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Variety of Bicommutative Algebras defined by identity $\alpha[(ab)c+(ba)c+(ca)b]+\beta[c(ba)+c(ab)+b(ac)]=0$

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THESIS

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Abstract

One of the important questions in modern algebra is to study algebras satisfying certain identities. There are two questions in the theory of polynomial identities. The first is to describe an algebra by a defined identity. The second is to describe identities in algebra. The study of identities will help us in the construction of basis of free algebra, in the study of the Hilbert sequence, the Specht problem and problems of the finite basis. In this work, we used two different research methods. First, the theory of representations of symmetric groups. Second, the theory of representations of linear groups. In this research paper, we have completely described the subvariety of variety of bicommutative algebras defined by the identity α[(ab)c+(ba)c+(ca)b]+β[a(bc)+a(cb)+b(ca)]=0.

Аңдатпа

Тепе-теңдіктерді қанағаттандыратын алгебраларды зерттеу заманауи алгебраның қызықты сұрақтарының бірі болып табылады. Полиномиалды тепе-теңдіктер теориясында екі үлкен сұрақ бар. Біріншісі, тепе-теңдікпен анықталған алгебраларды сипаттау. Екіншісі, алгебралардағы тепе-теңдіктерді сипаттау. Тепе-теңдіктерді зерттеу бізге еркін алгебралардың базисін құрастыруға, Гильберт қатарын зерттеуге, Шпехт есебі мен шекті базис сұрақтарына жауап беруге мүмкіндік береді. Есептерді шығару мақсатында екі түрлі зерттеу әдістерін қолдандық. Біріншісі, симметриялық топтардық көрсетілімдер теориясы. Екіншісі, толық сызықтық топтардық көрсетілімдер теориясы. Осы жұмыста $\alpha[(\mathrm{ab})\mathrm{c} + (\mathrm{ba})\mathrm{c} + (\mathrm{ca})\mathrm{b}] + \beta[\mathrm{a}(\mathrm{bc}) + \mathrm{a}(\mathrm{cb}) + \mathrm{b}(\mathrm{ca})] = 0$ тепе-теңдігімен анықталатын бикоммутативті алгебралар көпбейнесінің ішкі-көпбейнелерін толық сипатталды.

Аннотация

Одна из интересных вопросов в современной алгебре изучать алгебры удовлетворяющие некоторые тождества. В теории полиномных тождеств есть 2 больших вопросов. Первый, описывать алгебру опредляемым тождеством. Второй, описывать тождества в алгебре. Изучение тождеств поможет нам в конструкции базисов свободной алгебры, в изучении Гильбертовой последовательности, проблему Шпехта и ответить на вопросы конечного базиса. В этой работе мы использовали два разных методов исследования. Первый, теория представлений симметрических групп. Второй, теория представлений полных линейных групп. В этой исследовательской работе мы полностью описали внутреннее разнобразие разнообразие бикоммутативных алгебр определяемых тождеством $\alpha[(\mathrm{ab})\mathrm{c}+(\mathrm{ba})\mathrm{c}+(\mathrm{ca})\mathrm{b}]+\beta[\mathrm{a}(\mathrm{bc})+\mathrm{a}(\mathrm{cb})+\mathrm{b}(\mathrm{ca})]=0$.

Contents

1. Introduction

The study of algebras satisfying some identities is one of the most important issues in modern algebra. There are two basic questions in the theory of polynomial identities in algebras: 1) explain algebras with identities; 2) explain identities in algebra.

The study of algebras satisfying some identities is one of the most important issues in modern algebra. There are two basic questions in the theory of polynomial identities in algebras: 1) explain algebras with identities; 2) explain identities in algebra. Investigation the aforementioned questions leads to the study of free algebras, free algebra construction bases, finding Hilbert series, finding codimension sequences, finding codimension growth, finding Gelfand-Kirillov dimension, finding cocharacter sequences, finding colengths, investigating Specht problem, and so on.

Identity and algebra are mutually defining terms in the study of polynomial identities. Varieties of algebras determine their interconnectedness. Varieties of algebras' vocabulary enables easy transitions from identity to algebra and vice versa. As a result, learning different types of algebras is one of the most significant things you can do.

A.I. Malcev [4] and W.Specht [10] studied the representation theory of symmetric group to categorize polynomial identities of algebraic structures for the first time and independently in 1950. Every polynomial is known to be equivalent to a finite collection of multilinear polynomials if K is a field of characteristic 0. As a result, we concentrate on the module structures of multilinear components (parts) of free algebras in this paper. Because the multilinear sections of free algebras contain a wealth of relevant and crucial information about various algebras, they are particularly beneficial. The term "module structure" refers to modules over both the symmetric and generic linear groups.

In some circumstances, the best approaches for investigating multilinear components of free algebras are representation theory of the symmetric group and general linear group methods. Several free algebras have known S_n and GL_n -module structures. Associative algebra, Leibniz algebra, Zinbiel algebra, Lie algebra, right-symmetric algebra, Novikov algebra and other types of algebra are examples.

In some contexts, it is more convenient to use the procedures of general linear group representation theory to varieties of algebras than it is to apply the methods of symmetric group representation theory. The following works demonstrate this. For example, using methods from the theory of representations of the general linear group, there exist fully specified varieties of associative algebras with identity of degree three in [8]. Using methods from the theory of representations of generic linear groups, a criterion for the distributivity of the lattice of subvarieties of varieties of associative algebras is developed. Using methods from the theory of representations of general linear groups a criterion for the distributivity of the lattice of subvarieties of varieties of alternative algebras is presented as well as in [9].

Dzhumadil'daev and Tulenbaev [1] are the first to mention bicommutative algebras. If A is a bicommutative algebra, then A_2 is commutative and associative, according to [1]. Dzhumadil'daev, Ismailov, and Tulenbaev [2] computed diversity of irreducible Sn-representations in multilinear component decomposition using a combinatorial technique in 2011. Furthermore, the building of a basis of free bicommutative algebras, the description of basis element multiplication, supplied cocharacter and codimension sequences, and computed Hilbert series are all given. It is also shown that the bicommutative operad is not Koszul and that the development of the bicommutative algebra codimension sequence is equal to 2. An alternate proof of the formula for the cocharacter sequence of bicommutative algebra [6] was later published. The Specht issue for varieties of bicommutative algebras is solved affirmatively, according to Drensky and Zhakhayev [7]. The relationship between bicommutative algebra and free Novikov algebra filtration and grading provides us with a strong incentive to learn more about bicommutative algebra.

In the current research work we study variety of bicommutative algebras defined by identity

$$\alpha[(ab)c + (ba)c + (ca)b] - \beta[c(ba) + c(ab) + b(ac)] = 0,$$

where $\alpha, \beta \in K$.

Describe all varieties of algebras with a given system of identities is one of the most basic problems in the subject of polynomial identities in algebra. Our goal is to categorize all subvarieties of bicommutative algebras. The most common method of classification is to use lattices. This problem is, of course, similar to describing T-ideals. To create a lattice of subvarieties of a given variety of algebras we should point out next: 1) identify the module structure of $P_n(\mathfrak{N})$ over the symmetric group; 2) identify a consequence in $P_{n+1}(\mathfrak{N})$ for each irreducible S_n-module in $P_n(\mathfrak{N})$.

2. Preliminaries

In this section we will consider some definitions that we will use to formulate a theorem. Sources are indicated next to them.

Definition 2.1. *([5]) A vector space A is called an algebra if A is equipped with a binary operation $\cdot$, called multiplication, such that for any $a, b, c \in A$ and any $\alpha \in K$*

$$(a + b) \cdot c = a \cdot c + a \cdot b,$$

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$

$$\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b).$$

Definition 2.2. *([1]) An algebra $(A, \cdot)$ is called bicommutative if any $a, b, c \in A$ are satisfied the following identities*

$$a \cdot (b \cdot c) = b \cdot (a \cdot c) \tag{2.1}$$

$$(a \cdot b) \cdot c = (a \cdot c) \cdot b \tag{2.2}$$

Let X be a set of generators $x_1, \ldots, x_n$.

Definition 2.3. *([5]) Let $\mathfrak{B}$ be a class of algebras and let $F(X) \in \mathfrak{B}$ be an algebra generated by a set X. The algebra $F(X)$ is called a free algebra in the class $\mathfrak{B}$, if for any algebra $A \in \mathfrak{B}$, every mapping $X \to A$ can be extended to a homomorphism $F(X) \to A$.*

Let $K\{X\}$ be a free non-associative algebra generated by set $X = \{x_1, \ldots, x_n\}$.

Definition 2.4. *([5]) Let $f = f(x_1, ..., x_n) \in K\{X\}$ and let A be a non-associative algebra. We say that $f = 0$ is a polynomial identity for A if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$*

Definition 2.5. *([5]) Let $\{f_i(x_1, ..., x_n) \in K\{X\} \mid i \in I\}$ be a system of polynomials. The class $\mathfrak{V}$ of all non-associative algebras satisfying the polynomial identities f_i, $i \in I$, is called the variety defined by the system of polynomial identities $\{f_i \mid i \in I\}$. The variety $\mathfrak{M}$ is called a subvariety of $\mathfrak{V}$ if $\mathfrak{M} \subset \mathfrak{V}$*

Definition 2.6. *([5]) The set $T(\mathfrak{V})$ of all polynomial identities satisfied by the variety $\mathfrak{V}$ is called the T-ideal of $\mathfrak{V}$. We say, that the T-ideal $T(\mathfrak{V})$ is generated as a T-ideal by the defining set of identities $\{f_i | i \in I\}$ of the variety $\mathfrak{V}$.*

Definition 2.7. *([5]) Let V be a vector space over $\mathbb{R}$ and let G be a group. Then V is a G-module if a multiplication vg ($v \in V,\ g \in G$) is defined, satisfying the following conditions for all $u, v \in V,\ \lambda \in \mathbb{R}$ and $g, h \in G$.*

(1) $ug \in V$;
(2) $v(gh) = (vg)h$;
(3) $v1 = v$;
(4) $(\lambda v)g = \lambda(vg)$;
(5) $(u + v)g = ug + vg$.

Definition 2.8. *([3]) A polynomial $f = f(x_1, \ldots, x_n)$ is called linear in the variable x_i if x_i occurs in every monomial of f with degree 1. A polynomial f is called multilinear if it is linear in each variable.*

3. Variety of Bicommutative Algebras defined by identity $\alpha\big[\big(ab\big)c+\big(ba\big)c+\big(ca\big)b\big]+\beta\big[c\big(ba\big)+c\big(ab\big)+b\big(ac\big)\big]=0$

3.1 Main statements

Let $\mathfrak{M}$ be a variety of bicommutative algebras over a field K of characteristic 0 defined by identity

$$\alpha[(ab)c+(ba)c+(ca)b]+\beta[c(ba)+c(ab)+b(ac)]=0, \tag{3.1}$$

where $(\alpha,\beta)\neq(0,0)$, $\alpha,\beta\in K$.

Let $F(\mathfrak{N})$ be a free algebra in variety $\mathfrak{N}$, and let $F_n(\mathfrak{N})$ be a free algebra in $\mathfrak{N}$ generated by $X=\{x_1,\dots,x_n\}$.

Let P_n be a space of multilinear polynomials of $F_n(\mathfrak{M}\mathfrak{N})$ of degree n. Further we call the space of multilinear polynomials as multilinear component.

There four cases reliance on coefficients α and β
(I) $\alpha\beta(\alpha-\beta)(\alpha+\beta)\neq0$
(II) $\alpha=0,\ \beta\neq0$ $(\alpha\neq0,\ \beta=0$ (analogous)
(III) $\alpha+\beta=0$
(IV) $\alpha-\beta=0$

3.2 Case $\alpha\beta(\alpha+\beta)(\alpha-\beta)\neq0$

In this situation $\alpha\beta(\alpha+\beta)(\alpha-\beta)\neq0$ identity seems as:

$$\alpha[(ab)c+(ba)c+(ca)b]+\beta[c(ba)+c(ab)+b(ac)]=0 \tag{3.2}$$

Theorem 3.2.1. *As S_n module $\alpha\beta(\alpha+\beta)(\alpha-\beta)\neq0$*

$$P_1(\mathfrak{N})\cong S^{(1)},P_2(\mathfrak{N})\cong S^{(2)}\oplus S^{(1,1)},$$

$$P_3(\mathfrak{N}) \cong S^{(3)} \oplus 2S^{(2,1)}$$

$$P_n(\mathfrak{N}) \cong 0 \textit{ for } n \geq 4.$$

Proof. • $n = 3$. Number of basis elements of multilinear part of three degree free bicommutative algebras is equal to 6. They are:

$(**)*$	$*(**)$
$(ab)c$	$b(ac)$
$(ba)c$	$c(ab)$
$(ca)b$	$c(ba)$

$$3\alpha a(aa) + 3\beta(aa)a = 0$$

$$(aa)a = -\frac{\beta}{\alpha}a(aa)$$

By this new identity we can claim that the number of basis elements of the multilinear part of degree 3 of free algebras is 5.

• $n = 4$. Number of basis elements of multilinear part of four degree free bicommutative algebras is equal to 14. They are:

$((**)*)*$	$*((**)*)$	$*(*(**)$
$((ab)c)d$	$a((bc)d)$	$a(b(cd))$
$((bc)d)a$	$a((cb)d)$	$b(c(da))$
$((cd)a)b$	$a((db)c)$	$c(d(ab))$
$((da)b)c$	$b((ca)d)$	$d(a(bc))$
	$b((da)c)$	
	$c((da)b)$	

$$((aa)a)a = -\frac{\beta}{\alpha}(a(aa))a = -\frac{\beta}{\alpha}a((aa)a) = -\frac{\beta^2}{\alpha^2}a(a(aa) \tag{3.3}$$

Let consider that a:=a, b:=b and c:=a, then we can see that (3.3) identity takes the form:

$$\alpha[(ab)a+(ba)a+(aa)b]+\beta[a(ba)+a(ab)+b(aa)]=0$$

$$\alpha[2(aa)b+(ba)a]+\beta[2b(aa)+a(ab)]=0 \tag{3.4}$$

Then we should multiply (3.4) identity to a from right-side:

$$\alpha[2((aa)a)b+((ba)a)a]+\beta[2(b(aa))a+(a(ab))a]=0$$

$$\alpha[2((aa)a)b+((ba)a)a]+\beta[2b((aa)a)+a((ab)a)]=0$$

$$\alpha[2((aa)a)b+((ba)a)a]+\beta[2(-\frac{\beta}{\alpha})b(a(aa))+a((aa)b)]=0$$

$$\alpha[2((aa)a)b+((ba)a)a]+\beta[2(-\frac{\beta}{\alpha})b(a(aa))+(a(aa))b]=0$$

$$\alpha[2((aa)a)b+((ba)a)a]+\beta[2(-\frac{\beta}{\alpha})b(a(aa))+(-\frac{\alpha}{\beta})(a(aa))b]=0$$

$$\alpha((aa)a)b+\alpha((ba)a)a+2(-\frac{\beta^2}{\alpha^2})b(a(aa))=0 \tag{3.5}$$

To get new identity we should take that a:=aa, b:=b, c:=a. Then (3.5):

$$\alpha[((aa)b)a+(b(aa))a+(a(aa))b]+\beta[a(b(aa))+a((aa)b)+b((aa)a)]=0$$

$$\alpha[((aa)a)b+b((aa)a)+(-\frac{\alpha}{\beta}((aa)a)b]+\beta[b(a(aa))+(a(aa))b+(-\frac{\beta}{\alpha}b(a(aa))]=0$$

$$\alpha[(\frac{\beta-\alpha}{\beta})((aa)a)b+b((aa)a)]+\beta[(\frac{\alpha-\beta}{\alpha})b(a(aa))+(a(aa))b]=0$$

$$\alpha[(\frac{\beta-\alpha}{\beta})((aa)a)b+(-\frac{\beta}{\alpha})b(a(aa))]+\beta[(\frac{\alpha-\beta}{\alpha})b(a(aa))+(-\frac{\alpha}{\beta})((aa)a)b]=0$$

$$[\frac{\alpha}{\beta}(\beta-\alpha)+\beta(-\frac{\alpha}{\beta})]((aa)a)b+[\alpha(-\frac{\beta}{\alpha})+\beta(\frac{\alpha-\beta}{\alpha})]b(a(aa))=0$$

$$[\frac{\alpha\beta-\alpha^2-\alpha\beta}{\beta}]((aa)a)b+[\frac{-\alpha\beta+\alpha\beta-\beta^2}{\alpha}]=0$$

$$(-\frac{\alpha^2}{\beta})((aa)a)b+(-\frac{\beta^2}{\alpha})b(a(aa))=0$$

$$((aa)a)b=-\frac{\beta^3}{\alpha^3}b(a(aa)) \tag{3.6}$$

$$((aa)a)a = -\frac{\beta^3}{\alpha^3}a(a(aa)) = \frac{\beta^2}{\alpha^2}a(a(aa))$$

$$-\frac{\beta}{\alpha}a(a(aa)) = a(a(aa))$$

$$1 + \frac{\beta}{\alpha}a(a(aa)) = 0$$

$$a(a(aa)) = 0 \tag{3.7}$$

$$-\frac{\beta}{\alpha}(a(aa))v = -\frac{\beta^3}{\alpha^3}b(a(aa))$$

$$a((aa)b) = \frac{\beta^2}{\alpha^2}b(a(aa))$$

$$a((aa)b) = \frac{\beta^2}{\alpha^2}b((aa)a) * (-\frac{\alpha}{\beta})$$

$$a((ab)a) = -\frac{\beta}{\alpha}b((aa)a)$$

$$a((ab)a) = (-\frac{\beta}{\alpha}a((ba)a) \tag{3.8}$$

$$\alpha(-\frac{\beta^3}{\alpha^3}b(a(aa)) + \alpha((ba)a)a + 2(-\frac{\beta^2}{\alpha})b(a(aa)) = 0$$

$$\alpha((ba)a)a + 3(-\frac{\beta^2}{\alpha}b(a(aa)) = 0$$

$$((ba)a)a = 3\frac{\beta^2}{\alpha^2}b(a(aa)) \tag{3.9}$$

Let take that a:=ab, b:=a, c:=a. Then we can use this changes in (3.2):

$$\alpha[((ab)a)a+(a(ab))a+(a(ab))a]+\beta[a(a(ab))+a((ab)a)+a((ab)a)]=0$$

$$\alpha[((aa)a)b+2a((aa)b)]+\beta[a(a(ab))+2a((aa)b)]=0$$

$$\alpha((aa)a)b+\beta a(a(ab))+(2\alpha+2\beta)(a((aa)b))=0 \tag{3.10}$$

Let take that a:=a, b:=ba, c:=a. Then we can use this changes in (3.2):

$$\alpha[(a(ba))a+((ba)a)a+(aa)(ba)]+\beta[a((ba)a)+a(a(ba))+(ba)(aa)]=0$$

$$\alpha[b((aa)a)+((ba)a)a+b((aa)a)]+\beta[b((aa)a)+b(a(aa))+b((aa)a)]=0$$

$$\alpha[2b((aa)a)+((ba)a)a]+\beta[2b((aa)a)+b(a(aa))]=0$$

$$(2\alpha+2\beta)b((aa)a)+\alpha((ba)a)a+\beta b(a(aa))=0$$

$$(2\alpha+2\beta)b((aa)a)+\alpha((ba)a)a+\beta(-\frac{\alpha}{\beta})b((aa)a)=0$$

$$(\alpha+2\beta)b((aa)a)+\alpha((ba)a)a=0 \tag{3.11}$$

$$(\alpha+2\beta)(-\frac{\beta}{\alpha})b(a(aa))+\alpha((ba)a)a=0$$

$$2((ba)a)a+(\alpha+2\beta)(-\frac{\beta}{\alpha})\frac{\alpha^2}{3\beta^2}((ba)a)a=0$$

$$(\alpha+(\alpha+2\beta)(-\frac{\alpha}{3\beta}))((ba)a)a=0$$

$$\frac{3\alpha\beta+(\alpha+2\beta)(-\alpha)}{3\beta}((ba)a)a=0$$

$$3\alpha\beta-\alpha^2-2\alpha\beta)((ba)a)a=0$$

$$(\alpha\beta-\alpha^2)((ba)a)a=0$$

$$((ba)a)a=0$$

Finally, we take three main identities:

$\boxed{\text{((ba)a)a=0}}$ $\boxed{\text{b((aa)a)=0}}$ $\boxed{\text{a(a(ab))=0}}$

Suppose that a:=aa, b:=b, c:=b:

$$\alpha[((aa)b)b+(b(aa))b+(b(aa))b]+\beta[b(b(aa))+b((aa)b)+b((aa)b)]=0$$

$$\alpha[((aa)b)b+2b((aa)b)]+\beta[b(b(aa))+2b((aa)b)]=0$$

$$(2\alpha+2\beta)b((aa)b)+\alpha((aa)b)b+\beta b(b(aa))=0 \tag{3.12}$$

To get new identity we should multiply (3.4) identity to b from right-side.

$$\alpha[2((aa)b)b+((ba)a)b]+\beta[2(b(aa))b+(a(ab))b]=0$$

$$\alpha[2((aa)b)b+((bb)a)a]+\beta[2b((aa)b)+a((ab)b)]=0$$

$$\alpha[2((bb)a)a+((aa)b)b]+\beta[2b((aa)b)+b((ba)a)]=0 \tag{3.13}$$

Now, we should multiply (3.4) identity to b from left-side:

$$\alpha[2b((aa)b)+b((ba)a)]+\beta[2b(b(aa))+b(a(ab))]=0$$

$$\alpha[2b((aa)b)+b((ba)a)]+\beta[2b(b(aa))+a(a(bb))]=0 \tag{3.14}$$

$$\alpha[2b((aa)b)+a((ab)b)]+\beta[2a(a(bb))+b(b(aa))]=0 \tag{3.15}$$

In (1.6) identity there b:=bb:

$$\alpha[2(aa)(bb)+((bb)a)a]+\beta[2(bb)(aa)+a(a(bb))]=0$$

$$\alpha[2b((aa)b)+((bb)a)a]+\beta[2b((aa)b)+a(a(bb))]=0$$

$$(2\alpha+2\beta)b((aa)b)+\alpha((bb)a)a+\beta a(a(bb))=0 \tag{3.16}$$

$$(2\alpha+2\beta)b((aa)b)+\alpha((aa)b)b+\beta b(b(aa))=0 \tag{3.17}$$

To get new identity (3.17) - (3.16):

$$\alpha[((bb)a)a-((aa)b)b]+\beta[a(a(bb))-b(b(aa))]=0$$

$$((aa)b)b-((bb)a)a=\frac{\beta}{\alpha}[a(a(bb))-b(b(aa))] \tag{3.18}$$

To get new identity (3.15) - (3.14):

$$\alpha[a((ab)b)-b((ba)b)]+\beta[a(a(bb))-b(b(aa))]=0$$

$$\alpha[a((ab)b)-b((ba)a)]+\alpha[((aa)b)b-((bb)a)a]=0$$

$$a((ab)b)-b((ba)a)+((aa)b)b-((bb)a)a=0 \tag{3.19}$$

In (3.2) identity use changes a:=a, b:=b, c:=b:

$$\alpha[(ab)b+(ba)b+(ba)b]+\beta[b(ba)+b(ab)+b(ab)]=0$$

$$\alpha[(ab)b+2(bb)a]+\beta[b(ba)+2a(bb)=0] \tag{3.20}$$

Now, (3.20) identity multiply to a from right-side:

$$\alpha[((ab)b)a+2((bb)a)a]+\beta[(b(ba))a+2(a(bb))a]=0$$

$$\alpha[((aa)b)b+2((bb)a)a]+\beta[b((ba)a)+2b((aa)b)]=0 \tag{3.21}$$

$$\alpha[((bb)a)a+2((aa)b)b]+\beta[(a(ab)b)+2b((aa)b)]=0 \tag{3.22}$$

Now, (3.20) identity multiply to a from left-side:

$$\alpha[a((ab)a)+2a((bb)a)]+\beta[a(b(ba))+2a(a(bb))]=0$$

$$\alpha[a((ab)b)+2b((aa)b)]+\beta[b(b(aa))+2a(a(bb))]=0 \tag{3.23}$$

By permutation we can write (3.12) identity in following view:

$$(2\alpha+2\beta)b((aa)b)+2((bb)a)a+\beta a(a(bb))=0 \tag{3.24}$$

$$(2\alpha+2\beta)b((aa)b)+2((aa)b)b+\beta b(b(aa))=0 \tag{3.25}$$

By addition (3.24) and (3.25) we get new identity:

$$2(2\alpha+2\beta)b((aa)b)+2[((aa)b)b+((bb)a)a]+\beta[a(a(bb))+b(b(aa))]=0$$

Now, multiply (1) to d from right-side:

$$\alpha[((ab)c)d+((ba)c)d+((ca)b)d]+\beta[(c(ba))d+(c(ab))d+(b(ac))d]=0 \tag{3.26}$$

(3.27)

Then, multiply (3.2) to d from left-side:

$$\alpha[d((ab)c)+d((ba)c)+d((ca)b)]+\beta[d(c(ba))+d(c(ab))+d(b(ac))]=0$$

In (3.26) identity by changing a=b:=a and c=d:=b, we get:

$$\alpha[((aa)b)b+((aa)b)b+((ba)a)b]+\beta[b((aa)b)+b((aa)b)+a((ab)B)]=0$$

$$\alpha[2((aa)b)b+((bb)a)a]+\beta[2b((aa)b)+a((ab)b)]=0 \tag{3.28}$$

$$(2\alpha)((bb)a)a+\alpha((aa)b)b+(2\beta)b((aa)b)+\beta b((ba)a)=0$$

$$2\alpha b((aa)b)+\alpha a((ab)b)+2\beta a(a(bb))+\beta b(b(aa))=0$$

$\alpha((aa)b)b+\beta b(b(aa))+(2\alpha+2\beta)b((aa)b)+\alpha a((ab)b)+\beta b((ba)a)+2\alpha((bb)a)a+2\beta a(a(bb))=0$

$$\alpha a((ab)b)+\beta b((ba)a)+2\alpha((bb)a)a+2\beta a(a(bb))=0 \tag{3.29}$$

$$\alpha b((ba)a)+\beta a((ab)b)+2\alpha((aa)b)b+2\beta b(b(aa))=0 \tag{3.30}$$

By addition, we can add (3.29) to (3.30):
$(\alpha+\beta)a((ab)b)+(\alpha+\beta)b((ba)a)+2[\alpha((aa)b)b+\beta b(b(aa))]+2[\alpha((bb)a)a+\beta a(a(bb))]=0$ ☐

Theorem 3.2.2. *Let f be generator of an irreducible S_n-submodule of $P_n(\mathfrak{N})$. Then the consequences of higher degrees from the f are equivalent to the following identities:*

(a) f_{n+1} if $f=f_n,\ n=1,2;$
(b) g_{n+1} if $f=f_n,\ n=1,2.$
(c) g_{n+1} if $g=g_n,\ n=2$
(d) g'_{n+1} if $g=f_n,\ n=2$
(e) g'_{n+1} if $g=g_n,\ n=2$

To illustrate this theorem we can use this form:

$$\begin{array}{ccccccc}
f_3 & & & g_3 & & g'_3 & \\
 & \uparrow & \nearrow & \nwarrow & \uparrow & & \nearrow \\
f_2 & & & g_2 & & & \\
 & \uparrow & \nearrow & & & & \\
f_1 & & & & & &
\end{array}$$

Proof. Notice that $f_1=g_1$

As a result of this data, we can understand that the basic elements of multilinear part of three degree free algebra are:

$(ab)c$	$c(ba)$
$(ba)c$	$c(ab)$
$(ca)b$	

By following diagram we can illustrate these elements:

$$f_3 = \begin{array}{|c|c|c|}\hline x & x & x \\ \hline\end{array} = (xx)x$$

$$g_3 = \begin{array}{|c|c|}\hline x & x \\ \hline y & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = (xy - yx)x = (xy)x - (yx)x$$

(a) For $n = 1:\ f_1 = x = 0\ \rightarrow f_2 = xx = 0 \cdot x = 0.$

For $n = 2:\ f_2 = xx = 0\ \rightarrow f_3 = (xx)x = 0 \cdot x = 0.$
(b) For $n = 1$: $f_1 = x = 0 \rightarrow g_2 = xy - yx = 0 \cdot y - y \cdot 0 = 0$

For $n = 2$: $f_2 = xx = 0 \rightarrow g_3 = (xy)x - (yx)x.$

By substituting $x := x + y$ into f_2 we get:

$$xx = (x + y)(x + y) = xx + xy + yx + yy = 0$$

xx and yy are equal to 0 because of $f_2 = 0$. This means that $xy + yx = 0$ and we get:

$$xy = -yx$$

So $g_3 = (xy)x - (yx)x = (xy)x + (xy)x = 2(xx)y = 2f_2 \cdot y = 0$

(c) For $n = 2$: $g_2 = xy - yx = 0 \rightarrow g_3 = (xy)x - (yx)x = (xy - yx)x = g_2 \cdot x = 0 \cdot x = 0.$

(d) For $n = 2$: $f_2 = xx = 0 \rightarrow g'_3 = x(xy - yx)$

By substituting $x := x + y$ into f_2 we get:

$$xx = (x + y)(x + y) = xx + xy + yx + yy = 0$$

xx and yy are equal to 0 because of $f_2 = 0$. Therefore, $xy + yx = 0$ and we get:

$$xy = -yx$$

So $g'_3 = x(xy) - x(yx) = -x(yx) - x(yx) = -2x(yx) = -2y(xx) = 0$

(e) For $n = 2$: $g_2 = xy - yx \rightarrow g'_3 = x(xy - yx) = x \cdot 0 = 0$ $\square$

3.3 Case

$$\alpha \neq 0, \beta = 0$$

In this case we can represent identity as $(ab)c + (ba)c + (ca)b = 0$

Theorem 3.3.1. *As S_n-module*

$$P_1(\mathfrak{N}) \cong S^{(1)}, P_2(\mathfrak{N}) \cong S^{(2)} \oplus S^{(1,1)},$$

$$P_3(\mathfrak{N}) \cong S^{(3)} \oplus 2S^{(2,1)}, P_n(\mathfrak{N}) \cong S^{(n)} \oplus S^{n-1,1} n \geq 4.$$

n	*1*	*2*	*3*	*4*	*5*	*6*	...
dim(P(n))	*1*	*2*	*5*	*4*	*5*	*6*	...

Theorem 3.3.2. *Let f be generator of an irreducible S_n-submodule of $P_n(\mathfrak{N})$. Then the consequences of higher degrees from the f are equivalent to the following identities:*

(a) f_{n+1} *if* $f = f_n,\ n \geq 1$;
(b) g_{n+1} *if* $f = f_n,\ n \geq 2$;
(c) g_{n+1} *if* $g = g_n,\ n \geq 2$.
(d) g'_{n+1} *if* $f = f_n,\ n = 2$;
(e) g'_{n+1} *if* $g = g_n,\ n \geq 2$;

$$\begin{tikzcd}
\cdots & \cdots & \\
f_5 \arrow[u] & g_5 \arrow[u] & \\
f_4 \arrow[u] & g_4 \arrow[u] & \\
f_3 \arrow[u] \arrow[ur] & g_3 \arrow[u] & g'_3 \arrow[ul] \\
f_2 \arrow[u] \arrow[ur] & g_2 \arrow[u] \arrow[ur] & \\
f_1 \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

Proof. Notice that $f_1 = g_1$.

As a result of this data, we can understand that the basic elements of multilinear part of three degree free algebra are:

$$\begin{array}{ll} (ab)c & c(ba) \\ (ba)c & c(ab) \\ (ca)b & \end{array}$$

By following diagram we can illustrate these elements:

$$f_3 = \begin{array}{|c|c|c|}\hline x & x & x \\ \hline \end{array} = (xx)x$$

$$g_3 = \begin{array}{|c|c|}\hline x & x \\ \hline y & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = (xy - yx)x = (xy)x - (yx)x$$

As a result of this data, we can understand that the basic elements of multilinear part of four degree free algebra are:

$$\begin{array}{ll} ((ab)c)d & c((ab)d) \\ ((ba)c)d & c((ba)d) \end{array}$$

By following diagram we can illustrate these elements:

$$f_4 = \begin{array}{|c|c|c|c|}\hline x & x & x & x \\ \hline \end{array} = ((xx)x)x$$

As a result of this data, we can understand that the basic elements of multilinear part of five degree free algebra are:

$$(((ab)c)d)e$$

By following diagram we can illustrate these elements:

$$f_5 = \begin{array}{|c|c|c|c|c|}\hline x & x & x & x & x \\ \hline \end{array} = (((xx)x)x)x$$

(a) For $n = 1: \; f_1 = x = 0 \; \rightarrow f_2 = xx = 0 \cdot x = 0.$

For $n = 2: \; f_2 = xx = 0 \; \rightarrow f_3 = (xx)x = 0 \cdot x = 0.$

For $n = k$ we have $f_k = \underbrace{(...((}_{k-2 \text{ times}} \; xx \; \underbrace{)x)...x)}_{k-2 \text{ times}} \; x = 0 \rightarrow$

$f_{k+1} = \underbrace{(...((}_{k-1 \text{ times}} \; xx \; \underbrace{)x)...x)}_{k-1 \text{ times}} \; x = f_k \cdot x = 0 \cdot x = 0.$

(b) For $n = 1:\ f_1 = x = 0\ \to g_2 = xy - yx = 0 \cdot y - y \cdot 0 = 0$.

For $n = 2$: $f_2 = xx = 0 \to g_3 = (xy)x - (yx)x$.

By substituting $x := x + y$ into f_2 we obtain:

$$xx = (x+y)(x+y) = xx + xy + yx + yy = 0$$

xx and yy are equal to 0 since $f_2 = 0$. This means that $xy + yx = 0$ and we obtain:

$$xy = -yx$$

So $g_3 = (xy)x - (yx)x = (xy)x + (xy)x = 2(xx)y = 2f_2 \cdot y = 0$

(c) For $n = 2$: $g_2 = xy - yx = 0 \to g_3 = (xy)x - (yx)x = (xy - yx)x = g_2 \cdot x = 0 \cdot x = 0$.

For $n = 3$: $g_3 = (xy - yx)x = 0 \to g_4 = ((xy)x)x - ((yx)x)x = ((xy - yx)x)x = g_3 \cdot x = 0 \cdot x = 0$

For $n = k$ we have $g_k = \underbrace{(...((}_{k-2 \text{ times}}\ xy - yx\ \underbrace{)x)...x)}_{k-2 \text{ times}}\ x = 0 \to$ $g_{k+1} = \underbrace{(...((}_{k-1 \text{ times}}\ xy - yx\ \underbrace{)x)...x)}_{k-1 \text{ times}}\ x = g_k \cdot x = 0 \cdot x = 0$.

(d) For $n = 2$: $f_2 = xx = 0 \to g'_3 = x(xy) - x(yx)$.

By substituting $x := x + y$ into f_2 we obtain:

$$xx = (x+y)(x+y) = xx + xy + yx + yy = 0$$

xx and yy are equal to 0 since $f_2 = 0$. This means that $xy + yx = 0$ and we obtain:

$$xy = -yx$$

So $g'_3 = x(xy) - x(yx) = -x(yx) - x(yx) = -2x(yx) = -2f_2 \cdot y = 0$

(e) For $n = 2$: $g_2 = xy - yx = 0 \to g'_3 = x(xy) - x(yx) = x(xy - yx) = x \cdot g_2 = 0 \cdot x = 0$.

□

3.4 Case

$$\alpha = -\beta$$

In this case our identity seem as $(ab)c+(ba)c+(ca)b-c(ba)-c(ab)-b(ac)=0$.

Theorem 3.4.1. *As S_n-module*

$$P_1(\mathfrak{N}) \cong S^{(1)}, P_2(\mathfrak{N}) \cong S^{(2)} \oplus S^{(1,1)},$$

$$P_3(\mathfrak{N}) \cong S^{(3)} \oplus 2S^{(2,1)}, P_4(\mathfrak{N}) \cong S^{(4)} \oplus S^{(2,2)},$$

$$P_n(\mathfrak{N}) \cong S^{(n)} \textit{ for } n \geq 5.$$

Proof. To get new identity let that $c{=}cd$ in identity $(ab)c+(ba)c+(ca)b-c(ba)-c(ab)-b(ac)=0$.

$$(ab)(cd) + (ba)(cd) + ((cd)a)b - (cd)(ba) - (cd)(ab) - b(a(cd)) = 0$$

From [1] it is easy to see that (cd)(ab) = (c(ab))d = a((cd)b).

$$c((ab)d) + c((ba)d) + ((cd)a)b - b((cd)a) - a((cd)b) - b(a(cd)) = 0$$

$$((cd)a)b = b(a(cd))$$

We get new identity:

$$((cd)a)b = a(b(cd)) \tag{3.31}$$

Let's take that a: = b, b: = c, c: = d in identity $(ab)c+(ba)c+(ca)b-c(ba)-c(ab)-b(ac)=0$.

$$(bc)d + (cb)d + (db)c - d(cb) - d(bc) - b(cd) = 0 \tag{3.32}$$

The identity (3.32) multiply from the left side to a.

$$a(b(cd)) = a((bc)d) + a((cb)d) + a((db)c) - a(d(cb)) - a(d(bc)) \tag{3.33}$$

By the identity (3.31):

$$a(d(bc)) = b(a(dc)) = ((dc)a)b = ((db)a)c = a(d(cb)) = ((cb)a)d = ((cd)a)b = a(b(cd)) = ((ab)c)d$$

$$3((ab)c)d = a((bc)d) + a((cb)d) + a((db)c) \tag{3.34}$$

Let's take that $a := b$ and $b := a$ in the identity (3.34):

$$3((ba)c)d = b((ac)d) + b((ca)d) + b((da)c) \tag{3.35}$$

By substituting (3.35) from (3.34) and by identity (3.32) we can take that:

$$((ab)c)d = ((ac)b)d = d(b(ac)) = d(a(bc)) = ((bc)a)d = ((ba)c)d$$

$$a((cb)d) + a((db)c) = b((ca)d) + b((da)c) \tag{3.36}$$

Let's take identity $(ab)c + (ba)c + (ca)b - c(ba) - c(ab) - b(ac) = 0$ and multiply it from right side to a.

$$(a(bc))d = ((ab)c)d + ((ba)c)d + ((ca)b)d - (c(ab))d - (c(ba))d$$

By (3.34) identity we can take that:

$$a((bc)d) = \frac{1}{3}[a((bc)d) + a((cb)d) + a((db)c) + b((ac)d) + b((ca)d) + b((da)c) + c((ab)d) + c((ba)d) + c((da)b)] - (c(ab))d - (c(ba))d$$

So identity (3.31) can show as that:

$$3a((bc)d) = 2a((bc)d) + 2a((cb)d) + a((db)c) + 2b((ca)d) + b((da)c) + c((da)b) - 3c((ab)d) - 3c((ba)d)$$

$$a((bc)d) = a((db)c) - b((ca)d) + b((da)c) + c((da)b) - c((ab)d) \tag{3.37}$$

Let's consider monom $a((bc)d)$. By (3.36), it's obvious that

$$a((bc)d) + a((dc)b) = c((ba)d) + c((da)b)$$

$$a((bc)d) = c((ba)d) + c((da)b) - a((dc)b) \tag{3.38}$$

By adding two identity (3.37) and (3.38) we get:

$$2a((bc)d) = 2c((da)b) + b((da)c) - c((ab)d) \tag{3.39}$$

Let's consider two monoms $c((da)b)$ and $a((cb)d)$. By (3.36) it's obvious that:

$$c((da)b) + c((ba)d) = a((dc)b) + a((bc)d) \tag{3.40}$$
$$a((cb)d) + a((db)c) = b((ca)d) + b((da)c) \tag{3.41}$$

By adding (3.40) and (3.41) we take:

$$c((da)b) + a((cb)d) = a((bc)d) + b((da)c)$$

$$c((da)b) - a((bc)d) = b((da)c) - a((cb)d) \tag{3.42}$$

In identity (3.39) let's take that $a{:=}c,\ b{:=}d,\ c{:=}a$ and $d{:=}b$:

$$2c((da)b) = 2a((bc)d) + d((bc)a) - a((cd)b) \tag{3.43}$$

By subtracting (3.42) from (3.43) we can take that:

$$c((da)b) = b((ad)c) \tag{3.44}$$

$n{=}3$ B_3 has 6 basis elements. They are:

$$\begin{gathered}(ab)c\\(ba)c\\(ca)b\\a(bc)\\a(cb)\\b(ca)\end{gathered}$$

We get $(ca)b = c(ba) + c(ab) + b(ac) - (ab)c - (ba)c$ from (3.2). It is obvious that after this expression there remain 5 basis elements.

$$c(ba)$$
$$c(ab)$$
$$b(ac)$$
$$(ab)c$$
$$(ba)c$$

$$\mathbf{dim(3) = 5}$$

n=4 Let n=4. There are 14 basis elements of B_4. They are:

$((**)*)*$	$*((**)*)$	$*(*(**))$
$((ab)c)d$	$a((bc)d)$	$a(b(cd))$
$((bc)d)a$	$a((cb)d)$	$b(c(da))$
$((cd)a)b$	$a((db)c)$	$c(d(ab))$
$((da)b)c$	$b((ca)d)$	$d(a(bc))$
	$b((da)c)$	
	$c((da)b)$	

Let's consider elements of the type $((**)*)*$. By using (3.31) identity we get:

$$\begin{aligned} ((ab)c)d &= c(d(ab)) \\ ((bc)d)a &= d(a(bc)) \\ ((cd)a)b &= a(b(cd)) \\ ((da)b)c &= b(c(da)) \end{aligned}$$

As we see in the type $((**)*)*$ there are no elements left.
Let's consider elements type $*(*(**))$. By the identity (3.31) we can obtain element:

$$\begin{aligned} a(b(cd)) &= ((cd)a)b = ((ca)b)d = b(d(ca)) = b(c(da)) \\ a(b(cd)) &= ((cd)a)b = ((cb)d)a = d(a(cb)) = c(d(ab)) \\ a(b(cd)) &= ((ad)c)b = ((ac)d)b = d(b(ac)) = d(a(bc)) \end{aligned}$$

Based on this results we can see that here left one element:

$$a(b(cd))$$

Let's consider elements of the type $*((**)*)$. By the identity (3.44) we can obtain elements:

$$\begin{aligned} a((bc)d) &= d((ca)b) = c((da)b) \\ a((cb)d) &= d((bc)a) = b((da)c) \\ a((db)c) &= c((bc)a) = b((ca)d) \end{aligned}$$

In addition, by the identity (3.34) we can see that:
$3((ab)c)d = a((bc)d) + a((cb)d) + a((db)c)$
$3a(b(cd)) = a((bc)d) + a((cb)d) + a((db)c)$
$a((db)c) = 3a(b(cd)) - a((bc)d) - a((cb)d)$
As we see in the type *((**)* there are 2 elements left.
Based on this results we can see that here left 3 basis elements:

$$a(b(cd))$$
$$a((bc)d)$$
$$a((cb)d)$$

n=5 There are 30 basis elements of B_5. They are:

((**)*)*)*	*(((**)*)*)	*(*((**)*))	*(*(*(**)))
(((ab)c)d)e	a(((bc)d)e)	a(b((cd)e))	a(b(c(de)))
(((bc)d)a)e	a(((cb)d)e)	a(b((dc)e))	b(c(d(ea)))
(((cd)a)b)e	a(((db)c)e)	a(b((ec)d))	d(e(a(bc)))
(((da)b)c)e	b(((ca)d)e)	a(c((db)e))	e(a(b(cd)))
(((ea)b)c)d	b(((da)c)e)	a(c((eb)d))	c(d(e(ab)))
	c(((da)b)e)	a(d((eb)e))	
	b(((ea)c)d)	b(c((da))e)	
	a(((eb)c)d)	b(c((ea)d))	
	c(((ea)b)d)	b(d((ea)c))	
	d(((ea)b)c)	c(d((ea)b))	

Let's consider type *(*(*(**))). By the identity (3.30) we can obtain that *(*(*(**))) equal to (((**)*)*)*. What does mean that in type *(*(*(**))) has no element.

$$a(b(c(de))) = (((da)b)c)e$$
$$d(e(a(bc))) = (((ca)b)e)d$$
$$b(c(d(ea))) = (((ea)b)c)d$$
$$e(a(b(cd))) = (((cd)a)b)e$$
$$c(d(e(ab))) = (((ab)c)d)e$$

Let's consider type *(*((**)*)). By the identity (3.31) and (3.44):

$$a(b((ec)d)) = a(c((db)e))$$
$$a(b((dc)e)) = a(c((eb)d))$$
$$c(d((ea)b)) = a(b((cd)e))$$
$$a(d((cb)e)) = a(b((ec)d))$$
$$b(c((da)e)) = a(b((ec)d))$$
$$b(c((ea)d)) = a(b((dc)e))$$
$$b(d((ea)c)) = a(b((cd)e))$$
$$a(b((ec)d)) = (((bc)d)b)a$$

$$\begin{gathered} a(b((cd)e)) = (((bc)e)c)a \\ a(b((dc)e)) = (((bc)d)e)a \end{gathered}$$

As we see in type $*(*((**)*))$ there no one element.
Let's consider type $*(((**)*)*)$. By using formulas in (3.30):

$$\begin{gathered} a(((bc)d)e) = a(((cb)d)e) = b(((ca)d)e) = b(((da)c)e) = a(((eb)c)d) = \\ b(((ea)c)d) = c(((da)b)e) = c(((ea)b)d) = d(((ea)b)c) \\ a(((bc)d)e) = a(d(e(bc))) = (((ab)c)d)e \end{gathered}$$

As we see in $*(((**)*)*)$ no has element.

Let's consider type $(((**)*)*)*$. By identity (3.31) we can obtain that:

$$\begin{gathered} (((cd)e)a)b = (((cb)d)e)a \\ (((bc)d)e)a = (((cb)d)e)a \\ (((cb)d)e)a = (((ac)b)d)e = (((ab)c)d)e \end{gathered}$$

In type $(((**)*)*)*$ there are only 1 element.
Finally, we can see that in B_5 there one element. It is:

$$(((ab)c)d)e$$

$\square$

Theorem 3.4.2. *Let f be generator of an irreducible S_n-submodule of $P_n(\mathfrak{N})$. Then the consequences of higher degrees from the f are equivalent to the following identities:*
(a) f_{n+1} if $f = f_n,\ n \geq 1$;
(b) g_{n+1} if $f = f_n,\ n = 1, 2$;
(c) g_{n+1} if $g = g_n,\ n = 2, 3$.
(d) g'_{n+1} if $f = f_n,\ n = 3$;
(e) g'_{n+1} if $gg_n,\ n = 3$;

To illustrate this theorem we can use this form:

$$\begin{tikzcd}
\cdots & & \\
f_5 \arrow[u] & & \\
f_4 \arrow[u] & g_4 & \\
f_3 \arrow[u] \arrow[ur] & g_3 \arrow[u] & g'_3 \arrow[ul] \\
f_2 \arrow[u] \arrow[ur] & g_2 \arrow[u] \arrow[ur] & \\
f_1 \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

Proof. As a result of this data, we can understand that the basic elements of multilinear part of three degree free algebra are:

$$(ab)c \qquad\qquad c(ab)$$

$$(ba)c \qquad\qquad b(ac)$$

$$c(ba)$$

By following diagram we can illustrate these elements:

$$f_3 = \begin{array}{|c|c|c|}\hline x & x & x \\ \hline \end{array} = (xx)x \qquad\qquad g_3 = \begin{array}{|c|c|}\hline x & x \\ \hline y & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = (xy - yx)x = (xy)x - (yx)x$$

As a result of this data, we can understand that the basic elements of multilinear part of four degree free algebra are:

$$a((bc)d) \qquad\qquad a(b(cd))$$

$$a((cb)d)$$

By following diagram we can illustrate these elements:

$$f_4 = \begin{array}{|c|c|c|c|}\hline x & x & x & x \\ \hline \end{array} = ((xx)x)x \qquad\qquad f'_4 = \begin{array}{|c|c|c|c|}\hline x & x & x & x \\ \hline \end{array} = x(x(xx))$$

$$g_4 = \begin{array}{|c|c|c|}\hline x & x & x \\ \hline y & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} ((xy - yx)x)x = ((xy)x)x - ((yx)x)x$$

As a result of this data, we can understand that the basic elements of multilinear part of five degree free algebra are:

$$(((ab)c)d)e$$

By following diagram we can illustrate these elements:

$$f_5 = \begin{array}{|c|c|c|c|c|}\hline x & x & x & x & x \\ \hline \end{array} = (((xx)x)x)x$$

(a) For $n = 1: \;\; f_1 = x = 0 \;\; \rightarrow f_2 = xx = 0 \cdot x = 0.$

For $n = 2: \;\; f_2 = xx = 0 \;\; \rightarrow f_3 = (xx)x = 0 \cdot x = 0.$

For $n = k$ we have $f_k = \underbrace{(...((}_{k-2 \text{ times}} xx \underbrace{)x)...x)}_{k-2 \text{ times}} x = 0 \rightarrow$
$f_{k+1} = \underbrace{(...((}_{k-1 \text{ times}} xx \underbrace{)x)...x)}_{k-1 \text{ times}} x = f_k \cdot x = 0 \cdot x = 0.$

(b) $f_1 = x = 0 \rightarrow g_2 = xy - yx = 0 \cdot y - y \cdot 0 = 0$

For $n = 2$: $f_2 = xx = 0 \rightarrow g_3 = (xy - yx)x$

By substituting $x := x + y$ into f_2 we get:

$$xx = (x + y)(x + y) = xx + xy + yx + yy = 0$$

xx and yy are equal to 0 because of $f_2 = 0$. This means that $xy + yx = 0$ and we get:

$$xy = -yx$$

So $g_3 = (xy)x - (yx)x = (xy)x + (xy)x = 2(xx)y = 0$

(c) For $n = 2$: $g_2 = xy - yx = 0 \rightarrow g_3 = (xy - yx)x = 0 \cdot x = 0$

For $n = 3$: $g_3 = (xy - yx)x = 0 \rightarrow g_4 = ((xy)x)x - ((yx)x)x = ((xy - yx)x)x = g_3 \cdot x = 0 \cdot x = 0$

(d) For $n = 2$: $f_2 = xx = 0 \rightarrow g_3' = x(xy - yx)$ By substituting $x := x + y$ into f_2 we get:

$$xx = (x + y)(x + y) = xx + xy + yx + yy = 0$$

xx and yy are equal to 0 since $f_2 = 0$. This means that $xy + yx = 0$ and we get:

$$xy = -yx$$

So $g_3' = x(xy) - x(yx) = -x(yx) - x(yx) = -2y(xx) = -2 \cdot 0 = 0$

(e) For $n = 2$: $g_2 = xy - yx = 0 \rightarrow g_3' = x(xy - yx) = x \cdot 0 = 0$ $\square$

3.5 Case $\alpha = \beta$

In this case identity (1) take the form $(ab)c+(ba)c+(ca)b+c(ba)+c(ab)+b(ac) = 0$.

Theorem 3.5.1. *As S_n-module*

$$P_1(\mathfrak{N}) \cong S^{(1)}, P_2(\mathfrak{N}) \cong S^{(2)} \oplus S^{(1,1)},$$

$$P_3(\mathfrak{N}) \cong S^{(3)} \oplus 2S^{(2,1)}, P_4(\mathfrak{N}) \cong S^{(4)} \oplus S^{(2,2)},$$

$$P_n(\mathfrak{N}) \cong 0 \textit{ for } n \geq 5.$$

Proof

- Case $n = 3$

Proof. When $n = 3$ we have 6 basis elements of multilinear part of free bicommutative algebra. They are

$$\begin{array}{cc} (**)* & *(**) \\ (ab)c & b(ac) \\ (ba)c & c(ab) \\ (ca)b & c(ba) \end{array}$$

By identity $(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + b(ac) = 0$ we know

$$b(ac) = -((ab)c + (ba)c + (ca)b + c(ba) + c(ab))$$

So we received the following base elements

$$\begin{array}{ccc} (ab)c & b(ac) & (ca)b \\ (ba)c & c(ab) & \end{array}$$

As a result we obtain 5 basis elements of multilinear part of free relatively algebra of varieties of bicommutative algebra of degree 3.

- $n = 4$ When $n = 4$, we have 14 basis elements of multilinear part of free bicommutative algebra. There are 3 type of elements:

$$\begin{array}{ccc} ((**)*)* & *((**)*) & *(*(**)) \\ ((ab)c)d & a((bc)d) & a(b(cd)) \\ ((bc)d)a & a((cb)d) & b(c(da)) \\ ((cd)a)b & a((db)c) & c(d(ab)) \\ ((da)b)c & b((ca)d) & d(a(bc)) \\ & b((da)c) & \\ & c((da)b) & \end{array}$$

Let $a := a, b := a, c := a$ in the identity $(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + b(ac) = 0$:

$$\begin{gathered}(aa)a + (aa)a + (aa)a + a(aa) + a(aa) + a(aa) = 0\\ 3(aa)a = -3a(aa)\\ (aa)a = -a(aa)\end{gathered}$$

So we get identity:

$$(aa)a = -a(aa) \tag{3.45}$$

When we multiply a b to the left of identity (3.4):

$$\begin{gathered}b((aa)a) + b(a(aa)) = 0\\ a((ba)a) + a(a(ba)) = 0\\ a((ba)a) = -a(a(ba))\end{gathered}$$

Then we get a new identity:

$$a((ba)a) = -a(a(ba)) \tag{3.46}$$

And when we multiply a b to the right of identity (1.47):

$$\begin{gathered}((aa)a)b + (a(aa))b = 0\\ ((ab)a)a + (a(ab))a = 0\\ ((ab)a)a = -(a(ab))a\end{gathered}$$

Then we get a new identity:

$$((ab)a)a = -(a(ab))a \tag{3.47}$$

When we multiply an a to the left and right of identity (1.47), we can get two new identities such that:

$$a((aa)a) = -a(a(aa)) \tag{3.48}$$

$$((aa)a)a = -(a(aa))a \tag{3.49}$$

Let $a := (aa), b; = a, c := a$ in the identity $(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + b(ac) = 0$:

$$\begin{aligned}&((aa)a)a + (a(aa))a + (a(aa))a + a(a(aa) + a((aa)a) + a((aa)a) = 0\\ &= ((aa)a)a - ((aa)a)a - ((aa)a)a + a(a(aa)) - a(a(aa)) - a(a(aa)) =\\ &= -((aa)a)a - a(a(aa)) = 0\end{aligned}$$

So we get the following identity:

$$((aa)a)a = -a(a(aa)) \tag{3.50}$$

Let $a := (aa), b := a, c := b$ in the identity $(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + b(ac) = 0$:

$$\begin{gathered}
((aa)a)b + (a(aa))b + (b(aa))a + b(a(aa)) + b((aa)a) + a((aa)b) = 0 \\
((aa)a)b - ((aa)a)b + b((aa)a) - b((aa)a) + b((aa)a) + (a(aa))b = 0 \\
b((aa)a) + (a(aa))b = 0
\end{gathered}$$

$$((aa)a)b = -b(a(aa)) \tag{3.51}$$

Let $a := a, b := a, c := b$ in identity $(ab)c+(ba)c+(ca)b+c(ba)+c(ab)+b(ac) = 0$:

$$(aa)b + (aa)b + (ba)a + b(aa) + b(aa) + a(ab) = 0$$

$$2(aa)b + 2b(aa) + (ba)a + a(ab) = 0 \tag{3.52}$$

Then we multiple an a at the right of the identity (3.52):

$$\begin{aligned}
&2((aa)b)a + 2(b(aa))a + ((ba)a)a + (a(ab))a = 0 \\
&2((aa)a)b + 2b((aa)a) + ((ba)a)a + a((aa)b) = \\
&= 2((aa)a)b + 2b((aa)a) + ((ba)a)a + (a(aa))b = \\
&= 2((aa)a)b - 2b(a(aa)) + ((ba)a)a - ((aa)a)b = \\
&= ((aa)a)b - 2b(a(aa)) + ((ba)a)a = \\
&= ((aa)a)b - b(a(aa)) - b(a(aa)) + ((ba)a)a = 0 \\
&((aa)a)b - a(a(ab)) = ((ba)a)a - b(a(aa))
\end{aligned}$$

And multiple an a on the left of the identity (3.52):

$$\begin{aligned}
&2a((aa)b) + 2a(b(aa)) + a((ba)a) + a(a(ab)) = 0 \\
&2a((aa)b) + 2b(a(aa)) + b((aa)a) + a(a(ab)) = \\
&= 2a((aa)b) + 2b(a(aa)) - b(a(aa)) + a(a(ab)) = \\
&= 2(a(aa))b + b(a(aa)) + a(a(ab)) = \\
&= -2((aa)a)b + b(a(aa)) + a(a(ab)) = \\
&= -((aa)a)b + b(a(aa)) - ((aa)a)b + a(a(ab)) = 0 \\
&((aa)a)b - a(a(ab)) = -((aa)a)b + b(a(aa))
\end{aligned}$$

From these two equality we see that the left side is the same, so we can write as follows:

$$\begin{aligned}
&((ba)a)a - b(a(aa)) = ((aa)a)b - a(a(ab)) \\
&((ba)a)a + a(a(ab)) = ((aa)a)b + b(a(aa)) \\
&((ba)a)a + a(a(ab)) = 0
\end{aligned}$$

So we get new identity:

$$((ba)a)a = -a(a(ab)) \tag{3.53}$$

Let $a := a, b := (aa), c := b$ in the identity $(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + b(ac) = 0$:

$$\begin{aligned}
&(a(aa))b + ((aa)a)b + (ba)(aa) + b((aa)a) + b(a(aa)) + (aa)(ab) = 0\\
&(a(aa) + (aa)a)b + b((aa)a + a(aa)) + (ba)(aa) + (aa)(ab) = 0
\end{aligned}$$

$$\begin{aligned}
&(ba)(aa) + (aa)(ab) = 0\\
&a((ba)a) + (a(ab))a = 0\\
&b((aa)a) + (a(aa))b = 0\\
&-b(a(aa)) - ((aa)a)b = 0
\end{aligned}$$

$$b(a(aa)) = -((aa)a)b \tag{3.54}$$

Let $a := a, b := a, c := (bb)$ in the identity $(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + b(ac) = 0$:

$$\begin{aligned}
&(aa)(bb) + (aa)(bb) + ((bb)a)a + (bb)(aa) + (bb)(aa) + a(a(bb)) = 0\\
&b((aa)b) + b((aa)b) + ((bb)a)a + a((bb)a) + a((bb)a) + a(a(bb)) =\\
&= 2b((aa)b) + 2a((bb)a) + ((bb)a)a + a(a(bb)) =\\
&= 4a((bb)a) + ((bb)a)a + a(a(bb) = 0
\end{aligned}$$

$$4a((bb)a) + ((bb)a)a + a(a(bb)) = 0 \tag{3.55}$$

Let $a := a, b := b, c := (cd)$ in the identity $(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + b(ac) = 0$:

$$\begin{aligned}
&(ab)(cd) + (ba)(cd) + ((cd)a)b + (cd)(ba) + (cb)(ab) + b(a(cd)) = 0\\
&c((ab)d) + c((ba)d) + ((cd)a)b + b((cd)a) + a((cd)b) + b(a(cd)) = 0\\
&2c((ab)d) + 2c((ba)d) + ((cd)a)b + b(a(cd)) = 0
\end{aligned}$$

Then we multiple a b to the right of identity $(ab)c + (ba)c + (ca)b + c(ba) + c(ab) + b(ac) = 0$:

$$\begin{aligned}
&((ab)c)d + ((ba)c)d + ((ca)b)d + (c(ba))d + (c(ab))d + (b(ac))d = 0\\
&((ab)c)d + ((ba)c)d + ((ca)b)d + c((ba)d) + c((ab)d) + b((ac)d) = 0
\end{aligned}$$

and to the left:

$$d((ab)c) + d((ba)c) + d((ca)b) + d(c(ba)) + d(c(ab)) + d(b(ac)) = 0$$

$$\begin{cases} -((da)b)c + c((ba)d) + c((ab)d) + b((ac)d) = 0 \\ -c(b(ad)) + d((ab)c) + d((ba)c) + d((ca)b) = 0 \end{cases}$$

So we obtain new identity:

$$-((da)b)c = c(b(ad)) \tag{3.56}$$

In the identity (3.55) by identity (3.56) we can take new identity:

$$b((aa)b) = 0 \tag{3.57}$$

We multiple a b to the right of identity (3.52):

$$((aa)b)b + ((aa)b)b + ((ba)a)b + (b(aa))b + (b(aa))b + (a(ab)b = 0$$
$$2((aa)b)b + ((ba)a)b + 2(b(aa))b + (a(ab))b = 0$$

and we multiple a b to the left of the identity (3.52):

$$b((aa)b) + b((aa)b) + b((ba)a) + b(b(aa)) + b(b(aa)) + b(a(ab)) = 0$$
$$2b((aa)b) + 2b(b(aa)) + b((ba)a) + b(a(ab)) = 0$$

Then we add this two 2 identities

$$\begin{aligned} &2((aa)b)b + ((ba)a)b + 2(b(aa))b + (a(ab))b \\ &+ 2b((aa)b) + 2b(b(aa)) + b((ba)a) + b(a(ab)) \\ &= 4b((aa)b) + ((bb)a)a + a(a(bb)) + \\ &+ 2((aa)b)b + 2b(b(aa)) + a((ab)b) + b((ba)a) = 0 \end{aligned}$$

By identity (3.55), we obtain following identity:

$$2((aa)b)b + 2b(b(aa)) + a((ab)b) + b((ba)a) = 0$$

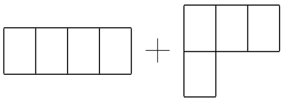
By identity (3.56), we get the next identity :

$$a((ab)b) + b((ba)a) = 0$$

So the last identitity:

$$a((ab)b) + b((ba)a) = 0 \tag{3.58}$$

In the first type we have two diagrams:

$((**)*)* =$ 

$((aa)a)a,\ \begin{array}{|c|c|c|c|}\hline \mathrm{a}&\mathrm{a}&\mathrm{a}&\mathrm{a}\\\hline\end{array}$

$$\begin{aligned}&a((aa)a)+a(a(aa))=0\\&(a(aa))a+a(a(aa))=0\\&-((aa)a)a+a(a(aa))=0\\&-((aa)a)a-((aa)a)a=0\\&-2((aa)a)a=0\\&((aa)a)a=0.\end{aligned}$$

In the second type we have three diagrams:

$$*((**)*)=\begin{array}{|c|c|c|c|}\hline \ &\ &\ &\ \\\hline\end{array}+\begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline \ \\\cline{1-1}\end{array}+\begin{array}{|c|c|}\hline \ &\ \\\hline \ &\ \\\hline\end{array}$$

$a((aa)a),\ \begin{array}{|c|c|c|c|}\hline \mathrm{a}&\mathrm{a}&\mathrm{a}&\mathrm{a}\\\hline\end{array}$

$$\begin{aligned}a((aa)a)&=-a(a(aa))\\a((aa)a)&=0.\end{aligned}$$

$a((ab)a)-a((ba)a),\ \begin{array}{|c|c|c|}\hline \mathrm{a}&\mathrm{a}&\mathrm{a}\\\hline \multicolumn{1}{|c|}{\mathrm{b}}\\\cline{1-1}\end{array}$

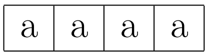
$$\begin{aligned}&a((ab)a)-a((ba)a)\\&=a((aa)b)-a((ba)a)\quad=(a(aa))b-b((aa)a)=-((aa)a)b+b(a(aa))=0\end{aligned}$$

$(ab-ba)(ab-ba),\ \begin{array}{|c|c|}\hline \mathrm{a}&\mathrm{a}\\\hline \mathrm{b}&\mathrm{b}\\\hline\end{array}$

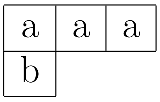
$$\begin{aligned}&(ab-ba)(ab-ba)\\&=(ab)(ab)-(ab)(ba)-(ba)(ab)+(ba)(ba)\\&=a((ab)b)-b((ab)a)-a((ba)b)+b((ba)a)\\&=a((ab)b)-2b((aa)b)+b((ba)a)\\&=a((ab)b)+b((ba)a)=0\end{aligned}$$

In the third type we have two diagrams:

$$*(*(**))=\begin{array}{|c|c|c|c|}\hline \ &\ &\ &\ \\\hline\end{array}+\begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline \ \\\cline{1-1}\end{array}$$

$a(a(aa))$, 

$$((aa)a)a = -a(a(aa)) \qquad\qquad a(a(aa)) = 0.$$

$a(a(ab)) - a(a(ba))$, 

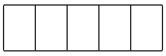
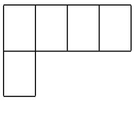
$$\begin{aligned}
&a(a(ab)) - a(a(ba))\\
&= a(a(ab)) - b(a(aa))\\
&= -((ba)a)a - b(a(aa))\\
&= -((ba)a)a + ((aa)a)b = 0
\end{aligned}$$

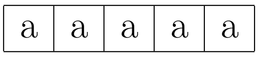
Based on result we get 3 basis elements of multilinear part of free relatively algebra of varieties of bicommutative algebra of degree 4. There are $((ab)c)d$, $((ba)c)d$, $((ca)b)d$.

- $n = 5$ When $n = 5$, we have 30 basis elements of multilinear part of free bicommutative algebra. There are 4 type of elements:

$(((**)*)*)*$	$*(*((**)*))$	$*(((**)*)*)$	$*(*(*(**)))$
$(((ab)c)d)e$	$a(b((cd)e))$	$a(((bc)d)e)$	$a(b(c(de)))$
$(((bc)d)e)a$	$a(b((dc)e))$	$a(((cb)d)e)$	$b(c(d(ea)))$
$(((cd)e)a)b$	$a(b((ec)d))$	$a(((db)c)e)$	$c(d(e(ab)))$
$(((de)a)b)c$	$a(c((db)e))$	$a(((eb)c)d)$	$d(e(a(bc)))$
$(((ea)b)c)d$	$a(c((eb)d))$	$b(((ca)d)e)$	$e(a(b(cd)))$
	$a(d((eb)c))$	$b(((da)c)e)$	
	$b(c((da)e))$	$b(((ea)c)d)$	
	$b(c((ea)d))$	$c(((da)b)e)$	
	$b(d((ea)c))$	$c(((ea)b)d)$	
	$c(d((ea)b))$	$d(((ea)b)c)$	

In the first type we have two diagrams:

$(((**)*)*)* =$  $+$ 

$(((aa)a)a)a$, 

As we got the identity $((aa)a)a = 0$ in $n = 4$, we can write like:

$$(((aa)a)a)a = 0 \cdot a = 0$$

$$(((ab)a)a)a-(((ba)a)a)a, \quad \begin{array}{|c|c|c|c|}\hline a & a & a & a \\ \hline b \\ \cline{1-1}\end{array}$$

$$\begin{aligned}&(((ab)a)a)a-(((ba)a)a)a\\ &=(((ab)a)a)a+(a(a(ab)))a\\ &=(((aa)a)a)b+(a(a(aa)))b\\ &=0\cdot b+0\cdot b=0\end{aligned}$$

By identities $((aa)a)a=0$ and $a(a(aa))=0$ that we proved in $n=4$

In the second type we have three diagrams:

$$*(*((**)*))=\begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \; & \; \\ \hline\end{array}+\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \; \\ \cline{1-1}\end{array}+\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \; \\ \cline{1-2}\end{array}$$

$$a(a((aa)a)), \quad \begin{array}{|c|c|c|c|c|}\hline a & a & a & a & a \\ \hline\end{array}$$

This part is same so we get:

$$a(a((aa)a))=a\cdot 0=0$$

$$a(a((ab)a))-a(a((ba)a)), \quad \begin{array}{|c|c|c|c|}\hline a & a & a & a \\ \hline b \\ \cline{1-1}\end{array}$$

$$\begin{aligned}&a(a((ab)a))-a(a((ba)a))\\ &=a(a((aa)b))-a(b((aa)a))=\\ &=a(a((aa)b))+a(b(a(aa)))=\\ &=a(a((aa)b))-a(((aa)a)b)=\\ &=(a(a(aa)))b-(a((aa)a))b=0\end{aligned}$$

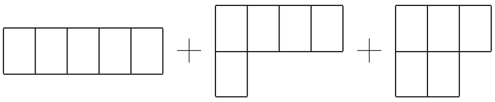
By identities $a((aa)a)=0$ and $a(a(aa))=0$ that we proved in $n=4$

$$a((ab-ba)(ab-ba)), \quad \begin{array}{|c|c|c|}\hline a & a & a \\ \hline b & b \\ \cline{1-2}\end{array}$$

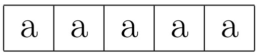
$$\begin{aligned}&a((ab-ba)(ab-ba))=\\ &=a((ab)(ab)-(ab)(ba)-(ba)(ab)+(ba)(ba))=\\ &=a(a((ab)b)-b((ab)a)-a((ba)b)+b((ba)a))=\\ &=a(a((ab)b)-2b((aa)b)+b((ba)a))=\\ &=a(a((ab)b)+b((ba)a))=0\end{aligned}$$

In the third type we have also three diagrams:

$$*(((**)*)*) =$$

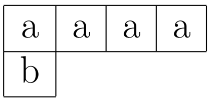


$a(((aa)a)a)$,



$$a(((aa)a)a) = a \cdot 0 = 0$$

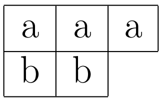
$a(((ab)a)a) - a(((ba)a)a)$,



$$\begin{aligned} &a(((ab)a)a) - a(((ba)a)a) \\ &= a((aa)a)b) - a(((ba)a)a) \\ &= (a((aa)a))b - b(((aa)a)a) = 0 \end{aligned}$$

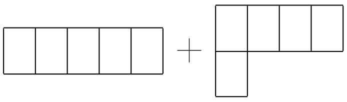
By identities $a((aa)a) = 0$ and $((aa)a)a = 0$ that we proved in $n = 4$

$a((ab - ba)(ab - ba))$,

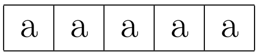


This diagram is the same with third diagram in the second type.
In the last type we have two diagrams:

$$*(*(*(**))) =$$



$a(a(a(aa)))$,



As we got the identity $((aa)a)a = 0$ in $n = 4$, we can write like:

$$(((aa)a)a)a = 0 \cdot a = 0$$

$$a(a(a(ab))) - a(a(a(ba))), \begin{array}{|c|c|c|c|} \hline \text{a} & \text{a} & \text{a} & \text{a} \\ \hline \text{b} \\ \cline{1-1} \end{array}$$

$$\begin{aligned} & a(a(a(ab))) - a(a(a(ba))) \\ & = a(a(a(ab))) + b(a(a(aa))) = \\ & = -a(((ba)a)a) + b(a(a(aa))) = \\ & = b(((aa)a)a) + b(a(a(aa))) = 0 \end{aligned}$$

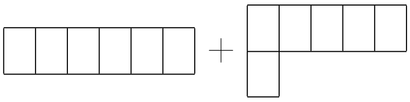
By identities $((aa)a)a = 0$ and $a(a(aa)) = 0$ that we proved in $n = 4$
Based on result we do not have base elements of multilinear part of free algebra of varieties of bicommutative algebra of degree 5.
It means that $(((ab)c)d)e = 0$, $a(((bc)d)e) = 0$, $a(b((cd)e)) = 0$, $a(b(c(de))) = 0$.

- Let $n = 6$. The number of basis elements of multilinear part of degree 6 of free bicommutative alebras is equal to 62. There are 5 types of elements:

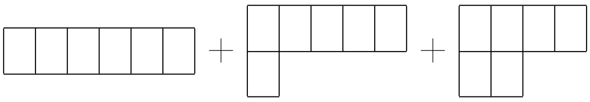
$((((**)*)*)*)*$	$*((((**)*)*)*)$	$*(*(((**)*)*))$	$*(*(*((**)*)))$	$*(*(*(*(**))))$
$((((ab)c)d)e)f$	$a((((bc)d)e)f)$	$a(b(((cd)e)f))$	$a(b(c((de)f)))$	$a(b(c(d(ef))))$
$((((bc)d)e)f)a$	$a((((cb)d)e)f)$	$a(b(((dc)e)f))$	$a(b(c((ed)f)))$	$b(c(d(e(fa))))$
$((((cd)e)f)a)b$	$a((((db)c)e)f)$	$a(b(((ec)d)f))$	$a(b(c((fd)e)))$	$c(d(e(f(ab))))$
$((((de)f)a)b)c$	$a((((eb)c)d)f)$	$a(b(((fc)d)e))$	$a(b(d((ec)f)))$	$d(e(f(a(bc))))$
$((((ef)a)b)c)d$	$a((((fb)c)d)e)$	$a(c(((db)e)f))$	$a(b(d((fc)e)))$	$e(f(a(b(cd))))$
$((((fa)b)c)d)e$	$b((((ca)d)e)f)$	$a(c(((eb)d)f))$	$a(b(e((fc)d)))$	$f(a(b(c(de))))$
	$b((((da)c)e)f)$	$a(c(((fb)d)e))$	$a(c(d((eb)f)))$	
	$b((((ea)c)d)f)$	$a(d(((eb)c)f))$	$a(c(d((fb)e)))$	
	$b((((fa)c)d)e)$	$a(d(((fb)c)e))$	$a(c(e((fb)d)))$	
	$c((((da)b)e)f)$	$a(e(((fb)c)d))$	$a(d(e((fb)c)))$	
	$c((((ea)b)d)f)$	$b(c(((da)e)f))$	$b(c(d((ea)f)))$	
	$c((((fa)b)d)e)$	$b(c(((ea)d)f))$	$b(c(d((fa)e)))$	
	$d((((ea)b)c)f)$	$b(c(((fa)d)e))$	$b(c(e((fa)d)))$	
	$d((((fa)b)c)e)$	$b(d(((ea)c)f))$	$b(d(e((fa)c)))$	
	$e((((fa)b)c)d)$	$b(d(((fa)c)e))$	$c(d(e((fa)b)))$	
		$b(e(((fa)c)d))$		
		$c(d(((ea)b)f))$		
		$c(d(((fa)b)e))$		

$$c(e(((fa)b)d))$$
$$d(e(((fa)b)c))$$

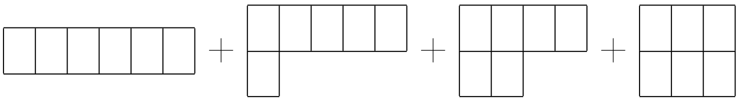
$((((**)*)*)*)*$ and $*(*(*(*(**))))$ type of elements we have diagrams like:



$*((((**)*)*)*)$ and $*(*(*((**)*)))$ type of elements we have the next diagrams:



And the last $*(*(((**)*)*))$ type of elements we have diagrams like:



As we have already proven that in $n = 5$ we do not have base elements, here we use the same methods and this will be easy to prove it.
Starting with the $n = 5, 6, 7, ...$ we do not have a basic element.
In this way our theorem is proved. $\square$

Theorem 3.5.2. *Let f be generator of an irreducible S_n-submodule of $P_n(\mathfrak{N})$. Then the consequences of higher degrees from the f are equivalent to the following identities:*
(a) f_{n+1} if $f = f_n,\ n = 1, 2, 3$;
(b) g_{n+1} if $f = f_n,\ n = 1, 2$;
(c) g_{n+1} if $g = g_n,\ n = 1, 2, 3$;
(d) g_{n+1} if $g'_n,\ n = 3$;

To illustrate this theorem we can use this form:

... ...

$$\begin{tikzcd}
f_4 & g_4 & {} & {} \\
f_3 \arrow[u] \arrow[ur] & g_3 \arrow[ul] & g'_3 \arrow[u] & {} \arrow[ul] \\
f_2 \arrow[u] \arrow[ur] & g_2 \arrow[ul] & {} \arrow[u] \arrow[ur] & {} \\
f_1 \arrow[u] \arrow[ur] & & &
\end{tikzcd}$$

Proof. As a result of this data, we can understand that the basic elements of multilinear part of three degree free algebra are:

$$\begin{array}{ll} (ab)c & c(ba) \\ (ba)c & c(ab) \\ (ca)b & \end{array}$$

By following diagram we can illustrate these elements:

$f_3 = \begin{array}{|c|c|c|}\hline x & x & x \\ \hline\end{array} = (xx)x$

$$g_3 = \begin{array}{|c|c|}\hline x & x \\ \hline y & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} = (xy - yx)x = (xy)x - (yx)x$$

(a) For $n = 1: \ f_1 = x = 0 \ \rightarrow f_2 = xx = 0 \cdot x = 0.$

For $n = 2: \ f_2 = xx = 0 \ \rightarrow f_3 = (xx)x = 0 \cdot x = 0.$

For $n = 3: \ f_3 = xx = 0 \ \rightarrow f_4 = ((xx)x)x$

(b) For $n = 1$: $f_1 = x = 0 \rightarrow g_2 = xy - yx = 0 \cdot y - y \cdot 0 = 0$

For $n = 2$: $f_2 = xx = 0 \rightarrow g_3 = (xy)x - (yx)x.$

By substituting $x := x + y$ into f_2 we get:

$$xx = (x + y)(x + y) = xx + xy + yx + yy = 0$$

xx and yy are equal to 0 because of $f_2 = 0$. This means that $xy + yx = 0$ and we get:

$$xy = -yx$$

So $g_3 = (xy)x - (yx)x = (xy)x + (xy)x = 2(xx)y = 2f_2 \cdot y = 0$

(c) For $n = 2$: $g_2 = xy - yx = 0 \to g_3 = (xy)x - (yx)x = (xy - yx)x = g_2 \cdot x = 0 \cdot x = 0$.

For $n = 3$: $g_3 = (xy - yx)x = 0 \to g_4 = ((xy)x)x - ((yx)x)x = ((xy - yx)x)x = g_3 \cdot x = 0 \cdot x = 0$

(d) For $n = 3$: $g_3' = x(xy - yx) = 0 \to g_4 = x(x(xy) - x(x(yx)) = x(x(xy - yx)) = x \cdot g_3' = x \cdot 0 = 0$ □

4. Conclusion

The major goal of this thesis was to classifying all subvarieties of the bicommutative algebras described by the identity below.

$$\alpha[(ab)c + (ba)c + (ca)b] + \beta[c(ba) + c(ab) + b(ac)] = 0.$$

In this thesis work we used 2 methods of classifying:

- The methods of linear algebra;
- The methods of the representation theory of groups.

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