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КУРСТЫҚ ЖҰМЫС

WEIGHTS OF PARTITIONS

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ABSTRACT

Studying Sn-module structure of any algebra is the one of the most important problem in algebra. The weight function gives a good classification of Sn-module structure of Novikov algebras. Image of weight function defines which Specht modules appear in the algebra, moreover, it is a good tool to determine isomorphism between submodules of Novikov algebras and permutation modules.

The main part of diploma gives some usefull and interesting properties of weight function. It is defined that if the great common divisor of all parts of a partition is more that one then the partition does not belong to the set of image of the weight.

Also, it is found a criteria minimal element with respect to dominance order in the image of weight. That gives huge help to define admissible partitions.

However, there remain some important questions in studying this function.

АННОТАЦИЯ

Изучение Sn-модуля любой алгебры является одним из наиболее важных задач алгебры.Весовая функция дает хорошую классификацию Sn-модульной структуры алгебры Новикова. Образ весовая функция определяет, какие модули Шпехта появляются в алгебре, более того, это хороший инструмент для определения изоморфизма подмодулей алгебры Новикова и перестановки модулей.

Основная часть диплома дает некоторые полезные и интересные свойства весовой функции. Например, что веса общий делитель, то веса не имеют обратного элемента.

Кроме того, он нашел критерий минимального элемента по отношению к господству порядка в образе веса. Это дает огромную помощь для определения допустимых разложений.

Однако остаются некоторые важные вопросы в изучении этой функции.

АННОТАЦИЯ

Салмақты функция мәндер облысы алгебрада қайсы Шпехт модульдері пайда болатының анықтайды, одан басқа, бұл Новиков алгебраның субмодулімен алмастырулар модулі арасында изоморфизм құрастыру үшін жақсы құрал.

Бұл ғылыми жұмыс салмақты функция туралы пайдалы және маңызды мағлұмат береді. Егер жіктеудің бүкіл қосындыларының ортақ бөлгіші бірден үлкен болса, онда сол жіктеу салмақты функцияның мәніне жатпайды.

Одан басқа, мәндер облысында мажорлау бойынша минимал элементін табу критерийі анықталған. Бұның бізге мүмкін болатын бөлшектерін анықтауда үлкен көмегі тиеді.

Бірақ, бұл функцияны зерттеуде бірқатар маңызды сұрақтар қалуда.

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1. INTRODUCTION

A Novikov algebra is a special kind of a pre-Lie algebra, or left-symmetric algebra, arising in many contexts in mathematics and physics.

Studying Sn-module structure of any algebra is the one of the most important problem in algebra. The weight function gives a good classification of sn-module structure of novikov algebras. Image of weight function defines which specht modules appear in the algebra, moreover, it is a good tool to determine isomorphism between submodules of novikov algebras and permutation modules.

2.1 NOVIKOV ALGEBRA

A Novikov algebra is a special kind of a pre-Lie algebra, or left-symmetric algebra, arising in many contexts in mathematics and physics. Pre-Lie algebras already have been introduced by Cayley in 1896 via rooted tree algebras. Vinberg classified convex homogeneous cones using pre-Lie algebras, Milnor and Auslander discovered the connection to affinely flat manifolds and their fundamental groups. Recently Connes, Kreimer and Kontsevich introduced pre-Lie algebras in mathematical physics, for quantum field theory and renormalization theory. Also Bakalov and Kac have used pre-Lie algebras in the study of vertex algebras. For a survey on this topic see [3]. On the other hand, Novikov algebras in particular were introduced in the study of Hamiltonian operators in the context of integrability of certain nonlinear partial differential equations.

Letkbe a field of characteristic zero. A Novikov algebra and, more generally, an LSA is defined as follows:

Definition

An algebra (A,·) over k with product $(\boldsymbol{x}, \boldsymbol{y}) \to \boldsymbol{x} \cdot \boldsymbol{y}$ is called a left-symmetric algebra(LSA), if the product is left-symmetric, i.e., if the identity

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z \qquad (1)$$

is satisfied for all $x, y, z \in A$.

The algebra is called Novikov, if in addition

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y \qquad (2)$$

is satisfied.

Denote by L(x), R(x) the left, respectively right multiplication operator in the algebra(A,·). Then an LSA is a Novikov algebra if the right multiplications commute

$$[R(x), R(y)]=0.$$

It is well known that LSAs are Lie-admissible algebras: the commutator

$$[x, y] = x \cdot y - y \cdot x \qquad (3)$$

defines a Lie bracket. The associated Lie algebra is denoted by g_A. The adjoint operator can be expressed by ad(x)=L(x)−R(x).

If A is a Novikov algebra, then we obtain, by expanding the condition

$$0 =[R(x), R(y)]=[L(x)-ad(x), L(y)-ad(y)],$$

the following operator identity:

$$L([x, y])+ad([x, y])-[ad(x), L(y)]-[L(x),ad(y)]=0. \qquad (4)$$

Definition 1.2. Anaffine structure on a Lie algebra g over k is a left-symmetric product $g \times g \to g$ satisfying (3) for all $x, y \in g$. If the product is Novikov, we say that gadmits a Novikov structure.

A given Lie algebra need not admit a Novikov structure, or an affine structure. The existence question for affine structures is very hard in general. It is more accessible for Novikov structures.

2.2 S_{n+1} submodules of Novikov algebras.

Abstract. An algebra with two identities

1) $a(bc) = b(ac)$ (right-commutative)
2) $a(bc) - (ab)c = a(cb) - (ac)b$ (left-symmetric)

is called Novikov.

A structure of multilinear part of a free Novikov algebra as a module of permutation group is studied.

Let V be a variety of algebras and $F_n(V)$ be its free algebra generated by n elements. All algebras and modules are considered over a field K of characteristic 0. Let $F_n(V)$ be multilinear part of it. It has natural structures of modules over permutation group S_n and over general linear group GL_n.

These module structures for many classes of algebras are known and well used in different branches of mathematics and physics. For example, if $V = Ass$ be a class of associative algebras, then $F_n(Ass)$ is $n!$-dimensional and as S_n-module is isomorphic to a regular module. The same is true for Leibniz and Zinbiel algebras. If $V = Lie$, then $dim F_n(Lie) = (n-1)!$ and as S_n-module is isomorphic to a module induced by 1-dimensional module of cyclic group of order n ([12]), ([15]). S_n module structure of anti-commutative algebras is studied in ([7]).

If $V = Bicom$ is a class of *bicommutative* algebras. It means

1) $a(bc) = b(ac)$,
2) $(ab)c = (ac)b$.

Then dim $F_n(Bicom) = 2^n - 2$ and as Sn-module is

$$F_n(Bicom) \cong \begin{cases} 1, & \text{if } n = 1, \\ 2\bigoplus_{i=1}^{\frac{n-1}{2}} Ind_{S_{(n-i,i)}}^{S_n}(1), & \text{if } n > 1 \text{ is odd}, \\ Ind_{S_{(\frac{n}{2},\frac{n}{2})}}^{S_n}(1) \oplus 2\bigoplus_{i=1}^{\frac{n-2}{2}} Ind_{S_{(n-i,i)}}^{S_n}(1), & \text{if } n \text{ is even}, \end{cases}$$

where 1 is the trivial Sn module ([5]).

In our paper we consider a variety of Novikov algebras. To define this class let us introduce non-associative non-commutative polynomials of degree three *rsym* (right-symmetric polynomial) and *lcom* (left-commutative polynomial) by

$$rsym = t_1(t_2t_3) - t_1(t_3t_2) - (t_1t_2)t_3 + (t_1t_3)t_2,$$

$$lcom = t_1(t_2t_3) - t_2(t_1t_3).$$

An algebra with identities $rsym = 0$ and $lcom = 0$ is called right- Novikov.

Example. Let $A = C[x]$ and $\circ\, b = (a)'b$, where $(a)'$ is partial derivation. Then $(A;\circ)$ is Novikov algebra.

Let $F_{n+1} = F_{n+1}(Nov)$ be free Novikov algebra generated by n + 1 elements $a_1, \dots, a_{n+1}$. Let $F_{n+1} = F_{n+1}(Nov)$ be its multilinear part.

The aim of our paper is to study S_{n+1} module structure of F_{n+1}.

Free Novikov algebras were described in [9]. In [10] bases of free Novikov algebras in terms of Young diagrams of degree n are constructed. In our paper this base is used intensively.

S_{n+1} submodules of Novikov algebras and permutation modules.

Now consider F_{n+1} as S_{n+1} modules with a natural action

$$\sigma X(a_1, \ldots, a_{n+1}) = X(a_{\sigma(1)}, \ldots, a_{\sigma(n+1)}).$$

Let F be a subspace of F_{n+1} generated by all base elements of a form $e_{\beta f}$ such that $\alpha > \beta$ and all $e_{\alpha f}$. Then

$$\mathcal{F}_{(n)} = F_{n+1}.$$

In this section we prove that F is a Sn+1 submodule of Fn+1.
Recall that if F is a Sn+1 submodule of Fn+1, then Fn+1=F is also a Sn+1 submodule such that

$$F_{n+1} \cong \mathcal{F}_\alpha \oplus (F_{n+1}/\mathcal{F}_\alpha)$$

as Sn+1 modules.

Defnition. Let $\alpha, \beta \vdash n$, β is called precedent of α if $\alpha > \beta$ and there is no $\gamma \vdash n$ such that $\alpha > \gamma > \beta$ and denote $\alpha \succ \beta$.

Since > is a total order on P(n); the precedent of any partition is uniquely determined. Define F_α by

$$F_\alpha = \mathcal{F}_\alpha / \mathcal{F}_\beta$$

where $\alpha \succ \beta$ and $F_{(1^n)} = \mathcal{F}_{(1^n)}$.
Then, we obtain

$$F_{n+1} \cong \bigoplus_{\alpha \vdash n} F_\alpha$$

as Sn+1-modules and

$$F_\alpha = \{e_{\alpha,f} + \mathcal{F}_\beta \,|\alpha \succ \beta, f - filling\}$$

As Sn+1 modules. Finally, we see that F_α is isomorphic to the permutation module corresponding to $w(\alpha)$, which is decomposed to Specht modules by Young's rule.

2.3 Kostka numbers

In mathematics, the Kostka number $K_{\lambda\mu}$ (depending on two integer partitions λ and μ) is a non-negative integer that is equal to the number of semistandard Young tableaux of shape λ and weight μ. They were introduced by the mathematician Carl Kostka in his study of symmetric functions (Kostka (1882)).

For example, if λ = (3, 2) and μ = (1, 1, 2, 1), the Kostka number $K_{\lambda\mu}$ counts the number of ways to fill a left-aligned collection of boxes with 3 in the first row and 2 in the second row with 1 copy of the number 1, 1 copy of the number 2, 2 copies of the number 3 and 1 copy of the number 4 such that the entries increase along columns and do not decrease along rows. The three such tableaux are shown

1	3	3
2	4	

1	2	3
3	4	

1	2	4
3	3	

So there is three semistandard Young tableaux, $K_{(3,2)(1,1,2,1)} = 3$.

Kostka coefficients appear in combinatorics and representation theory and they are very important from a physical point of view. The Kostka number $K_{\pi,\mu}$ is the number of semi-standard Young tableaux of shape π and content μ (see Section 2 below). It is also the multiplicity with which the weight μ appears in the irreducible representation of $sl_n(\mathbb{C})$ with highest weight π. Similarly, in the representation theory of symmetric groups we encounter the Kostka numbers as the multiplicity of irreducible character χ_μ in the induction of the principal character of the Young subgroup S_π to S_m. The Kostka coefficients are also important in the study of Schur functions. This work is devoted to computing these numbers in some special cases using an elementary method

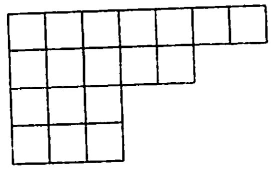
Let m be a positive integer. By a partition of m, we mean an s-tuple π = $[a_1, a_2, \ldots, a_s]$ of positive integers with

$$a_1 \geq a_2 \geq \cdots \geq a_s$$

and

$$a_1 + a_2 + \cdots + a_s = m.$$

We say that s is the height of π and we denote it by h(π). To any partition π of m we associate a Ferrer's diagram consisting of m boxes arranged in s rows in such a way that the ith row contains a_i, boxes. For example, the following is the Ferrer's diagram associated with the partition π = [7, 5, 3, 3]:



Now, let μ = $[b_1, b_2, \dots, b_r]$ be another partition of m. We say that π majorizes μ, if for any i we have

$$b_1 + b_2 + \cdots + b_i \leq a_1 + a_2 + \cdots + a_i .$$

In this case, we write $\mu \trianglelefteq \pi$. This is a partial ordering on the set of all partitions of m. The partition [m] is the maximum element and the partition $[1^m] = [1, 1, \dots, 1]$ is the minimum element with respect to this ordering. If $\pi = [a_1, a_2, \dots, a_s]$ and $\mu = [b_1, b_2, \dots, b_r]$ are two arbitrary partitions of m, then by a semi-standard Young tableau of shape π and content μ, we mean any distribution of the numbers 1, 2, . . . , m in the boxes of the associated Ferrer's diagram of π in such a way that

1. every row is non-decreasing;
2. every column is (strictly) increasing; and
3. for any $1 \leq i \leq m$, the multiplicity of i in the distribution is b_i.

Example 2.1. Let m = 5, π = [3, 2] and μ = [2, 2, 1]. Then the only semi-standard Young tableaux of shape π and content μ are the following two diagrams:

1	1	2
2	3	

1	1	3
2	2	

The number of all semi-standard Young tableaux of shape π and content μ is denoted by $K_{\pi,\mu}$ and it is called the Kostka coefficient or the Kostka number. This combinatorial notion has a wide range of interpretations in the representation theory of Lie groups and Lie algebras as well as the representation theory of the symmetric group.

The Kostka numbers are also very important in the study of symmetric functions, especially Schur functions. It is well-known that $K_{\pi,\mu} \neq 0$ if and only if $\mu \trianglelefteq \pi$.

2.4 Partition of number n into k different parts.

A partition of a positive integer n is a non-increasing sequence whose sum equals n. We define p(n) as the number of partitions of n and for convenience, we define p(0)=1. Various properties of p(n) have been studied in many ways. For the literature, consult [27]. In this note, we will investigate $p(k,n)$ which counts the number of partitions of n into k different parts.

For example, there are 7 partitions of 5 as follows:
5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.
Therefore, $p(1,5) = 2, p(2,5) = 5$ and $p(k,5) = 0$ for all $k > 2$.

Since many important partition functions such as the ordinary partitions and over partitions can be expressed in terms of $p(k,n)$, it is interesting to find how to find $p(k,n)$. We will express $p(k,n)$ in terms of the number of divisor function. The form of its generating function looks very interesting in the sense that the generating function looks reminiscent of the Jacobi–Trudi identity for Schur polynomials and the algorithm to find coefficients in the generating function is reminiscent of the Pieri rule. For the reference on symmetric functions, consult [32].

Before stating the generating series for $p(k,n)$, we need to introduce some notation. We denote the partition λ of neither as $1^{l_1}, 2^{l_2}, \ldots$ as a sum of a non-increasing sequence. Here, l_i in the first representation is the multiplicity of the part i. Thus, for the partition λ of n, it is $n = \sum j l_j$.

If λ is a partition of n, then we denote $\lambda \vdash n$ and we define the weight of a partition |λ| as the number being partitioned. In other words, if $\lambda \vdash n$, then $|\lambda| = n$. The following Lambert series $D(j,q)$ will play a key role in this article as it will serve as a building block for the generating function for $p(k,n)$:

$$D(j,q) = \sum_{i=1}^{\infty} \frac{q^{ji}}{(1-q^i)^j}.$$

Then, for a given partition $\lambda = 1^{l_1}, 2^{l_2}, \ldots$, we define $D(\lambda, q)$ as

$$D(\lambda,q) = \prod_{j=1}^{\infty} D(j,q)^{\ell_j}.$$

We define $P(k,q) = \sum_{n=1}^{\infty} p(k,n) q^n$ as a generating function for $p(k,n)$.

Now we are ready to claim first theorem.

Theorem 1.For all positive integers k,

$$P(k,q) = \sum_{\lambda \vdash k} d_\lambda D(\lambda,q), \tag{1.1}$$

where the sum runs all partitions of k and d_λ is a constant defined by

$$\frac{(-1)^{|\lambda| - \#(\lambda)}}{\prod j^{\ell_j} \ell_j!}.$$

Here, $\#(\lambda)$ denotes the number of parts in the partition λ.

Remark. If λ is a partition of n and λ has k different parts, then n should be larger than or equal to $1+2+3+\cdots+k=\frac{k(k+1)}{2}$.
Therefore, $p(k,n)=0$ if $n<\frac{k(k+1)}{2}$.
Therefore, the coefficient of q^k in P(k,q) is 0 if $k\geq 2$.
Since the coefficient of q^k of $D(\lambda,q)$ is 1 if $\lambda\vdash k$, we see that

$$\sum_{\lambda\vdash k} d_\lambda = 0,$$

for all $k\geq 2$. Moreover, we see that $k!\,|d\lambda|$ is the number of permutations of $|\lambda|$ with cycle type λ.
Therefore, we obtain

$$\sum_{\lambda\vdash k} |d_\lambda| = 1,$$

for all $k\geq 1$.
For example, here are some $P(k,q)$ for small values of k:

$$P(1,q)=D(1^1,q),$$

$$P(2,q)=\frac{1}{2}D(1^2,q)-\frac{1}{2}D(2^1,q),$$

$$P(3,q)=\frac{1}{6}D(1^3,q)-\frac{1}{2}D(2^1 1^1,q)+\frac{1}{3}D(3^1,q).$$

Before stating our next proposition which gives the coefficients of D(λ,q), we need to introduce a notation. For a fixed positive integer j, we define

$$d^{(j)}(n)=\sum_{d|n}(n-1)(n-2)\cdots(n-j+1), \tag{1.2}$$

for $j\geq 2$ and $d^{(1)}(n)=d(n)=\sum_{d|n} d$
Remark. We can express $d^{(j)}(n)$ as a linear combination of $\sigma_j(n)$'s.
Here, $\sigma_j(n)=\sum_{d|n} d^j$
Proposition 2. Let $\lambda=j_1+j_2+\cdots+j_l$ be a partition of k.
If we denote the n-th coefficient of power series expansion for D(λ,q) as τ(λ,n), then

$$\tau(\lambda,n)=\sum_{n_1+n_2+\cdots+n_\ell=n} d^{(j_1)}(n_1)d^{(j_2)}(n_2)\cdots d^{(n_\ell)}(n_\ell),$$

where the n_i's are positive integers and the empty sum is defined as 0.

By combining Theorem 1 and Proposition 2, we obtain an expression for p(k,n).
Theorem 3. For a fixed positive integer k, we can express p(k,n) in terms of $\sigma_j(n)$.
Here are some examples for the first few k's:

$$p(1,n) = d(n), \tag{1.3}$$

$$p(2,n) = \frac{1}{2}\sum_{k=1}^{n-1} d(k)d(n-k) + \frac{1}{2}(\sigma_1(n) - \sigma_0(n)), \tag{1.4}$$

$$p(3,n) = \frac{1}{6}\sum_{\substack{n_1+n_2+n_3=n \\ n_i>0}} d(n_1)d(n_2)d(n_3) - \frac{1}{2}\sum_{k=1}^{n-1}\sigma_1(k)d(n-k) + \frac{1}{2}\sum_{k=1}^{n-1} d(k)d(n-k)$$

$$+ \frac{1}{6}\sigma_2(n) - \frac{1}{2}\sigma_1(n) + \frac{1}{3}d(n). \tag{1.5}$$

In particular, we obtain that

$$\sum_{k=1}^{n-1} d(k)d(n-k) = \sigma_1(n) - \sigma_0(n) + 2p(2,n). \tag{1.6}$$

The left side of (1.6) is an additive convolution of a multiplicative function d(n). Unlike the Dirichlet convolution of a multiplicative function (in this case, we can use L-series to investigate the convolution), an exact formula for such a convolution is not well known. Moreover, since there is no known closed formula for $\sum_{k=1}^{n-1} d(k)d(n-k)$ is very interesting.

Since $p(n) = \sum_{k(k+1)\le 2n} p(k,n)$ for all $n \ge 1$, the following corollary is clear.

Corollary 4.We can express p(n) in terms of $\sigma_j(n)$.

On the other hand, Fine [29, Section 22] showed that

$$p(1,n) = \sum_{\lambda\vdash n} \ell_1 - 2\ell_1\ell_2 + 3\ell_1\ell_2\ell_3 - \cdots.$$

By employing Fine's argument, we can see the following proposition.

Proposition 5.For a fixed positive integer k,

$$p(k,n) = \sum_{\lambda\vdash n}\binom{k}{k}\ell_1\ell_2\cdots\ell_k - \binom{k+1}{k}\ell_1\ell_2\cdots\ell_{k+1} + \binom{k+2}{k}\ell_1\ell_2\cdots\ell_{k+2} - \cdots.$$

Combining these two representations forp(k,n) (Theorem 3 and Proposition 5), we can derive many mysterious formulas relating the sum involving power of divisor functions and the sum over partitions. For example,

$$d(n) = \sum_{\lambda\vdash n} \ell_1 - 2\ell_1\ell_2 + 3\ell_1\ell_2\ell_3 - \cdots, \tag{1.7}$$

$$\sum_{k=1}^{n-1} d(k)d(n-k) - \sigma_1(n) + d(n) = 2\sum_{\lambda\vdash n}\binom{2}{2}\ell_1\ell_2 - \binom{3}{2}\ell_1\ell_2\ell_3 + \binom{4}{2}\ell_1\ell_2\ell_3\ell_4 - \cdots \tag{1.8}$$

$$\sum_{\substack{n_1+n_2+n_3=n \\ n_i>0}} d(n_1)d(n_2)d(n_3) - 3\sum_{k=1}^{n-1}(\sigma_1(k) - d(k))d(n-k) + \sigma_2(n) - 3\sigma_1(n) + 2d(n)$$

$$= 6\sum_{\lambda\vdash n}\binom{3}{3}\ell_1\ell_2\ell_3 - \binom{4}{3}\ell_1\ell_2\ell_3\ell_4 + \binom{5}{3}\ell_1\ell_2\ell_3\ell_4\ell_5 - \cdots.$$

Remark.(1.7) is given in [29, Section 22].
For example, there are 5 partitions of 5 into 2 different parts. In the left side of (1.8), $\sum_{k=1}^{4} d(k)d(5-k) = 14$, $\sigma_1(5) = 6$, and $d(5) = 2$. Therefore, the left side of (1.8) equals 10. The right side of (1.8) is 10 since the summand in right side of (1.8) is 4 if the partition λ is $1^1 2^2$, and 6 if the partition λ is $1^3 2^1$, and 0, otherwise. Therefore, we can see that each side of (1.8) gives twice of p(2,5) as we expected.
As we have already seen, we can express p(n)in terms of p(k,n). Beside of the ordinary partition function p(n), the overpartition function $\bar{p}(n)$ can also be expressed in terms of p(k,n). By using this connection between $\bar{p}(n)$ and p(k,n), we will investigate values of the sum

$$A(n) = \sum_{k=1}^{n-1} d(k)d(n-k) \tag{1.9}$$

modulo 8. An overpartition of n is a partition of n in which we may overline the first occurrence of the part. For example, there are 8 overpartitions of 3:

$$3,\ \bar{3},\ 2+1.\ \bar{2}+1.\ 2+\bar{1}.\ \bar{2}+\bar{1},\ 1+1+1,\ \bar{1}+1+1.$$

Overpartitions have played an important role in basic hypergeometric series and related fields.
Theorem 6. The arithmetic density of the set $\{n \in N | A(n) \equiv 0 (mod\ 8)\}$ is 1, where A(n) is a convolution of two divisor functions defined in (1.9).
2. Proofs of the results.

We start this section with a simple observation. If d divides n, we can write n as the sum of n/d copies d. Therefore, each divisor of n corresponds to the partitions of n into 1 different part. Hence, we see that $p(1,n) = d(n)$. Before proceeding further, we note that $D(k^1, q)$ generates the number of representations of n as $n = (m_1 + m_2 + \cdots + m_k)l$, where m_i and n_i are positive integers, and the different orders of m_j's are counted as different.

Proof of Theorem 1.

For a generating function for p(2,n), since P(1,q) generates the partitions into 1 different part, by multiplying D(1,q) by P(1,q), we obtain a generating series for the number of representations for n as $n = m_1 n_1 + m_2 n_2$ such that m_i and n_j are positive integers and that n_1 and n_2 might be equal.

Since we want the case when n1 and n2 are different from each other, we need to subtract the case $n_1 = n_2$. This case is generated by D(2,q).

Hence, $P(1,q)D(1,q) - D(2,q)$ generates the number of representations for n as $n = m_1 n_1 + m_2 n_2$, where n1 and n2 are different positive integers at this time. Because we regard different orders of representations as the same partition, we find that

$$P(2,q) = \frac{1}{2}\left(P(1,q)D(1^1,q) - D(2^1,q)\right).$$

Similarly, we see that $P(2,q)D(1^1,q)$ generates the number of representations of n as $n = m_1 n_1 + m_2 n_2 + m_3 n_3$, where ni andmj are positive integers such thatn1 and n2 are different and different places of n3 in the summation are counted as different.

The case when n3 is the same as either n1 or n2 is generated by $P(1,q)D(2^1,q)$. However, $P(1,q)D(2^1,q)$ generates the case $n1 = n2 = n3$. Therefore, we need to subtract $D(3^1,q)$ to cancel this case. In summary, $P(2,q)D(1^1,q) - P(1,q)D(2\text{^}1,q) + D(3^1,q)$ generates the number of representations of n as $n = m_1n_1 + m_2n_2 + m_3n_3$, where n_i and m_j are positive integers such that n1, n2 and n3 are different and different places of n3 in the summations are counted as different. Thus, we arrive at

$$P(3,q) = \frac{1}{3}\left(P(2,q)D(1^1,q) - P(1,q)D(2^1,q) + D(3^1,q)\right).$$

By the same argument, we obtain the following recurrence relation for P(k,q):

$$P(k,q) = \frac{1}{k}\left(P(k-1,q)D(1^1,q) - P(k-2,q)D(2^1,q) + \cdots + (-1)^{k-1}D(k^1,q)\right). \quad (2.1)$$

Before continuing, we need to define an operation for the partitions. If j is a part of the partition λ, we define $\lambda - \{j\}$ as the partition, which has all parts of λ except one j. Then, for dλ, we find the following recurrence relation from (2.1).

Proposition 7. For each positive integer k, $d_{k^1} = \frac{(-1)^{k-1}}{k}$.

Moreover, for a given partition λ, $d_\lambda = \sum_j \frac{(-1)^{j-1}}{|\lambda|} d_{\lambda-\{j\}}$ where the sum runs over all different parts of λ.

By using the recurrence relation given in Proposition 7, we can calculated λ for a given partition λ by using induction, which concludes Theorem 1.

A proof of Proposition 2 follows easily by expanding D(j,q) as a geometric series.

Proof of Proposition 2. To prove Proposition 2, we need to determine the n-th coefficient of D(j,q). By expanding the geometric sum, we find that

$$D(j,q) = \sum_{i=1}^{\infty} \frac{q^{ji}}{(1-q^i)^j} = \frac{1}{(j-1)!}\sum_{n=1}^{\infty} d^{(j)}(n)q^n,$$

where $d^{(k)}(n)$ is defined as (1.2). Hence, from the definition of $D(\lambda,q)$, Proposition 2 follows.

Now, we turn to the values of $A(n) = \sum_{k=1}^{n-1} d(k)d(n-k)$. From now on, a property holds almost always means that the set which satisfies the property has arithmetic density 1. As we already mentioned in the introduction, we will investigate $A(n)$ via the connection between $p(k,n)$ and $\bar{p}(n)$.To this end, we relate $\bar{p}(n)$ with $p(k,n)$ as follows. Since there are two choices (we may overline or may not overline) for each different part, we observe that

$$\bar{p}(n) = \sum_{k=1}^{\infty} 2^k p(k,n),$$

for all $n \geq 1$. Therefore, we find that

$$\bar{p}(n) \equiv 2p(1,n) + 4p(2,n) + 8p(3,n) \pmod{16}, \quad (2.2)$$

for all $n \geq 1$.

Before going further, we introduce the following lemma, which provides divisible properties we need. Though the following lemma is a summary of the results proven in K. Mahlburg [31] and the author [30], we give a brief sketch for the completeness.

Lemma 8. Let $r_k(n)$ be the number of representations of n as the sum of k squares of integers and $r_{1,2}(n)$ be the number of representations of n as $x^2 + 2y^2$, where x and y are integers. Then, $r_1(n), r_2(n)$ and $r_{1,2}(n)$ are usually 0. Moreover, for any fixed integer j, $r_3(n)$, $d(n)$, $\sigma_1(n)$, and $\sigma_2(n)$ are almost always divisible by 2^j.

Finally, p(n) is almost always divisible by 16. Sketch of proof. It is clear that $r_1(n)$ is usually 0. By the famous result of Landau, $r_2(n)$ is usually 0. An analogous proof for the fact r1,2(n) is almost always 0 is given in [30, Lemma 3].

Regarding r3(n), by the famous result of Gauss and by the divisibility property of Hurwitz class num-ber,r3(n)is divisible by 2^j provided there are at least j distinct odd primes dividing the square-free part of n. Moreover, it is clear that d(n), σ1(n), and σ2(n) are also divisible by 2^j if n has at least j distinct odd primes in its square-free part. Let $Bj(x)$ be the number of integers $n \leq x$ having at most j odd primes in its square-free part. Then, it is well known that $\frac{Bj(x)}{x}$ tends to 0 as x goes to infinity (for example, consult [30, Lemma 2]).

Therefore, r3(n), d(n), σ1(n) and σ2(n) are almost always divisible by 2^j. Concerning p(n), it is now proven that p(n)is almost always divisible by 128 [30, Theorem 1].

Now we are ready to give a proof of Theorem 6.

Proof of Theorem 6. From (1.4), we see that

$$A(n) = 2p(2, n) - \sigma_1(n) + d(n).$$

Since $\sigma_1(n)$ and $d(n)$ are almost always divisible by 8, proving p(2,n) is almost always divisible by 4 is enough for the proof for Theorem 6. Since $p(1, n) = d(n)$ and $p(n)$ are almost always divisible by 16 by Lemma 8, we can conclude that p(2,n) is almost always divisible by 4 if p(3,n) is almost always divisible by 2.

From (1.5), we obtain that

$$6p(3, n) = \sum_{\substack{n_1+n_2+n_3=n \\ n_i>0}} d(n_1)d(n_2)d(n_3)$$

$$-3\sum_{k=1}^{n-1}(\sigma_1(k) - d(k))d(n-k) + \sigma_2(n) - 3\sigma_1(n) + 2d(n). \quad (2.3)$$

We are going to show that the right side of (2.3) is usually divisible by 4. First, we investigate the sum

$$\sum_{x+y+z=n} d(x)d(y)d(z)$$

modulo 4.

Note that d(x) is odd if and only if x is a square of a positive integer and σ1(n) is odd if and only if $n = m^2$ or $2m^2$. When $x + y + z = n$, we have three cases: (1) $x = y = z$, (2) two of them are the same and the other is different from the other two, and (3) all of them are different. When $x = y = z$, $d(x)^3$ is not even only if x is a square. In other words, unless $n = 3m^2$, $d(x)^3$ is divisible by 4.
Note that the set $\{n \in N | n = 3m^2\}$ has arithmetic density 0, so $\sum_{x+y+z=n} d(x)d(y)d(z)$ is usually divisible by 4. For the second case, there are three possible choices of pairs which are the same and for convenience, say x=y and $x \neq z$.
Then, $d(x)^2d(z)$ might not be divisible by 4 if x is a square. For the last case, by considering permutations of x, y, z, we see that $\sum_{x+y+z=n} d(x)d(y)d(z)$ is not divisible by 4 only if $n = x^2 + y^2 + z^2$.In summary, we arrive at

$$\sum_{\substack{x+y+z=n \\ x,y,z>0}} d(x)d(y)d(z) \equiv 3 \sum_{\substack{2m^2+z=n \\ m,z>0}} d(m^2)^2 d(z) + 2 \sum_{\substack{x^2+y^2+z^2=n \\ x>y>z}} 1 \pmod 4, \tag{2.4}$$

for almost all n. By Lemma 8, we see that $\sum_{n=x^2+y^2+z^2} 1$ is almost always divisible by 2. Moreover, by employing a similar argument, we observe that

$$\sum_{\substack{x+y=n \\ x,y>0}} (\sigma_1(x) - d(x))d(y) \equiv \sum_{\substack{2m^2+z=n \\ m,z>0}} \sigma_1(2m^2)d(z) \pmod 4, \tag{2.5}$$

for almost all n. By adding (2.4) and (2.5), we find that

$$\sum_{\substack{n_1+n_2+n_3=n \\ n_i>0}} d(n_1)d(n_2)d(n_3) - 3\sum_{k=1}^{n-1}(\sigma_1(k) - d(k))d(n-k)$$

is almost always divisible by 4 since the number of representations of n as $2x^2 + y^2$ is almost always 0 from Lemma 8. Since σ2(n), σ1(n) and d(n) are almost always divisible by 4, we have seen that p(3,n)is almost always even, which finishes the proof of Theorem 6.

3.WEIGHT FUNCTION

3.1. DEFINITION

Def. The weight function is

$$w(\alpha) = sort(n+1-i_1-i_2-\cdots-i_k, i_1, i_2, \ldots, i_k)$$

where $\alpha = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n})$.

If α denotes partition

$$n = 1*\alpha_1 + 2*\alpha_2 + \cdots + n*\alpha_n,$$

then $w(\alpha) \vdash n+1$ denotes sorted partition of

$$n+1 = \alpha_1 + \alpha_2 + \cdots + \alpha_n + (n+1-\alpha_1-\alpha_2-\cdots-\alpha_n)$$

About weight function we can notice that $w: P(n) \to P(n+1)$.

It means that if $\alpha \vdash n$, then $w(\alpha) \vdash n+1$.

For example, $n = 5$

$$(5) \to (5,1)$$
$$(4,1) \to (4,1^2)$$
$$(3,2) \to (4,1^2)$$
$$(3,1^2) \to (3,2,1)$$
$$(2^2,1) \to (3,2,1)$$
$$(2,1^3) \to (3,2,1)$$
$$(1^5) \to (5,1)$$

Therefore, we see that $Im(w) = \{(5,1),\ (4,1^2),\ (3,2,1)\}$

Definition. If $\gamma \in Im(w)$, then partition γ is called *weight* partition.

Theorem.1. The Kostka number $K_{\alpha\beta} \neq o$ if and only if $\alpha \trianglerighteq \beta$. [33, p17]

Theorem.2. F_{n+1} is multi-linear part of free Novikov algebra in degree n+1.
S^β is a Specht module corresponding to a partition β. [33,p.3]

$$F_{n+1} \cong \bigoplus_{\beta \vdash n+1} \Big(\sum_{\alpha \vdash n} K_{\beta w(\alpha)}\Big) S^\beta$$

The main task is find out for which partitions β multiplicity of S^β is non-zero. According to theorem.1. it is equivalent next problem:

Which partitions β dominate any of *weight* partitions?

Because if there exist partition $\alpha \vdash n$ such that $w(\alpha) \in Im(w)$ and $\beta \trianglerighteq w(\alpha)$, then by theorem.1.

$$K_{\beta w(\alpha)} \neq o \Rightarrow \sum_{\alpha \vdash n} K_{\beta w(\alpha)} > 0$$

3.2 PROPERTIES OF WEIGHT PARTITIONS

Theorem.3. If all summands of partition β has a common divisor (greater than 1), then β is *non-weight*.
Definition. *Length* of partition is number of summands in that partition.
Theorem.4. The *length* of partition $\beta \in Im(w)$ do not exceed $\left[\frac{\sqrt{8n+1}+1}{2}\right]$.
Def. The summand $(n+1-\alpha_1-\alpha_2-\dots-\alpha_n)$ in partition that $w(\alpha) \vdash n+1$ is called *new summand* for $w(\alpha)$.
Theorem.5. *New* summand for weight partition is one of two largest summand of that partition.
Theorem.6. If $\beta = (\beta_1, \beta_2, \dots, \beta_k) \vdash Im(w)$ and β_1 is largest in β, then

$$\beta_1 \geq 1 + \beta_3 + 2\beta_4 + \dots + (k-2)\beta_k$$

Proofs of the theorems.
Theorem.3. If all summands of partition β has a common divisor (greater than 1), then β is *non-weight.*
Proof of theorem.3. We solve it by contradiction method. Assume that $\beta \in Im(w)$. Let β mean partition $n+1 = \beta_1 + \beta_2 + \dots + \beta_k$ (1.1) and without lost of generality let β_k be *new* for β.
All this means that there exists $\{r_1, r_2, \dots, r_{k-1}\} \subset \{1,2,\dots,n\}$:

$$n = r_1\beta_1 + r_2\beta_2 + \dots + r_{k-1}\beta_{k-1} \quad (1.2)$$

Also we have that $\exists d \in N, d > 1 \ \forall i \ \beta_i \vdots d$.
Since all of β_i are divisible by d from (1.1) we can say that $n+1 \vdots d$ and from (1.2) that $\vdots d$.
But it's impossible for $d > 1$. Contradiction. Q.E.D.

Corollary. Also we can say that if all summands of partition β except one have a common factor (greater than 1), then the partition can be good if only that summand is *new* for β.

Theorem.4. The *length* of partition $\beta \in Im(w)$ do not exceed $\left[\frac{\sqrt{8n+1}+1}{2}\right]$.
Proof of theorem.4. Let $w(\alpha) = \beta$ mean partition $n+1 = \beta_1 + \beta_2 + \dots + \beta_k$ (2.1), where k is maximum.
So we have to prove that $k \leq \frac{\sqrt{8n+1}+1}{2} < k+1$.
Also notice, that the number of weights of $\alpha \vdash n$ is $k-1$.
It means that $\alpha \vdash n$ can have $k-1$ distinct summands at maximum, so

$$1 + 2 + \dots + k > n \geq 1 + 2 + \dots + (k-1)$$

$$\frac{(k+1)k}{2} > n \geq \frac{(k-1)k}{2}$$

$$(2k+1)^2 = 4k^2 + 4k + 1 > 8n + 1 \geq 4k^2 - 4k + 1 = (2k-1)^2$$

$$2k + 1 > \sqrt{8n+1} \geq 2k - 1$$

$$k+1>\frac{\sqrt{8n+1}+1}{2}\geq k$$

Q.E.D.

Theorem.5. *New* summand for weight partition is one of two largest summand of that partition.

Proof of theorem.5. W.L.O.G. we can give an order to summands of partition

$$n+1=\beta_1+\beta_2+\cdots+\beta_k \quad (3.1)$$

We say that $\beta_1\geq\beta_2\geq\cdots\geq\beta_k$ (3.2).

So we have to prove that the *new summand* for β is β_1 *or* β_2.

Using identities (3.1) and $=r_1\beta_1+r_2\beta_2+\cdots+r_k\beta_k-r_t\beta_t$, where β_t is *new* for β, we get

$$\underbrace{r_1\beta_1+r_2\beta_2+\cdots+r_k\beta_k-r_t\beta_t}_{n}+1=\beta_1+\beta_2+\cdots+\beta_k$$

$$r_t\beta_t=1+(r_1-1)\beta_1+(r_2-1)\beta_2+\cdots+(r_k-1)\beta_k$$

$$\beta_t-1=\sum_{i=1,i\neq t}^{k}(r_i-1)\beta_i \quad (3.3)$$

Since r_i are distinct naturals and $\beta_1\geq\beta_2\geq\cdots\geq\beta_k$ by the property of rearrangement inequality we have

$$\sum_{i=1,i\neq t}^{k}(r_i-1)\beta_i\geq\sum_{i\neq t}^{k}(i-1)\beta_i \quad (3.4)$$

And if β_t is not β_1 or β_2 then

$$\sum_{i=1,i\neq t}^{k}(i-1)\beta_i\geq 0*\beta_1+1*\beta_2 \quad (3.5)$$

So conclusion of statements (3.3) , (3.4) and (3.5) is

$$\beta_t-1\geq\beta_2$$

What is impossible, because from (3.2) we know that $\beta_2\geq\beta_t$. Q.E.D

Theorem.6. If $\beta=(\beta_1,\beta_2,\ldots,\beta_k)\vdash Im(w)$ and β_1 is largest in β, then holds

$$\beta_1\geq 1+\beta_3+2\beta_4+\cdots+(k-2)\beta_k$$

Proof of theorem.6. According to th.5. β_t (new) is β_1 or β_2.

If $\beta_t=\beta_1$ statements (3.3) and (3.4) give us

$$\beta_1-1\geq\beta_3+2\beta_4+\cdots+(k-2)\beta_k$$

If $\beta_t=\beta_2$ statements (3.3) and (3.4) give us

$$\beta_2-1\geq\beta_3+2\beta_4+\cdots+(k-2)\beta_k$$

Since $\beta_1\geq\beta_2$ both cases give us needen inequality.

5. CONCLUSION

This scientific work is research of weight function. Here is found next properties of image of this function:
Summands of weight partition are respectively prime;
Maximum number of summands in weight partition;
Necessary inequality for summands of weight partition;
Some propertis of *new* summand of weight function.

5. REFERENCE

[1] A.A. Balinskii, S.P. Novikov, Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras, Sov.Math. Dokl. 32 (1985) 228–231.

[2] Y. Benoist, Une nilvariété non affine, J. Differential Geom. 41 (1995) 21–52.

[3] D. Burde, Left-symmetric algebras, or pre-Lie algebras in geometry and physics, Central Eur. J. Math. 4 (3) (2006)323–357.

[4] D. Burde, K. Dekimpe, Novikov structures on solvable Lie algebras, J. Geom. Phys. 56 (9) (2006) 1837–1855.

[5] I.M. Gel'fand, I. Dorfman, Hamiltonian operators and algebraic structures related to them, Funct. Anal. Appl. 13(1980) 248–262.

[6] E. Zelmanov, On a class of local translation invariant Lie algebras, Sov. Math. Dokl. 35 (1987) 216–218

[7] M. Bremner, Classifying varieties of anti-commutative algebras, Nova Journal of Math., Game Theory and Algebra 4 (1996), no. 2, 119-127.

[8] Byungchan Kim On the number of partitions of n into k different parts, Journal of Number Theory, 132 (2012), pp.1306-1313.

[9] A.S. Dzhumadil'daev, L ofwall C.Trees, free right-symmetric algebras, free Novikov algebras and identities, Homology, Homotopy and Appl.,4 (2002), No.2(1), pp.165-190.

[10] A.S. Dzhumadil'daev, Codimension growth and non-Koszulity of Novikov operad, Comm. Algebra., 39 (2011), No. 8, pp.2943-2952.

[11] A.S. Dzhumadil'daev, N.A. Ismailov, K.M. Tulenbaev Free bicommutative algebras, Serdica Math. J., 37 (2011), No. 1, pp.25-44.

[12]Richard P. Stanley, Enumerative Combinatorics, Volumes 1 and 2. Cambridge University Press, 1999 ISBN 0-521-56069-1

[13] G.E. Andrews, The Theory of Partitions, Addison–Wesley, Reading, MA, 1976; reissued by Cambridge University Press, Cam-bridge, 1984.

[14] S. Corteel, J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. 356 (2004) 1623–1635.

[15] N.J. Fine, Basic Hypergeometric Series and Applications, Math. Surveys Monogr., vol. 27, American Mathematical Society, Providence, RI, 1988.

[16] B. Kim, Overpartition function modulo 128, Integers 8 (2008), A #38.

[17] K. Mahlburg, The overpartition function modulo small powers of 2, Discrete Math. 286 (2004) 263–267.

[18] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Stud. Adv. Math. vol. 62, Cambridge University Press, Cambridge, 1999.

[19] A.S. Dzhumadil'daev, N.A. Ismailov, Free Novikov Algebras As Sn Modules, unpublished.