in Figure 4.4.

#### Remark:

**The probability that a continuous random variable  $x$  assumes a single value is always zero.**

This is because the area of a line, which represents a single point, is zero. (Fig.4.5)

In general, if  $a$  and  $b$  are two of the values that  $X$  can assume, then,  $P(a) = 0$  and  $P(b) = 0$ .

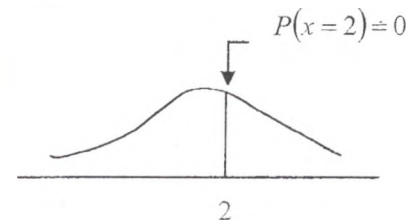


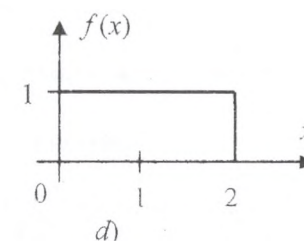
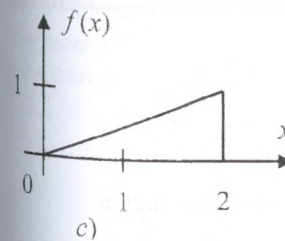
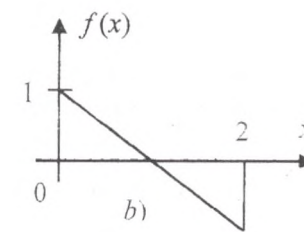
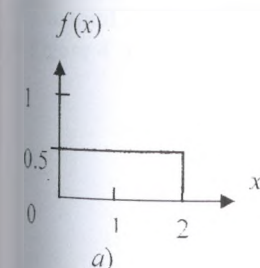
Fig.4.5 Probability of a single value of  $x$  is zero

When determining the probability of an interval  $a$  to  $b$ , we need not be concerned if either or both end points are included in the interval. Since the probabilities of  $X = a$  and  $X = b$  are both equal to 0.

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b).$$

#### Exercises

1. Which of the functions sketched in  $a-d$  could be a probability density function for a continuous random variable? Why or why not?



2. Determine the following probabilities from the curve  $f(x)$  diagrammed in exercise 1(a).

- a)  $P(0 < X < 0.5)$                       b)  $P(0.5 < X < 1)$   
 c)  $P(1.5 < X < 2)$                       d)  $P(X = 1)$

3. For the curve  $f(x)$  graphed in exercise 1(c) which of the two intervals  $(0 < X < 0.5)$  or  $(1.5 < X < 2)$  is assigned a higher probability?

4. The time it takes for a TV repair master to finish his job (in hours) has a density function of the form

$$f(x) = \begin{cases} c(x-1)(x-2) & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

- a) Determine the constant  $c$ .  
 b) What is the probability that a TV repair master will finish the job in less than 75 minutes? Between  $1\frac{1}{2}$  and 2 hours?

5. Suppose that the loss in a certain investment, in thousands of dollars, is a continuous random variable  $X$  that has a density function of the form

$$f(x) = \begin{cases} k(2x - 3x^2) & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$$

- a) Calculate the value of  $k$ .  
 b) Find the probability that the loss is at most \$500.

6. Let the random variable  $X$  has probability density function

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

- a) Draw the probability density function  
 b) Show that the density function has the properties of a proper probability density function  
 c) Find the probability that  $X$  takes a value between 0.5 and 1.5.

### Answers

2. a) 0.25; b) 0.25; c) 0.25; d) 0; 3. The interval 1.5 to 2 has higher probability; 4. a) -6; b)  $5/32$ ;  $1/2$ ; 5. a)  $-1/2$ ; b)  $3/16$ ; 6. c) 0.75.

### 4.3. The normal distribution

The normal distribution is one of the many probability distributions that a continuous random variable can possess. The normal distribution is the most important and most widely used of all the probability distributions. A large number of phenomena in the real world are normally distributed either exactly or approximately.

The probability density function for a normally distributed random variable  $X$  is:

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } -\infty < x < \infty$$

where  $\mu$  and  $\sigma^2$  are any number such that  $-\infty < \mu < \infty$  and  $0 < \sigma^2 < \infty$ .  $e=2.71828\dots$  and  $\pi = 3.14159\dots$  are constants.

A normal probability distribution, when plotted, gives a bell-shaped curve such that

1. The total area under the curve is 1.0.
2. The curve is symmetric about the mean.
3. The two tails of the curve extend indefinitely.

1. The total area under a normal curve is 1.0 or 100%, as shown in Figure 4.6.

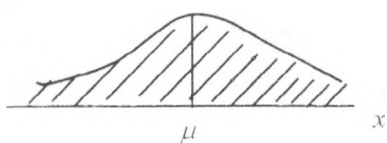


Fig.4.6. Total area under a normal curve

2. A normal curve is symmetric about the mean, as shown in Figure 4.7. Consequently 0.5 of the total area under a normal curve lies on the left side of the mean and 0.5 lies on the right side of the mean

Each of the two shaded areas is 0.5

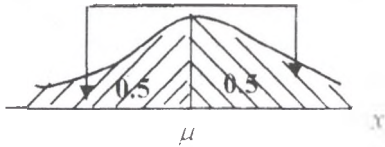


Fig.4.7. A normal curve is symmetric about the mean

3. The tails of a normal distribution curve extend indefinitely in both directions without touching or crossing the horizontal axis.

Each of the two shaded areas is very close to zero

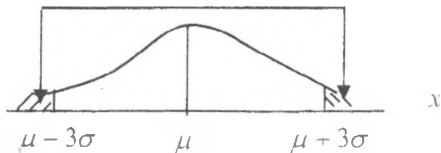


Fig.4.8.

Although a normal curve never meets the horizontal axis, beyond the points represented by  $\mu - 3\sigma$  and  $\mu + 3\sigma$  it becomes so close to this axis that the area under the curve beyond these points in both directions can be taken as virtually zero. These areas are shown in Figure 4.8.

**Remark:**

There is not just one normal distribution curve but rather a family of normal distribution curves. Each different set of values of  $\mu$  and  $\sigma$  gives a different normal distribution.

The value of  $\mu$  determines the center of a normal distribution on the horizontal axis. The three distribution curves drawn in Figure 4.9 have the same mean but different standard deviations.

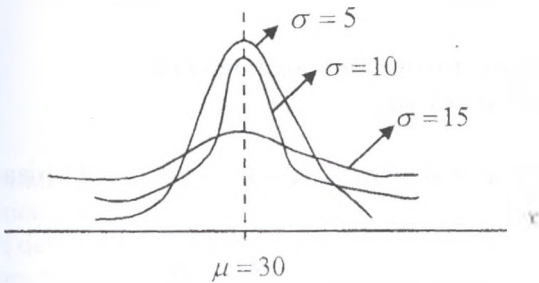


Fig.4.9. Three normal distribution curves with the same mean but different standard deviations

The value of  $\sigma$  gives the spread of the normal distribution curve. The three normal distribution curves in Figure 4.10 have different means but the same standard deviation.

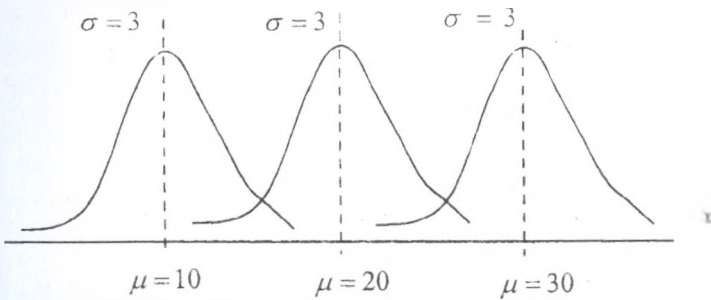


Fig.4.10. Three normal distribution curves with different means but the same standard deviation.

#### Properties of the normal distribution:

Suppose that the random variable  $X$  follows a normal distribution. Then the following properties hold:

1. The mean of the random variable is  $\mu$ :
2. The variance of the random variable is  $\sigma^2$

$$\text{Var}(X) = E(X - \mu)^2 = \sigma^2$$

3. By knowing the mean and standard deviation (or variance) we can define the normal distribution by using the notation:

$$X \sim N(\mu, \sigma^2)$$

### 4.3.1. Cumulative distribution function of the normal distribution

Suppose that  $X$  is a normal random variable with mean  $\mu$ , and variance  $\sigma^2$ , that is  $X \sim N(\mu, \sigma^2)$ .

Then the cumulative distribution function

$$F(x_0) = P(x \leq x_0)$$

This is the area under the normal probability density function to the left of  $x_0$  as illustrated in Figure 4.11.

As for any proper density function, the total area under the curve is 1, that is

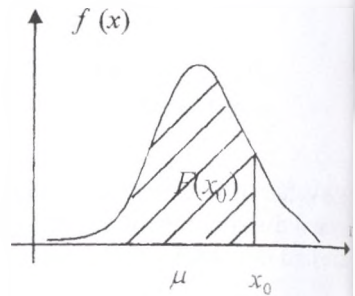


Fig.4.11

$$F(-\infty) = 0 \text{ and } F(\infty) = 1$$

Let  $a$  and  $b$  be two possible values of  $X$ , with  $a < b$ . Then

$$P(a < X < b) = F(b) - F(a)$$

The probability is the area under corresponding density function between  $a$  and  $b$  as shown in Figure 4.12.

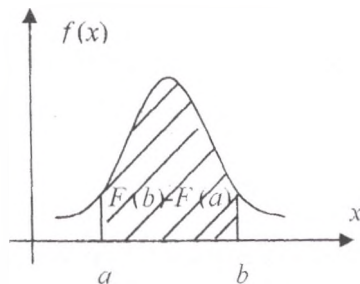


Fig.4.12. Shaded area indicates the probability that  $X$  lies between  $a$  and  $b$

#### 4.4. The standard normal distribution

The **standard normal distribution** is a special case of the normal distribution. The particular normal distribution that has a mean of 0 and a standard deviation of 1 is called the standard normal distribution. It is customary to denote the standard normal variable by  $Z$ .

**Definition:**

The standard normal distribution has a bell-shaped density with  
mean =  $\mu = 0$   
standard deviation =  $\sigma = 1$

The standard normal distribution is denoted by  $Z \sim N(0, 1)$ . (Fig.4.13)

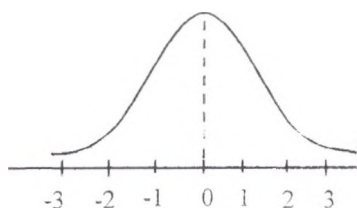


Fig.4 13. The standard normal curve

We will denote the cumulative distribution function of  $Z$  by  $F(z)$ , and for two numbers  $a$  and  $b$  with  $a < b$

$$P(a < Z < b) = F(b) - F(a)$$

Now let us see the procedure for finding probabilities associated with a continuous random variable. We wish to determine the probability of a random variable having a value in a specified interval from  $a$  to  $b$ . Thus we have to find the area under the curve in the interval from  $a$  to  $b$ . Finding areas under the standard normal distribution curve appears at first glance to be much more difficult. The mathematical technique for obtaining these areas is beyond the scope of the text, but fortunately tables are available which provide the areas or probability values for the standard normal distribution. The cumulative distribution function of the standard normal

distribution is tabulated in Table 2 in the Appendix. This table gives values of

$$F(x) = P(Z \leq z)$$

for nonnegative values of  $z$ .

For example the cumulative probability for a  $Z$  value of 1.13 from Table 2

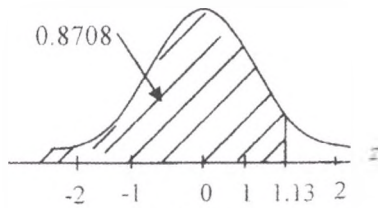


Fig.4.14

$$F(1.13) = 0.8708$$

This is the area, shown in Figure 4.14, for  $Z$  less than 1.13.

Because of the symmetry of the normal distribution the probability  $P(Z > -1.13)$  is also equal to 0.8708.

In general, values of the cumulative distribution function for negative values of  $z$  can be inferred using the symmetry of the probability density function. To find the cumulative probability for a negative  $z$  (for example  $z = -2.25$ ) defined as

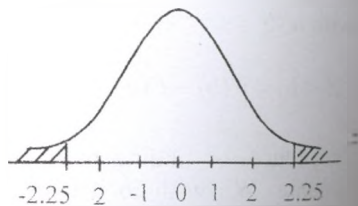


Fig.4.15

$$F(-z_n) = P(Z \leq -z_n) = F(-2.25)$$

We use the complement of the probability for  $Z = -2.25$  as shown in Figure 4.15. From the symmetry we can see that

$$F(-2.25) = 1 - P(Z \leq 2.25) = 1 - F(2.25) = 1 - 0.9878 = 0.0122$$

We can see that the area under the curve to the left of  $Z = -2.25$  is equal to the area to the right of  $Z = 2.25$  because of the symmetry of the normal distribution.

**Example:** Find  $P(Z > -1.35)$

**Solution:**

We see that because of the symmetry the probability or area to the right of  $-1.35$  (Fig.4.16) is the same as the area to the left of  $1.35$  (Fig.4.17.)

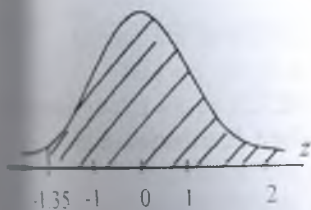


Fig.4.16

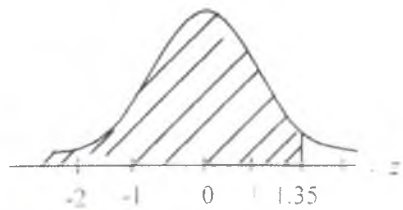


Fig.4.17

$$P(z > -1.35) = P(z < 1.35) = F(1.35) = 0.9115.$$

**Example:**

Let the random variable  $Z$  follow a standard normal distribution. The probability is 0.25 that  $Z$  is greater than what number?

**Solution:**

We need to find such a point  $z_0$  that  $P(Z > z_0) = 0.25$ . (Fig.4.18)

The area to the left of  $z_0$  is  $1 - 0.25 = 0.75$ .

$$F(z_0) = 0.75 \text{ and } z_0 = 0.675.$$

$$P(z > 0.675) = 0.25.$$

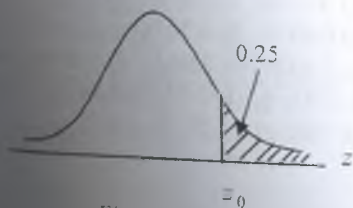


Fig.4.18

### Exercises

- 1.** Find the area under the standard normal curve to the left of  
a)  $z = 1.17$ ;                      b)  $z = 0.16$ ;  
c)  $z = -1.83$ ;                      d)  $z = -2.3$
- 2.** Find the area under the standard normal curve to the right of  
a)  $z = 1.17$ ;                      b)  $z = 0.60$ ;  
c)  $z = -1.13$ ;                      d)  $z = 1.55$ ;
- 3.** Find the area under the standard normal curve over the interval  
a)  $z = -0.65$  to  $z = 0.65$ ;    b)  $z = -1.04$  to  $z = 1.04$ ;  
c)  $z = 0.32$  to  $z = 2.65$ ;    d)  $z = -0.755$  to  $z = 1.254$  (interpolate)
- 4.** For the random variable  $Z$  follows a standard distribution, find  
a)  $P(Z < 0.42)$ ;                      b)  $P(Z < -0.42)$ ;  
c)  $P(Z > 1.69)$ ;                      d)  $P(Z > -1.69)$ ;  
e)  $P(-1.2 < Z < 2.1)$ ;                      f)  $P(0.5 < Z < 0.8)$ ;  
g)  $P(-1.62 < Z < -0.51)$ ;                      h)  $P(Z < 1.64)$ ;  
i)  $P(-2.34 < Z < -1.09)$
- 5.** Find the  $z$ -value in each of the following cases:  
a)  $P(Z < z) = 0.1736$ ;                      b)  $P(Z > z) = 0.20$ ;  
c)  $P(-z < Z < z) = 0.8$ ;                      d)  $P(-0.6 < Z < z) = 0.50$
- 6.** Let the random variable  $Z$  follow a standard normal distribution.  
a) The probability is 0.80 that  $Z$  is less than what number?  
b) The probability is 0.35 that  $Z$  is less than what number?  
c) The probability is 0.3 that  $Z$  is greater than what number?  
d) The probability is 0.75 that  $Z$  is greater than what number?

### Answers

- 1.** a) 0.8790; b) 0.5636; c) 0.0336; d) 0.0107; **2.** a) 0.1210; b) 0.2743;  
c) 0.8708; d) 0.0606; **3.** a) 0.4844; b) 0.7016; c) 0.3705; d) 0.6700;  
**4.** a) 0.6628; b) 0.3372; c) 0.0455; d) 0.9545; e) 0.8670; f) 0.0966;  
g) 0.2524; h) 0.9495; i) 0.1283; **5.** a) -0.94; b) 0.84; c) 1.28 and -1.28;  
d) 0.755; **6.** a) 0.842; b) -0.386; c) 0.524; d) -0.675.

#### 4.5. Standardizing a normal distribution

Fortunately, no new tables are required for probability calculations regarding the general normal distributions. Any normal distribution can be converted to the standard normal by the following relation:

**Rule:**

Let  $X$  be a normally distributed random variable with mean  $\mu$  and variance  $\sigma^2$ . Then random variable

$$Z = \frac{X - \mu}{\sigma}$$

has a standard normal distribution.

It follows that if  $a$  and  $b$  are any numbers with  $a < b$ , then

$$P(a < X < b) = P\left(\frac{a - \mu}{\sigma} < Z < \frac{b - \mu}{\sigma}\right) = F\left(\frac{b - \mu}{\sigma}\right) - F\left(\frac{a - \mu}{\sigma}\right),$$

where  $Z$  is the standard normal random variable and  $F(z)$  denotes its cumulative distribution function.

**Example:**

If  $X \sim N(12, 4)$  find the probability that  $X$  is greater than 16.

**Solution:**

The probability that  $X$  assumes a value greater than 16 is given by the area under the area under the normal curve to the right of  $x = 16$ , as shown in Figure 4.19.

For  $x = 16$ ;

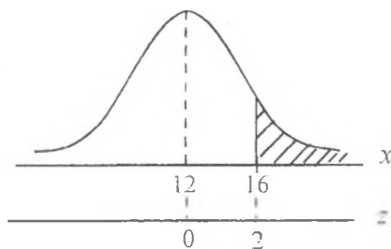


Fig.4.19. Finding  $P(X > 16)$

$$z = \frac{X - \mu}{\sigma} = \frac{16 - 12}{2} = 2.00.$$

The required probability is given by the area to the right of  $z = 2.00$ .

$$P(X > 16) = P\left(Z > \frac{16 - 12}{2}\right) = P(Z > 2) = 1 - F(2.00) = 1 - 0.9772 = 0.0228.$$

**Example:**

Let  $X$  be a continuous random variable that has a normal distribution with  $\mu = 50$  and  $\sigma = 8$ . Find the probability  $P(30 \leq X \leq 39)$ .

**Solution:**

The probability  $P(30 \leq X \leq 39)$  is given by the area from  $x = 30$  to  $x = 39$  under the normal curve. (Fig. 4.20)

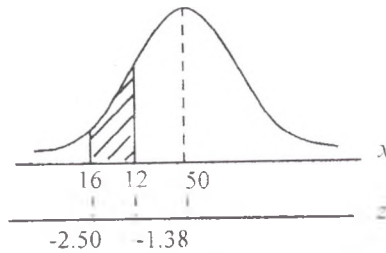


Fig. 4.20

The  $z$ -values corresponding to  $x = 30$  and  $x = 39$  are

$$\text{For } x = 30; z = \frac{30 - 50}{8} = -2.50;$$

$$\text{For } x = 39; z = \frac{39 - 50}{8} = -1.38.$$

We calculate:

$$P(30 \leq X \leq 39) = P(-2.50 \leq Z \leq -1.38) = 0.9938 - 0.9162 = 0.0776.$$

**Example:**

The number of calories in a salad on the lunch menu is normally distributed with mean  $\mu = 200$  and standard deviation  $\sigma = 5$ . Find the probability that the salad you select will contain

- more than 208 calories;
- between 190 and 200 calories.

**Solution:**

Letting  $X$  denote the number of calories in the salad, we have the standardized variable

$$Z = \frac{X - 200}{5}$$

$$\begin{aligned} \text{a) } P(X > 208) &= P\left(Z > \frac{208 - 200}{5}\right) = P(Z > 1.6) = 1 - P(Z \leq 1.6) = \\ &= 1 - 0.9452 = 0.0548. \end{aligned}$$

$$\begin{aligned} \text{b) } P(190 \leq X \leq 200) &= P\left(\frac{190 - 200}{5} \leq Z \leq \frac{200 - 200}{5}\right) = \\ &= P(-2.0 \leq Z \leq 0) = F(0) - F(-2.0) = 0.5 - (1 - F(2.0)) = \\ &= 0.5 - 1 + F(2.0) = 0.5 - 1 + 0.9772 = 0.4772. \end{aligned}$$

**Exercises**

1. If  $X \sim N(80, 16)$ , find

- $P(X < 75)$ ;
- $P(X > 86)$ ;
- $P(73 \leq X \leq 89)$ ;
- $P(81 < X < 84)$

2. If  $X$  is normally distributed with a mean of 30 and a standard deviation of 5, find

- $P(X < 26)$ ;
- $P(X \geq 37)$ ;
- $P(20 \leq X \leq 40)$ ;
- $P(20 < X < 40)$

3. If  $X$  has a normal distribution with  $\mu = 150$  and  $\sigma = 5$ , find  $b$  such that

- $P(X < b) = 0.9920$ ;
- $P(X > b) = 0.0197$ ;
- $P(X < b) = 0.063$

4. Scores on a university entrance examination follow a normal distribution with a mean of 500 and a standard deviation of 100. Find the probability that a student will score

- a) over 650;
- b) less than 250;
- c) between 325 and 675;
- d) If the university admits students who score over 670, what proportion of the student pool would be eligible for admission?
- e) What should be the limit if only the top 15% are to be eligible?

5. According to the children's growth chart that doctors use as a reference, the heights of two-year-old boys are nearly normally distributed with a mean of 85 cm inches and a standard deviation of 5 cm. If a two year-old boy is selected at random, what is the probability that he will be between 75 cm and 92 cm tall?

6. The weights of apples served at a restaurant are normally distributed with a mean of 125 grams and standard deviation of 8 grams. What is the probability that the next person served will be given an apple that weights less than 120 grams?

7. The National bank is reviewing its service charge and interest-paying policies on checking accounts. The bank has found that the average daily balance on personal checking accounts is \$55000, with a standard deviation of \$15000. In addition, the average daily balances have been found to be normally distributed.

- a) What percentage of personal checking account customers carry average daily balances in excess of \$80000?
- b) What percentage of the bank's customers carry daily balances below \$20000?
- c) What percentage of the bank's customer carry average daily balances between \$30000 and \$70000?
- d) The bank is considering paying interest to customers carrying average daily balances in excess of a certain amount. If the bank does not want to pay interest to more than 5% of its customers, what is the minimum average daily balance it should be willing to pay interest on?

8. The sales of high-bright toothpaste are believed to be approximately normally distributed, with a mean of 10 000 tubes per week and a standard deviation of 1500 tubes per week.

- a) What is the probability that more than 12000 tubes will be sold in any given week?

b) In order to have a 0.95 probability that the company will have sufficient stock to cover the weekly demand, how many tubes should be produced?

**9.** The attendance at football games at a certain stadium is normally distributed, with a mean of 45000 and a standard deviation of 3000.

a) What percentage of the time should attendance be between 44000 and 48000?

b) What is the probability of exceeding 50000?

c) Eighty percent of the time the attendance should be at least how many?

**10.** The lifetime of a color television picture tube is normally distributed, with a mean of 7.8 years and a standard deviation of 2 years.

a) What is the probability that a picture tube will last more than 10 years?

b) If the firm guarantees the picture tube for 2 years, what percentage of the television sets sold will have to be replaced because of picture tube failure?

c) If the firm is willing to replace the picture tubes in a maximum of 1% of the television set sold, what guarantee period can be offered for the television picture tubes?

**11.** It is estimated that the scores on the university entrance test are distributed normally with mean of 80 and standard deviation of 5.

a) For a randomly chosen participant taking this test, what is the probability of a score more than 72?

b) For a randomly chosen participant taking this test, what is the probability of a score between 73 and 85?

c) What is the minimum score needed in order to be in the top 5% of all participants taking this test?

d) What is the minimum score is needed to enter to the university if only the best 70% of all participants will pass this test?

e) Two participants are chosen at random. What is the probability that at least one of them scores more than 85?

### Answers

**1.** a) 0.1056; b) 0.0668; c) 0.9477; d) 0.2426; **2.** a) 0.2119; b) 0.0808;

c) 0.9544; d) 0.9544; **3.** a) 162.05; b) 160.3; c) 142.35; **4.** a) 0.0668;

b) 0.0062; c) 0.9198; d) 4.46%; e) 603.6; **5.** 0.8964; **6.** 0.266; **7.** a) 4.75% ;

b) 0.99%; c) 79.38%; d) \$79675; **8.** a) 0.0918; b) 12468; **9.** a) 0.4706;

b) 0.0475; c) 42450; **10.** a) 0.1357; b) 0.19%; c) 3.14 years or approximately

3 years; **11.** a) 0.9452; b) 0.7605; c) 88.225; d) 77.375; e) 0.2922.

#### 4.6. The normal distribution approximation to the binomial distribution

Whenever the number of trials in a binomial experiment is small it is easy to find probabilities of the various values of  $x$ , the number of successes, by using formula

$$P(x) = C_x^n \cdot p^x \cdot q^{n-x}$$

As the number of trials increases, however, the effort involved in answering questions about probabilities associated with the experiment quickly becomes laborious.

For instance, suppose that we want to know the probability that in fifteen tosses of a fair coin we toss at least nine heads. You will undoubtedly agree that  $n=15$  is not a large number of trials. However, in order to find  $P(x \geq 9)$  we say

$$P(x \geq 9) = P(x=9 \text{ or } 10 \text{ or } 11 \text{ or } 12 \text{ or } 13 \text{ or } 14 \text{ or } 15) =$$

$$= C_9^{15} \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^6 + \dots + C_{15}^{15} \left(\frac{1}{2}\right)^{15}$$

So we have seven probabilities to compute, after which we must perform the addition. This is not practically difficult, but it takes a fair amount of time. To find only one of these probabilities, for example, we have

$$P(x=10) = C_{10}^{15} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^5;$$

$$C_{10}^{15} = \frac{15!}{10!5!} = \frac{11 \cdot 12 \cdot 13 \cdot 14 \cdot 15}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 3003$$

$$\left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^{15} = \frac{1}{32768}.$$

Therefore

$$C_{10}^{15} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^5 = 3003 \cdot \frac{1}{32768} = 0.092.$$

Thus, you see that if we were to calculate other six such probabilities we would expend a considerable amount of time and energy.

In such cases, the normal distribution can be used to approximate the binomial probability. Note that, for a binomial problem, the exact probability is obtained by using the binomial formula. If we apply the normal distribution to solve a binomial problem, the probability that we obtain is an approximation to the exact probability.

**Example:**

According to an estimate, 50% of the people have at least one credit card. If a random sample of 30 persons is taken, what is the probability 19 of them will have at least one credit card?

**Solution:**

Let  $n$  be the total number of persons in the sample,  $x$  be the number of persons in the sample who have at least one credit card, and  $p$  be the probability that a person has at least one credit card. Then, this is a binomial problem with

$$\begin{array}{lll} n=30; & p=0.50; & q=1-p=0.50; \\ x=19; & \text{and} & n-x=30-19=11. \end{array}$$

Using the binomial formula, the exact probability that 19 persons in a sample of 30 have at least one credit card is

$$P(x=19) = C_{19}^{30} \cdot 0.5^{19} \cdot 0.5^{11} = 0.0509$$

Now let us solve this problem using the normal distribution as an approximation to the binomial distribution. For this example,

$$n \cdot p = 30 \cdot 0.5 = 15 \text{ and}$$

$$n \cdot p \cdot q = 7.5.$$

Using the normal distribution as an approximation to the binomial involves the following steps:

**Step 1:**

*Compute  $\mu$  and  $\sigma$  for the binomial distribution.*

To use the normal distribution, we need to know the mean and standard deviation of the distribution. Hence, the first step in using the normal

approximation to the binomial distribution is to compute the mean and standard deviation of the binomial distribution. As we know the mean and standard deviation of the binomial distribution are given by

$$\mu = n \cdot p \text{ and } \sigma = \sqrt{n \cdot p \cdot q}.$$

Using these formulas, we obtain

$$\mu = n \cdot p = 30 \cdot 0.50 = 15;$$

$$\sigma = \sqrt{n \cdot p \cdot q} = \sqrt{30 \cdot 0.5 \cdot 0.5} = 2.74.$$

**Step2:**

*Convert the discrete random variable to a continuous random variable.*

The normal distribution applies to a continuous random variable, whereas the binomial distribution applies to a discrete random variable. The second step is to convert the discrete random variable to a continuous random variable by making the **correction for continuity**.

To make the correction for continuity, we use the interval 18.5 to 19.5 for 19 persons.

**Step3:**

*Compute the required probability using the normal distribution.*

The area under the normal curve between  $x=18.5$  and  $x=19.5$  will give us the (approximate) probability that 19 persons possess at least one credit card. We calculate this probability as follows:

$$\text{For } x=18.5: \quad z = \frac{18.5 - 15}{2.74} = 1.28;$$

$$\text{For } x=19.5; \quad z = \frac{19.5 - 15}{2.74} = 1.64.$$

The required probability is given by the area under the standard normal curve between  $z=1.28$  and  $z=1.64$ . (Fig 4.21).

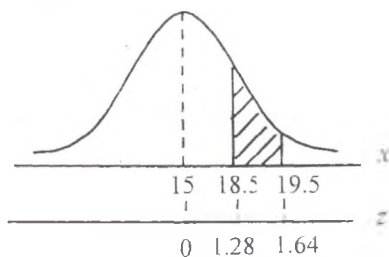


Fig.4.21

The required probability is

$$P(18.5 \leq x \leq 19.5) = P(1.28 \leq z \leq 1.64) = 0.9495 - 0.8997 = 0.0498.$$

Thus, based on the normal approximation, the probability that 19 persons in a sample of 30 will possess at least one credit card is approximately 0.0498. Earlier, using the binomial formula, we obtained the exact probability 0.0509. The error due to using the normal approximation is  $0.0509 - 0.498 = 0.0011$ . Thus, the exact probability is underestimated by 0.0011 if the normal approximation is used.

**Definition:**

Let  $x$  be the number of successes from  $n$  independent trials, each with probability of success  $p$ . Then number of successes,  $x$ , is a binomial random variable and if  $n \cdot p \cdot q > 9$  (where  $q = 1 - p$ ) a good approximation is

$$P(a \leq x \leq b) = P\left(\frac{a - np}{\sqrt{n \cdot p \cdot q}} \leq Z \leq \frac{b - np}{\sqrt{n \cdot p \cdot q}}\right) \quad (4.1)$$

or if  $5 < n \cdot p \cdot q < 9$  we can use the **continuity correction factor** to obtain

$$P(a \leq x \leq b) = P\left(\frac{a - 0.5 - np}{\sqrt{n \cdot p \cdot q}} \leq Z \leq \frac{b + 0.5 - np}{\sqrt{n \cdot p \cdot q}}\right) \quad (4.2)$$

where  $Z$  is a standard normal random variable.

**Example:**

Let  $X$  have a binomial distribution with  $p = 0.6$  and  $n = 150$ . Approximate the probability that

- a)  $x$  lies between 82 and 101;
- b)  $x$  is greater than 97.

**Solution:**

Since  $n \cdot p \cdot q = 150 \cdot 0.6 \cdot 0.4 = 36 > 9$ , then we will use approximation without using the continuity correction.

Since  $n \cdot p = 90$ ,  $\sigma = \sqrt{n \cdot p \cdot q} = \sqrt{36} = 6$  we obtain:

$$\text{a) } P(82 \leq x \leq 101) = P\left(\frac{82 - 90}{6} \leq Z \leq \frac{101 - 90}{6}\right) = P(-1.33 \leq Z \leq 1.83) =$$

$$= F(1.83) - (1 - F(1.33)) = F(1.83) - 1 + F(1.33) = 0.9664 - 1 + 0.9082 = 0.8746.$$

$$\begin{aligned} \text{b) } P(x > 97) &= P\left(Z > \frac{97 - 90}{6}\right) = P(Z > 1.17) = \\ &= 1 - F(1.17) = 1 - 0.8790 = 0.1210. \end{aligned}$$

**Example:**

A large-scale survey conducted two years ago revealed that 30% of the adult population were regular users of alcoholic beverages. If this is still the current rate, what is the probability that in a random sample of 40 adults the number of users of alcoholic beverages will be

- a) less than 15
- b) 10 or more?

**Solution:**

For this example

$$n = 40, \quad p = 0.3; \quad q = 0.7.$$

Since  $n \cdot p \cdot q = 40 \cdot 0.3 \cdot 0.7 = 8.4 < 9$  we must use continuity correction factor to obtain necessary probabilities.

$$\text{a) } P(x < 15) = P\left(Z < \frac{15 + 0.5 - 12}{\sqrt{40 \cdot 0.3 \cdot 0.7}}\right) = P\left(Z < \frac{3.5}{2.9}\right) = P(Z < 1.21) = 0.8869.$$

The probability that 15 out of 40 adults use alcoholic beverages regularly is 0.8869.

$$\begin{aligned} \text{b) } P(x > 10) &= P\left(Z > \frac{10 - 0.5 - 12}{\sqrt{40 \cdot 0.3 \cdot 0.7}}\right) = P\left(Z > -\frac{2.5}{2.9}\right) = \\ &= P(Z > -0.86) = F(0.86) = 0.8051. \end{aligned}$$

**Exercises**

**1.** For a binomial probability distribution,  $n = 80$  and  $p = 0.50$ . Let  $x$  be the number of successes in 80 trials.

- a) Find the mean and standard deviation of the binomial distribution.
- b) Find  $P(x \geq 37)$  using the normal approximation.
- c) Find  $P(41 \leq x \leq 44)$  using the normal approximation

**2.** For a binomial probability distribution,  $n = 120$  and  $p = 0.6$ . Let  $x$  be the number of successes in 120 trials.

- a) Find the mean and standard deviation of the binomial distribution.

- b) Find  $P(x \leq 70)$  using the normal approximation.
- c) Find  $P(67 \leq x \leq 71)$  using the normal approximation
3. Find the following binomial probabilities using the normal approximation
- a)  $n=70$ ;  $p=0.30$ ;  $P(x=18)$
- b)  $n=200$ ;  $p=0.70$ ;  $P(133 \leq x \leq 145)$
- c)  $n=40$ ;  $p=0.25$ ;  $P(x \geq 12)$
- d)  $n=50$ ;  $p=0.10$ ;  $P(x \leq 7)$
4. According to the statistics, 19% of cars in the city were at least 12 years old in 2004. Assume that this result holds true for the current population of all cars in the city. Random sample of 500 cars are selected at random. Find the probability that
- a) exactly 92 cars are at least 12 years old.
- b) 100 or more cars are at least 12 years old.
- c) 90 to 98 cars are at least 12 years old?
5. According to a survey, 30% of credit card holders pay off their balances in full each month. Assume that this result holds true for the current population of credit card holders.
- a) Find the probability that in a random sample of 400 credit card holders, exactly 125 pay off their balances in full each month.
- b) Find the probability that in a random sample of 400 credit card holders, at least 110 pay off their balances in full each month.
- c) What is the probability that in a random sample of 400 credit card holders, 115 to 130 pay off their balances in full each month?
6. A fast food chain store conducted a taste survey before marketing a new hamburger. The results of the survey showed that 70% of the people who tried this hamburger liked it. Encouraged by this result, the company decided to market the new hamburger. Assume that 70% of all people like this hamburger. On a certain day, 40 customers bought this hamburger.
- a) Find the probability that exactly 32 of the 40 customers will like this hamburger.
- b) What is the probability that 25 or less of the 40 customers will like this hamburger?
- c) What is the probability that 31 to 34 of the 40 customers will like this hamburger?

- 7.** According to a survey, 20.8% of the lawyers and judges are women.
- Find the probability that in a random sample of 200 lawyers and judges, exactly 35 are women.
  - Find the probability that in a random sample of 200 lawyers and judges, at most 45 are women.
  - What is the probability that in a random sample of 200 lawyers and judges, 43 to 50 are women?
- 8.** Of the customers visiting the stereo section of a large electronic store, only 25% make a purchase. If 45 customers visit the stereo section tomorrow, find the probability that more than 10 will make a purchase.
- 9.** The unemployment rate in a city is 7.9%. A sample of 100 persons is selected from the labor force. Approximate the probability that
- less than 11 unemployed persons are in the sample
  - more than 9 unemployed persons are in the sample.
  - between 8 and 12 unemployed persons are in the sample.

### Answers

- 1.** a)  $\mu = 40$ ;  $\sigma = 4.472$ ; b) 0.7486; c) 0.2262; **2.** a)  $\mu = 72$ ;  $\sigma = 5.367$ ; b) 0.3557; c) 0.2485; **3.** a) 0.0764; b) 0.6393; c) 0.2912; d) 0.8810; **4.** a) 0.0413; b) 0.2843; c) 0.3488; **5.** a) 0.0378; b) 0.8621; c) 0.5709; **6.** a) 0.0525; b) 0.1949; c) 0.1824; **7.** a) 0.0371; b) 0.7224; c) 0.3331; **8.** 0.7257; **9.** a) 0.9082; b) 0.4129; c) 0.5150.

#### 4.7. The exponential probability distribution

The **exponential probability distribution** is another important probability density function. This probability distribution is closely related to the Poisson probability distribution.

The exponential probability distribution has only one parameter  $\lambda$ , which denotes the average number of occurrences per unit of time.

**Remark:**

The exponential distribution differs from the normal distribution in two important ways

1. it is restricted to random variables with positive values;
2. its distribution is not symmetric.

**Definition:**

The exponential random variable  $X(x>0)$  has a probability density function

$$f(x) = \lambda \cdot e^{-\lambda x} \text{ for } x > 0$$

where  $\lambda$  is the mean number of occurrences per unit time,  $x$  is the number of time units until the next occurrence, and  $e = 2.71828\dots$ , then  $X$  is said to follow an **exponential probability distribution**. It can be shown that  $\lambda$  is the same parameter used for the Poisson distribution and that the mean time between occurrences is  $1/\lambda$ .

The cumulative distribution function is

$$F(x) = 1 - e^{-\lambda x} \text{ for } x > 0$$

The distribution has mean  $1/\lambda$  and variance  $1/\lambda^2$ .

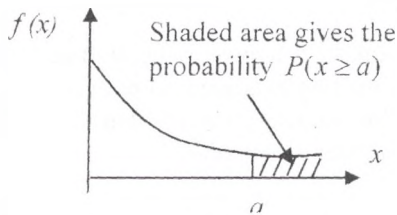


Fig.4.22

The probability  $P(x \geq a)$  for the exponential probability distribution is given by the area in the tail of the exponential probability distribution curve beyond  $x = a$ , as it shown in Figure 4.22.

As we know from earlier discussion, for a continuous random variable  $x$ ,  $P(x > a)$  is equal to  $P(x \geq a)$ . Hence for an exponential probability distribution,

$$P(x \geq a) = P(x > a) = e^{-\lambda a}$$

By using the complementary probability rule, we obtain:

$$P(x \leq a) = 1 - P(x > a) = 1 - e^{-\lambda a}$$

The probability that the  $x$  is between two successive occurrences is in the interval " $a$ " to " $b$ " is

$$\begin{aligned} P(a \leq x \leq b) &= P(x \leq b) - P(x \leq a) = (1 - e^{-\lambda b}) - (1 - e^{-\lambda a}) = \\ &= 1 - e^{-\lambda b} - 1 + e^{-\lambda a} = e^{-\lambda a} - e^{-\lambda b}. \end{aligned}$$

### Probabilities for the exponential probability distribution.

For the exponential probability distribution with the mean number of occurrences per unit of time equal to  $\lambda$ ,

$$P(x \geq a) = e^{-\lambda a}$$

$$P(x \leq a) = 1 - e^{-\lambda a}$$

$$P(a \leq x \leq b) = e^{-\lambda a} - e^{-\lambda b}.$$

#### **Example:**

A processing machine breaks down an average of once in four weeks. What is the probability that the next breakdown will not occur for at least six weeks after the previous breakdown? Assume that the time between breakdowns has an exponential distribution.

#### **Solution:**

Let  $x$  denote the lapse time between any two successive breakdowns of this machine. We are to find the probability

$$P(x \geq 6 \text{ weeks})$$

Because the unit of time for  $x$  is in the weeks, we must define the mean number of breakdowns  $\lambda$  per week. Since there is one breakdown in four weeks,

$\lambda =$  the mean number of breakdowns per week  $= \frac{1}{4} = 0.25$ .

The required probability is calculated using the formula  $P(x \geq a) = e^{-\lambda a}$ .  
In our example,  $\lambda = 0.25$  and  $a = 6$  weeks. The required probability is

$$P(x \geq 6 \text{ weeks}) = e^{-0.25 \cdot 6} = e^{-1.5} = 0.2231.$$

The value of  $e^{-1.5}$  can be found from the Table 1 of the Appendix.

**Example:**

A teller at the bank serves, on average, 30 customers per hour. Assume that the service time for a customer has an exponential distribution.

- a) What is the probability that the next customer will take five minutes or more to be served?
- b) Find the probability that the next customer will take two minutes or less to be served.
- c) What is the probability the next customer will take two to four minutes to be served?

**Solution:**

Let  $x$  be the time taken by this teller to serve a customer. We must find the mean number of customers served per minute by this teller to define  $\lambda$  per unit of time (minute). The teller serves on average 30 customers per 60 minutes. Hence

$$\lambda = 30/60 = 0.5 \text{ customers served per minute.}$$

- a) We need to find the probability  $P(x \geq 5)$ . (Fig. 4.23.)

In this case  $a = 5$ .

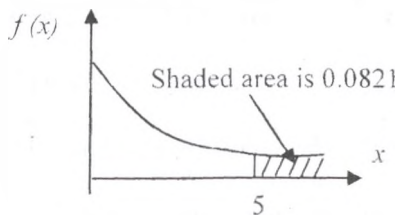


Fig.4.23. The probability for  $P(x \geq 5)$

$$P(x \geq 5) = e^{-\lambda a} = e^{-0.5 \cdot 5} = e^{-2.5} = 0.082085 = 0.0821$$

The probability is 0.0821 that a customer will take more than five minutes to be served.

b) We are to find  $P(x \leq 2)$ . This probability will be calculated using the formula  $P(x \leq 2) = 1 - e^{-\lambda a}$ . In this case  $a = 2$  minutes.

So

$$\begin{aligned} P(x \leq 2) &= 1 - P(x > 2) = 1 - e^{-\lambda a} = 1 - e^{-0.5 \cdot 2} = \\ &= 1 - e^{-1.0} = 1 - 0.367879 = 0.6321. \end{aligned}$$

Thus, the probability is 0.6321 that a customer will be served in two minutes or less.

$$\begin{aligned} \text{c) } P(2 \leq x \leq 4) &= P(x \geq 2) - P(x \geq 4) = e^{-0.5 \cdot 2} - e^{-0.5 \cdot 4} = \\ &= e^{-1} - e^{-2} = 0.367879 - 0.135335 = 0.2325. \end{aligned}$$

Thus, the probability that the teller will take two to four minutes to serve a customer is 0.2325.

### Exercises

**1.** Let  $x$  be a continuous random variable that possesses an exponential probability distribution with  $\lambda = 1.0$ . Find the following probabilities

a)  $P(x \geq 3)$ ;                      b)  $P(x \leq 4)$ ;                      c)  $P(2 \leq x \leq 6)$

**2.** The life of a pie is exponentially distributed with a mean of three days. What is the probability that a pie that is baked just now will still be good after four days?

**3.** At the supermarket, a customer has to wait an average of four minutes in line before being served. The time a customer has to wait is exponentially distributed.

a) What is the probability that a customer will have to wait for more than eight minutes in line?

b) What is the probability that a customer will have to wait for three to seven minutes in line?

**4.** On average, 20 telephone calls are received per hour at an office. The time between calls received at this office is exponentially distributed.

a) What is the probability that no calls will come in during the next 10 minutes?

b) What is the probability that the next call will come in within 4 minutes?

5. Aysel works for a toy company and assembles five toys per hour on average. The assembly time for this toy follows an exponential distribution.

a) Find the probability that the next toy will take more than 15 minutes to assemble.

b) What is the probability that the next toy will take less than 8 minutes to assemble?

c) What is the probability that the next toy will take 10 to 16 minutes to assemble?

6. A student, working in a part time job, sells life insurance policies. The past data show that he (she) sells, on an average, 10 life insurance policies per 4-week period. Assume that the time between successive sales of life insurance policies by student has an exponential distribution.

a) What is the probability that the next life insurance policy will not be sold for two weeks?

b) What is the probability that the next life insurance policy will be sold within one week?

c) What is the probability that the next life insurance policy will be sold in one to two weeks?

### Answers

1. a) 0.0498; b) 0.9817; c) 0.1329; 2. 0.2725; 3. a) 0.1353; b) 0.2986;

4. a) 0.0369; b) 0.7275; 5. a) 0.2725; b) 0.5034; c) 0.1768; 6. a) 0.0067;

b) 0.9179; c) 0.0754.

## Chapter 5. Sampling distributions

### 5.1. Sampling and sampling distributions

Suppose that we want to select a sample of  $n$  objects from a population of  $N$  objects.

A **simple random sample** is selected such that every object has an equal probability of being selected and the objects are selected independently—the selection of one object does not change the probability of selecting any other objects.

It is important that a sample represents the population as a whole. If a marketing manager wants to assess reactions to a new food product, she would not sample only her friends and neighbors. People must be selected randomly and independently. Random selection is our insurance policy against allowing personal influence the selection.

We use sample information to make inferences about the parent population. The distribution of all values of interest in this population can be represented by a random variable. It would be too ambitious to attempt to describe the entire population distribution based on a small random sample of observations. However, we may well be able to make quite firm inferences about important characteristics of the population distribution, such as the population mean and variance.

#### Sampling distribution:

The probability distribution of  $\bar{x}$  is called its sampling distribution. It lists

the various values that  $\bar{x}$  can assume and the probability for each value of  $\bar{x}$ . In general, the probability distribution of a sample statistic is called its **sampling distribution**.

Let us consider sampling distribution in example.

#### Example:

Suppose that there are only five employees working for a small company. The following data give the annual salaries (in thousands of dollars) of these employees:

17; 24; 35; 35; 43

Let  $X$  denote the annual salary of an employee. We can write the frequency distribution of annual salaries as in table 5.1

Table 5.1

$X$	$f$
17	1
24	1
35	2
43	1

Population frequency  
distribution

Table 5.2

$X$	$P(X)$
17	$1/5=0.2$
24	$1/5=0.2$
35	$2/5=0.4$
43	$1/5=0.2$

Population  
probability distribution

Dividing the frequencies of classes by the population size we obtain the relative frequencies, which can be used as probabilities of those classes. Table 5.2, which lists the probabilities of various  $X$  values, presents the probability distribution of the population.

Now, let us consider all possible samples of three salaries each, that can be selected, without replacement, from the population. The total number of possible samples, given by the combination

$$\text{Total number of samples} = C_3^5 = \frac{5!}{3!2!} = 10.$$

Suppose we assign letters A, B, C, D, and E to the salaries of five employees so that

$$A=17; B=24; C=35; D=35; E=43.$$

Then 10 possible samples of three salaries are

ABC, ABD, ABE, ACD, ACE,  
ADE, BCD, BCE, BDE, CDE.

These 10 samples and their respective means are listed in Table 5.3.

Note that the values of means of samples in Table 5.3 are rounded to two decimal places.

Table 5.3

Sample	Salaries in the sample	$\bar{X}$
ABC	17, 24, 35	25.33
ABD	17, 24, 35	25.33
ABE	17, 24, 43	28.00
ACD	17, 35, 35	29.00
ACE	17, 35, 43	31.67
ADE	17, 35, 43	31.67
BCD	24, 35, 35	31.33
BCE	24, 35, 43	34.00
BDE	24, 35, 43	34.00
CDE	35, 35, 43	37.67

All possible samples and their means when the sample size is 3.

By using the values of  $\bar{X}$  given in Table 5.3, we record the frequency distribution of  $\bar{X}$  in Table 5.4.

Table 5.4

$\bar{X}$	$f$
25.33	2
28.00	1
29.00	1
31.33	1
31.67	2
34.00	2
37.67	1

Frequency distribution of  $\bar{X}$  when the sample size is 3.

By dividing the frequencies of various values of  $\bar{X}$  by the sum of all frequencies, we obtain the relative frequencies of classes, which can be used as probabilities of classes. These probabilities are listed in Table 5.5.

This table gives the sampling distribution of  $\bar{X}$ .

Table 5.5.

$\bar{X}$	$P(\bar{X})$
25.33	$2/10=0.20$
28.00	$1/10=0.10$
29.00	$1/10=0.10$
31.33	$1/10=0.10$
31.67	$2/10=0.20$
34.00	$2/10=0.20$
37.67	$1/10=0.10$
	$\sum P(x) = 1.0$

Sampling distribution of  $\bar{X}$   
when the sample size is 3.

If we draw just one sample of three salaries from the population of five salaries, we may draw any of 10 possible samples. Hence, the sample mean

$\bar{X}$  can assume any of the values listed in Table 5.5 with the corresponding probability. This probability function is graphed in Figure 5.1. For example, the probability that the mean of a randomly drawn sample of three salaries is 31.67 is 0.20. This can be written as

$$P(\bar{X} = 31.67) = 0.20$$

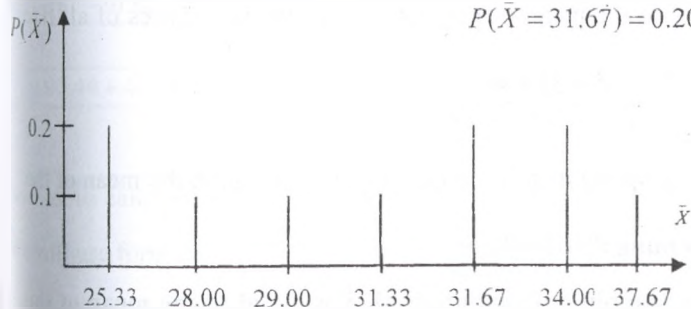


Fig 5.1. Probability function of sampling distribution for means of samples of three observations selected from the population of 17, 24, 35, 35, 43

### 5.1.1. Mean and standard deviation of $\bar{X}$

The mean and standard deviation calculated for the sampling distribution of  $\bar{X}$  are called the **mean** and **standard deviation** of  $\bar{X}$  and denoted by  $\mu_{\bar{X}}$  and  $\sigma_{\bar{X}}$  respectively.

Let us calculate the mean of the 10 values of  $\bar{X}$  listed in Table 5.3.

$$\begin{aligned}\mu_{\bar{X}} &= \frac{\sum \bar{X}}{10} = \\ &= \frac{25.33 + 25.33 + 28.00 + 29.00 + 31.67 + 31.67 + 31.33 + 34 + 34 + 37.67}{10} = \\ &= \frac{308}{10} = 30.80\end{aligned}$$

Alternatively, we can calculate the mean of the sampling distribution of  $\bar{X}$  listed in Table 5.5 as

$$\begin{aligned}\mu_{\bar{X}} &= \\ &= \sum X \cdot P(\bar{X}) = 25.33 \cdot 0.2 + 28 \cdot 0.1 + 29 \cdot 0.1 + 31.33 \cdot 0.1 + 31.67 \cdot 0.2 + \\ &+ 34 \cdot 0.2 + 37.67 \cdot 0.1 = 30.80.\end{aligned}$$

Now let us calculate the mean of population: the annual salaries of all five employees:

$$\mu = \frac{17 + 24 + 35 + 35 + 43}{5} = \frac{154}{5} = 30.80$$

The mean of the sampling distribution of  $\bar{X}$  always equal to the mean of the population.

**Mean of the sampling distribution:**

**The mean of the sampling distribution of  $\bar{X}$  is equal to the mean of the population.**

Hence

$$\bar{X} = \mu.$$

Hence, if we take all possible samples (of the same size) from a population and calculate their means, the mean  $\bar{X}$  of all these sample means will be the same as the population mean.

The sample mean  $\bar{X}$  is called an estimator of the population mean  $\mu$ . When the expected value (or mean) of a sample statistic is equal to the value of the corresponding population mean, that sample statistic is said to be an **unbiased estimator**. For the sample mean  $\bar{X}$ ,  $\mu_{\bar{X}} = \mu$ . Hence,  $\bar{X}$  is an unbiased estimator of  $\mu$ .

Let us talk about population standard deviation and standard deviation of the sampling distribution of  $\bar{X}$ .

First, let us find standard deviation of salaries of five employees.

We will use

$$\sigma = \sqrt{\frac{\sum (X_i - \mu)^2}{N}}$$

to obtain standard deviation of population.

$$\begin{aligned} \sigma &= \sqrt{\frac{(17-30.8)^2 + (24-30.8)^2 + (35-30.8)^2 + (35-30.8)^2 + (43-30.8)^2}{5}} \\ &= \sqrt{\frac{190.44 + 46.24 + 17.64 + 17.64 + 144.84}{5}} = \sqrt{\frac{420.8}{5}} = \sqrt{84.16} = 9.174 \end{aligned}$$

Now let us calculate the value of  $\sigma_{\bar{X}}$ .

We will use formula

$$\sigma_{\bar{X}} = \sqrt{\sum (\bar{X})^2 \cdot P(X) - (\mu_{\bar{X}})^2}.$$

We obtain

$$\begin{aligned}\sigma_{\bar{X}} &= \sqrt{\sum (25.33)^2 \cdot 0.2 + (28)^2 \cdot 0.1 + \dots + (37.67)^2 \cdot 0.1 - (30.80)^2} = \\ &= \sqrt{(128.32 + 78.4 + 84.1 + 98.16 + 200.6 + 231.2 + 141.90) - 948.64} = \\ &= \sqrt{962.68 - 948.64} = \sqrt{14.04} = 3.747.\end{aligned}$$

As we see, the standard deviation  $\sigma_{\bar{X}}$  of  $\bar{X}$  is not equal to the standard

deviation  $\sigma$  of the population distribution. The standard deviation of  $\bar{X}$  is equal to the standard deviation of the population divided by the square root of the sample size.

That is,

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

We also call  $\sigma_{\bar{X}}$  as a standard error of  $\bar{X}$ .

We use the above formula for standard error if the sample size is a small in comparison to the population size. The sample size is considered to be small compared to the population size if the sample size is equal to or less than 5% of the population size, that is, if

$$n/N \leq 0.05$$

If this condition does not satisfied, we use the following formula to calculate  $\sigma_{\bar{X}}$ .

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}}$$

The term  $\sqrt{\frac{N-n}{N-1}}$  is often called a **finite population correction factor**.

In most practical applications, the sample size is usually small compared to the population size.

Consequently, in most cases the formula used for calculating  $\sigma_{\bar{X}}$  is

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

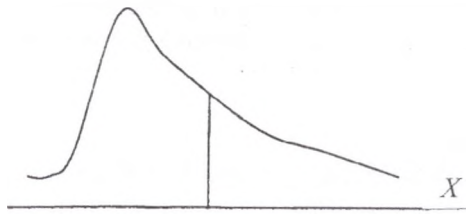
### 5.1.2. Central limit theorem

**Theorem:** Whatever the population, the distribution of  $\bar{X}$  is approximately normal when  $n$  is large. In random sampling from the population with mean  $\mu$  and standard deviation  $\sigma$ , when  $n$  is large, the distribution of  $\bar{X}$  is approximately normal with mean  $\mu$  and standard deviation  $\sigma_{\bar{X}}$ . Consequently,

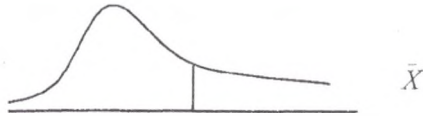
$$Z = \frac{\bar{X} - \mu}{\sigma_{\bar{X}}} = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

is approximately  $N \sim (0,1)$

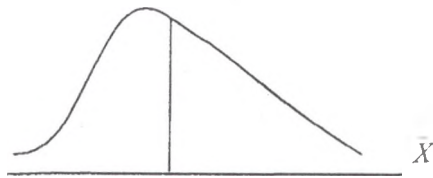
Population that is not normally distributed



Sampling distribution of  $\bar{X}$  for  $n=5$



Sampling distribution of  $\bar{X}$  for  $n=20$ .



Sampling distribution of  $\bar{X}$  for  $n=50$

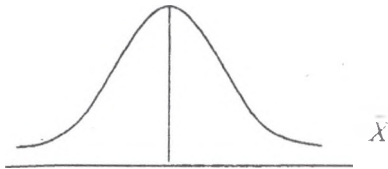


Fig. 5.2. Population distribution and sampling distributions of  $\bar{X}$ .

**Remark:**

The sample size is usually considered to be large if  $n \geq 30$ .

The central limit theorem states that the distribution of  $\bar{X}$  is normally distributed and can be used to compute a random variable,  $Z$ , with mean 0 and variance 1:

$$Z = \frac{\bar{X} - \mu}{\sigma_{\bar{X}}} = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

**Example:**

The amount of telephone bills for all households in a large city have a distribution that is skewed to the right with a mean of \$30 and a standard deviation equal to \$7. Calculate the mean and standard deviation of  $\bar{X}$  and describe the shape of its sampling distribution when the sample size is

a)  $n = 35$ ;

b)  $n = 70$

**Solution:**

Although the population distribution is not normal, in each case the sample size is large ( $n > 30$ ).

Hence the central limit theorem can be applied.

a) Let  $\bar{X}$  be the mean value of telephone bills paid by a sample of 35 households. Then the sampling distribution of  $\bar{X}$  is approximately normal with

$$\mu_{\bar{X}} = \mu = \$30 \quad \text{and}$$

$$\sigma_{\bar{X}} = \sigma / \sqrt{n} = 7 / \sqrt{35} = 1.18.$$

Figure 5.3 shows the population distribution and sampling distribution of  $\bar{X}$

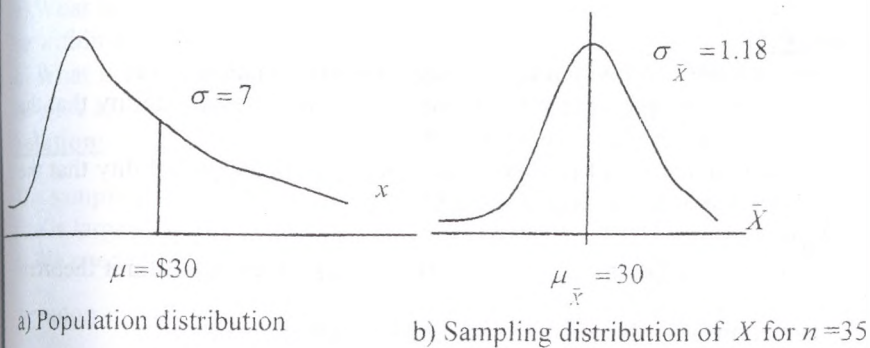


Fig. 5.3.

b) Let  $\bar{X}$  be the mean value of telephone bills paid by a sample of 70 households. Then the sampling distribution of  $\bar{X}$  is approximately normal with

$$\mu_{\bar{X}} = \mu = \$30 \quad \text{and}$$

$$\sigma_{\bar{X}} = \sigma / \sqrt{n} = 7 / \sqrt{70} = 0.84.$$

Figure 5.4 shows the population distribution and sampling distribution of  $\bar{X}$ .

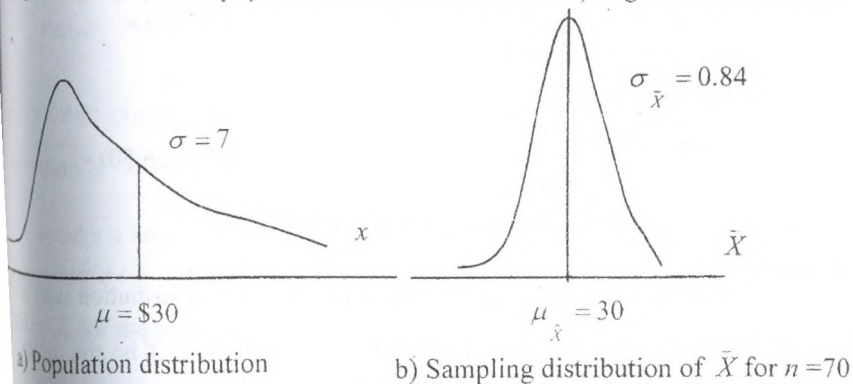


Fig. 5.4.

**Example:**

Consider a population with mean 75 and standard deviation of 11.

a) If a random sample of size 64 is selected, what is the probability that the sample mean will be between 73 and 78?

b) If a random sample of size 80 is selected, what is the probability that the sample mean will be between 68 and 83?

**Solution:**

We have  $\mu = 75$  and  $\sigma = 11$ . Since  $n = 64$  is large, the central limit theorem tells us that the distribution of  $\bar{X}$  is approximately normal.

a) To calculate  $P(73 \leq \bar{X} \leq 78)$  we convert to the standardized variable

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

The  $Z$ -values corresponding to 73 and 78 are

$$\text{For 73: } \frac{73 - 75}{11 / \sqrt{64}} = \frac{-2}{11/8} = -1.45$$

$$\text{For 78: } \frac{76 - 75}{11 / \sqrt{64}} = \frac{1}{11/8} = 0.73$$

Consequently,

$$P(73 \leq \bar{X} \leq 78) = P(-1.45 < Z < 0.73) = F(0.73) - (1 - F(1.45)) = F(0.73) - 1 + F(1.45) = 0.7673 - 1 + 0.9265 = 0.6938.$$

b) We now have  $n = 80$ .

$$P(68 \leq \bar{X} \leq 83) = P\left(\frac{68 - 75}{11 / \sqrt{80}} \leq Z \leq \frac{83 - 75}{11 / \sqrt{80}}\right) = P(-5.69 \leq Z \leq 6.50) = F(6.50) - (1 - F(5.69)) = F(6.50) - 1 + F(5.69) \approx 1.$$

**Example:**

The prices of all houses in a large city have a probability distribution with a mean of \$80 000 and a standard deviation of \$15 000. Let  $X$  be the mean price of a sample of 200 houses selected from this city.

a) What is the probability that the mean price obtained from this sample will be within \$2 000 of the population mean?

b) What is the probability that the mean price obtained from this sample will be more than the population mean, by \$1 500 or more?

**Solution:**

The sampling distribution of  $\bar{X}$  is approximately normal because the sample size is large ( $n > 30$ ).

a) We need to find the probability

$$\begin{aligned} P(78000 \leq \bar{X} \leq 82000) &= P\left(\frac{78000 - 80000}{15000/\sqrt{200}} \leq Z \leq \frac{82000 - 80000}{15000/\sqrt{200}}\right) = \\ &= P(-1.89 \leq Z \leq 1.89) = F(1.89) - (1 - F(1.89)) = \\ &= 0.9706 - 1 + 0.9706 = 0.9412. \end{aligned}$$

b) The probability that the mean price obtained from the sample of 200 houses will be more than the population mean by \$1 500 or more is written as

$$\begin{aligned} P(\bar{X} \geq 81500) &= P\left(Z \geq \frac{81500 - 80000}{15000/\sqrt{200}}\right) = P(Z \geq 1.42) = \\ &= 1 - F(1.42) = 1 - 0.9222 = 0.0778. \end{aligned}$$

**Exercises**

1. A large population has mean 90 and standard deviation 8. Calculate  $\sigma_{\bar{X}}$  and  $\sigma_{\bar{Y}}$  for a random sample of size

a)  $n=9$                       and              b)  $n=36$ .

2. A large population has a standard deviation 10. What is the standard deviation of  $\bar{X}$  for a random sample of size

a)  $n=25$ ;              b)  $n=100$ ;              and              c)  $n=400$

3. Consider a large population with  $\mu = 60$  and  $\sigma = 12$ .

Assuming  $n/N \leq 0.05$ , find the mean and standard deviation of the sample

mean  $\bar{X}$  for the sample size of

a) 18;    b) 90

**4.** A population of  $N=5000$  has a  $\sigma = 20$ . In each of the following cases which formula you use to calculate  $\sigma_{\bar{X}}$  and why?

Using the appropriate formula, calculate  $\sigma_{\bar{X}}$  for each of these cases.

a)  $n=300$ ;      b)  $n=200$ ;      c)  $n=500$ ;      d)  $n=100$ .

**5.** For a population with  $\mu = 125$  and  $\sigma = 18$

a) For a sample selected from this population,  $\mu_{\bar{X}} = 125$  and  $\sigma_{\bar{X}} = 3.6$ .

Find the sample size. Assume  $n/N \leq 0.05$ .

b) For a sample selected from this population,  $\mu_{\bar{X}} = 125$  and  $\sigma_{\bar{X}} = 2.25$ .

Find the sample size. Assume  $n/N \leq 0.05$ .

**6.** The following data show the number of automobiles owned in a population of five families:

2;      1;      0;      2;      3

a) List the ten possible samples of size 3 for this population. (Use sampling without replacement).

b) Using the ten  $\bar{X}$  values, compute the mean and variance of  $\bar{X}$ .

c) Compute the mean and variance of the population. Compare your results to those in part b. Interpret your findings.

**7.** For a population,  $N = 205000$ ,  $\mu = 66$  and  $\sigma = 7$ , find the  $Z$  values for each of the following for  $n=49$ .

a)  $\bar{X} = 68.44$ ;      b)  $\bar{X} = 58.75$ ;      c)  $\bar{X} = 62.35$

**8.** A population has a mean of 58 and a standard deviation of 12. Assuming  $n/N \leq 0.05$ , find the following probabilities for a sample size of 50.

a)  $P(53.7 \leq \bar{X} \leq 56.3)$ ;      b)  $P(\bar{X} \leq 59.2)$

**9.** Let  $X$  be a continuous random variable that has a distribution with  $\mu = 90$  and  $\sigma = 16$ . Assuming  $n/N \leq 0.05$ , find the probability that the sample mean  $\bar{X}$  for a random sample of 64 taken from this population will be

a) less than 82.3;  
b) more than 86.7.

10. The amount of electric bills for all households in a city have a skewed probability distribution with a mean of \$65 and a standard deviation of \$25. Find the probability that the sample mean amount of electric bills for a random sample of 75 households selected from this city will be

- a) more than \$70;
- b) between \$58 and \$63;
- c) within \$6 of the population mean;
- d) more than the population mean by at least \$5?

11. The balances of all saving accounts at a local bank has a mean equal \$45 360 and standard deviation equal to \$5 900. Find the probability that the mean of a sample of 80 saving accounts selected from this bank will be

- a) less than \$46 500;
- b) between \$40 000 and \$49 200;
- c) within \$1 800 of the population mean;
- d) lower than the population mean by \$1 200 or more.

12. The mean and standard deviation of the population are 55 and 7, respectively. Sample of 40 observations is selected randomly from this population.

- a) What is the probability that sample mean  $\bar{X}$  will be between 54 and 56?
- b) Find the shortest interval centered at 55, where  $\bar{X}$  will lie with probability 0.95.

13. Consider a random sample of size  $n = 100$  from a population that has a standard deviation of  $\sigma = 20$ .

- a) Find the probability that the sample mean  $\bar{X}$  will lie within 2 units of the population mean—that is,  $P(-2 \leq \bar{X} - \mu \leq 2)$ .
- b) Find the number  $k$  so that  $P(-k \leq \bar{X} - \mu \leq k) = 0.90$ .
- c) What is the probability that  $\bar{X}$  will differ from  $\mu$  by more than 4 units?

### Answers

10. a)  $\mu_{\bar{X}} = 65; \sigma_{\bar{X}} = 2.67$ ; b)  $\mu_{\bar{X}} = 65; \sigma_{\bar{X}} = 2.67$ ; c) 0.5; d) 0.5

11. a)  $\mu_{\bar{X}} = 45360; \sigma_{\bar{X}} = 662.5$ ; b)  $\mu_{\bar{X}} = 45360; \sigma_{\bar{X}} = 662.5$ ; c) 0.5; d) 0.5

**4.** a)  $\sigma_{\bar{x}} = 1.120$ ; b)  $\sigma_{\bar{y}} = 1.414$ ; c)  $\sigma_{\bar{x}} = 0.849$ ; d)  $\sigma_{\bar{y}} = 2.000$ ; **5.** a)  $n = 25$ , b)  $n = 64$ ; **7.** a)  $Z = 2.44$ ; b)  $Z = -7.25$ ; c)  $Z = -3.65$ ; **8.** a) 0.1530; b) 0.7611; **9.** a) 0.0001; b) 0.9505; **10.** a) 0.0418; b) 0.2373; c) 0.9624; d) 0.0418; **11.** a) 0.9582; b) 1.0; c) 0.9936; d) 0.0344; **12.** a) 0.6318; b) [52.82: 57.18]; **13.** a) 0.6826; b)  $k = 3.29$ ; c) 0.0456.

## 5.2. Sampling distribution of a sample proportion

### 5.2.1. Population and sample proportions

The concept of proportion is the same as the concept of relative frequency discussed in Chapter 2 and the concept of probability of success in a binomial distribution. The relative frequency of a category or class gives the proportion of the sample or population that belongs to that category or class. Similarly, the probability of success in a binomial problem represents the proportion of the sample or population that possesses a given characteristic.

The **population proportion**, denoted by  $p$ , is obtained by taking the ratio of the number of elements in a population with a specific characteristic to the total number of elements in the population.

The **sample proportion**, denoted by  $\hat{p}$  (read as "p hat") gives a similar ratio for a sample.

**Definition:** The population and sample proportions, denoted by  $p$  and  $\hat{p}$ , respectively, are calculated as

$$p = \frac{x}{N} \quad \text{and} \quad \hat{p} = \frac{x}{n}$$

where

$N$  = total number of elements in the population;

$n$  = total number of elements in the sample;

$x$  = number of elements in the population or sample that possesses a specific characteristic.

**Example:**

Suppose a total of 393 217 families live in a city and 123 017 of them own at least one car. Then,

$$N = \text{population size} = 393\,217$$

$$x = \text{families in the population who own car} = 123\,017.$$

The proportion of families in this city who own car is

$$p = \frac{x}{N} = \frac{123017}{393217} = 0.31.$$

Now, suppose that a sample of 560 families is taken from this city and 215 of them have at least one car. Then

$$n = \text{sample size} = 560$$

$$x = \text{families in the sample who own car} = 215.$$

The sample proportion is

$$p = \frac{x}{n} = \frac{215}{560} = 0.38.$$

### 5.2.2. Sampling distribution of $p$ . Its mean and standard deviation

Just like the sample mean,  $\bar{X}$ , the sample proportion  $p$  is also a random variable. Hence, it possesses a probability distribution, which is called its **sampling distribution**.

It can be shown by relying on the definition of the mean that the mean value of  $p$  - that is, the mean of all possible values of  $p$  is equal to the population proportion  $p$  just as the mean of the sampling distribution.

**Definition:**

The mean of the sample proportion  $p$  is denoted by  $\mu_p$ , and is equal to the population proportion  $p$ . Thus,

$$E(\hat{p}) = \mu_p = p.$$

The mean of all possible  $p$  values is equal to the population proportion  $p$ .

Since  $E(\hat{p}) = p$ , the sample proportion is an unbiased estimator of the population proportion.

Now we are interested in determining the standard deviation of the  $\hat{p}$  values.

Just as in the case of sample mean,  $\bar{X}$ , the standard deviation of  $\hat{p}$  depends on whether the sample size is a small proportion of the population or not.

**Definition:**

The standard deviation of the sample proportion  $\hat{p}$  is denoted by  $\sigma_{\hat{p}}$  and defined as

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{p \cdot q}{n}}$$

where

$p$  – is the population proportion,  
 $q = 1 - p$ , and  $n$  – is the sample size.

This formula is valid when  $n/N \leq 0.05$ , where  $N$  – is the population size.

If  $n/N > 0.05$ , then  $\sigma_{\hat{p}}$  is calculated as follows:

$$\sigma_{\hat{p}} = \sqrt{\frac{p \cdot q}{n}} \cdot \sqrt{\frac{N-n}{N-1}}$$

where  $\sqrt{\frac{N-n}{N-1}}$  is called the finite population correction factor.

**5.2.3. Form of the sampling distribution of  $\hat{p}$**

Now that we know the mean and standard deviation of  $\hat{p}$ , and we want to consider the form of the sampling distribution of  $\hat{p}$ . Applying the central limit theorem as it relates to the  $\hat{p}$  random variable, we have the following

**Definition:**

According to the central limit theorem, the sampling distribution of  $\hat{p}$  is approximately normal for a sufficiently large sample size. The random variable

$$Z = \frac{\hat{p} - p}{\sigma_{\hat{p}}}$$

is approximately distributed as a standard normal. This approximation is good if  $n \cdot p \cdot q > 9$ .

**Summary**

Let  $\hat{p}$  be the sample proportion of success in a random sample from a population with proportion of success  $p$ . Then

1. The sampling distribution of  $\hat{p}$  has mean  $p$

$$E(\hat{p}) = \mu_{\hat{p}} = p$$

2. The sampling distribution of  $\hat{p}$  has a standard deviation

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{p \cdot q}{n}} \quad \text{if } n/N \leq 0.05$$

$$\sigma_{\hat{p}} = \sqrt{\frac{p \cdot q}{n}} \cdot \sqrt{\frac{N-n}{N-1}} \quad \text{if } n/N > 0.05$$

3. The  $Z$  value for a value of  $\hat{p}$  is

$$Z = \frac{\hat{p} - p}{\sigma_{\hat{p}}}$$

Once again, the last approximation is good if  $n \cdot p \cdot q > 9$ .

**Example:**

The firm makes deliveries of a large number of products to its customers. It is known that 75% of all the orders it receives from its customers are delivered on time. Let  $p$  be the proportion of orders in a random sample of 120 that are delivered on time. Find the probability that the value of  $p$  will be

- a) between 0.73 and 0.80;
- b) less than 0.72.

**Solution:**

From the given information,

$$p = 0.75, \quad q = 1 - p = 1 - 0.75 = 0.25,$$

where  $p$  is the proportion of orders in the population.

The mean of the sample proportion  $p$  is

$$\mu_p = p = 0.75$$

The standard deviation of  $p$  is

$$\sigma_p = \sqrt{\frac{p \cdot q}{n}} = \sqrt{\frac{0.75 \cdot 0.25}{210}} = 0.0299.$$

Let us find  $n \cdot p \cdot q$ .

$$n \cdot p \cdot q = 210 \cdot 0.75 \cdot 0.25 = 39.38.$$

Since  $n \cdot p \cdot q = 39.38 > 9$ , we can infer from the central limit theorem that the sampling distribution of  $p$  is approximately normal.

Next, the two values of  $p$  are converted to their respective  $Z$  values by

$$Z = \frac{p - P}{\sigma_p}$$

a) For  $p = 0.73$ : 
$$Z = \frac{0.73 - 0.75}{0.0299} = -0.67.$$

For  $\hat{p} = 0.80$ ;  $Z = \frac{0.80 - 0.75}{0.0299} = 1.67$

The required probability is (Figure 5.5).

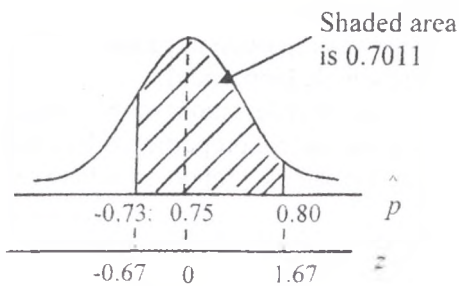


Fig.5.5  $P(0.73 < \hat{p} < 0.80)$

$$P(0.73 < \hat{p} < 0.80) = P(-0.67 < Z < 1.67) = F(1.67) - (1 - F(0.67)) = 0.9525 - 1 + 0.7486 = 0.7011.$$

Thus, the probability is 0.7011 that between 73% and 80% of orders of the sample of 210 orders will be delivered on time.

$$\begin{aligned} \text{b) } P(\hat{p} < 0.72) &= P\left(Z < \frac{0.72 - 0.75}{0.0299}\right) = P(Z < -1.01) = \\ &= 1 - F(1.01) = 1 - 0.8438 = 0.1567. \end{aligned}$$

Thus, the probability that less than 72% of the sample of 210 orders will be delivered on time is 0.1567.

### Exercises

1. For a population,  $N=40\ 000$  and  $p = 0.65$ , find the  $Z$  value for each of the following for  $n = 200$

- a)  $p = 0.59$ ;    b)  $p = 0.72$ ;    c)  $p = 0.43$ ;    d)  $p = 0.73$

2. 83% of the households of a large city own VCRs. Let  $p$  be the population proportion of the households who own VCRs in a random sample of 400 households.

Find the probability that the value of  $\hat{p}$  will be

- a) between 0.85 and 0.88
- b) more than 0.80

**3.** A doctor believes that 80% of all patients having a particular disease will be fully recovered within 3 days after receiving a new drug. Assume that a random sample of 230 patients is selected.

- a) What is the mean of the sample proportion of patients?
- b) What is the variance of the sample proportion?
- c) What is the standard error (standard deviation) of the sample proportion?
- d) What is the probability that the sample proportion is less than 0.75?
- e) What is the probability that the sample proportion is between 0.78 and 0.85?

**4.** Sixty percent of adults favor some kind of government control on the prices of medicines.

- a) Find the probability that the proportion of adults in a random sample of 200 who favor some kind of government control on the prices of medicines is
  - i) less than 0.55;
  - ii) between 0.57 and 0.68.

b) What is the probability that the proportion of adults in a random sample of 200 who favor some kind of government control is within 0.04 of the population proportion?

c) What is the probability that the sample proportion is greater than the population proportion by 0.06 or more?

**5.** Stress on the job is a major concern of a large number of people who go into managerial positions. Eighty percent of all managers of companies

suffer from stress. Let  $p$  be the proportion in a sample of 100 managers of companies who suffer from stress.

a) What is the probability that this sample proportion is within 0.08 of the population proportion?

b) What is the probability that this sample proportion is not within 0.08 of the population proportion?

c) What is the probability that this sample proportion is lower than the population proportion by 0.10 or more?

d) What is the probability that this sample proportion is greater than the population proportion by 0.11 or more?

**6.** A private university has 1250 students. Of these, 357 concerned about the GPA. A random sample of 265 students was taken.

- a) What is the standard error (standard deviation) of the sample proportion of students who are concerned about the GPA?
- b) What is the probability that the sample proportion is less than 0.35?
- c) What is the probability that the sample proportion is between 0.25 and 0.33?
7. A plant has total of 736 employees. Of these, 342 are married. A random sample of 170 employees was taken.
- a) What is the mean of the sample proportion of married employees?
- b) What is the standard error of the sample proportion of married employees?
- c) What is the probability that the sample proportion is greater than 0.37?
- d) What is the probability that the sample proportion is between 0.43 and 0.53?
8. Suppose that 78% of all adults like sport.
- a) Find the probability that the proportion of adults who like sport in a random sample of 400 is
- i) more than 0.81;                      ii) between 0.75 and 0.82
- iii) less than 0.80;                    iv) between 0.73 and 0.76
- b) What is the probability that the proportion of adults in a random sample of 400 who like sport is within 0.05 of the population proportion?
- c) What is the probability that the proportion of adults in a random sample of 400 who like sport is lower than the population proportion by 0.04 or more?

### Answers

1. a) -1.78; b) 2.08; c) -6.53; d) 2.37; 2. a) 0.1426; b) 0.9429; 3. a) 0.80; b) 0.000696; c) 0.0264; d) 0.0294; e) 0.7470; 4. a) i) 0.0735; ii) 0.7974; b) 0.754; c) 0.0418; 5. a) 0.9544; b) 0.0456; c) 0.0062; d) 0.0030; 6. a) 0.0246; b) 0.9956; c) 0.8906; 7. a) 0.4647; b) 0.0336; c) 0.9976; d) 0.8223; 8. a) i) 0.0735; ii) 0.8949; iii) 0.8289; iv) 0.8202; b) 0.9844; c) 0.0268.

## 5.2. Sampling distribution of a sample variance

Like inferences about the population means and population proportions, the population variability may also be point of interest.

In this section we consider inferences for the standard deviation  $\sigma$  of a population under the assumption that the population distribution is normal.

To make inferences about  $\sigma^2$  the natural choice of a statistic is its sample analogue, which is sample variance

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$$

One simple explanation of using  $(n-1)$  as a divisor in the formula for  $S^2$  is that in a random sample of  $n$  observations we have  $n$  different values of degrees of freedom. But we know that there are only  $(n-1)$  different values that can be uniquely defined. In addition can be shown that the expected value of the  $S^2$  computed in this way is the population variance.

The population variance and sample variance are related to a probability distribution known as the **Chi-square distribution** whose form depends on  $(n-1)$ .

### Definition:

Let  $X_1, X_2, \dots, X_n$  be a random sample from a normal population with mean  $\mu$  and standard deviation  $\sigma$ . Then the distribution

$$\chi_v^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} = \frac{(n-1) \cdot S^2}{\sigma^2}$$

is called the Chi-square distribution with  $\nu = n-1$  degrees of freedom. Unlike a normal or Student's  $t$  distribution, the probability density curve of a  $\chi^2$  distribution is an asymmetric curve stretching over the positive side of the line and having a long right tail. The form of the curve depends on the

value of the degrees of freedom. (Fig. 5.6). In this figure  $\nu = n-1$ . Table 3 in Appendix provides the upper  $\alpha$  points of  $\chi_v^2$  distributions for

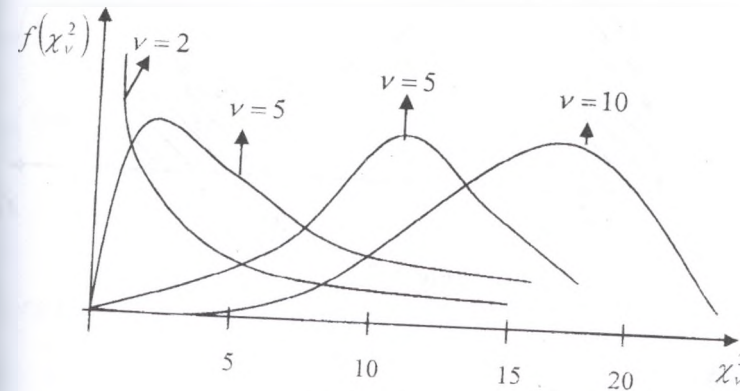


Fig. 5.6

various values of  $\alpha$  and the degree of freedom  $\nu$ . As in the both cases in the normal and the normal distributions, the upper  $\alpha$  point  $\chi_{\nu, \alpha}^2$  denotes the  $\chi_v^2$  value such that the area to the right is  $\alpha$ .

The lower  $\alpha$  point read from the column  $\chi_{\nu, 1-\alpha}^2$  in the table, have area  $(1-\alpha)$  to the right.

For example, the lower 0.05 point is obtained from the table by using the  $\chi_{\nu, 0.95}^2$  column. Whereas the upper 0.05 point is obtained by reading the column  $\chi_{\nu, 0.05}^2$ . (Fig. 5.7).

### Example:

Find  $P(\chi_{17}^2 > \chi_{17, 0.05}^2)$ .

### Solution:

The upper 0.05 point (is read from the column labeled  $\alpha = 0.05$ ), we find that  $\chi_{17, 0.05}^2 = 27.59$ .

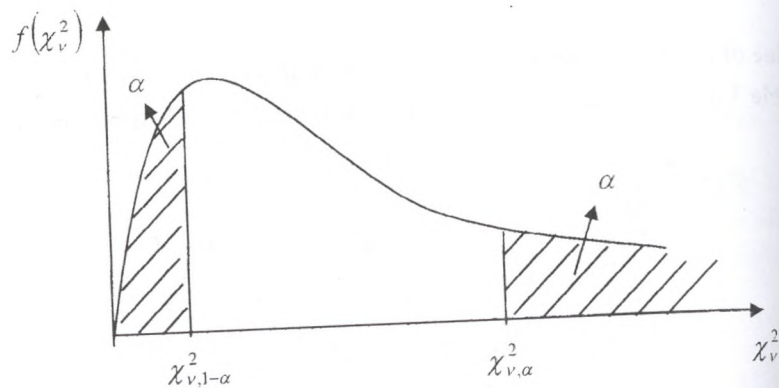


Fig. 5.7.

**Example:**

Find  $P(\chi_{17}^2 < \chi_{17,0.05}^2)$ .

**Solution:**

$$P(\chi_{17}^2 < \chi_{17,0.05}^2) = P(\chi_{17}^2 > \chi_{17,1-0.05}^2) = P(\chi_{17}^2 > \chi_{17,0.95}^2) = 8.67.$$

**Example:**

Assume that from normally distributed population with variance 2.5, sample of 26 observations are selected.

- What is the probability that sample variance is greater than 2.8?
- What is the probability that sample variance is less than 2.2?

**Solution:**

$$n = 26; \quad v = n - 1 = 25$$

$$a) P\left(\frac{(n-1) \cdot S^2}{\sigma^2} > \frac{25 \cdot 2.8}{2.5}\right) = P(\chi_{25}^2 > 28) = \text{just greater than } 0.10.$$

$$b) P\left(\frac{(n-1) \cdot S^2}{\sigma^2} < \frac{25 \cdot 2.8}{2.5}\right) = P(\chi_{25}^2 < 22) = \text{greater than } 0.90.$$

**Example:**

It is known that students scores on the final exam follow a normal distribution with standard deviation 6.6. A random sample of 25 students is taken.

- What is the probability that sample standard deviation of scores is greater than 4?

- What is the probability that sample standard deviation of scores is less than 8?

**Solution:**

$$n = 25; \quad v = 25 - 1 = 24$$

$$a) P\left(\frac{(n-1) \cdot S^2}{\sigma^2} > \frac{24 \cdot 4^2}{6.6^2}\right) = P(\chi_{24}^2 > 8.82) = \text{more than } 0.99.$$

$$b) P\left(\frac{(n-1) \cdot S^2}{\sigma^2} < \frac{24 \cdot 8^2}{6.6^2}\right) = P(\chi_{24}^2 < 35.26) = \text{between } 0.90 \text{ and } 0.95.$$

**Exercises**

1. Using the table for the  $\chi_v^2$  distribution, find

- The upper 5 % point when  $n = 9$
- The upper 1 % point when  $n = 18$
- The lower 2.5 % point when  $n = 12$
- The lower 1 % point when  $n = 24$ .

2. Find the probability of

- $\chi_{19}^2 > 30.14$
- $\chi_5^2 < 0.831$
- $3.25 < \chi_v^2 < 15.99$ , when  $n = 11$
- $3.49 < \chi_v^2 < 17.53$ , when  $n = 9$

3. The monthly salaries of part time working students follow a normal distribution with standard deviation of 3 000 tg. A random sample of 24 observations was taken.

- Find the probability that sample standard deviation is more than 4 000 tg.
- What is the probability that sample standard deviation is less than 2 000 tg?

4. A random sample of 10 households was taken. Suppose that the electric bill they pay follow a normal distribution.

- The probability is 0.1 that sample variance is greater than what percentage of the population variance?
- Find any pair of numbers,  $a$  and  $b$ , such that the following statement holds: The probability is 0.90 that the sample variance is between  $a$  % and  $b$  % of the population variance.

5. A production process follows a normal distribution. A sample of 15 produced items was selected.

a) The probability is 0.95 that the sample variance is more than what percentage of the population variance?

b) The probability is 0.99 that the sample variance is greater than what percentage of the population variance?

c) Find any pair of numbers,  $a$  and  $b$ , such that the following statement holds: The probability is 0.95 that the sample variance is between  $a$  % and  $b$  % of the population variance.

#### Answers

1. a) 15.51; b) 33.41; c) 3.82; d) 10.20; 2. a) 0.05; b) 0.025; c) 0.875; d) 0.875; 3. a) between 0.025 and 0.010; b) between 0.01 and 0.025; 4. a) 1.6311; or 163.11 %; b)  $a = 0.37$  and  $b = 1.88$ ; the probability is 0.90 that the sample variance is between 37 % and 188 % of the population variance; 5. a) 46.92% ; b) 33.28 %; c)  $a = 40.21\%$  and  $b = 186.57\%$  -the probability is 0.95 that the sample variance is between 40.21 % and 186.57 % of the population variance.

## Chapter 6 Interval estimation

### 6.1. Introduction

The problem of statistical inference arises when we wish to make generalization about a population when only a sample will be available. Once a sample is observed, its main features can be determined by the methods of descriptive summary discussed in previous chapters. Our principal concern is with not just the particular data set, but what can be said about the population based on the information extracted from analyzing the sample data.

Statistical inference deals with drawing conclusions about population parameters from an analysis of the sample data.

The value(s) assigned to a population parameters based on the value of a sample statistic is called **an estimate** of the population parameters. For example, suppose the manager selects a sample of 50 new employees and finds that the mean time  $\bar{x}$  taken to learn the job for these employees is 10 hours. If manager assigns this value to the population mean, then 10 hours will be called an estimate of  $\mu$ . Thus, the sample mean  $\bar{x}$  is an estimator of the population mean  $\mu$ , and the sample proportion  $p$  is an estimator of the population proportion  $p$ .

An estimate may be a point estimate or an interval estimate.

#### Definition:

The value of a sample statistic that is used to estimate population parameters is called a **point estimate**.

Each sample taken from a population is expected to yield a different value of the sample statistics. Thus, the value assigned to a population parameter based on the point estimate depends on which of the sample is drawn. Consequently, the point estimate assigns a value to a population parameters almost always differs from the true value of the population parameters.

In the case of **interval estimation**, instead of assigning a simple value to a population parameter, an interval is constructed around the point estimate and then a probability statement that this interval contains the corresponding population parameter is made.

**Definition:**

In interval estimation, an interval is constructed around the point estimate, and it is stated that this interval likely to contain the corresponding population parameter.

**6.2. Confidence interval and confidence level**

Since interval estimators have been described as "likely" to contain the true, but unknown value of the population parameters, it is necessary to phrase such term as probability statement.

Suppose that a random sample is selected and based on the sample information, it is possible to find two random variables  $a$  and  $b$ . Then interval extending from  $a$  and  $b$  either includes the population parameter or it does not contain population parameter. However, suppose that the random samples are repeatedly selected from the population and similar intervals are found. In the long run, a certain percentage of this interval will contain the unknown value. According to the frequency concept of probability, an interpretation of such intervals follows:

If the population is repeatedly samples and intervals calculated, then in the long run 90% (or some other percentages) of the intervals would contain the true value of the unknown parameter. The interval from  $a$  and  $b$  is said to be 90% (or some other percentages) confidence interval estimator for population parameters.

**Definition:**

Let  $\theta$  be unknown parameter. Suppose that based on sample information, random variables  $a$  and  $b$  are found such that

$$P(a < \theta < b) = 1 - \alpha,$$

where  $\alpha$  -is any number between 0 and 1.

The interval from  $a$  to  $b$  is called  $100 \cdot (1 - \alpha)\%$  **confidence interval** for  $\theta$ .

The quantity  $(1 - \alpha)$  is called the **confidence level** of the interval.

If the population were repeatedly sampled a very large number of times, the true value of the parameter  $\theta$  would be contained in  $100 \cdot (1 - \alpha)\%$  of intervals calculated this way.

The confidence interval calculated in this way is written as  $a < \theta < b$  with  $100 \cdot (1 - \alpha)\%$  confidence.

Let us find the confidence intervals with any required confidence level  $(1 - \alpha)$ , where  $\alpha$  is any number such that  $0 < \alpha < 1$ .

We will use the notation  $z_\alpha$  for the number such that

$$P(Z > z_\alpha) = \alpha.$$

A notation  $z_\alpha$  indicates the value in the standard normal table cuts off a right tail area of  $\alpha$ . (Fig. 6.1).

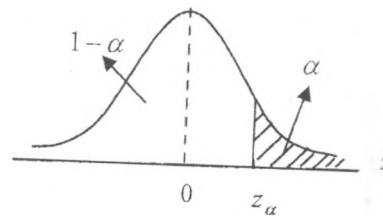


Fig.6.1.  $P(Z > z_\alpha) = \alpha$

For example, if  $\alpha = 0.13$ , then  $1 - \alpha = 0.87$ .

So,

$$F(z_\alpha) = F(z_{0.13}) = 0.87$$

and from the standard normal distribution table we obtain  $z_{0.13} \approx 1.125$ .

Therefore

$$P(Z > 1.125) = 0.13.$$

Now suppose that a  $100 \cdot (1 - \alpha)\%$  confidence interval is required. (Fig.6.2).

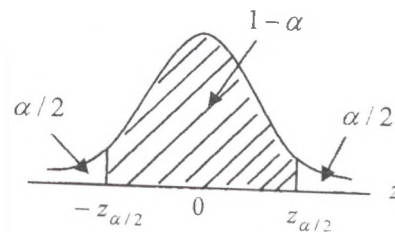


Fig.6.2  $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$

We have

$$P(Z > z_{\alpha/2}) = \alpha/2$$

By the symmetry about the mean

$$P(Z < -z_{\alpha/2}) = \alpha/2.$$

And it follows that

$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha,$$

where the random variable  $Z$  follows a standard normal distribution.

**Example:**

Find the value of  $z_{\alpha/2}$  if  $\alpha = 0.1$ .

**Solution:**

$$\alpha = 0.1$$

$$\alpha/2 = 0.05.$$

$$P(Z > z_{\alpha/2}) = P(Z > z_{0.05}) = 0.05.$$

$$P(Z > z_{0.05}) = F(z_{0.05}) = 0.95 \text{ and}$$

$$z_{0.05} = 1.645 \text{ (Fig. 6.3).}$$

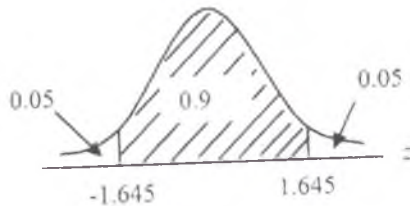


Fig.6.3

### 6.3. Confidence intervals for the mean of population that is normally distributed: population variance known

Let  $X_1, X_2, \dots, X_n$  be a random sample of  $n$  observation from a normal population with unknown  $\mu$  and known variance  $\sigma^2$ . Let  $\bar{x}$  be the sample mean. Then  $100 \cdot (1 - \alpha)\%$  confidence interval for the population mean with known variances is given by

$$\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}},$$

where  $z_{\alpha/2}$  is the number for which  $P(Z > z_{\alpha/2}) = \alpha/2$  and the random variable  $Z$  has a standard normal distribution.

#### Example:

Given a random sample of 36 observations from a normal population for which  $\mu$  is unknown and  $\sigma = 8$ , the sample mean is found to be  $\bar{x} = 45.3$ . Construct a 95% confidence interval for  $\mu$ .

#### Solution:

$$100 \cdot (1 - \alpha)\% = 95\%$$

$$1 - \alpha = 0.95$$

$$\alpha = 0.05 \quad \text{and} \quad \alpha/2 = 0.025$$

$$P(Z > z_{\alpha/2}) = P(Z > z_{0.025}) = 0.025$$

$$P(Z > z_{0.025}) = F(z_{0.025}) = 0.975$$

$$z_{\alpha/2} = z_{0.025} = 1.96$$

From

$$\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

we obtain that

$$45.3 - 1.96 \cdot \frac{8}{\sqrt{36}} < \mu < 45.3 + 1.96 \cdot \frac{8}{\sqrt{36}}$$

$$42.69 < \mu < 47.91$$

So, (42.68, 47.91) is a 95% confidence interval for  $\mu$ .

It means, if sample of 36 observations are drawn repeatedly and independently from the population, then over a very large number of repeated trials, 95% of these intervals will contain the value of the true population mean.

### Exercises

1. Find  $z_{\alpha/2}$  for each of the following confidence levels  
a) 88%;      b) 94%;      c) 96%;      d) 99%
2. The standard deviation for a population is  $\sigma = 12.6$ . A sample of 36 observations selected from this population gave a mean equal to 74.8.  
a) Make a 90 % confidence interval for  $\mu$ .  
b) Construct a 95 % confidence interval for  $\mu$ .  
c) Determine a 99 % confidence interval for  $\mu$ .
3. The standard deviation for a population is  $\sigma = 8.3$ . A sample of 121 observations selected from this population gave a mean equal to 84.5.  
a) Make a 99 % confidence interval for  $\mu$   
b) Construct a 95 % confidence interval for  $\mu$   
c) Determine a 90 % confidence interval for  $\mu$
4. The standard deviation for a population is  $\sigma = 6.30$ . A random sample selected from this population gave a mean equal to 78.90.  
a) Make a 99 % confidence interval for  $\mu$  assuming  $n = 36$   
b) Make a 99 % confidence interval for  $\mu$  assuming  $n = 81$   
c) Make a 99 % confidence interval for  $\mu$  assuming  $n = 100$

Explain your results.

5. Given that a sample of size 16 from a normal distribution yielded  $\bar{x} = 25$ . The population variance is known to be 64. Find  
a) 90 % confidence interval for  $\mu$   
b) 95 % confidence interval for  $\mu$   
c) 99 % confidence interval for  $\mu$ .

## Answers

1. a) 1.555; b) 1.88; c) 2.055; d) 2.575; 2. a) (71.35; 78.26);  
b) (70.68; 78.92); c) (69.39; 80.21); 3. a) 82.56 to 86.44; b) 83.03 to 85.97;  
c) 83.26 to 85.74; 4. a) 76.19 to 81.61; b) 77.09 to 80.71; c) 77.27 to 80.53;  
5. a) (21.71, 28.29); b) (21.08, 28.92); c) (19.85, 30.15).

### 6.4. Confidence intervals for the mean of population that is normally distributed: large sample size

Let  $X_1, X_2, \dots, X_n$  be a random sample of  $n$  observation from a normal population with unknown  $\mu$  and unknown variance  $\sigma^2$ . Let  $\bar{x}$  be the sample mean. If the sample size  $n$  is large ( $n \geq 30$ ), then according to the central limit theorem, for a large sample the sampling distribution of the sample mean  $\bar{x}$  is (approximately) normal irrespective of the shape of the population from which the sample is drawn. Therefore, when the sample size is 30 or larger, we will use the normal distribution to construct a confidence interval for  $\mu$ .

If the variance of the population is unknown, it should be estimated by the sample variance,  $s_x^2$ , and  $\sigma$  replaced by  $s_x$  in confidence interval formula for the case when population variance is known.

#### Remark:

For large sample sizes,  $s_x$  is usually close to the true value of  $\sigma$ .

An approximate  $100 \cdot (1 - \alpha)\%$  confidence interval for the population mean with unknown variance is given by

$$\bar{x} - z_{\alpha/2} \cdot \frac{s_x}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \cdot \frac{s_x}{\sqrt{n}}$$

where  $\bar{x}$  is based on a sample of at least thirty observations,  $z_{\alpha/2}$  is the number for which  $P(Z > z_{\alpha/2}) = \alpha/2$  and the random variable  $Z$  has a standard normal distribution.

**Example:**

A sample of 64 observations from a large population yielded the sample values,  $\bar{x} = 172$  and  $s_x^2 = 299$ . Find an approximate 99 % confidence interval for  $\mu$ .

**Solution:**

First we find the standard deviation of  $\bar{x}$ . Because  $\sigma$  is not known, we will use  $s_x$  as an estimator of  $\sigma$ .

$$s_x = \sqrt{s_x^2} = \sqrt{299} = 17.29.$$

Then

$$100 \cdot (1 - \alpha)\% = 99\%$$

$$1 - \alpha = 0.99$$

$$\alpha = 0.01 \quad \text{and} \quad \alpha/2 = 0.005$$

$$P(Z > z_{\alpha/2}) = P(Z > z_{0.005}) = 0.005$$

$$P(Z > z_{0.005}) = F(z_{0.005}) = 0.995$$

$$z_{\alpha/2} = z_{0.005} = 2.58$$

Substituting all the values in the formula, the 99 % confidence interval for  $\mu$  is

$$\bar{x} - z_{\alpha/2} \cdot \frac{s_x}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \cdot \frac{s_x}{\sqrt{n}}$$

$$172 - 2.58 \cdot \frac{17.29}{8} < \mu < 172 + 2.58 \cdot \frac{17.29}{8}$$

$$166.4 < \mu < 177.6$$

An approximate 99 % confidence interval for  $\mu$  is (166.4, 177.6).

**Example:**

Radiation measurements on a sample of 69 microwave ovens produced  $\bar{x} = 0.13$  and  $s_x = 0.04$ . Determine a 94 % confidence interval for the mean radiation.

**Solution:**

Again since  $n = 69 > 29$  we can use  $s_x$  as an estimator of  $\sigma$ .

Then  $100 \cdot (1 - \alpha)\%$  confidence interval for the population mean with unknown variance is given by

$$\bar{x} - z_{\alpha/2} \cdot \frac{s_x}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \cdot \frac{s_x}{\sqrt{n}}$$

$$1 - \alpha = 0.94$$

$$\alpha = 0.06 \quad \text{and} \quad \alpha/2 = 0.03$$

$$P(Z > z_{\alpha/2}) = P(Z > z_{0.03}) = 0.03$$

$$P(Z > z_{0.03}) = F(z_{0.03}) = 0.97$$

$$z_{\alpha/2} = z_{0.03} = 1.88$$

Substituting all the values in the formula we obtain

$$0.13 - 1.88 \cdot \frac{0.04}{\sqrt{69}} < \mu < 0.13 + 1.88 \cdot \frac{0.04}{\sqrt{69}}$$

$$0.121 < \mu < 0.139.$$

Thus, we can state with 94 % confidence that average radiation measure of microwave ovens is between (0.121, 0.139).

### Exercises

1. Determine a 90 % confidence interval for  $\mu$  if  $n = 48$ ,  $\bar{x} = 86.5$ , and  $s_x = 7.9$ .
2. Determine a 98 % confidence interval for  $\mu$  if  $n = 150$ ,  $\bar{x} = 0.865$ , and  $s_x = 0.057$ .
3. A sample of size 50 from a population yielded the sample values  $\bar{x} = 190$ , and  $s_x^2 = 800$ . Find a 95 per cent confidence interval for  $\mu$ .
4. For a sample data set,  $\bar{x} = 16$ , and  $s_x = 5.3$ 
  - a) Construct a 95 % confidence interval for  $\mu$  assuming  $n = 50$ .
  - b) Construct a 90 % confidence interval for  $\mu$  assuming  $n = 50$ . Is the width of the 90 % confidence interval smaller than the width of the 95 % confidence interval calculated in part a? If yes, why it is so?
  - c) Find a 95% confidence interval for  $\mu$  assuming  $n = 100$ . Is the width of the 95 % confidence interval for  $\mu$  with  $n = 100$  smaller than the width of the 95 % confidence interval for  $\mu$  with  $n = 50$  calculated in part a? If so, why?

5.

- a) A sample of 100 observations taken from a population produced a sample mean equal to 55.32 and a standard deviation equal to 8.4. Make a 90 % confidence interval for  $\mu$ .
- b) Another sample of 100 observations taken from the same population produced a sample mean equal to 57.40 and a standard deviation equal to 7.5. Make a 90 % confidence interval for  $\mu$ .
- c) A third sample of 100 observations taken from the same population produced a sample mean equal to 56.25 and a standard deviation equal to 7.9. Make a 90 % confidence interval for  $\mu$ .
- d) The true population mean for this population is 55.80. How many of the confidence intervals constructed in a-c cover this population mean and how many do not?

6. The mean annual salaries of managers at the certain company is \$80 722 for males and \$65 258 for females. These mean salaries are based on samples of 400 male and 200 female managers. Assume that the standard deviation of the annual salaries of male managers is \$11 500 and standard deviation of the female managers is \$8 400.

- a) Construct a 95 % confidence interval for the mean annual salary of male managers.
- b) Construct a 95 % confidence interval for the mean annual salary of female managers.

7. From a random sample of 70 high school seniors, the sample mean and standard deviation of the math scores are found to be 96 and 17 respectively. Determine a 96% confidence interval for the mean math score of all seniors in the school.

8. With a random sample of size  $n = 144$ , someone proposes

$$\left( \bar{x} - 0.12 \cdot s_x; \bar{x} + 0.12 \cdot s_x \right)$$

to be a confidence interval for  $\mu$ . What then is the level of confidence?

#### Answers

1. (84.63, 88.38); 2. (0.854, 0.876); 3. (182.2, 197.8); 4. a) (14.53, 17.47);  
b) (14.76, 17.24); c) (14.96, 17.04); 5. a) 53.94 to 56.70; b) 56.17 to 58.63;  
c) 54.95 to 57.55; 6. a) (\$79 595, \$81 849); b) (\$64 093, \$66 422);  
7. 91.83 to 100.17; 8. 0.8502 or about 85 %.

## 6.5. Confidence intervals for the mean of a normal distribution: population variance unknown: small sample size

In previous topics we discussed inferences about a population mean when a large sample is available. Those methods are deeply rooted in the central limit theorem, which guarantees that the distribution of  $\bar{X}$  is approximately normal.

Many investigations require statistical inferences to be drawn from small samples ( $n < 30$ ). Since the sample mean  $\bar{x}$  will still be used for inferences about  $\mu$ , we must address the question, "what is the sampling distribution of

$\bar{X}$  when  $n$  is not large?". Unlike the large sample situations, here we do not have an unqualified answer, and central limit theorem is no longer applicable.

### 6.5.1. Student's $t$ distribution

Consider a sampling situation where the population has a normal distribution with unknown  $\sigma$ . Because  $\sigma$  is unknown, an intuitive approach is to estimate  $\sigma$  by the sample standard  $S$ . Just as we did in the large sample situation, we consider the ratio

$$t = \frac{\bar{x} - \mu}{S / \sqrt{n}}$$

This random variable does not follow a standard normal distribution. Its distribution is known as **Student's  $t$  distribution**.

The graph of the  $t$ -distribution resembles the graph of the standard normal distribution: they both are symmetric, bell shaped curves with mean equal to zero. The graph of the Student's  $t$  distribution is lower at the center and higher at the extremities than the standard normal curve. (Fig. 6.4).

The new notation  $t$  is required in order to distinguish it from the standard normal variable  $Z$ . As the number of degrees of freedom increases, the difference between  $t$  distribution and the standard normal distribution becomes smaller and smaller.

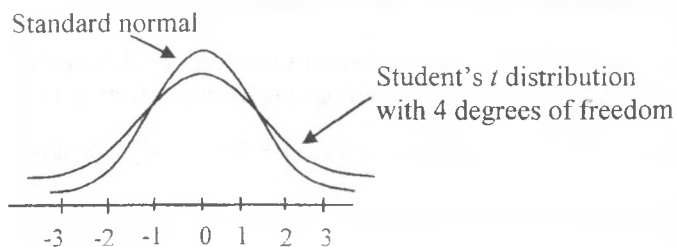


Fig.6.7.

The qualification “with  $(n - 1)$  degrees of freedom” is necessary, because with each different sample size or value of  $(n - 1)$ , there is a different  $t$  distribution.

**Definition:**

The **number of degrees of freedom** is defined as the number of observations that can be chosen freely.

**Example:**

Suppose we know that the mean number of 5 values is 25. Consequently, the sum of these 5 values is 125 ( $5 \cdot 25 = 125$ ). Now how many values out of 5 can be chosen freely so that the sum of these 5 values is 125? The answer is that we can freely choose  $5 - 1 = 4$  values. Suppose we choose 15, 35, 45, and 10 as the 4 values. Given these 4 values and the information that the mean of the 5 values is 25, the 5<sup>th</sup> value is

$$125 - (15 + 35 + 45 + 10) = 15$$

Thus, once we have chosen 4 values, the fifth value is automatically determined. Consequently, the number of degrees of freedom for this example is

$$d.f. = n - 1 = 5 - 1 = 4$$

We subtract 1 from  $n$  because we lose one degree of freedom to calculate the mean.

The  $t$ -table in the Appendix (see table 4) is arranged to give the value  $t$  for several frequently used values of  $\alpha$  and for a number of values  $(n - 1)$ .

**Definition:**

A random variable having the standard distribution with  $\nu$  ( Greek letter nu) Degrees of freedom will be denoted by  $t_\nu$  (Fig. 6.8). Then  $t_{\nu,\alpha}$  is defined as the number for which

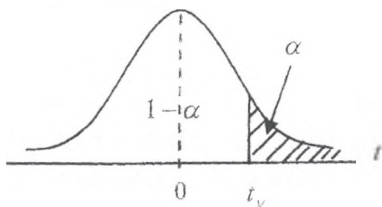


Fig.6.8

$$P(t_\nu > t_{\nu,\alpha}) = \alpha$$

**Example:** Find  $t_{5,0.10}$

**Solution:**

In words it means we need to find a number that is exceeded with the probability 0.10 by a Student's  $t$  random variable with 5 degrees of freedom.

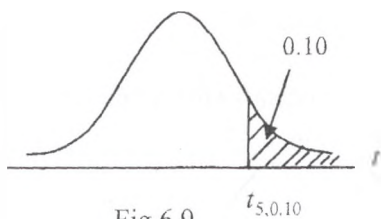


Fig.6.9

$$P(t_5 > t_{5,0.10}) = 0.10$$

From table 4 of the Appendix we read that  $t_{5,0.10} = 1.476$ . (Fig. 6.9).

Similarly, to  $z_{\alpha/2}$  for Student's  $t$  distribution the value  $t_{\nu,\alpha/2}$  is defined as

$$P(t > t_{\nu,\alpha/2}) = \alpha/2.$$

### 6.5.2. Confidence interval for $\mu$ : small samples

Let us turn our attention to finding  $100 \cdot (1 - \alpha)\%$  confidence interval for  $\mu$  when sample size is small. Using

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}}$$

We can derive the formula for  $100 \cdot (1 - \alpha)\%$  confidence interval for the case when a small sample is selected from a normally distributed population with mean  $\mu$  and unknown variance. It is given by

$$\bar{x} - t_{n-1, \alpha/2} \cdot \frac{S}{\sqrt{n}} < \mu < \bar{x} + t_{n-1, \alpha/2} \cdot \frac{S}{\sqrt{n}}$$

where  $t_{n-1, \alpha/2}$  is the number for which

$$P(t_{n-1} > t_{n-1, \alpha/2}) = \alpha/2$$

The random variable  $t_{n-1}$  has a Student's  $t$  distribution with  $\nu = (n - 1)$  degrees of freedom. (Fig. 6.10).

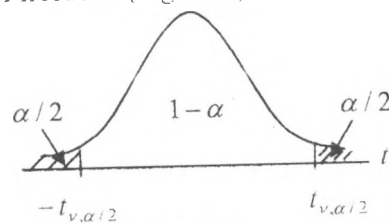


Fig.6.10

#### Remark:

If the sample is available, then standard deviation can be calculated as

$$S = \sqrt{S^2}, \text{ where } S^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} \quad \text{or}$$

$$S^2 = \frac{1}{n-1} \left[ \sum x_i^2 - n \cdot (\bar{x})^2 \right]$$

#### Example:

For the  $t$  distribution with  $n=10$ , find the number  $b$  such that  $P(-b < t < b) = 0.80$

#### Solution:

The probability in the interval  $(-b, b)$  is 0.80. (Fig. 6.11).

We must have a probability of 0.10 to the right of  $b$  and a probability of 0.10 to the left of  $-b$ .

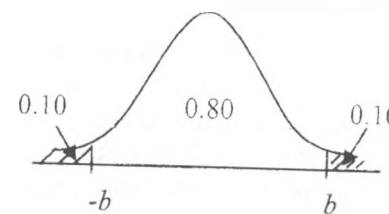


Fig.6.11

So

$$P(t_9 > t_{9,0.10}) = 0.10$$

$$t_{9,0.10} = 1.383$$

$$b = 1.383 \text{ and } -b = -1.383.$$

#### Example:

A random sample of 25 busses shows a sample mean of 225 passengers carried per day per bus. The sample standard deviation is computed to be 60 passengers. Find a 90% confidence interval for the mean number of passengers carried per bus during a 1-day period.

#### Solution:

A 90% confidence interval for the mean  $\mu$  is given by

$$\bar{x} - t_{n-1, \alpha/2} \cdot \frac{S}{\sqrt{n}} < \mu < \bar{x} + t_{n-1, \alpha/2} \cdot \frac{S}{\sqrt{n}}$$

$$n = 25, \text{ so } \nu = n - 1 = 25 - 1 = 24$$

$$100 \cdot (1 - \alpha)\% = 90\%$$

$$1 - \alpha = 0.90$$

$$\alpha = 0.10 \text{ and } \alpha/2 = 0.05$$

$$t_{n-1, \alpha/2} = t_{24, 0.05}$$

$$P(t_{24} > t_{24, 0.05}) = 0.05$$

$$t_{24, 0.05} = 1.711$$

After substitution we obtain

$$225 - 1.711 \cdot \frac{60}{5} < \mu < 225 + 1.711 \cdot \frac{60}{5}$$

$$204.5 < \mu < 245.5 \text{ or } (204.5, 245.5).$$

We are 90 % confident the mean number of passengers carried per day by bus is between 204.5 and 245.5, because 90 % of the intervals calculated in this manner will contain the true mean number of passengers carried per day per bus.

### Exercises

**1.** In each case, find the number  $b$  so that

- a)  $P(t < b) = 0.95$  when  $n = 7$
- b)  $P(-b < t < b) = 0.95$  when  $n = 16$
- c)  $P(t > b) = 0.01$  when  $n = 9$
- d)  $P(t > b) = 0.99$  when  $n = 12$

**2.** For each of the following, find the area in the appropriate tail of the  $t$  distribution

- a)  $t = 2.060$  and  $n = 26$
- b)  $t = -3.686$  and  $n = 17$
- c)  $t = -2.650$  and  $n = 15$
- d)  $t = 2.845$  and  $n = 22$

**3.** Find the value of  $t$  from  $t$  - distribution table for each of the following:

- a) Confidence level = 99 % and  $d.f. = 18$
- b) Confidence level = 95 % and  $n = 26$
- c) Confidence level = 90 % and  $d.f. = 15$

**4.** The mean number of the sample of 25 bolts produced on a specific machine per day was found to be 47 with a standard deviation of 2.4. Assume that the number of bolts produced per day on this machine has a normal distribution.

Construct a 90 % confidence interval for the population mean  $\mu$ .

**5.** A random sample of 16 cars, which were tested for fuel consumption, gave a mean of 26.4 miles per gallon with a standard deviation of 2.3 miles per gallon. Assuming that the miles per gallon given by cars have a normal distribution, find a 99 % confidence interval for the population mean  $\mu$ .

**6.** A sample of eight adults was taken, and these adults were asked about the time they spend per week on sport activities. Their responses (in hours) are as follows:

45; 12; 31; 16; 28; 14; 18; 26

Make a 95 % confidence interval for the mean of time spent per week by all adults on sport activities.

7. A sample of 10 customers who visited a supermarket was taken. The following data give the money (in dollars) they spent during that visit: 74; 89; 121; 63; 146; 47; 91; 28; 84; 76. Assuming that the money spent by all customers at this supermarket has a normal distribution, construct a 90 % confidence interval for the population mean.

8. The mean time taken to design a home plan by 20 designers was found to be 185 minutes with a standard deviation of 23 minutes. Assume that the time taken by all designers this home plan is normally distributed. Construct a 99 % confidence interval for the population mean  $\mu$ .

### Answers

1. a) 1.943; b) 2.131; c) 2.896; d) -2.718; 2. a)  $\alpha = 0.025$ ; right tail; b)  $\alpha < 0.005$ ; left tail; c) less than  $\alpha = 0.01$ ; left tail; d) less than  $\alpha = 0.005$ ; right tail; 3. a) 2.878; b) 2.060; c) 1.753; 4. a) (46.18; 47.82); 5. (24.71; 28.09); 6. (14.52; 32.98); 7. 62.24 to 101.56; 8. (170.29; 199.71).

## 6.6. Confidence intervals for population proportion: Large samples

The reason leading to estimation of a mean also applies to the problem of estimation of a population proportion.

Suppose that  $n$  elements are randomly selected from the large population. And let  $n$  consists of  $X$  elements possessing some characteristic.

Common sense suggests the sample proportion

$$p = \frac{X}{n}$$

as an estimator of  $p$ .

When the sample size  $n$  is only small fraction of the population size, the sample count  $X$  has the binomial distribution with mean  $n \cdot p$  and standard deviation  $\sqrt{n \cdot p \cdot q}$ .

When  $n$  is large, the binomial variable  $X$  is well approximated by a normal distribution with mean  $n \cdot p$  and standard deviation  $\sqrt{n \cdot p \cdot q}$ .

That is

$$Z = \frac{X - n \cdot p}{\sqrt{n \cdot p \cdot q}}$$

is approximately standard normal.

If we divide both numerator and denominator by  $n$  we will get a statement about proportions:

$$Z = \frac{(X - n \cdot p) / n}{\sqrt{n \cdot p \cdot q} / n} = \frac{p - p}{\sqrt{p \cdot q} / n}$$

As we see the denominator of  $Z$  contains  $p$  and  $q$ . If sample size is large that  $n \cdot p \cdot q > 9$ , then a good approximation is obtained if  $p$  replaces the point estimator  $p$  in the denominator.

Hence, for large sample size, ( $n \cdot p \cdot q > 9$ ) the distribution of

$$Z = \frac{p - p}{\sqrt{p \cdot q} / n}$$

is approximately standard normal.

We can use this result to obtain  $100 \cdot (1 - \alpha)\%$  confidence interval for the population proportion. (Fig.6.12)

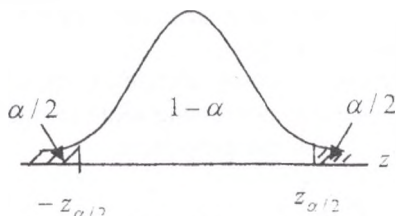


Fig.6.12

Using Fig.6.12 we obtain

$$1 - \alpha = P(-z_{\alpha/2} < Z < z_{\alpha/2}) = P(-z_{\alpha/2} < \frac{p - p}{\sqrt{p(1-p)/n}} < z_{\alpha/2}) =$$

$$= P \left( -z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < \hat{p} - p < z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right) =$$

$$= P \left( \hat{p} - z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

**Definition:**

If sample of  $n$  observations selected from the population is large enough that  $n \cdot p \cdot q > 9$ , then a  $100 \cdot (1 - \alpha)\%$  confidence interval for the population proportion is given by

$$\hat{p} - z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where  $\hat{p} = \frac{X}{n}$  is the sample proportion and  $z_{\alpha/2}$  is the number for which

$$P(Z > z_{\alpha/2}) = \frac{\alpha}{2}$$

**Example:**

From a country labor force a random sample of 800 persons was selected and 75 people were found unemployed out of random sample of 800 persons. Compute 90 % confidence interval for the rate of unemployment in the country.

**Solution:**

The confidence interval for the population proportion is obtained from

$$\hat{p} - z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

The observed  $\hat{p} = \frac{75}{800} = 0.09375$  and  $1 - \hat{p} = 0.90625$ .

Since  $n \cdot p \cdot q = 800 \cdot 0.09375 \cdot 0.90625 = 67.97 > 9$ , we say that sample size is large and a normal approximation to the sample proportion  $p$  is justified. Then

$$100 \cdot (1 - \alpha)\% = 90\%$$

$$1 - \alpha = 0.90$$

$$\alpha = 0.1 \quad \text{and} \quad \alpha / 2 = 0.05$$

$$P(Z > z_{\alpha/2}) = P(Z > z_{0.05}) = 0.05$$

$$P(Z > z_{0.05}) = F(z_{0.05}) = 0.95$$

$$z_{\alpha/2} = z_{0.05} = 1.645$$

After substituting, we obtain

$$0.09375 - 1.645 \cdot \sqrt{\frac{0.09375 \cdot 0.90625}{800}} < p < 0.09375 + 1.645 \cdot \sqrt{\frac{0.09375 \cdot 0.90625}{800}}$$

$$0.0768 < p < 0.1107$$

Therefore, a 90 % confidence interval for the rate of unemployment in the country is ( 0.0768; 0.1107), or (7.68 %; 11.07 %).

Because our procedure will produce true statements 90 % of the time, we can be 90 % confident that the rate of unemployment is between 7.68 and 11.07.

### Exercises

**1.** Check if the sample size is large enough to use the normal distribution to make a confidence interval for  $p$  for each of the following cases

a)  $n = 60$  and  $p = 0.30$

b)  $n = 180$  and  $p = 0.027$

c)  $n = 200$  and  $p = 0.73$

d)  $n = 65$  and  $p = 0.05$

**2.** A sample of 500 observations selected from a population gave a sample proportion equal to 0.72.

a) make a 90 % confidence interval for  $p$ .

b) construct a 95 % confidence interval for  $p$ .

c) make a 99 % confidence interval for  $p$ .

Interpret your results.

**3.** A sample selected from a population gave a sample proportion equal to 0.73

- a) make a 98 % confidence interval for  $p$  assuming  $n = 90$
- b) construct a 98 % confidence interval for  $p$  assuming  $n = 500$
- c) construct a 98 % confidence interval for  $p$  assuming  $n = 100$

Interpret your results.

**4.** A sample of 87 university students revealed that 53 carried their books and notes in a backpack. Obtain a 95 % confidence interval for the population of students who use backpacks.

**5.** The Beverage Company has been experiencing problems with the automatic machine that places labels on bottles. A sample of 300 bottles resulted in 27 bottles with improperly applied labels. Using these data, develop a 90 % confidence interval for the population proportion of bottles with improperly applied labels.

**6.** If 65 persons in a random sample of 180 required lawyer services, then find and interpret 96 % confidence interval for proportion of persons in the population who required a lawyer services.

**7.** Let sample proportion  $p = 0.7$ . How large a sample should be taken to be 95 % sure that the error of estimation does not exceed 0.02 when estimating a proportion?

**8.** A sample of 20 managers was taken and they were asked whether or not they usually take work home. The responses are given below:

Yes	Yes	No	No	No	Yes	No	No
No	No	Yes	Yes	No	Yes	Yes	No
No	No	No	Yes				

Make a 99 % confidence interval for the percentage of all managers who take work home.

### Answers

- 1.** a) Yes, sample size is large; b) No, sample size is not large; c) Yes, sample size is large; d) No, sample size is not large; **2.** a) ( 0.687; 0.753); b) (0.681; 0.759); c) ( 0.668; 0.772); **3.** a) (0.621; 0.839); b) (0.684; 0.776); c) (0.627; 0.833); **4.** (0.506; 0.712); **5.** (0.063; 0.117); **6.** (0.286; 0.434); **7.**  $n = 2017$ ; **8.** (0.117; 0.683).

## 6.7. Confidence intervals for the difference between means of two normal populations

Let  $\mu_X$  be the mean of the first population and  $\mu_Y$  be the mean of the second population. Suppose we want to make a confidence interval for the difference between these two population means, that is,  $\mu_X - \mu_Y$ .

Let  $\bar{X}$  be the mean of a sample from the first population and  $\bar{Y}$  be the mean of a sample taken from the second population. Then  $(\bar{X} - \bar{Y})$  is the sample statistic that is used to make an interval estimate. We will consider several cases.

### 6.7.1. Confidence intervals for the difference between means: paired samples

In the case of two dependent samples, two data values—one in each sample—are collected from the same source and these are called **paired** or **matched pairs**.

Suppose that  $n$  matched pairs of observations, denoted by  $(x_1, y_1); (x_2, y_2) \dots \dots (x_n, y_n)$ , are selected from two populations with means  $\mu_X$  and  $\mu_Y$ .

Our aim is to find  $100 \cdot (1 - \alpha)\%$  confidence interval for  $(\mu_X - \mu_Y)$ .

To find interval estimation we apply following steps:

1. Find  $n$  differences  $d_i = x_i - y_i$
2. Find  $\bar{d}$
3. Calculate  $S_d$

If the population distribution of differences is assumed to be normal, then  $100 \cdot (1 - \alpha)\%$  confidence interval for the difference between means is given by

$$\bar{d} - t_{n-1, \alpha/2} \cdot \frac{S_d}{\sqrt{n}} < \mu_X - \mu_Y < \bar{d} + t_{n-1, \alpha/2} \cdot \frac{S_d}{\sqrt{n}}$$

where  $t_{n-1, \alpha/2}$  is the number for which

$$P(t_{n-1} > t_{n-1, \alpha/2}) = \frac{\alpha}{2}$$

The random variable  $t_{n-1}$  has a Student's  $t$  distribution with  $(n - 1)$  degrees of freedom.

**Example:**

A company claims that its special exercise program significantly reduces weight. A random sample of seven persons were put on exercise program. The following table gives the weights (in kg) of those seven persons before and after the program

Before	68	81	98	86	110	92	80
After	62	76	86	79	103	87	82

Make a 95 % confidence interval for the mean of the population paired differences. Assume that the population of paired differences is (approximately) normally distributed.

**Solution:**

Let  $d$  be the difference between the weights before and after the program. The necessary calculations are shown in the following table

Before	After	Difference $d$	$d^2$
68	62	6	36
81	76	5	25
98	86	12	144
86	79	7	49
110	103	7	49
92	87	5	25
80	82	-2	4
		$\sum d = 40$	$\sum d^2 = 332$

The values of  $\bar{d}$  and  $S_d$  are calculated as follows:

$$\bar{d} = \frac{\sum d}{n} = \frac{40}{7} = 5.71$$

$$S_d = \sqrt{\frac{1}{n-1} \left[ \sum d^2 - n \cdot (\bar{d})^2 \right]} = \sqrt{\frac{1}{6} (332 - 7 \cdot 5.71^2)} = \sqrt{17.30} = 4.16.$$

Then

$$100 \cdot (1 - \alpha)\% = 90\%$$

$$1 - \alpha = 0.90$$

$$\alpha = 0.1 \quad \text{and}$$

$$\alpha/2 = 0.05$$

$$t_{n-1, \alpha/2} = t_{6, 0.05} = 1.943.$$

In the end, 90 % confidence interval for  $(\mu_X - \mu_Y)$  is

$$5.71 - 1.943 \cdot \frac{4.16}{\sqrt{7}} < \mu_X - \mu_Y < 5.71 + 1.943 \cdot \frac{4.16}{\sqrt{7}}$$

$$2.6 < \mu_X - \mu_Y < 8.82$$

Thus, we can state with 90 % confidence that the mean difference between the weights before and after exercise program is between 2.6 and 8.82 kg.

### Exercises

**1.** Find the following confidence interval for the difference between two population means assuming that the populations of paired differences are normally distributed

a)  $n = 10$        $\bar{d} = 23.6$ ;       $S_d = 12.6$ ;      confidence level = 99 %

b)  $n = 26$        $\bar{d} = 13.2$ ;       $S_d = 4.8$ ;      confidence level = 95 %

c)  $n = 14$        $\bar{d} = 46.2$ ;       $S_d = 13.6$ ;      confidence level = 90 %

**2.** A company attempts to evaluate the potential for a new bonus plan by selecting a random sample of 5 salespersons to use the bonus plan for a week period. The weekly sales volumes before and after the bonus plan implementation shown below

Weekly sales Before	Weekly sales After
15	18
12	14
18	19
15	18
16	18

Construct a 90 % confidence estimate for the mean increase in weekly sales that can be expected if a new bonus plan is implemented.

**3.** A company claims that the course they offer significantly increases the writing speed of secretaries. The following table gives the scores of eight secretaries before and after they attended this course.

Before	81	75	89	91	65	70	90	69
After	97	72	93	110	78	69	115	75

Make a 90 % confidence interval for the mean ( $\mu_x - \mu_y$ ) of the population paired differences, where a paired differences is equal to the score before attending the course minus the score after attending the course.

**4.** A company sent 7 of its employees to attend a course in building self-confidence. The following table gives the scores of these employees before and after attending the course

Before	8	5	4	9	6	8	5
After	10	7	5	11	6	7	9

Construct a 95 % confidence interval for the mean of population paired differences where a paired difference is equal to the score of an employee before attending the course minus after attending this course.

### Answers

- 1.** a) (10.65; 36.55); b) (11.26; 15.14); c) (39.76; 52.64); **2.** 1.40 to 3.00;  
**3.** (-16.54; -3.21); **4.** (-2.93; 0.07).

### 6.7.2. Confidence intervals for the difference between means of two normal populations with known variances

Suppose that the random variable  $\bar{X}$  is based on a random sample of size  $n_X$  from a normal population with mean  $\mu_X$  and known variance  $\sigma_X^2$ .

Also suppose that the random variable  $\bar{Y}$  is based on a random sample of size  $n_Y$  from a normal population with mean  $\mu_Y$  and known variance  $\sigma_Y^2$ .

The difference between population means has a mean  $(\mu_X - \mu_Y)$  and

variance  $\left(\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}\right)$ .

Therefore, the random variable

$$Z = \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}}$$

has a standard normal distribution.

We can use this fact to obtain  $100 \cdot (1 - \alpha)\%$  confidence interval for the difference between the population means.

**Definition:**

When the variances  $\sigma_X^2$  and  $\sigma_Y^2$  of two normal are known, then  $100 \cdot (1 - \alpha)\%$  confidence interval for  $(\mu_X - \mu_Y)$  is given by

$$(\bar{X} - \bar{Y}) - z_{\alpha/2} \cdot \sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}} < \mu_X - \mu_Y < (\bar{X} - \bar{Y}) + z_{\alpha/2} \cdot \sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}$$

**Example:**

A sample of size 13 from a normal population with variance 100

yielded  $\bar{X} = 31.4$ . A sample of size 7 from a second normal population with variance 80 yielded  $\bar{Y} = 38.1$ . Find a 95 % confidence interval for  $(\mu_X - \mu_Y)$ .

**Solution:**

$$100 \cdot (1 - \alpha)\% = 95\%$$

$$1 - \alpha = 0.95$$

$$\alpha = 0.05 \quad \text{and} \quad \alpha / 2 = 0.025$$

$$z_{\alpha/2} = z_{0.025} = 1.96$$

The 95 % confidence interval for  $(\mu_X - \mu_Y)$  is

$$(31.4 - 38.1) - 1.96 \cdot \sqrt{\frac{100}{13} + \frac{80}{7}} < \mu_X - \mu_Y < (31.4 - 38.1) + 1.96 \cdot \sqrt{\frac{100}{13} + \frac{80}{7}}$$

$$-15.27 < \mu_X - \mu_Y < 1.87$$

**Remark:**

When  $n_X$  and  $n_Y$  are both large, the normal approximation remains valid if

$\sigma_X^2$  and  $\sigma_Y^2$  are replaced by their estimators  $S_X^2$  and  $S_Y^2$ . When  $n_X$  and  $n_Y$  are greater than 30, an approximate  $100 \cdot (1 - \alpha)\%$  confidence interval for  $(\mu_X - \mu_Y)$  is given by

$$(\bar{X} - \bar{Y}) - z_{\alpha/2} \cdot \sqrt{\frac{S_X^2}{n_X} + \frac{S_Y^2}{n_Y}} < \mu_X - \mu_Y < (\bar{X} - \bar{Y}) + z_{\alpha/2} \cdot \sqrt{\frac{S_X^2}{n_X} + \frac{S_Y^2}{n_Y}}$$

where  $z_{\alpha/2}$  is the number for which

$$P(Z > z_{\alpha/2}) = \alpha / 2$$

and  $Z$  follows standard normal distribution.

**Example:**

A sample of 50 yogurt cups produced by the company showed that they contain an average of 146 calories per cup with a standard deviation of 6.4 calories. A sample of 60 such yogurt cups produced by its competitor showed that they contained an average of 143 calories per cup with a standard deviation of 7.2 calories. Make a 97 % confidence interval for the

difference between the mean number of calories in yogurt cups produced by the two companies.

**Solution:**

We can refer to the respective samples as sample 1 and sample 2.

Let  $\mu_X$  and  $\mu_Y$  be the means of populations 1 and 2 respectively, and

let  $\bar{X}$  and  $\bar{Y}$  be the means of the respective samples.

From the given information:

$$n_X = 50; \quad \bar{X} = 146; \quad S_X = 6.4$$

$$n_Y = 60; \quad \bar{Y} = 143; \quad S_Y = 7.2$$

Since both sample sizes are large ( $n_X > 30, n_Y > 30$ ) we can replace  $\sigma_X^2$  and  $\sigma_Y^2$  by  $S_X^2$  and  $S_Y^2$  respectively.

Then  $100 \cdot (1 - \alpha)\%$  confidence interval for  $(\mu_X - \mu_Y)$  is given by

$$(\bar{X} - \bar{Y}) - z_{\alpha/2} \cdot \sqrt{\frac{S_X^2}{n_X} + \frac{S_Y^2}{n_Y}} < \mu_X - \mu_Y < (\bar{X} - \bar{Y}) + z_{\alpha/2} \cdot \sqrt{\frac{S_X^2}{n_X} + \frac{S_Y^2}{n_Y}}$$

$$1 - \alpha = 0.97$$

$$\alpha = 0.03 \quad \text{and} \quad \alpha / 2 = 0.015$$

$$z_{\alpha/2} = z_{0.015} = 2.17$$

Finally, substituting all the values in the confidence interval formula, we obtain 97 % confidence interval for  $(\mu_X - \mu_Y)$  as

$$(146 - 143) - 2.17 \cdot \sqrt{\frac{6.4^2}{50} + \frac{7.2^2}{60}} < \mu_X - \mu_Y < (146 - 143) + 2.17 \cdot \sqrt{\frac{6.4^2}{50} + \frac{7.2^2}{60}}$$

$$0.18 < \mu_X - \mu_Y < 5.82 .$$

Thus, with 97 % confidence we can state that the difference in the mean calories of the two population of yogurt cups produced by two different companies is between 0.18 and 5.82.

### Exercises

1. A random sample of size 10 from a normal population with variance 50 gave a mean 43.2. A second random sample of size 18 from a normal population with variance 72 gave a mean 48.7. Find a 99 per cent confidence interval for the difference between two population means.

2. A random sample of size 100 yielded the sample values  $\bar{X} = 509$ ,  $S_X^2 = 950$ . A random sample size 100 from another population yielded

$\bar{Y} = 447$ ,  $S_Y^2 = 875$ . Find a 95 % confidence interval for  $(\mu_X - \mu_Y)$ .

3. An urban planning group is interested in estimating the difference between mean household incomes for two cities. Independent samples of households in two cities provide the following results:

City 1	City 2
$n_1 = 32$	$n_2 = 36$
$\bar{x}_1 = \$500$	$\bar{x}_2 = \$375$
$S_1 = \$150$	$S_2 = \$130$

Develop an interval estimate of the difference between mean incomes in the two cities. Show the results for confidence coefficients of 0.90 and 0.95.

4. The management at the National Bank investigates mean waiting time for all customers at its two branches. They took a sample of 200 customers from the branch A and found that they waited an average of 4.60 minutes with a standard deviation of 1.2 minutes before being served. Another sample of 300 customers taken from the branch B showed that these customers waited an average of 4.85 minutes with a standard deviation of 1.5 minutes before being served. Make a 97 % confidence interval for the difference between the two population means.

5. Rural and urban students are to be compared on the basis of their scores on a nationwide university entrance test. Two random samples of sizes 80 and 95 are selected from rural and urban students. The summary statistics from the test scores are

	Rural	Urban
Sample size	80	95
Mean	78.6	85.7
Standard deviation	9.1	8.3

Establish a 96 % confidence interval for the difference in population mean scores between urban and rural students.

**6.** A business consultant wanted to investigate if providing day care facilities on premises by companies reduces the absentee rate of working mothers from companies that provide day care facilities on premises. Sample of 50 mothers selected from the companies that provide day care facilities was taken. These mothers missed an average of 6.4 days from work last year with a standard deviation of 1.20 days. Another sample of 50 such mothers taken from companies that do not provide day care facilities on premises showed that these mothers missed an average of 9.3 days last year with a standard deviation of 1.83 days. Construct a 98 % confidence interval for the difference between the two population means.

#### Answers

**1.** (-13.24; 2.24); **2.** (53.63; 70.37); **3.** (68,68; 181.32); (57.89; 192.11);  
**4.** (-0.51; 0.01); **5.** (4.37; 9.83); **6.** (-3.62 to -2.18 days).

### 6. 8. Confidence interval for the difference between the population means: unknown population variances that are assumed to be equal

Let us consider the  $100(1 - \alpha)\%$  confidence interval estimation procedure for the difference between the means of the populations when the population have normal distributions with equal variances, i.e.,  $\sigma_x^2 = \sigma_y^2$ . We will again be assuming that independent random samples are selected from the populations. In this case the sampling distribution of  $(\bar{X} - \bar{Y})$  is normal regardless of the sample sizes involved. The mean of the sampling distribution is  $(\mu_x - \mu_y)$ .

Because of the equal variances  $\sigma_x^2 = \sigma_y^2 = \sigma^2$ , we can write

$$\sigma_{\bar{X} - \bar{Y}} = \sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}} = \sqrt{\sigma^2 \left( \frac{1}{n_x} + \frac{1}{n_y} \right)}$$

If the variance  $\sigma^2$ , is known, then confidence interval population means can be found easily. However, if  $\sigma^2$  is unknown, the two samples variances,  $S_x^2$  and  $S_y^2$ , can be combined to compute the following estimate of  $\sigma^2$ :

$$S^2 = \frac{(n_x - 1) \cdot S_x^2 + (n_y - 1) \cdot S_y^2}{n_x + n_y - 2}$$

The process of combining the results of the two independent samples to provide an estimate of the population variance is referred to as **polling**, and  $S^2$  is referred to as **polled** estimator of  $\sigma^2$ .

#### Definition:

Suppose that two samples of sizes  $n_x$  and  $n_y$  are selected from normally distributed population with means  $\mu_x$  and  $\mu_y$ , and a common, but unknown variance  $\sigma^2$ . If sample means are  $\bar{X}$  and  $\bar{Y}$ , sample variances are  $S_x^2$  and  $S_y^2$ , then  $100(1 - \alpha)\%$  confidence interval for  $(\mu_x - \mu_y)$  is given by

$$\left(\bar{X} - \bar{Y}\right) - t_{n_x+n_y-2, \alpha/2} \cdot S \cdot \sqrt{\frac{n_x+n_y}{n_x \cdot n_y}} < \mu_X - \mu_Y < \left(\bar{X} - \bar{Y}\right) + t_{n_x+n_y-2, \alpha/2} \cdot S \cdot \sqrt{\frac{n_x+n_y}{n_x \cdot n_y}}$$

where  $S$  is given by

$$S = \sqrt{\frac{(n_x - 1) \cdot S_x^2 + (n_y - 1) \cdot S_y^2}{n_x + n_y - 2}}$$

and  $t_{n_x+n_y-2, \alpha/2}$  is the number for which

$$P(t_{n_x+n_y-2} > t_{n_x+n_y-2, \alpha/2}) = \frac{\alpha}{2}$$

The random variable  $t$  follows to the Student's  $t$  distribution with  $(n_x + n_y - 2)$  degrees of freedom.

**Example:**

Independent random samples of checking account balances for customers at two branches of National Bank show the following results:

Bank branches	Number of checking accounts	Sample mean balance	Sample standard deviation
Bank A	12	$\bar{X} = \$1000$	$S_x = \$150$
Bank B	10	$\bar{Y} = \$920$	$S_y = \$120$

Find a 90 % confidence interval estimate for the difference between the mean checking account balances of the two branches.

**Solution:**

100(1 -  $\alpha$ )% confidence interval is

$$\left(\bar{X} - \bar{Y}\right) - t_{n_x+n_y-2, \alpha/2} \cdot S \cdot \sqrt{\frac{n_x+n_y}{n_x \cdot n_y}} < \mu_X - \mu_Y < \left(\bar{X} - \bar{Y}\right) + t_{n_x+n_y-2, \alpha/2} \cdot S \cdot \sqrt{\frac{n_x+n_y}{n_x \cdot n_y}}$$

$$S^2 = \frac{11 \cdot 150^2 + 9 \cdot 120^2}{12 + 10 - 2} = 18855$$

$$S = 137.3$$

$$n_x + n_y - 2 = 20$$

$$100(1 - \alpha)\% = 90\%$$

$$\alpha / 2 = 0.05$$

$$P(t_{n_x+n_y-2} > t_{n_x+n_y-2, \alpha/2}) = P(t_{n_x+n_y-2} > t_{20, 0.05}) = 0.05$$

$$t_{20, 0.05} = 1.725$$

Thus, the interval estimation becomes

$$(1000 - 920) - 1.725 \cdot 137.3 \cdot \sqrt{\frac{22}{120}} < \mu_X - \mu_Y < (1000 - 920) + 1.725 \cdot 137.3 \cdot \sqrt{\frac{22}{120}}$$

$$-\$21.40 < \mu_X - \mu_Y < \$181.40$$

At a 90 % level of confidence the interval estimate for the difference in mean account balances of two branches of Bank is - \$21.40 to \$181.40 .

The fact that the interval includes a negative range of values indicates that the actual difference in the two means  $(\mu_X - \mu_Y)$  may be negative.

Thus  $\mu_Y$  could be actually be larger than  $\mu_X$  .

### Exercises

1. The following information was obtained from two independent samples selected from two normally distributed populations with unknown but equal variances.

$$n_x = 20; \quad \bar{X} = 33.75; \quad S_x = 5.25$$

$$n_y = 23; \quad \bar{Y} = 28.50; \quad S_y = 4.55$$

Construct a 99 % confidence interval for  $(\mu_X - \mu_Y)$ .

2. The National Bank is interested in estimating the difference between the mean credit card balances of its two branches. Independent random samples of credit card customers provide the following results:

Branch A

$$n_1 = 32$$

$$\bar{X}_1 = \$500$$

$$S_1 = \$150$$

Branch B

$$n_2 = 36$$

$$\bar{X}_2 = \$375$$

$$S_2 = \$130$$

Construct an interval estimate of the difference between mean balances. Use a confidence coefficient of 0.99.

3. Production quantities of two workers are shown below. Each data value indicates the amount of items produced during a randomly selected 1 day period

Worker 1:	20;	18;	21;	22;	20
Worker 2:	22;	18;	20;	23;	24

Develop a 90 % confidence interval estimate for the difference between the mean production rates of the two workers.

4. The mean salary of male professors at the universities was 54 340 tg and that of female professors was 48 080 tg. For convenience, assume that these two means are based on random samples of 28 male and 26 female professors. Assume that the standard deviations of the two samples are 3 100 tg and 2 800 tg, respectively.

Construct a 90 % confidence interval for the difference between the two population means.

5. The following summary statistics are recorded for independent random samples from two populations:

Sample 1

$$n_x = 7$$

$$\bar{X} = 86.2$$

$$S_x = 14.2$$

Sample 2

$$n_y = 8$$

$$\bar{Y} = 74.7$$

$$S_y = 5.5$$

Stating any assumptions that you need, determine a 98 % confidence interval for  $(\mu_x - \mu_y)$ .

6. A company is interested in buying one of two different kinds of machines. Company tested the two machines for production purposes. The first machine was run for 8 hours and produced an average 123 items per hour with a standard deviation of 9 items. The second machine was run for 10 hours and produced an average of 114 items per hour with a standard

deviation of 6 items. Assume that the production per hour for each machines approximately normally distributed. Also assume that the standard deviation of the hourly production of the two populations is equal. Then find a 95 % confidence interval for the difference between the two population means.

### Answers

1. (2.21; 9.29); 2. (\$37.57; \$212.43); 3. (Worker 2- Worker 1): 0.87; units to 3.27 units); 4. (4909.87; to 7610.13); 5. (-2.84; 25.84); 6. (1.50 to 16.50 items).

### 6. 9. Confidence interval for the difference between the population proportions: (large samples)

As it was discussed earlier, for a large sample the sample proportion  $p$  is approximately normally distributed with mean  $p$  and standard deviation

$$\sqrt{\frac{p \cdot (1 - p)}{n}}$$

Suppose that a random sample of size  $n_X$  observations from a population

with proportion of "success"  $p_X$  has a sample proportion of success  $p_X$ , and that an independent random sample of size  $n_Y$  observations from a

population with proportion of "success"  $p_Y$  yields sample proportion  $p_Y$ .

Since  $n_X$  and  $n_Y$  both are large, their sample proportions  $p_X$  and  $p_Y$  are approximately normally distributed with means  $p_X$  and  $p_Y$ , and standard deviations

$$\sqrt{\frac{p_X(1 - p_X)}{n_X}} \text{ and } \sqrt{\frac{p_Y(1 - p_Y)}{n_Y}} \text{ respectively.}$$

Then the random variable  $p_X - p_Y$ , and the variance

$$\sigma_{p_X - p_Y}^2 = \frac{p_X(1-p_X)}{n_X} + \frac{p_Y(1-p_Y)}{n_Y}$$

The standardized random variable

$$Z = \frac{\left( \hat{p}_X - \hat{p}_Y \right) - (p_X - p_Y)}{\sqrt{\frac{p_X(1-p_X)}{n_X} + \frac{p_Y(1-p_Y)}{n_Y}}}$$

is approximately standard normal.

In order to find confidence interval for  $(p_X - p_Y)$ , we must either know or estimate the quantity of

$$\frac{p_X(1-p_X)}{n_X} + \frac{p_Y(1-p_Y)}{n_Y}$$

We can estimate the population proportion  $p_X$  by the sample proportion  $\hat{p}_X$  and we can estimate the population proportion  $p_Y$  by the sample proportion  $\hat{p}_Y$ . Then

$$\frac{p_X(1-p_X)}{n_X} + \frac{p_Y(1-p_Y)}{n_Y} \approx \frac{\hat{p}_X(1-\hat{p}_X)}{n_X} + \frac{\hat{p}_Y(1-\hat{p}_Y)}{n_Y}$$

Then an approximate  $100(1-\alpha)\%$  confidence interval is given by

$$\left( \hat{p}_X - \hat{p}_Y \right) - z_{\alpha/2} \cdot A < p_X - p_Y < \left( \hat{p}_X - \hat{p}_Y \right) + z_{\alpha/2} \cdot A$$

where

$$A = \sqrt{\frac{\hat{p}_X(1-\hat{p}_X)}{n_X} + \frac{\hat{p}_Y(1-\hat{p}_Y)}{n_Y}}$$

**Example:**

Mike and Tom like to throw darts. Mike throws 100 times and hits the target 54 times; Tom throws 100 times and hits the target 49 times. Find a 95% confidence interval for  $(p_X - p_Y)$ , where  $p_X$  represents the true proportion of hits in Mike's tosses, and  $p_Y$  represents the true proportion of hits in Tom's tosses.

**Solution:**

$$100(1 - \alpha)\% = 95\% \quad \text{and} \quad z_{0.025} = 1.96$$

$$\hat{p}_X = \frac{54}{100} = 0.54$$

$$\hat{p}_Y = \frac{49}{100} = 0.49$$

$$A = \sqrt{\frac{\hat{p}_X(1 - \hat{p}_X)}{n_X} + \frac{\hat{p}_Y(1 - \hat{p}_Y)}{n_Y}} = \sqrt{\frac{0.54 \cdot 0.46}{100} + \frac{0.49 \cdot 0.51}{100}} = 0.0706$$

$$(0.54 - 0.49) - 1.96 \cdot 0.0706 < p_X - p_Y < (0.54 - 0.49) + 1.96 \cdot 0.0706$$

$$-0.088 < p_X - p_Y < 0.188$$

Thus, with 95% confidence we can state that the difference between the proportions of Mike's and Tom's tosses is between -0.088 and 0.188.

**Exercises**

- Find a 90% confidence interval for  $p_X - p_Y$ , if a sample of size 200 yielded  $\hat{p}_X = 0.70$  and a sample of size 300 yielded  $\hat{p}_Y = 0.65$ .
- Construct a 99% confidence interval for  $p_X - p_Y$  if

$$n_X = 300; \quad \hat{p}_X = 0.53$$

$$n_Y = 200; \quad \hat{p}_Y = 0.59$$

**3.** A sample of 400 observations taken from the first population gave  $x_1 = 150$ . Another sample of 700 observations taken from the second population gave  $x_2 = 225$ . Make a 96 % confidence interval for  $p_1 - p_2$ .

**4.** A sample of 500 items produced by a supplier A possessed 270 defective items. A random sample of 360 items produced by supplier B possessed 162 defective items. Compute a 95 % confidence interval estimate for the difference in proportion defective from the two suppliers.

**5.** Assume that 66 % of single women and 81.9 % of single men own cars. Also assume that these estimates are based on random samples of 1640 single women and 1800 single men. Develop a 99 % confidence interval for the difference between the two population proportions.

**6.** The management of a market wanted to investigate if the percentage of men and women who prefer to buy national brand products over the store brand products are different. A sample of 500 men shoppers at supermarkets showed that 175 of them prefer to buy national brand products over the store brand products. Another sample of 800 women shoppers showed that 360 of them prefer to buy national brand products over the store brand products. Construct a 95 % confidence interval for the difference between the proportions of all men and women shoppers at supermarket who prefer to buy national brand products over the store brand products.

**7.** A sample of 600 females was selected from ethnic group A and a sample of 700 from ethnic group B. Each female was asked "Did you get married before you were 22?". 246 of females from group A and 266 of females from group B answered "yes". Find a 95 % confidence interval for the two population proportions.

**8.** According to a survey, 1010 adults conducted and 74.2 % of male and 88.8 % of women said that they are concerned about living near a nuclear power plant. Assume that there were 520 men and 490 women in this sample. Construct a 99 % confidence interval for the difference between the proportions of all men and all women who are concerned about living near power plant.

#### Answers

**1.** (-0.02; 0.12); **2.** (-0.18; 0.06); **3.** (-0.006; 0.116); **4.** (0.023; 0.157);  
**5.** (-0.20; -0.12); **6.** (-0.154; 0.046); **7.** (-0.02; 0.08); **8.** (-0.21; -0.08).

#### 6. 10. Confidence interval for the variance of a normal distribution

We may often need to estimate confidence interval for the population variance (or standard deviation).

Like every sample statistic, the sample variance is a random variable and it possesses a sampling distribution. If all the possible samples of a given size are taken from a population and their variances are calculated, the probability distribution of these variances is called the sampling distribution of the sample variance.

The random variable

$$\chi_{n-1}^2 = \frac{(n-1) \cdot S^2}{\sigma^2}$$

follows a Chi-square distribution with  $(n-1)$  degrees of freedom.

To find the formula for calculating Confidence intervals for the variance, it is necessary to introduce new notations.

We will denote  $\chi_{v,\alpha}^2 = \chi_{n-1,\alpha}^2$  the number for which  $P(\chi_{n-1}^2 > \chi_{n-1,\alpha}^2) = \alpha$  (Fig.6.13)

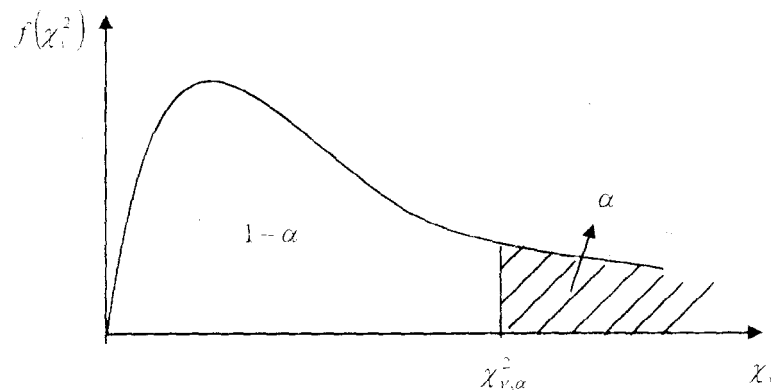


Fig. 6.13.

Similarly, it follows that  $\chi^2_{n-1, \frac{\alpha}{2}}$  is defined as  $P\left(\chi^2_{n-1} > \chi^2_{n-1, \frac{\alpha}{2}}\right) = \frac{\alpha}{2}$

And  $\chi^2_{n-1, 1-\frac{\alpha}{2}}$  is defined as  $P\left(\chi^2_{n-1} > \chi^2_{n-1, 1-\frac{\alpha}{2}}\right) = 1 - \frac{\alpha}{2}$ .

Then it follows that  $P\left(\chi^2_{n-1, 1-\frac{\alpha}{2}} < \chi^2_{n-1} < \chi^2_{n-1, \frac{\alpha}{2}}\right) = \frac{\alpha}{2}$

In the end, as it shown in Fig.6.14

$$P\left(\chi^2_{n-1, 1-\frac{\alpha}{2}} < \chi^2_{n-1} < \chi^2_{n-1, \frac{\alpha}{2}}\right) = 1 - \alpha$$

Using usual procedure we obtain  $100(1 - \alpha)\%$  confidence interval for population variance as

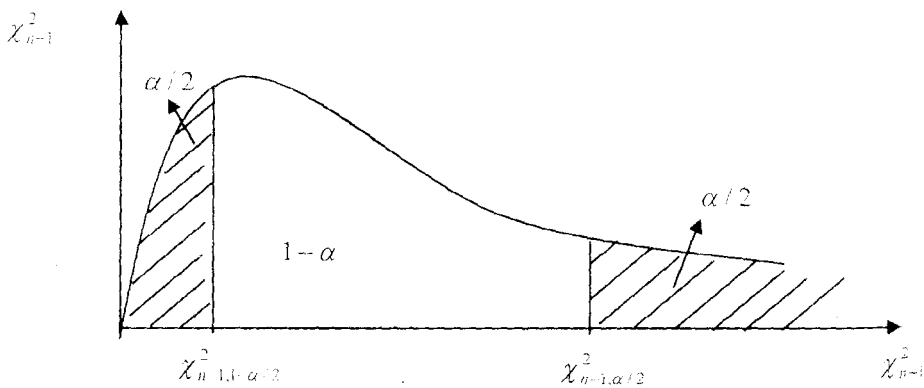


Fig. 6.14

$$\frac{(n-1) \cdot S^2}{\chi^2_{n-1, \frac{\alpha}{2}}} < \sigma^2 < \frac{(n-1) \cdot S^2}{\chi^2_{n-1, 1-\frac{\alpha}{2}}}$$

where  $\chi^2_{n-1}$  follows a Chi-square distribution with  $(n-1)$  degrees of freedom.

**Example:**

The variance in drug weights is very critical in the pharmaceutical industry. For a specific drug, with weights measured in grams, a sample of 18 units provided a sample variance of  $S^2 = 0.36$

a) Construct a 90 % confidence interval estimate for the population variance for the weights of this drug.

b) Construct a 90 % confidence interval estimate for the population standard deviation for the weights of this drug.

**Solution:**

From the given information

$$n = 18; \quad v = n - 1 = 17; \quad S^2 = 0.36;$$

and for a 90 % confidence interval,  $\alpha = 0.1$ ;  $\alpha / 2 = 0.05$ .

100 · (1 -  $\alpha$ )% confidence interval for  $\sigma^2$  is given by

$$\frac{(n-1) \cdot S^2}{\chi^2_{n-1, \frac{\alpha}{2}}} < \sigma^2 < \frac{(n-1) \cdot S^2}{\chi^2_{n-1, 1-\frac{\alpha}{2}}}$$

a) From the Chi-square table we obtain that

$$\chi^2_{n-1, \alpha/2} = \chi^2_{17, 0.05} = 27.59;$$

$$\chi^2_{n-1, 1-\alpha/2} = \chi^2_{17, 0.95} = 8.67$$

After substitution we obtain

$$\frac{17 \cdot 0.36}{27.59} < \sigma^2 < \frac{17 \cdot 0.36}{8.67}$$

$$0.22 < \sigma^2 < 0.71$$

b) We can obtain the confidence interval for the population standard deviation  $\sigma$  by taking the positive square root of the two limits of the above confidence interval for the population variance. Thus, 90 % confidence interval estimate for the population standard deviation is

$$\sqrt{0.22} < \sigma < \sqrt{0.71}$$

$$0.47 < \sigma < 0.84$$

Hence, the standard deviation of all investigated drug is between 0.47 and 0.84 grams at a 90 % confidence level.

### Exercises

**1.** The sample of 15 bus arrivals showed a sample variance at  $S^2 = 4.2$  min.  
a) Construct a 95 % confidence interval of the variance for the population of arrival times.

b) Suppose that the sample variance of  $S^2 = 3.5$  min. had been obtained from a sample of 26 bus arrivals. Determine a 95 % confidence interval of the variance for the population of arrival times.

**2.** From production process a random sample of 25 a certain brand of light bulbs was taken. The variance of the lives of these bulbs was found to be 4710 hours. Assume that the lives of all such bulbs are approximately normally distributed.

a) Make a 99 % confidence interval for the variance and standard deviation of the lives of all such bulbs.

**3.** The time required to complete a certain operation by the sample of 25 employees of auditing company has a standard deviation of 3.1 min. Construct a 98 % confidence interval for  $\sigma$ .

**4.** From a data set of  $n=10$  observation, one has calculated the 95 % confidence interval for  $\sigma$  and obtained the result (0.81; 2.15).

Calculate a 90 % confidence interval for  $\sigma$ .

**5.** Given the sample data

12, 18, 9, 15, 14

Construct a 95 % confidence interval for  $\sigma$ .

**6.** A sample of 7 observations taken from a population produced the following data

10, 8, 13, 15, 6, 8, 13

Make the 98 % confidence intervals for the population variance and standard deviation.

**7.** A random sample of 25 customers taken from the bank gave the variance of the waiting times equal to 7.9 min. Construct 99 % confidence intervals for the population variance.

**8.** Suppose that based on a random sample of size 10 from a normal distribution, one has found the 95 % confidence interval for the population mean to be (36.2; 45.8). Using this result determine a 95 % confidence interval for the population standard deviation.

### Answers

**1.** a) (2.25; 10.44); b) (2.15; 6.67); **2.** a) (2481.1; 11429.7) and (49.8; 106.9); **3.** (2.32; 4.61) **4.** (0.86; 1.94); **5.** (2.25; 9.66); **6.** (3.91; 75.36) and (1.98; 8.68); **7.** (4.16; 19.18); **8.** (4.62; 12.25).

### 6.11. Sample size determination

The reason why we always conduct a sample observations and not a census is that almost always we have limited resources at our disposal. In our calculations, if a smaller sample can serve our purpose, then we will be wasting our resources by taking a larger sample. For example, suppose we want to estimate the mean life of certain type of lights bulbs. If a sample of 50 light bulbs can give us the type of confidence interval that we are looking for, then we will be wasting money and time if we take a sample of much larger size, say 800 light bulbs. In such cases if we know the confidence interval that we want, then we can find the (approximate) size of the sample that will produce the required result.

#### 6.11.1. Sample size determination for the estimation of mean

Suppose that sample of  $n$  observations is taken from a normally distributed population with mean  $\mu$  and known variance  $\sigma^2$ . We know that  $100 \cdot (1 - \alpha)\%$  confidence interval for the population mean  $\mu$  is given by

$$\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

where  $\bar{x}$  is the sample mean and  $z_{\alpha/2}$  is the appropriate cutoff point of the standard normal distribution. This confidence interval is centered on the

sample mean and extends a distance of  $L$ , the margin of error (also called the sample error, the bound, or the interval half width) is given by

$$L = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

Suppose that we predetermine the size of  $L$  and want to find the size of the sample that will yield this margin error. From the above expression, the following formula is obtained that determines the required sample size  $n$ .

**Definition:**

Given the confidence level and standard deviation of the population (or population variance), the sample size that will produce a predetermined margin error  $L$  of the confidence interval estimate of  $\mu$  is

$$n = \frac{(z_{\alpha/2})^2 \cdot \sigma^2}{L^2}$$

**Remark 1:**

If we do not know  $\sigma$ , we can take a sample and find sample standard deviation. Then we can use  $S$  for  $\sigma$  in the formula.

**Remark 2:**

$n$  must be rounded to the next higher integer, because a sample size can not be fractional.

**Example:**

Suppose that we want to estimate the mean family size for all country families at 99 % confidence level. It is known that the standard deviation  $\sigma$  for the sizes of all families in the country is 0.45.

How large a sample should we select if we want its estimate to be within 0.02 of the population mean?

**Solution:**

We want the 99 % confidence interval for the mean family size to be

$$\bar{x} \pm 0.02.$$

Hence, the margin of errors is to be 0.02, that is

$$L = 0.02$$

The value of  $z_{\alpha/2}$  for a 99 % confidence level is 2.58.

The value of  $\sigma$  is given to be 0.45. Therefore, substituting all values in the formula and simplifying, we obtain

$$n = \frac{(z_{\alpha/2})^2 \cdot \sigma^2}{L^2} = \frac{(2.58)^2 \cdot (0.45)^2}{(0.02)^2} = 3369.8 \approx 3370$$

Thus, the required sample size is 3370. If we will take a sample of 3370 families, compute the mean family size for this sample, and then margin of a 99% confidence interval around this sample, the margin of error of the estimate will be approximately 0.02.

### 6.11.2. Sample size determination for the estimation of proportion

Just as we did with the mean, we can also determine the sample size for estimating the population proportion  $p$ .

We know that  $100 \cdot (1 - \alpha)\%$  confidence interval for  $p$  is given by

$$p - z_{\alpha/2} \cdot \sqrt{\frac{p(1-p)}{n}} < p < p + z_{\alpha/2} \cdot \sqrt{\frac{p(1-p)}{n}}$$

where  $p$  is the sample proportion.

This interval is centered on the sample proportion and extends a distance  $L$ :

$$L = z_{\alpha/2} \cdot \sqrt{\frac{p(1-p)}{n}}$$

This result can not be used directly to determine the sample size  $n$  necessary

to obtain a confidence interval of some specific width, since it involves  $p$ ,

which is not known. But whatever the outcome,  $p(1-p)$  can not be bigger

than 0.25, its value when the sample proportion is 0.5. Thus, the largest

possible value for  $L$  is given by

$$L = z_{\alpha/2} \cdot \sqrt{\frac{0.25}{n}} = \frac{0.5 \cdot z_{\alpha/2}}{\sqrt{n}}$$

Using basic algebra, we obtain

$$\sqrt{n} = \frac{0.5 \cdot z_{\alpha/2}}{L}$$

and squaring yields

$$n = \frac{0.25 \cdot (z_{\alpha/2})^2}{L^2}$$

**Definition:**

Let a random sample be selected from a normal population.  $100 \cdot (1 - \alpha)\%$  confidence interval for the population proportion, extending a distance of at most  $L$  on each side of the sample proportion, can be guaranteed if the sample size is

$$n = \frac{0.25 \cdot (z_{\alpha/2})^2}{L^2}$$

**Example:**

A public health survey is to be designed to estimate the proportion  $p$  of a population having defective vision. How many persons should be examined if the public health doctor wishes to be 98 % certain that error of estimation is below 0.05?

**Solution:**

The public health doctor wants the 98 % confidence interval to be  $p \pm 0.05$ . Therefore  $L = 0.05$ . The value of  $z_{\alpha/2}$  for a 98 % confidence level is 2.33. After substituting we obtain that the required sample size is

$$n = \frac{0.25 \cdot (z_{\alpha/2})^2}{L^2} = \frac{0.25 \cdot (2.33)^2}{(0.05)^2} = 542.89 \approx 543.$$

Thus, if the doctor takes a sample of 543 persons, the estimate of  $p$  will be within 0.05 of the population proportion.

**Exercises**

**1.** Determine the sample size for the estimate of  $\mu$  for the following:

- |                 |                     |                        |
|-----------------|---------------------|------------------------|
| a) $L = 0.17$ ; | $\sigma = 0.90$ ;   | confidence level = 99% |
| b) $L = 4.1$ ;  | $\sigma = 23.45$ ;  | confidence level = 95% |
| c) $L = 25.9$ ; | $\sigma = 122.25$ ; | confidence level = 90% |

**2.** Determine the most conservative sample size for estimation of the population proportion for the following:

- |                  |                         |
|------------------|-------------------------|
| a) $L = 0.025$ ; | confidence level = 99 % |
| b) $L = 0.045$ ; | confidence level = 96 % |
| c) $L = 0.015$ ; | confidence level = 90 % |

3. A sample of 50 workers' average weekly earnings gave  $\sigma = \$35$ . Determine the sample size that is needed for estimating the population mean weekly earnings with a 98 % error margin of \$ 3.50.
4. How large a sample should be taken to be 95 % sure that the error of estimation does not exceed 0.02 when estimating a population proportion?
5. A food service manager wants to be 95 % confident that the error in the estimate of the mean number of sandwiches dispensed over the lunch hour is 10 or less. What sample size should be selected if  $\sigma = 40$ ?
6. One department manager wants to estimate at 90 % confidence level the mean amount spent by all customers at this store. He knows that the standard deviation of amounts spent by customers at this store is \$ 27. What sample size he chooses so that the estimate is within \$ 3 of the population mean?
7. A teacher wants to estimate the proportion of all students who own mobile telephones. How large should the sample size be so that the 99 % confidence interval for the population proportion has a maximum error of 0.03?
8. A private university wants to determine a 99 % confidence interval for the mean number of hours that students spend per week doing homework. How large a sample should be selected so that the estimate is within 1 hour of the population mean? Assume that the standard deviation for the time spent per week doing homework by students is 3 hours.

#### Answers

1. a) 186; b) 126; c) 61; 2. a) 2653; b) 437; c) 3007; 3. 543; 4. 2401; 5. 62; 6. 220; 7. 1842; 8. 60.

# APPENDIX

$\lambda$	$e^{-\lambda}$
0.00	1.000000
0.10	.904837
0.20	.818731
0.30	.740818
0.40	.670320
0.50	.606531
0.60	.548812
0.70	.496585
0.80	.449329
0.90	.406570
1.00	.367879
1.10	.332871
1.20	.301194
1.30	.272532
1.40	.246579
1.50	.223130
1.60	.201897
1.70	.182684
1.80	.165299
1.90	.149569
2.00	.135335
2.10	.122456
2.20	.110803
2.30	.100259
2.40	.090518
2.50	.082085

Table 1: Values of  $e^{-x}$

$\lambda$	$e^{-\lambda}$	$\lambda$	$e^{-\lambda}$	$\lambda$	$e^{-\lambda}$
2.60	.074274	5.10	.006097	7.60	.000501
2.70	.067206	5.20	.005517	7.70	.000453
2.80	.060810	5.30	.004992	7.80	.000410
2.90	.055023	5.40	.004517	7.90	.000371
3.00	.049787	5.50	.004087	8.00	.000336
3.10	.045049	5.60	.003698	8.10	.000304
3.20	.040762	5.70	.003346	8.20	.000275
3.30	.036883	5.80	.003028	8.30	.000249
3.40	.033373	5.90	.002739	8.40	.000225
3.50	.030197	6.00	.002479	8.50	.000204
3.60	.027324	6.10	.002243	8.60	.000184
3.70	.024724	6.20	.002029	8.70	.000167
3.80	.022371	6.30	.001836	8.80	.000151
3.90	.020242	6.40	.001661	8.90	.000136
4.00	.018316	6.50	.001503	9.00	.000123
4.10	.016573	6.60	.001360	9.10	.000112
4.20	.014996	6.70	.001231	9.20	.000101
4.30	.013569	6.80	.001114	9.30	.000091
4.40	.012277	6.90	.001008	9.40	.000083
4.50	.011109	7.00	.000912	9.50	.000075
4.60	.010052	7.10	.000825	9.60	.000068
4.70	.009095	7.20	.000747	9.70	.000061
4.80	.008230	7.30	.000676	9.80	.000056
4.90	.007447	7.40	.000611	9.90	.000050
5.00	.006738	7.50	.000553	10.00	.000045

1.21	.8869	1.46	.9279	1.71	.9564	1.96	.9750	2.21	.9864	2.46	.9931
1.22	.8888	1.47	.9292	1.72	.9573	1.97	.9756	2.22	.9868	2.47	.9932
1.23	.8907	1.48	.9306	1.73	.9582	1.98	.9761	2.23	.9871	2.48	.9934
1.24	.8925	1.49	.9319	1.74	.9591	1.99	.9767	2.24	.9875	2.49	.9936
1.25	.8944	1.50	.9332	1.75	.9599	2.00	.9772	2.25	.9878	2.50	.9938
1.26	.8962	1.51	.9345	1.76	.9608	2.01	.9778	2.26	.9881	2.51	.9940
1.27	.8980	1.52	.9357	1.77	.9616	2.02	.9783	2.27	.9884	2.52	.9941
1.28	.8997	1.53	.9370	1.78	.9625	2.03	.9788	2.28	.9887	2.53	.9943
1.29	.9015	1.54	.9382	1.79	.9633	2.04	.9793	2.29	.9890	2.54	.9945
1.30	.9032	1.55	.9394	1.80	.9641	2.05	.9798	2.30	.9893	2.55	.9946
1.31	.9049	1.56	.9406	1.81	.9649	2.06	.9803	2.31	.9896	2.56	.9948
1.32	.9066	1.57	.9418	1.82	.9656	2.07	.9808	2.32	.9898	2.57	.9949
1.33	.9082	1.58	.9429	1.83	.9664	2.08	.9812	2.33	.9901	2.58	.9951
1.34	.9099	1.59	.9441	1.84	.9671	2.09	.9817	2.34	.9904	2.59	.9952
1.35	.9115	1.60	.9452	1.85	.9678	2.10	.9821	2.35	.9906	2.60	.9953
1.36	.9131	1.61	.9463	1.86	.9686	2.11	.9826	2.36	.9909	2.61	.9955
1.37	.9147	1.62	.9474	1.87	.9693	2.12	.9830	2.37	.9911	2.62	.9956
1.38	.9162	1.63	.9484	1.88	.9699	2.13	.9834	2.38	.9913	2.63	.9957
1.39	.9177	1.64	.9495	1.89	.9706	2.14	.9838	2.39	.9916	2.64	.9959
1.40	.9192	1.65	.9505	1.90	.9713	2.15	.9842	2.40	.9918	2.65	.9960
1.41	.9207	1.66	.9515	1.91	.9719	2.16	.9846	2.41	.9920	2.66	.9961
1.42	.9222	1.67	.9525	1.92	.9726	2.17	.9850	2.42	.9922	2.67	.9962
1.43	.9236	1.68	.9535	1.93	.9732	2.18	.9854	2.43	.9925	2.68	.9963
1.44	.9251	1.69	.9545	1.94	.9738	2.19	.9857	2.44	.9927	2.69	.9964
1.45	.9265	1.70	.9554	1.95	.9744	2.20	.9861	2.45	.9929	2.70	.9965

Table 2: Cumulative distribution function of the standard normal distribution

z	F(z)	z	F(z)	z	F(z)	z	F(z)	z	F(z)	z	F(z)
.00	.5000	.10	.5398	.20	.5793	.30	.6179	.40	.6554	.50	.6915
.10	.5398	.11	.5438	.21	.5832	.31	.6217	.41	.6608	.51	.6955
.20	.5793	.12	.5478	.22	.5875	.32	.6255	.42	.6651	.52	.6995
.30	.6179	.13	.5517	.23	.5918	.33	.6293	.43	.6693	.53	.7035
.40	.6554	.14	.5557	.24	.5960	.34	.6331	.44	.6735	.54	.7075
.50	.6915	.15	.5596	.25	.6001	.35	.6368	.45	.6776	.55	.7115
.60	.7277	.16	.5636	.26	.6041	.36	.6405	.46	.6817	.56	.7154
.70	.7643	.17	.5675	.27	.6080	.37	.6441	.47	.6857	.57	.7193
.80	.8013	.18	.5714	.28	.6119	.38	.6477	.48	.6897	.58	.7232
.90	.8388	.19	.5753	.29	.6158	.39	.6512	.49	.6936	.59	.7271
1.00	.8764	.20	.5793	.30	.6197	.40	.6547	.50	.6975	.60	.7310
1.10	.9141	.21	.5832	.31	.6236	.41	.6582	.51	.7014	.61	.7349
1.20	.9518	.22	.5871	.32	.6275	.42	.6617	.52	.7053	.62	.7388
1.30	.9893	.23	.5910	.33	.6314	.43	.6652	.53	.7092	.63	.7427
1.40	1.0267	.24	.5949	.34	.6353	.44	.6687	.54	.7131	.64	.7466
1.50	1.0641	.25	.5988	.35	.6392	.45	.6722	.55	.7170	.65	.7505
1.60	1.1014	.26	.6027	.36	.6431	.46	.6757	.56	.7209	.66	.7544
1.70	1.1387	.27	.6066	.37	.6470	.47	.6792	.57	.7248	.67	.7583
1.80	1.1760	.28	.6105	.38	.6509	.48	.6827	.58	.7287	.68	.7622
1.90	1.2133	.29	.6144	.39	.6548	.49	.6862	.59	.7326	.69	.7661
2.00	1.2506	.30	.6183	.40	.6587	.50	.6897	.60	.7365	.70	.7700
2.10	1.2879	.31	.6222	.41	.6626	.51	.6932	.61	.7404	.71	.7739
2.20	1.3252	.32	.6261	.42	.6665	.52	.6967	.62	.7443	.72	.7778
2.30	1.3625	.33	.6300	.43	.6704	.53	.7002	.63	.7482	.73	.7817
2.40	1.3998	.34	.6339	.44	.6743	.54	.7037	.64	.7521	.74	.7856
2.50	1.4371	.35	.6378	.45	.6782	.55	.7072	.65	.7560	.75	.7895
2.60	1.4744	.36	.6417	.46	.6821	.56	.7107	.66	.7599	.76	.7934
2.70	1.5117	.37	.6456	.47	.6860	.57	.7142	.67	.7638	.77	.7973
2.80	1.5490	.38	.6495	.48	.6899	.58	.7177	.68	.7677	.78	.8012
2.90	1.5863	.39	.6534	.49	.6938	.59	.7212	.69	.7716	.79	.8051
3.00	1.6236	.40	.6573	.50	.6977	.60	.7247	.70	.7755	.80	.8090

Table 3: Chi-square distribution

$\alpha$												
	.995	.990	.975	.950	.900	.100	.050	.025	.010	.005		
	0.0393	0.0157	0.0982	0.0393	0.0158	2.71	3.84	5.02	6.63	7.88		
	0.0100	0.0201	0.0506	0.103	0.211	4.61	5.99	7.38	9.21	10.60		
	0.072	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.34	12.84		
	0.207	0.297	0.484	0.711	1.064	7.78	9.49	11.14	13.28	14.86		
	0.412	0.554	0.831	1.145	1.61	9.24	11.07	12.83	15.09	16.75		
	0.676	0.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55		
	0.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28		
	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.96		
	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59		
	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19		
	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.73	26.76		
	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30		
	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82		
	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14	31.32		
	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80		
	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27		
	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41	35.72		
	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16		
	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19	38.58		
	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00		
	8.03	8.90	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40		
	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80		
	9.26	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18		
	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56		
	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93		
	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64	48.29		
	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.64		
	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99		
	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34		
	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67		
	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77		
	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15	79.49		
	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95		
	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.4	104.2		
	51.17	53.54	57.15	60.39	64.28	96.58	101.9	106.6	112.3	116.3		
	59.20	61.75	65.65	69.16	73.29	107.6	113.1	118.1	124.1	128.3		
	67.33	70.06	74.22	77.93	82.36	118.5	124.3	129.6	135.8	140.2		

Table 2: Cumulative distribution function of the standard normal distribution (continue)

z	F(z)	z	F(z)	z	F(z)	z	F(z)	z	F(z)	z	F(z)	z	F(z)
2.71	.9966	2.91	.9982	3.11	.9991	3.31	.9995	3.51	.9998	3.71	.9999	3.91	1.0000
2.72	.9967	2.92	.9982	3.12	.9991	3.32	.9996	3.52	.9998	3.72	.9999	3.92	1.0000
2.73	.9968	2.93	.9983	3.13	.9991	3.33	.9996	3.53	.9998	3.73	.9999	3.93	1.0000
2.74	.9969	2.94	.9984	3.14	.9992	3.34	.9996	3.54	.9998	3.74	.9999	3.94	1.0000
2.75	.9970	2.95	.9984	3.15	.9992	3.35	.9996	3.55	.9998	3.75	.9999	3.95	1.0000
2.76	.9971	2.96	.9985	3.16	.9992	3.36	.9996	3.56	.9998	3.76	.9999	3.96	1.0000
2.77	.9972	2.97	.9985	3.17	.9992	3.37	.9996	3.57	.9998	3.77	.9999	3.97	1.0000
2.78	.9973	2.98	.9986	3.18	.9993	3.38	.9996	3.58	.9998	3.78	.9999	3.98	1.0000
2.79	.9974	2.99	.9986	3.19	.9993	3.39	.9997	3.59	.9998	3.79	.9999	3.99	1.0000
2.80	.9974	3.00	.9986	3.20	.9993	3.40	.9997	3.60	.9998	3.80	.9999		
2.81	.9975	3.01	.9987	3.21	.9993	3.41	.9997	3.61	.9998	3.81	.9999		
2.82	.9976	3.02	.9987	3.22	.9994	3.42	.9997	3.62	.9999	3.82	.9999		
2.83	.9977	3.03	.9988	3.23	.9994	3.43	.9997	3.63	.9999	3.83	.9999		
2.84	.9977	3.04	.9988	3.24	.9994	3.44	.9997	3.64	.9999	3.84	.9999		
2.85	.9978	3.05	.9989	3.25	.9994	3.45	.9997	3.65	.9999	3.85	.9999		
2.86	.9979	3.06	.9989	3.26	.9994	3.46	.9997	3.66	.9999	3.86	.9999		
2.87	.9979	3.07	.9989	3.27	.9995	3.47	.9997	3.67	.9999	3.87	.9999		
2.88	.9980	3.08	.9990	3.28	.9995	3.48	.9997	3.68	.9999	3.88	.9999		
2.89	.9981	3.09	.9990	3.29	.9995	3.49	.9998	3.69	.9999	3.89	1.0000		
2.90	.9981	3.10	.9990	3.30	.9995	3.50	.9998	3.70	.9999	3.90	1.0000		

**Table 4: Cut-off point of Student's  $t$  distribution**

$v$	$\alpha$				
	0.100	0.050	0.025	0.010	0.005
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.160
12	1.356	1.782	2.179	2.681	3.057
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
40	1.303	1.684	2.021	2.423	2.704
60	1.296	1.671	2.000	2.390	2.660
$\infty$	1.282	1.645	1.960	2.326	2.576

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