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Generalization of Hurwitz Theorem

In this paper I consider the proof of the main theorem of my diploma thesis.

Let $$f(z)=a_0+a_1 z+\ldots+a_n z^n \qquad (1)$$

be a polynomial with real coefficients. Let's consider the following polynomials

$$p_r(z)=f(z+r), \qquad (2)$$

$$q_r(z)=f(-z+r). \qquad (3)$$

The following lemmas will help us:

Lemma 3.1.1

a). Coefficients of polynomial (2) are connected with coefficients of polynomial (1) by the formulas:

$$b_k=\sum_{l=k}^{n} C_l^{l-k} r^{l-k} a_l, \qquad k=0,1,\ldots,n,$$

where $C_l^i=\dfrac{l!}{i!(l-i)!}$.

b). Coefficients of polynomial (3) are connected with coefficients of polynomial (1) by the formulas:

$$c_k=(-1)^k\sum_{l=k}^{n} C_l^{l-k} r^{l-k} a_l, \qquad k=0,1,\ldots,n.$$

Proof.

a). By definition

$$p_r(z)=f(z+r)=\sum_{l=0}^{n} a_l (z+r)^l=\sum_{l=0}^{n} a_l (z+r)^l. \qquad (4)$$

Let's consider polynomials $(z+r)^l$. It is clear that if $l<k$ then the polynomial $(z+r)^l$ does not contain z^k, i.e. z^k the zero coefficient. So z^k has nonzero coefficients after opening brackets in $(z+r)^l$ if and only if $k\leq l\leq n$. By Newton binomial formula we have:

$$(z+r)^k=C_k^0 z^k+kz^{k-1}r+\ldots+r^k$$

....

$$(z+r)^l=z^l+\ldots+C_l^{l-k} r^{l-k} z^k+\ldots+r^l$$

....

$$(z+r)^n=z^n+\ldots+C_n^{n-k} r^{n-k} z^k+\ldots+r^n.$$

So the corresponding coefficients of z^k will be

$$C_k^0$$

....

$$C_l^{l-k} r^{l-k} \qquad (5)$$

....

$$C_n^{n-k} r^{n-k}.$$

Then from (4) and (5) we have

$$b_k=\sum_{l=k}^{n} C_l^{l-k} r^{l-k} a_l.$$

b). Again by definition we have

$$q_r(z)=f(-z+r)=\sum_{l=0}^{n} a_l (-z+r)^l. \qquad (6)$$

If we repeat the explanations we use in a) we receive that coefficients of z^k in the polynomial $q_r(z)$ are equal to

$$c_k=(-1)^k C_l^{l-k} r^{l-k} a_l, \quad k=0,1,\ldots,n.$$

Now we may formulate and prove the main theorem of this paper.

Theorem 3.1.1

Roots of the polynomial $f(z)$ belong to

а). The semi-plane $\operatorname{Re} z < r$ if and only if the polynomial $p_r(z)$ is the Hurwitz polynomial.

б). The semi-plane $\operatorname{Re} z > r$ if and only if the polynomial $q_r(z)$ is the Hurwitz polynomial.

в). The domain $r_1 < \operatorname{Re} z < r_2$ if and only if the polynomials $q_{r_1}(z)$ and $p_{r_2}(z)$ are the Hurwitz polynomials.

Proof.

а). Let α be a root of the polynomial $f(z)$, i.e. $f(\alpha) = 0$. Then no problem to see that $\alpha - r$ is the root of the polynomial $p_r(z) = f(z + r)$ as

$$p_r(\alpha - r) = f(\alpha - r + r) = f(\alpha) = 0.$$

So if $\alpha_1, \ldots, \alpha_n$ are all roots of the polynomial $f(z)$ then $\alpha_1 - r, \ldots, \alpha_n - r$ are all roots of the polynomial $p_r(z)$. So the polynomial $p_r(z)$ is the Hurwitz polynomial if and only if

$$\operatorname{Re}(\alpha_k - r) = \operatorname{Re}\alpha_k - r < 0, \quad k = 0, 1, \ldots, n,$$

or

$$\operatorname{Re}\alpha_k < r, \quad k = 0, 1, \ldots, n.$$

b). If α is a root of the polynomial $f(z)$ then $-\alpha + r$ is the root of the polynomial $q_r(z)$ as

$$q_r(-\alpha + r) = f(-(-\alpha + r) + r) = f(\alpha) = 0.$$

So if $\alpha_1, \ldots, \alpha_n$ are all roots of the polynomial $f(z)$ then $-\alpha_1 + r, \ldots, -\alpha_n + r$ are all roots of the polynomial $q_r(z)$. So $q_r(z)$ is the Hurwitz polynomial if and only if

$$\operatorname{Re}(-\alpha_k + r) = -\operatorname{Re}\alpha_k + r < 0, \quad k = 0, 1, \ldots, n$$

or

$$\operatorname{Re}\alpha_k > r, \quad k = 0, 1, \ldots, n.$$

в). The proof of this item immediately follows from items a) and b) of our theorem.

Now let's consider the following general problem:

Let $f(z)$ be a polynomial with real coefficients, $y = kx + b$ be a line in the complex plane $z = x + yi$. Find the necessary and sufficient conditions the coefficients of the polynomial $f(z)$ satisfy such that all roots of the polynomial belong to

а). In the semi-plane $y > kx + b$.

б). In the semi-plane $y < kx + b$.

The following example shows that in general case this problem has no solution.

Example. Let's consider the line $y = 0$. If $f(z)$ is any polynomial with real coefficients and α is a complex root of $f(z)$, i.e. $\operatorname{Im}\alpha \neq 0$ then by theorem1.12. it follows that conjugate number $\bar{\alpha}$ is the root of $f(z)$ too. So all roots of any polynomial with real coefficients can not belong to the semi-plane $y > 0$ or to the semi-plane $y < 0$.

References

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Резюме

В этой работе доказывается, что если и то корни многочлена принадлежат:
а). полуплоскости тогда и только тогда, когда многочлен Гурвица.
б). полуплоскости тогда и только тогда, когда многочлен Гурвица.
в). области, если и только если полиномы и полиномы Гурвица.

Resume

In this paper we prove that if $f(z) = a_0 + a_1 z + \ldots + a_n z^n$, $p_r(z) = f(z + r)$ *and* $q_r(z) = f(-z + r)$
then roots of the polynomial $f(z)$ *belong to*
а). The semi-plane $\operatorname{Re} z < r$ *if and only if the polynomial* $p_r(z)$ *is the Hurwitz polynomial.*
б). The semi-plane $\operatorname{Re} z > r$ *if and only if the polynomial* $q_r(z)$ *is the Hurwitz polynomial.*
в). The domain $r_1 < \operatorname{Re} z < r_2$ *if and only if the polynomials* $q_{r_1}(z)$ *and* $p_{r_2}(z)$ *are the Hurwitz polynomials.*

Özet

Bu yazıda ve eğer kanıtlamak sonra polinom köklerini ait
а). yarı-düzlem ve eğer polinom Hurwitz polinom yalnızca.
б). yarı-düzlem ve eğer polinom Hurwitz polinom yalnızca.
в). etki alanı ancak ve ancak polinomları ve eğer vardır Hurwitz polinomları.

Тіл – халықтың мәдениеті мен қуатының көрінісі.

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