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Distribution of periods of the continued fractions for quadratic irrationals

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Distribution of periods of the continued fractions for quadratic irrationals

THESIS

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Supervisor: **Shirali Kadyrov**

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Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Moldir Tlepova

June 2023

Abstract

This dissertation thesis focuses on the distribution patterns of the periods of continued fractions for quadratic irrational numbers. The main objective of this thesis is to investigate and examine the patterns and distribution of the periods of continued fractions associated with quadratic irrational numbers of the form $\alpha = n\sqrt{d}$, where n is a positive integer and d is a square-free integer. We denote the length of the period of the continued fraction as $D(\alpha)$. We establish a relationship between the length of the periodic part of the continued fraction $D(\alpha)$ and the quadratic irrational number $a = n\sqrt{d}$ by introducing an exact and explicit formula for the irrational number α for specific cases when $D(\alpha) = 2$ and $D(\alpha) = 3$.

By utilizing the equation $\alpha = n\sqrt{d}$ and considering cases where the length of the periodic part is 2, we have successfully demonstrated that when $n \geq 1$ is the solution of the Pell's equation in the form of $n^2 - dx^2 = 1$, the corresponding quadratic irrational $\alpha = n\sqrt{d}$ will indeed have a periodic part of length 2. Through the proof of this theorem, we have established that for any value of d greater than or equal to 1, there exists an infinite set of positive integers n for which the length of the period $D(n\sqrt{d})$ is precisely 2.

Аңдатпа

Бұл диссертациялық жұмыс квадраттық иррационал сандар үшін үздіксіз бөлшектердің периодтарының ұзындықтарының таралу заңдылықтарын қарастырады. Атап айтқанда, диссертациялық жұмыстың мақсаты - $\alpha = n\sqrt{d}$ квадраттық иррационал санына арналған үздіксіз бөлшектің периодтарын зерттеу, мұнда n - оң бүтін сан, ал d - квадраттан бос бүтін сан. Үздіксіз бөлшек периодтарының ұзындығын $D(\alpha)$ деп белгілейміз. α иррационал санының $D(\alpha) = 2$ және $D(\alpha) = 3$ кезіндегі анық формуласын анықтау арқылы үздіксіз бөлшек периоды $D(\alpha)$ мен квадраттық иррационал сан $a = n\sqrt{d}$ арасындағы өзара байланысты анықтадық.

$\alpha = n\sqrt{d}$ иррационал санының периодының ұзындығы 2-ге тең кезіндегі формуласын қолданып, біз оң бүтін сан $n \geq 1$ Диофанттық Пелл теңдеуін $n^2 - dx^2 = 1$ қанағаттандырған кезде, әрқашан $D(n\sqrt{d}) = 2$ екенін дәлелдедік. Теореманы дәлелдеу арқылы әрбір $d \geq 1$ үшін $D(n\sqrt{d}) = 2$ болатындай шексіз көп n бар екенін көрсеттік.

Аннотация

Эта диссертационная работа посвящена закономерностям распределения периодов непрерывных дробей для квадратичных иррациональных чисел. В частности, целью диссертации является анализ распределения периодов непрерывных дробей для квадратичного иррационального числа $\alpha = n\sqrt{d}$, где n - положительное целое число, а d - целое число свободное от квадратов. Мы обозначим длину периода непрерывной дроби как $D(\alpha)$. Мы установили взаимосвязь между длиной периода непрерывной дроби $D(\alpha)$ и квадратичным иррациональным числом $a = n\sqrt{d}$, вводя явную формулу для иррационального числа α для конкретных случаев, когда $D(\alpha) = 2$ и $D(\alpha) = 3$.

Используя выведенную формулу для $\alpha = n\sqrt{d}$ с длиной периода, равной 2, мы доказали, что $D(n\sqrt{d}) = 2$ всякий раз, когда положительное целое число $n \geq 1$ удовлетворяет Диофантовое уравнение Пелля вида $n^2 - dx^2 = 1$. Доказав теорему, мы показали, что для каждого $d \geq 1$ существует бесконечно много n таких, что $D(n\sqrt{d}) = 2$.

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Chapter 1

Introduction

The continued fraction expansion provides a unique representation of real numbers, unraveling their inner structure through an infinite sequence of nested fractions.

The given form below represents the representation of *simple continued fractions* for any irrational number α .

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}} = [a_0; a_1, a_2, a_3, a_4, \dots], \quad (1)$$

where a_0 is an integer and $a_1, a_2, a_3 \dots$ are positive integers.

In the thesis, we consider continued fractions specifically for quadratic irrational numbers. For any quadratic irrational number α , it is well-known that the continued fraction expansion of α eventually becomes periodic. This classical result, known as Lagrange's theorem demonstrates that when expressing a quadratic irrational, such as the square root of a non-square integer, as a continued fraction, the sequence of partial quotients repeats periodically. This means that after a certain point; the same sequence of partial quotients will appear again and continue indefinitely [1].

Moreover, a French mathematician Galois showed that for a quadratic irrational $\alpha = \sqrt{d}$, where d is a non-square integer, the recurring portion of its continued fraction representation is a palindrome. This can be succinctly denoted in the following format:

$$\sqrt{d} = [a_0; \overline{a_1, a_2, \dots, a_1, 2a_0}], \quad (2)$$

where the periodic part is over-lined [2]. Here, the word *palindrome* means that the sequence of the periodic part reads identically both backward and forward. To represent the length of the recurring portion of the continued fraction for a quadratic irrational α , we introduce a new notation, denoted as $D(\alpha)$. We will discuss the background of continued fractions for quadratic irrationals in more

detail in Chapter 2.

Continued fractions for quadratic irrationals are used to solve Pell's Diophantine equation, which is named after mathematician John Pell. Pell's equation is a Diophantine equation of the form $x^2 - Py^2 = 1$, where P is a non-square positive integer.

In the context of Pell's equation, continued fractions provide a method to find solutions. If \sqrt{P} is a quadratic irrational solution to Pell's equation, its continued fraction expansion provides convergents that are increasingly accurate approximations of \sqrt{P} . These convergents have the property that they satisfy Pell's equation with relatively small values of x and y . By considering the convergents of the continued fraction, it is possible to generate an infinite sequence of solutions to Pell's equation.

Using these properties of continued fractions for quadratic irrationals, nowadays, this topic proceeds to be relevant and finds a way for expansion and development. In Chapter 3 we analyse and consider more recent works on this topic. One of them and the one that becomes the basis for this thesis is a recent research work by mathematicians Filip Gawron and Tomasz Kobos. The authors in their research article explored the distribution of the lengths of periods, denoted as $D(n\sqrt{d})$, for quadratic irrationals of the form $n\sqrt{d}$, where $n \geq 1$ and d is a square-free integer. They established that the sequence $(D(n\sqrt{d}))_{n=1}^{\infty}$ possesses infinitely many limit points. This intriguing discovery has raised the question of whether, for every non-square integer $d \geq 1$ and every positive integer $k \geq 1$, at least one of the numbers k or $k + 1$ appears as a limit point in the sequence $(D(n\sqrt{d}))_{n=1}^{\infty}$ [3].

Inspired by their research and the previous inquiry, a new question arises: Is it valid to claim that for any $d \geq 1$, there exists an infinite set of n values such that $D(n\sqrt{d}) = 2$? By addressing this inquiry, we arrive at the following outcome.

Theorem 1.0.1. *Let d be a positive non-square integer. Then, if the length of the period $D(n\sqrt{d}) = 2$, $n \geq 1$ satisfies the Pell's Diophantine equation $n^2 - dx^2 = 1$.*

By proving Theorem 1.0.1 we can answer positively to the question above that for every $d \geq 1$ there exist infinitely many n such that $D(n\sqrt{d}) = 2$.

But before the proof of the main Theorem 1.0.1 we need to establish a relationship between the length of the periodic part in the continued fraction expansion of $n\sqrt{d}$ and the equation representing $N = n^2d$. We consider some special cases when continued fractions of $n\sqrt{d}$ has a period 2. Our result is as follows.

Theorem 1.0.2. *Let the length of the period for the continued fraction of the quadratic irrational \sqrt{N} be $D(\sqrt{N}) = 2$, where $n\sqrt{d} = \sqrt{N}$. For any positive integer $a \geq 1$ and positive integer b that is a divisor of $2a$, the equation for N is given by $N = a^2 + \frac{2a}{b}$. However, we have the condition $2a \neq b$.*

The result states that if the length of the period is 2, the equation for N can

be expressed as $N = a^2 + \frac{2a}{b}$, where a is a positive integer greater than or equal to 1, and b is a positive integer that divides $2a$. Using the equation of N for $D(\sqrt{N}) = 2$ from this Theorem 1.0.2 we prove Theorem 1.0.1.

In Chapter 5, we present a series of numerical algorithms related to continued fractions. Specifically, we focus on the study of quadratic irrational numbers with periodic parts and derive a concise form of their continued fraction expansions. By considering all non-square integers within the range of 1 to 1000, we systematically analyze the square roots of these numbers and express them in continued fraction form. This investigation allows us to ascertain the length of the periodic part associated with each quadratic irrational.

Through our numerical analysis, we aim to gain insights into the patterns and properties exhibited by quadratic irrationals. By examining their continued fraction representations, we can elucidate the nature of their periodicity and explore the underlying mathematical structure governing these numbers.

We present the proofs of the theorems above and examples in Chapter 4. Conclusion and discussion about possible future directions for further research are presented in Chapter 6.

Chapter 2

Background

This chapter introduces fundamental concepts and principles associated with continued fractions. The initial section provides a general introduction to continued fractions. The subsequent section focuses specifically on continued fractions for quadratic irrational numbers. Lastly, we explore the practical applications of continued fractions and examine their connection to Pell's equation in the final section.

2.1 Continued fractions: An Overview

We already gave a definition of *Simple Continued fractions* in Introduction part. So let us recall what simple continued fraction is.

Definition 2.1.1. A continued fraction with all numerators equal to 1 is called *Simple Continued fraction*. And it takes the following form:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}}, \quad (2.1.1)$$

where a_0 is an integer and a_1, a_2, a_3, \dots are positive integers [1].

Remark. Throughout the thesis, we will use the term Continued fraction for Simple continued fractions.

Definition 2.1.2. $a_0, a_1, a_2, a_3, \dots$ are called *terms* or *partial quotients* of the continued fraction. For example, a_3 is the third term.

The continued fraction expansion of a real number can be infinite, where the fraction continues indefinitely, or finite, where the fraction terminates after a certain number of terms.

So the continued fraction that we presented in Definition 2.1.1 is an expression of Infinite Continued fraction. An infinite continued fraction has an infinite number of terms, extending indefinitely.

The infinite continued fraction provides an exact representation of certain numbers and is often used to approximate irrational numbers. A continued fraction expansion for any irrational number is unique. If two continued fractions have the same values, then the irrational numbers are equal [5].

Example:

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{1 + \frac{1}{292 + \frac{1}{1 + \frac{1}{\dots}}}}}}}}$$

Continued fractions can be represented shortly in this form for infinite continued fractions:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\dots}}}}$$

Definition 2.1.3. A finite continued fraction has a finite number of terms, terminating after a certain number of terms. It is represented as:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3}}} \tag{2.1.2}$$

Finite continued fractions provide rational approximations to rational numbers and have various applications in numerical computations.

Continued fractions can be represented shortly in this form for finite continued fractions:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots + \frac{b_n}{a_n}}}}} = [a_0; a_1, a_2, a_3, \dots, a_n].$$

Continued fractions can be expressed in different mathematical forms. Let's explore two common formulations:

General Form

In the general form, a continued fraction for a real number x can be written as:

$$x = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{\ddots}}}} = [b_0; b_1, b_2, b_3, \dots], \quad (2.1.3)$$

where the square brackets indicate the continued fraction representation of x . The terms $b_0, b_1, b_2, b_3, \dots$ are called the partial quotients of the continued fraction.

Recursive Form

Another way to represent a continued fraction is using recursive formulas. Let's consider a continued fraction with n terms:

$$x = b_0 + \frac{1}{x_1}, \quad x_1 = b_1 + \frac{1}{x_2}, \quad x_2 = b_2 + \frac{1}{x_3}, \quad \dots, \quad x_{n-1} = b_{n-1} + \frac{1}{b_n}. \quad (2.1.4)$$

Here, $b_0, b_1, b_2, \dots, b_n$ are the partial quotients, x_1, x_2, \dots, x_n are the auxiliary fractions.

Finite Continued Fractions

Here we present some basic theorems on Finite Continued fractions that will be useful for some properties of Infinite Continued fractions.

Theorem 2.1.1. *Any rational number can be expressed using a finite simple continued fraction.*

Theorem 2.1.2. *A finite simple continued fraction can represent every rational number.*

Remark. It is important to observe that finite continued fractions representing rational numbers do not possess a unique representation. A rational number can be expressed as a continued fraction in two distinct manners: one with an odd count of terms and the other with an even count of terms.

Definition 2.1.4. The continued fraction $[a_0; a_1, a_2, a_3, \dots, a_t]$ is referred to as the t^{th} convergent of a finite continued fraction $[a_0; a_1, a_2, a_3, a_4, \dots, a_n]$, where $t < n$ and it is a positive integer. It is denoted as A_t .

Theorem 2.1.3. Let $a_0, a_1, a_2, \dots, a_n$ be real numbers, where a_1, a_2, \dots, a_n are non-negative integers. Consider the sequences $r_0, r_1, r_2, \dots, r_n$ and $g_0, g_1, g_2, \dots, g_n$, defined recursively for $t = 2, 3, \dots, n$ as follows:

$$\begin{aligned} r_0 &= a_0, & g_0 &= 1 \\ r_1 &= a_0 a_1 + 1, & g_1 &= a_1 \\ & & & \vdots \\ r_k &= a_t r_{t-1} + r_{t-2}, & g_t &= a_t g_{t-1} + g_{t-2}. \end{aligned}$$

Then the t^{th} convergent $A_t = [a_0; a_1, a_2, a_3, \dots, a_t]$ can be expressed as

$$A_t = \frac{r_t}{g_t}.$$

Proof of the theorem above we can find in [6].

Corollary 1. Let $A_t = \frac{r_t}{g_t}$ be the t^{th} convergent referred to the continued fraction $[a_0, a_1, a_2, \dots, a_n]$. Then r_t and g_t are relatively prime.

Corollary 2. For finite continued fraction $[a_0, a_1, a_2, \dots, a_n]$ we denote $A_t = \frac{r_t}{g_t}$ as the t -th convergent. Then, the following equation holds for $1 \leq t \leq n$:

$$A_t - A_{t-2} = \frac{a_t(-1)^t}{g_t g_{t-2}}.$$

Furthermore, for $2 \leq t \leq n$, we have:

$$A_t - A_{t-2} = \frac{a_t(-1)^t}{g_t g_{t-2}}.$$

Theorem 2.1.4. Let $A_t = \frac{r_t}{g_t}$ be the t^{th} refer as a convergent of the continued fraction $[a_0, a_1, a_2, \dots, a_n]$. The following inequalities hold:

$$A_1 > A_3 > A_5 > \dots$$

$$A_0 < A_2 < A_4 < \dots$$

Moreover, every even-ordered convergent A_{2i} , where $i = 0, 1, 2, \dots$, is smaller than every odd-ordered convergent A_{2i+1} , where $i = 0, 1, 2, \dots$.

Infinite Continued Fractions

Since our thesis deals only with infinite continued fractions, let us introduce some properties of these continued fractions.

Theorem 2.1.5. Consider an infinite sequence of integers a_0, a_1, a_2, \dots , where a_1, a_2, a_3, \dots are non-negative integers. Let $A_t = [a_0; a_1, a_2, a_3, \dots, a_t]$. The sequence A_t converges to a limit value α .

Proof of Theorem 2.1.5 you can find in [6].

Theorem 2.1.6. Let a_0, a_1, a_2, \dots be integers, where a_1, a_2, a_3, \dots are positive integers. Then the value of the continued fraction $[a_0; a_1, a_2, a_3, a_4, \dots]$ is irrational.

The proof of this theorem is cited from [6].

Proof. By Theorem 2.1.5, we can write $\alpha = [a_0; a_1, a_2, a_3, a_4, \dots]$. Let

$$A_t = \frac{r_t}{g_t} = [a_0; a_1, a_2, a_3, \dots, a_t]$$

denote the t^{th} convergent of α . According to Theorem 2.1.4, $A_{2n} < \alpha < A_{2n+1}$. Therefore,

$$0 < \alpha - A_{2n} < A_{2n+1} - A_{2n}.$$

By Corollary 2, we can show that

$$A_{2n+1} - A_{2n} = \frac{1}{g_{2n+1}g_{2n}}.$$

Substituting this result, we have

$$0 < \alpha - A_{2n} = \alpha - \frac{r_{2n}}{g_{2n}} < \frac{1}{g_{2n+1}g_{2n}}.$$

Multiplying the inequality by g_{2n} yields

$$0 < \alpha g_{2n} - r_{2n} < \frac{1}{g_{2n+1}}.$$

Let's assume that α is not an irrational number. In that case, we can express

α as a fraction $\alpha = \frac{a}{b}$, where a and b are integers and $b \neq 0$. Consequently, we can establish the following inequality:

Multiplying all sides of the inequality by b , we obtain

$$0 < ag_{2n} - br_{2n} < \frac{b}{g_{2n+1}}.$$

We can observe that the expression $ag_{2n} - br_{2n}$ is always an integer for any positive integer n . Nonetheless, due to the fact that $g_{2n+1} > 2n + 1$, there is a positive integer n for which $g_{2n+1} > b$. Consequently, the fraction $\frac{b}{g_{2n+1}}$ is less than 1. However, it is not feasible for the expression $ag_{2n} - br_{2n}$ to be less than 1, which presents a contradiction. Thus, we can deduce that α is an irrational number. □

Remark. Every irrational number can be represented by infinite continued fraction and it is unique.

2.2 Continued fractions for quadratic irrationals

Quadratic Irrationals

Definition 2.2.1. A real number α is considered a **quadratic irrational** if it is irrational and can be expressed as the root of a quadratic polynomial with integer coefficients, represented as

$$Q\alpha^2 + W\alpha + R = 0, \tag{2.2.1}$$

where Q , W , and R are integers.

Example: Let's take $\alpha = 2 + \sqrt{3}$. To demonstrate the irrationality of α , we can prove that it is a root of the equation:

$$\alpha^2 - 4\alpha + 1 = 0.$$

Substituting α into this equation gives

$$(2 + \sqrt{3})^2 - 4(2 + \sqrt{3}) + 1 = 7 + 4\sqrt{3} - 8 - 4\sqrt{3} + 1 = 0.$$

Lemma 1. A real number α is a quadratic irrational if and only if there exist integers F , M , and G , where $M > 0$ and $G \neq 0$, such that M is not a perfect square and

$$\alpha = \frac{F + \sqrt{M}}{G}.$$

Proof. If α is a quadratic irrational, it is irrational, and we can find integers Q , W , and R such that $Q\alpha^2 + W\alpha + R = 0$. Applying the quadratic formula, we

obtain

$$\alpha = \frac{-W \pm \sqrt{W^2 - 4QR}}{2Q}. \quad (2.2.2)$$

Since α is a real number, we have $W^2 - 4QR > 0$. Moreover, since α is irrational, $W^2 - 4QR$ is not a perfect square and $Q \neq 0$. By choosing either $F = -W$, $M = W^2 - 4QR$, and $G = 2Q$, or $F = M$, $M = W^2 - 4QR$, and $G = -2Q$, we can express α in the desired form. \square

Definition 2.2.2. Consider a quadratic irrational number denoted as $\alpha = \frac{F + \sqrt{M}}{G}$.

And we denote the conjugate of α as α' , it is represented as $\alpha' = \frac{F - \sqrt{M}}{G}$.

Lemma 2. If the quadratic irrational number α satisfies the equation $Q\alpha^2 + W\alpha + R = 0$ as one of its roots, then the other root of this equation can be expressed as the conjugate α' .

Proof. Using the quadratic formula, we can represent the two roots of the equation $Q\alpha^2 + W\alpha + R = 0$ as

$$\frac{-W \pm \sqrt{W^2 - 4QR}}{2Q}.$$

If α corresponds to one of these roots, then the other root, denoted as α' , can be obtained by changing the sign of $\sqrt{W^2 - 4QR}$. \square

Continued fractions for quadratic irrationals

Quadratic irrationals give rise to a distinct category of continued fractions, which emerge when representing the square root of a non-square integer in continued fraction form. These continued fractions have attracted significant attention due to their captivating characteristics and their profound connections to the field of number theory.

Definition 2.2.3. A continued fraction $[a_0; a_1, a_2, a_3, \dots]$ is classified as **periodic** if there exists a positive integer m such that for sufficiently large n , $a_n = a_{n+m}$. The positive integer m is denoted as the **period** of the continued fraction.

Joseph Louis Lagrange, a prominent mathematician during the 18th century, made significant advancements in the field of continued fractions. In 1770, Lagrange made a profound discovery, known as Lagrange's theorem, which states that the continued fraction representation of any quadratic irrational number eventually becomes periodic. This theorem has had a profound influence on our understanding of quadratic irrationals and their representations using continued fractions [1].

Theorem 2.2.1. Lagrange's Theorem. *An irrational number is a quadratic irrational if and only if its continued fraction representation is periodic.*

In other words, when expressing a quadratic irrational, such as the square root of a non-square integer, as a continued fraction, the sequence of partial quotients

repeats periodically. After a certain point, the same sequence of partial quotients reappears and continues indefinitely.

Proof. Let us consider a quadratic irrational number α and its periodic continued fraction of the form:

$$\alpha = [a_0; a_1, a_2, a_3, \dots, a_{s-1}, \overline{a_s, a_{s+1}, \dots, a_{s+k}}].$$

Now, we define β as follows:

$$\beta = [\overline{a_s; a_{s+1}, \dots, a_{s+k}}].$$

By considering the continued fraction expansion of β , we can write:

$$\beta = [a_s; a_{s+1}, \dots, a_{s+k}, \beta].$$

According to Theorem 2.1.3, we can express β in terms of its convergents as:

$$\beta = \frac{\beta r_k + r_{k-1}}{\beta g_k + g_{k-1}},$$

where $\frac{r_k}{g_k}$ and $\frac{r_{k-1}}{g_{k-1}}$ are convergences of $[a_s; a_{s+1}, \dots, a_{s+k}]$. Since the continued fraction expansion of β is not finite, it follows that β is not rational. From the equation above, we can derive:

$$g_k \beta^2 + (g_{k-1} - r_k) \beta - r_{k-1} = 0.$$

Thus, we can conclude that β is a quadratic irrational. Now, consider the continued fraction expansion of α :

$$\alpha = [a_0; a_1, a_2, \dots, a_{s-1}, \beta].$$

Once again, utilizing Theorem 2.1.3, we can represent α in relation to its convergents as:

$$\alpha = \frac{\beta r_{s-1} + r_{s-2}}{\beta g_{s-1} + g_{s-2}},$$

where $\frac{r_{s-1}}{g_{s-1}}$ and $\frac{r_{s-2}}{g_{s-2}}$ denote the convergents of $[a_0; a_1, a_2, \dots, a_{s-1}]$. As β represents a quadratic irrational, it follows that α is also a quadratic irrational (since

we are already aware that α is irrational based on its infinite simple continued fraction expansion).

□

To illustrate this concept, let us consider the quadratic irrational $\sqrt{14}$:

$$\sqrt{14} = 3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{6 + \frac{1}{1 + \frac{1}{2 + \dots}}}}}}. \quad (2.2.3)$$

In this example, we observe that the partial quotients 1, 2, 1, 6 repeat in a cyclic manner. According to Lagrange's theorem, such periodicity is a fundamental characteristic of quadratic irrationals.

Lagrange's theorem marked a significant breakthrough in the study of continued fractions. It provided a profound understanding of the behavior of quadratic irrationals and their representations as continued fractions. This result laid the groundwork for further investigations and advancements in the field of continued fractions, empowering mathematicians to explore the properties and applications of these mathematical expressions. The theorem remains a cornerstone of continued fraction theory and continues to inspire ongoing research in this fascinating area of mathematics.

Évariste Galois, a renowned mathematician of the 19th century, made a notable impact on the investigation of continued fractions for quadratic irrationals, leading to enhanced insights into their periodic nature and symmetry. Galois established a significant correlation between the periodic portion of the continued fraction expansion of quadratic irrationals and the characteristics of quadratic fields [2]. This breakthrough not only deepened our understanding of the patterns exhibited by these continued fractions but also established meaningful connections to other mathematical fields such as quadratic fields and algebraic number theory.

Theorem 2.2.2. Galois's Theorem. *For quadratic irrationals of the form \sqrt{d} , where d is a square-free integer, the periodic part of their continued fraction expansion is a palindrome. We can express the continued fraction expansion of a quadratic irrational \sqrt{d} as:*

$$\sqrt{d} = [a_0, \overline{a_1; a_2; \dots; a_2; a_1}, 2a_0], \quad (2.2.4)$$

where a_0 is an integer and the sequence of partial quotients a_1, a_2, \dots, a_n is overlined to indicate its periodic and palindromic nature.

A **palindrome** is a sequence that reads the same forwards and backwards, reflecting a remarkable symmetry within the continued fraction representation.

This remarkable insight by Galois not only brought to light the inherent symmetries within continued fractions for quadratic irrationals but also opened doors to exploring their connections with various other areas of mathematics, such as quadratic fields and algebraic number theory.

For instance, let us consider the example we showed above:

$$\sqrt{14} = 3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{6 + \frac{1}{1 + \frac{1}{2 + \dots}}}}} = [3; \overline{1, 2, 1, 6}]. \quad (2.2.5)$$

Now let us illustrate some another example:

$$\begin{aligned} \sqrt{31} = 5 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3 + \frac{1}{5 + \frac{1}{3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{10 + \dots}}}}}}} \\ = [5; \overline{1, 1, 3, 5, 3, 1, 1, 10}]. \end{aligned} \quad (2.2.6)$$

Here, we can observe that the periodic part of this quadratic irrational example satisfies the equation form established by Galois in 2.2.1.

Galois' remarkable findings have significantly deepened our understanding of continued fractions for quadratic irrationals, illuminating the underlying symmetries and paving the way for further investigations in related areas of mathematics. By exploring the properties of these continued fractions, mathematicians continue to uncover fascinating connections and advance our knowledge of these intriguing mathematical structures.

2.3 Pell's equation

Pell's equation, a prominent problem in number theory, has captivated mathematicians for centuries. Named after John Pell, an eminent English mathematician of the 17th century, it represents a quadratic Diophantine equation of the form $x^2 - Py^2 = 1$, where P is a positive non-square integer, and x and y are positive integers [7].

One remarkable property of Pell's equation is its inherent abundance of solutions in positive integers (x, y) for a given nonsquare integer N . These solutions, known as fundamental solutions, serve as a foundation for generating an infinite sequence of solutions. In fact, if (x_1, y_1) represents a fundamental solution, then all solutions can be expressed as $(x_n, y_n) = (x_1, y_1)^n$, where n is a positive integer.

Example:

Let us consider the equation

$$x^2 - 8y^2 = 1.$$

It's obvious that the fundamental solution for this equation is $(x_1, y_1) = (3, 1)$. By knowing the fundamental solution, we can easily find other solutions using the formula:

$$(x_1 + y_1\sqrt{P})^n = x_n + y_n\sqrt{P},$$

where $n \geq 2$.

Next, let us find a solution for the case when $n = 2$:

$$(x_1 + y_1\sqrt{8})^2 = (3 + \sqrt{8})^2 = 17 + 6\sqrt{8}.$$

So, the second solution for the equation $x^2 - 8y^2 = 1$ is $(x_2, y_2) = (17, 6)$.

By the same method we calculate the third solution:

$$(x_1 + y_1\sqrt{8})^3 = (3 + \sqrt{8})^3 = 99 + 35\sqrt{8}.$$

The third solution is $(x_3, y_3) = (99, 35)$.

$$\begin{aligned}
(x_1, y_1) &= (3, 1), \\
(x_2, y_2) &= (17, 6), \\
(x_3, y_3) &= (99, 35), \\
(x_4, y_4) &= (577, 204), \\
(x_5, y_5) &= (3363, 1189), \\
&\vdots
\end{aligned}$$

For the equation $x^2 - 8y^2 = 1$ using the method $(x_1 + y_1\sqrt{P})^n = x_n + y_n\sqrt{P}$ we can find infinitely many solutions.

A productive approach to solving Pell's equation lies in the utilization of continued fractions. These fractions provide an efficient method for approximating irrational numbers and exhibit a close connection to Pell's equation. Specifically, the convergents of the continued fraction expansion of \sqrt{P} yield solutions to Pell's equation. By obtaining the convergents of \sqrt{P} and evaluating them within the equation, the solutions (x, y) can be determined. This intricate relationship between continued fractions and Pell's equation was elucidated by the renowned mathematician Joseph Louis Lagrange.

The key insight connecting Pell's equation and periodic continued fractions resides in the observation that the periodic part of the continued fraction expansion of the square root of P generates solutions to Pell's equation. To illustrate this connection, let us consider an example. Suppose we have the Pell's equation $x^2 - 2y^2 = 1$, where $P = 2$. The square root of 2 admits a continued fraction expansion:

$$\sqrt{2} = [1; 2, 2, 2, 2, \dots]$$

In this instance, the partial quotient 2 repeats indefinitely. Truncating the continued fraction expansion at the periodic part yields the periodic continued fraction $[1; \overline{2}]$. The periodic part is enclosed in parentheses to indicate the repeating pattern. Remarkably, the convergent of the periodic continued fraction expansion of $\sqrt{2}$, denoted as $[1; \overline{2}]$, satisfies Pell's equation $x^2 - 2y^2 = 1$, where the values of the numerator and denominator fulfill the equation.

To delve deeper into this phenomenon, let us examine the periodic continued fraction expansion $[1; \overline{2}]$ more closely:

$$[1; \overline{2}] = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}$$

If we denote the n th convergent of the periodic continued fraction as $[a_0; \overline{a_1, a_2, \dots, a_k}]$, the numerator and denominator of this convergent can be computed using a recurrence relation akin to that employed for regular continued fractions:

$$x_n = a_n \cdot x_{n-1} + x_{n-2}$$

$$y_n = a_n \cdot y_{n-1} + y_{n-2}$$

with initial values $x_0 = a_0$, $x_1 = a_0 \cdot a_1 + 1$, $y_0 = 1$, and $y_1 = a_1$. By substituting the values of a_n and utilizing the initial conditions, the convergents of $[1; \overline{2}]$ can be computed, revealing that the corresponding numerator and denominator values satisfy Pell's equation.

Conversely, the connection between Pell's equation and periodic continued fractions can be employed in reverse. Given a solution (x, y) of Pell's equation $x^2 - Py^2 = 1$, the associated periodic continued fraction expansion of \sqrt{P} can be determined systematically. This approach facilitates the generation of periodic continued fractions from solutions of Pell's equation.

We have some property of Pell's equations that is related to the main theorem we want to formulate in Chapter 4.

Theorem 2.3.1. *Assuming d is a non-square integer, where d is a non-negative integer, let us consider the continued fraction expansion of \sqrt{d} and denote the m -th convergent as $\frac{r_m}{g_m}$, where $m = 1, 2, 3, \dots$. The length of the period of this continued fraction is represented by D . The following results can be observed:*

If D is an even number, the positive solutions to the Pell's Diophantine equation $x^2 - dy^2 = 1$ can be expressed as $x = r_{jD-1}$ and $y = g_{jD-1}$ for $j = 1, 2, 3, \dots$. However, there are no solutions to the Pell's Diophantine equation $x^2 - dy^2 = -1$.

If D is an odd number, the solutions to the equation $x^2 - dy^2 = 1$ can be expressed as $x = r_{2jD-1}$ and $y = g_{2jD-1}$ for $j = 1, 2, 3, \dots$. Additionally, the positive solutions to the equation $x^2 - dy^2 = -1$ can be represented as $x = r_{2j-1}$ and $y = g_{2j-1}$ for $j = 1, 2, 3, \dots$.

For instance, let us consider the quadratic irrational $\sqrt{13}$. The continued fraction expansion for this number is given by

$$\sqrt{13} = [3; \overline{1, 1, 1, 1, 6}].$$

Therefore, solution of this equation $x^2 - 13y^2 = 1$ can be expressed as $x = r_{10j-1}$, $y = g_{10j-1}$ for $j = 1, 2, 3, \dots$, where $\frac{r_{10j-1}}{g_{10j-1}}$ represents the $(10j - 1)^{th}$ convergent of the simple continued fraction expansion of $\sqrt{13}$. The smallest positive solution is obtained when $j = 1$, giving $r_9 = 649$ and $g_9 = 180$.

Similarly, the positive solutions of the Pell's Diophantine equation $x^2 - 13y^2 = -1$ will be $x = r_{10j-6}$ and $y = g_{10j-6}$ for $j = 1, 2, 3, \dots$, and the smallest positive

solution occurs when $j = 1$, resulting in $r_4 = 18$ and $g_4 = 5$.

Furthermore, periodic continued fractions are intimately linked with quadratic irrationals. When \sqrt{P} is a quadratic irrational, the corresponding continued fraction expansion of this quadratic irrational number is either periodic or non-repeating. The periodic continued fraction expansions correspond to solutions of Pell's equation, whereas the non-repeating continued fraction expansions correspond to irrational numbers that do not satisfy Pell's Diophantine equation.

The exploration of periodic continued fractions has yielded significant advancements in the theory of Pell's equation. For example, Lagrange's theorem states that the period of the continued fraction representation of \sqrt{P} is connected to the fundamental solution of Pell's equation. Lagrange demonstrated that the period equals $2k$, where k represents the smallest positive integer such that (x_k, y_k) is a solution to Pell's equation. This outcome offers valuable understanding of the characteristics and traits of periodic continued fractions related to Pell's equation.

Chapter 3

Literature Review

3.1 History

Studying the history of mathematics is necessary in order to understand and contribute to the discipline. Unlike other subject areas, maths persistently builds upon previous discoveries. Once something has been proved conclusively, it is considered closed and future work can be built on it. Therefore, in order to study a specific field of maths, such as, algebra, topology or continued fractions, it is important to first delve into its past. By doing this, one can build upon past accomplishments rather than repeating them. The origin of continued fractions is difficult to determine as they have been present in mathematics for the past 2,000 years, but their true foundations were established in the late 1600s and early 1700s.

The historical beginning of continued fractions is typically associated with the time when Euclid's Algorithm was created. Euclid's Algorithm was originally designed to determine the greatest common denominator (gcd) of two numbers. However, by manipulating the algorithm algebraically, one can obtain the simple continued fraction representation of the rational number $\frac{p}{q}$ instead of the gcd of p and q . It is uncertain whether Euclid or his contemporaries actually utilized this algorithm in such a way, but the fact that it is closely connected to continued fractions marks the initial advancement of continued fractions [8].

For example, let us determine the greatest common divisor of 27 and 129 using Euler's method:

$$129 = 4 \cdot 27 + 21,$$

$$27 = 1 \cdot 21 + 6,$$

$$21 = 3 \cdot 6 + 3,$$

$$6 = 2 \cdot 3 + 0.$$

So, the greatest common divisor of 27 and 129 is 3. Now let us divide both

sides of above equations in the form

$$a = p \cdot q + r$$

by q and we derive these following expressions

$$\begin{aligned} \frac{129}{27} &= 4 + \frac{21}{27}, \\ \frac{27}{21} &= 1 + \frac{6}{21}, \\ \frac{21}{6} &= 3 + \frac{3}{6}, \\ \frac{6}{3} &= 2 + \frac{0}{3}. \end{aligned}$$

Here, every fraction on the right hand side is reciprocal of the fraction in the next equation. And then by substitution we obtain

$$\frac{129}{27} = 4 + \frac{21}{27} = 4 + \frac{1}{1 + \frac{6}{21}} = 4 + \frac{1}{1 + \frac{1}{3 + \frac{3}{6}}} = 4 + \frac{1}{1 + \frac{1}{3 + \frac{1}{2}}}.$$

As we can see we derived a continued fraction using Euler's method for the greatest common divisor.

The ancient features of continued fractions can be seen also in the approximation of quadratic irrationals and in the works of Fibonacci.

The formal concept of continued fractions began to take shape during the Renaissance period, particularly from the works of mathematicians such as Rafael Bombelli and John Wallis. Bombelli's algebraic treatise, "L'Algebra" (1572), discussed the representation of square roots as infinite continued fractions, providing the foundation for later developments [9]. He released the second edition of "L'Agebra Opera" in 1579, where he provided an approach for finding the square root of 13 in this book, which can be written using the following current notation:

$$\sqrt{13} = 3 + \frac{4}{6 + \frac{4}{6 + \dots}}$$

Wallis, in his book "Arithmetica Infinitorum" (1655), introduced the continued fraction notation and demonstrated its use in approximating square roots and other irrational numbers. Moreover, in this book he introduced *continued fraction* as a new term [9, 10].

The history of continued fractions dates back to ancient times, but it was in the 18th and 19th centuries that significant advancements were made in the theory of continued fractions. Mathematicians such as Leonhard Euler, Joseph Lagrange, and Carl Friedrich Gauss made substantial contributions to the field, laying the foundation for further exploration.

The pioneering work of Christiaan Huygens, a mathematician and astronomer from the Netherlands, demonstrated the practical applications of continued fractions. In his written work, Huygens demonstrated the practical application of convergents in continued fractions for accurately approximating gear ratios. This enabled him to select gears with precise numbers of teeth, an essential aspect in his construction of a mechanical planetarium [10, 11].

Leonhard Euler explored the properties of convergents in continued fractions and derived several important theorems. In 1737, he published his first mathematical article on the topic, titled *De fractionibus continuis* [10]. In this work, Euler proved the significant theorem that every real number can be uniquely represented by a simple continued fraction. He also showed that continued fractions are finite for rational numbers and infinite for irrational numbers. Additionally, Euler obtained a solution for the quadratic equation $x^2 = ax + b$ in the form of continued fractions:

$$x = a + \frac{b}{a + \frac{b}{a + \frac{b}{a + \dots}}}$$

Furthermore, Euler presented the number e as a continued fraction expansion and proved that both e and e^2 are irrational numbers [10]. The continued fraction representation for e is as follows:

$$e - 1 = 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \dots}}}}}}$$

Johan Heinrich Lambert investigated trigonometric functions [10] and represented them in the form of continued fractions:

$$\tan x = \frac{1}{\frac{1}{x} - \frac{1}{\frac{3}{x} - \frac{1}{\frac{5}{x} - \frac{1}{\frac{7}{x} - \dots}}}}$$

Joseph Lagrange, in his work "Théorie des fonctions analytiques" (1798), established the periodicity of continued fractions for quadratic irrationals, a result now known as Lagrange's theorem [1]. We will discuss this theorem in more detail and its contribution to the development of periodic continued fractions in Section 2.2.

Carl Friedrich Gauss, renowned for his profound insights into number theory, made significant contributions to the study of continued fractions, particularly in relation to the theory of quadratic forms.

In the 19th century, continued fractions found applications in number theory and Diophantine approximation. Augustin-Louis Cauchy and J.J. Sylvester developed the theory of simultaneous approximations and utilized continued fractions to study rational approximations to real numbers. These developments paved the way for further investigations into the distribution of partial quotients and the approximation properties of continued fractions.

Moving into the 20th century, continued fractions continued to find applications in various fields, including number theory, analysis, and cryptography. The work of Srinivasa Ramanujan, a self-taught mathematical genius, introduced novel techniques for generating remarkable continued fraction expansions. Continued fractions also became instrumental in the design of cryptographic algorithms, such as the RSA algorithm, due to their intricate and non-repeating nature.

In summary, the rich history and extensive literature on continued fractions reflect the enduring fascination and fruitful developments in this field. In modern times, continued fractions continue to play a crucial role in various areas of mathematics and serve as a powerful tool for approximations, Diophantine equations, and number theory. In the next section, we will discuss the role of continued fractions for quadratic irrationals and explore the contributions of mathematicians in the 20th and 21st centuries.

3.2 Recent works

The study of continued fractions has long been a prominent topic in number theory, captivating the attention of numerous mathematicians. In 1972, mathematicians P. Chowla and S. Chowla delved into the investigation of periodic continued fractions, exploring various problems and shedding light on the sums of partial quotients in these continued fractions. Notably, at the conclusion of their research paper, they

posed an intriguing question: "Is it possible to find an infinite set of positive integers d such that the length of the periodic part $D(\sqrt{d}) = l$, where l is a given integer greater than or equal to 1?" They conjectured that the answer to this question is affirmative [12].

In 1988, Christian Friesen undertook the study of this question and provided a positive proof. In his work, he established a more general result, addressing palindromic sequences $(a_1, a_2, a_3, \dots, a_3, a_2, a_1)$ that satisfy certain conditions, primarily pertaining to parity. Friesen demonstrated that infinitely many integers $d \geq 1$ exist such that $\sqrt{d} = [a_0, a_1, a_2, a_3, \dots, a_3, a_2, a_1, 2a_0]$. Remarkably, the original question posed by Chowla and Chowla emerged as a special case of Friesen's result, which he established and proved as a corollary (see Corollary 1 in [13]).

Subsequently, in 1989, Franz Halter-Koch refined the statement of Friesen's theorem, introducing additional conditions concerning the prime factorizations of d . These conditions further elucidated the nature of the integers satisfying $\sqrt{d} = [a_0, a_1, a_2, a_3, \dots, a_3, a_2, a_1, 2a_0]$ and their relation to the prime valuations of d . Halter-Koch's work provided a more precise understanding of the underlying intricacies involved in Friesen's result, enhancing the mathematical community's comprehension of the topic [14].

The investigations carried out by Chowla, S. Chowla, Friesen, and Halter-Koch significantly contributed to the understanding of periodic continued fractions and their connections to integer values of square roots. Their collective efforts shed light on the rich properties and intricate patterns exhibited by these mathematical objects. Theorems and conjectures in this area continue to stimulate further research, allowing mathematicians to delve deeper into the fascinating realm of continued fractions and their diverse applications in number theory.

In the 21st century, Balková and Hrusková compiled a comprehensive overview of existing findings on continued fractions, with a particular focus on quadratic numbers. They provided descriptions of the periodicity and specific forms of continued fractions for \sqrt{N} , where N is a natural number. The authors established various equations linking N to different values of $D(\sqrt{N})$. For instance, the continued fraction of \sqrt{N} exhibits a period of length 1 if and only if $N = n^2 + 1$, resulting in $\sqrt{N} = [n, 2n]$. They presented numerous observations and proofs for various cases and numbers. Balková and Hrusková also investigated the equation $D(\sqrt{d}) = k$ for small values of $k \geq 1$ in their work [15].

Among the many studies involving transformations and conditions applied to a number d , Rada and Starosta conducted one such investigation. They explored the behavior of the length of the continued fraction expansion resulting from a specific transformation of \sqrt{d} . The transformation in question is a Möbius transformation, denoted by:

$$h_N(x) = \frac{ax + b}{cx + d}$$

The Möbius transformation linked to the matrix N , also known as a linear frac-

tional transformation, is responsible for this mapping. When given a quadratic irrational number x , it becomes apparent that $h_N(x)$ is likewise a quadratic irrational number within the same field. The primary result of their research was obtaining lower and upper bounds for the period of the continued fraction expansion of $h_N(x)$ in relation to the period of the continued fraction expansion of x . Essentially, they established boundaries for $D(h(\sqrt{x}))$ based on $D(\sqrt{x})$ [16].

As discussed in Chapter 1, this dissertation thesis builds upon the work of mathematicians Filip Gawron and Tomasz Kobos [3]. In their article, they focused on the simplest version of the Möbius transformation, specifically $x \rightarrow nx$, where $n \geq 1$. For the transformed irrational number $n\sqrt{d}$, they constructed a set A_d representing the limit points of the sequence $(D(n\sqrt{d}))_{n=1}^{\infty}$. Their main finding is summarized by the result below:

Theorem 3.2.1. *Let d be a non-negative and a non-square integer. Then, the set A_d is infinite. In other words, the sequence $(D(n\sqrt{d}))_{n=1}^{\infty}$ has infinitely many distinct limit points.*

Gawron and Kobos successfully demonstrated the existence of infinitely many limit points for the sequence $(D(n\sqrt{d}))_{n=1}^{\infty}$ when d is a positive integer. Furthermore, they posed several new open questions for future research in this area.

Chapter 4

Main results

4.1 Continued fractions of $n\sqrt{d}$ with periodic part with length 2

In this thesis, our focus is on quadratic irrational numbers of the form $n\sqrt{d}$. This particular transformation has been previously mentioned in references [3] and [16], where the authors employed Möbius transformations. Specifically, they utilized the Möbius transformation defined as:

$$h_N(x) = \frac{ax + b}{cx + d}.$$

A detailed discussion on this topic can be found in Chapter 3 of this thesis. In this chapter, we also employ the simplest Möbius transformation in the form $x \rightarrow nx$.

Using this modification, we aim to analyze the periods of continued fractions for quadratic irrationals of the form $n\sqrt{d}$ with lengths 2 and 3. Our objective is to derive an exact formula for the irrational number $n\sqrt{d}$, where d is not a perfect square and n is a positive integer. In the course of studying and analyzing this question, we have formulated the following theorems.

Theorem 4.1.1. *Let the length of the period for the continued fraction of the quadratic irrational \sqrt{N} be $D(\sqrt{N}) = 2$, where $n\sqrt{d} = \sqrt{N}$. For any positive integer $a \geq 1$ and positive integer b that is a divisor of $2a$, the equation for N is given by $N = a^2 + \frac{2a}{b}$. However, we have the condition $2a \neq b$.*

In this theorem, we establish a relationship between the length of the period in the continued fraction expansion of \sqrt{N} and the equation representing N . The result states that if the length of the period is 2, the equation for N can be expressed as $N = a^2 + \frac{2a}{b}$, where a is a positive integer greater than or equal to 1, and b is a positive integer that divides $2a$.

Proof. Let $D(\sqrt{N}) = 2$. Here $n\sqrt{d} = \sqrt{N}$.

By the definition of continued fractions and Galois' property that the periodic part of continued fractions for quadratic irrationals is a palindrome, we can express \sqrt{N} in the following form:

$$\sqrt{n^2d} = \sqrt{N} = a + \frac{1}{b + \frac{1}{2a + \frac{1}{b + \frac{1}{2a + \dots}}}} = [a; \overline{b, 2a}]. \quad (4.1.1)$$

Now, let us add the integer a to both sides of Equation 4.1.1 and present a new equation x .

$$x = a + \sqrt{N} = 2a + \frac{1}{b + \frac{1}{2a + \frac{1}{b + \frac{1}{2a + \dots}}}} = 2a + \frac{1}{b + \frac{1}{x}} = 2a + \frac{x}{bx + 1}. \quad (4.1.2)$$

From Equation 4.1.2, we can derive the quadratic equation in the following form:

$$x = a + \sqrt{N} = 2a + \frac{x}{bx + 1},$$

$$x^2 - 2ax - \frac{2a}{b} = 0. \quad (4.1.3)$$

By solving this quadratic equation, we find 2 basic roots and derive the formula for the integer N :

$$x_1 = a + \sqrt{a^2 + \frac{2a}{b}},$$

$$x_2 = a - \sqrt{a^2 + \frac{2a}{b}}.$$

Since we defined $x = a + \sqrt{N}$ in the beginning, from the root x_1 , we can obtain the formula for the integer N :

$$x = a + \sqrt{a^2 + \frac{2a}{b}} = a + \sqrt{N} \implies N = n^2d = a^2 + \frac{2a}{b}. \quad (4.1.4)$$

In order for N to be an integer, b must be a divisor of $2a$. □

Example 4.1.1: Let us take $a = 2$ and $b = 2$, then $N = 6$. Also, from the Table 4.1 below, we observe that $D(\sqrt{6}) = 2$.

Example 4.1.2: Let us take $a = 4$ and $b = 2$, then $N = 20$. According to the Table 4.1 below, $D(\sqrt{20}) = 2$.

This theorem establishes a precise relationship between the length of the period in the continued fraction expansion of quadratic irrationals and the equation representing the quadratic irrational. It provides a concrete formula for N based on the values of a and b satisfying the conditions outlined in Theorem 4.1.1. The examples illustrate the application of the theorem and demonstrate the correspondence between the chosen values of a , b , and the resulting length of the period.

\sqrt{N}	Continued fraction
3	$[1, \overline{1; 2}]$
6	$[2, \overline{2; 4}]$
8	$[2, \overline{1; 4}]$
11	$[3, \overline{3; 6}]$
12	$[3, \overline{2; 6}]$
15	$[3, \overline{1; 6}]$
18	$[4, \overline{4; 8}]$
20	$[4, \overline{2; 8}]$
24	$[4, \overline{1; 8}]$

Table 4.1: $D(\sqrt{N} = 2)$

As established in Theorem 4.1.1, we derived formulas for N when $D(\sqrt{N}) = 2$. Now, we extend our analysis to determine formulas for N when $D(\sqrt{N}) = 3$, $D(\sqrt{N}) = 4$, and we can continue to determine formulas for quadratic irrationals with any length of periods.

Theorem 4.1.2. *Let the length of the period for the continued fraction of the quadratic irrational \sqrt{N} be $D(\sqrt{N}) = 3$, where $n\sqrt{d} = \sqrt{N}$. For any positive integer $a \geq 1$ and positive integer b that is a divisor of $2ab + 1$, the equation for N is given by $N = a^2 + \frac{2ab + 1}{b^2 + 1}$. However, we have the condition $2a \neq b$.*

Proof. Let $D(\sqrt{N}) = 3$. Here, $n\sqrt{d} = \sqrt{N}$.

By the definition of continued fractions and Galois' property that the periodic part of continued fractions for quadratic irrationals is a palindrome, we can express

\sqrt{N} in the following form:

$$\sqrt{n^2d} = \sqrt{N} = a + \frac{1}{b + \frac{1}{c + \frac{1}{2a + \frac{1}{b + \frac{1}{c + \frac{1}{2a + \dots}}}}}} = [a; \overline{b, c, 2a}]. \quad (4.1.5)$$

Now, let us add the integer a to both sides of Equation 4.1.5 and introduce a new variable x .

$$\begin{aligned} x = a + \sqrt{N} &= 2a + \frac{1}{b + \frac{1}{c + \frac{1}{2a + \frac{1}{b + \frac{1}{c + \frac{1}{2a + \dots}}}}}} \\ &= 2a + \frac{1}{b + \frac{1}{c + \frac{1}{x}}} \\ &= 2a + \frac{cx + 1}{bcx + b + x}. \end{aligned} \quad (4.1.6)$$

From Equation 4.1.6, we can derive the quadratic equation:

$$\begin{aligned} x = a + \sqrt{N} &= 2a + \frac{cx + 1}{bcx + b + x} \\ (bc + 1)x^2 + (b - 2a - c)x - 1 &= 0. \end{aligned} \quad (4.1.7)$$

By solving this quadratic equation, we find two basic roots and derive the formula for the integer N :

$$x_1 = a + \sqrt{a^2 + \frac{2ab + 1}{b^2 + 1}}$$

$$x_2 = a - \sqrt{a^2 + \frac{2ab + 1}{b^2 + 1}}$$

Since we defined $x = a + \sqrt{N}$ at the beginning, from the root x_1 , we can obtain the formula for the integer N :

$$x = a + \sqrt{a^2 + \frac{2ab + 1}{b^2 + 1}} = a + \sqrt{N} \implies N = n^2d = a^2 + \frac{2ab + 1}{b^2 + 1}. \quad (4.1.8)$$

For N to be an integer, we must have $2ab + 1 \neq b^2 + 1$, which implies that $2a \neq b$. \square

Example 4.1.3: Let us take $a = 6$ and $b = 2$, then $N = 41$. Referring to the Table 4.2 below, we can confirm that $D(\sqrt{41}) = 3$.

Example 4.1.4: Let us take $a = 11$ and $b = 2$, then $N = 130$. Referring to the Table 4.2 below, we can confirm that $D(\sqrt{130}) = 3$.

\sqrt{N}	Continued fraction
41	$[6, 2; 2; 12]$
130	$[11, 2; 2; 22]$
269	$[16, 2; 2; 32]$
370	$[19, 4; 4; 38]$
458	$[21, 2; 2; 42]$
697	$[26, 2; 2; 52]$
986	$[31, 2; 2; 62]$
1313	$[36, 4; 4; 72]$
1325	$[36, 2; 2; 72]$

Table 4.2: $D(\sqrt{N} = 3)$

In that manner, we can develop such formulations of explicit formulas for any length of period. In Theorem 4.1.3 below we demonstrate it by giving a formula when $D(n\sqrt{d}) = D(\sqrt{N}) = 4$.

Theorem 4.1.3. *Let the length of the period for continued fraction of quadratic irrational \sqrt{N} be $D(\sqrt{N}) = 4$, where $n\sqrt{d} = \sqrt{N}$. Then for any positive integer a greater than or equal to 1, and positive integer b a divisor of $2abc + 2a + c$, the equation for N will be $N = a^2 + \frac{2abc + 2a + c}{b^2c + 2b}$. But $2abc + 2a + c \neq b^2c + 2b$.*

Example 4.1.5: Let us take $a = 2$, $b = 1$ and $c = 1$, then $N = 7$. Referring to the Table 4.3 below, we can confirm that $D(\sqrt{7}) = 4$.

Example 4.1.6: Let us take $a = 5$, $b = 1$ and $c = 1$, then $N = 32$. Referring to the Table 4.3 below, we can confirm that $D(\sqrt{32}) = 4$.

\sqrt{N}	Continued fraction
7	$[2, 1; 1; 1; 4]$
14	$[3, 1; 2; 1; 6]$
23	$[4, 1; 3; 1; 8]$
28	$[5, 3; 2; 3; 10]$
32	$[5, 1; 1; 1; 10]$
33	$[5, 1; 2; 1; 10]$
34	$[5, 1; 4; 1; 10]$
47	$[6, 1; 5; 1; 12]$
55	$[7, 2; 2; 2; 14]$

Table 4.3: $D(\sqrt{N} = 4)$

4.2 Proof of the Theorem

In the previous section we presented the Theorem 4.1.1, where we established formula for $n\sqrt{d} = \sqrt{N} = a^2 + \frac{2a}{b}$ when $D(n\sqrt{d}) = 2$. Now we pose a new question whether it is true that for every $d \geq 1$ there exist infinitely many n such that $D(n\sqrt{d}) = 2$. Answering this question, we obtain the following result. We have already presented this theorem in Chapter 1. But let us recall it again.

Theorem 4.2.1. *Let d be a positive square-free integer. Then, if the length of the period $D(n\sqrt{d}) = 2$, a positive integer n satisfies the Pell's Diophantine equation $n^2 - dx^2 = 1$.*

Our primary objective is to address the question of whether it holds true that for every d , there exist infinitely many n such that $D(n\sqrt{d}) = 2$. So this theorem helps us to answer to the question.

Proof. By referring to Theorem 4.1.1, we are already aware that when $D(n\sqrt{d}) = 2$,

$$n^2d = N = a^2 + \frac{2a}{b}. \quad (4.2.1)$$

Let us consider a specific set of quadratic irrational numbers where $\frac{2a}{b} = d$, implying $a = dx$ and $b = 2x$.

Substituting these values into Equation 4.2.1, we obtain

$$n^2d = a^2 + d, \quad (4.2.2)$$

$$n^2d = d^2x^2 + d. \quad (4.2.3)$$

Dividing both sides of Equation 4.2.3 by d , we have:

$$n^2 - dx^2 = 1. \quad (4.2.4)$$

Notably, Equation 4.2.4 is a form of Pell's equation $x^2 - Py^2 = 1$, where P is not a perfect square. It is well-known that Pell's equation of this form has infinitely many solutions [7]. Therefore, for any d , there exist infinitely many values of n that satisfy the equation $n^2 - dx^2 = 1$.

□

Consequently, by virtue of Theorem 4.2.1, we can affirmatively respond to the aforementioned question: for every d , there exist infinitely many n such that $D(n\sqrt{d}) = 2$.

Example 1: Let us consider $d = 2$. In this case, $D(n\sqrt{2}) = 2$, and the equation becomes

$$n^2 - 2x^2 = 1. \quad (4.2.5)$$

To solve this equation, we can employ the method described in Chapter 2 for solving Pell's Diophantine equations. By knowing the fundamental solution (n_1, x_1) , we can easily find other solutions using the formula

$$(n_1 + x_1\sqrt{d})^m = n_m + x_m\sqrt{d},$$

where $m \geq 2$.

The fundamental solution for Equation 4.2.5 is $(3, 2)$. Therefore, for $n_1 = 3$, we have $n\sqrt{d} = 3\sqrt{2} = \sqrt{18}$. For this quadratic irrational $\sqrt{18}$, we indeed have $D(\sqrt{18}) = 2$.

$$\sqrt{18} = 4 + \frac{1}{4 + \frac{1}{8 + \frac{1}{4 + \dots}}}$$

Next, let us find the solution for $m = 2$:

$$n_2 + x_2\sqrt{2} = (3 + 2\sqrt{2})^2 = 17 + 12\sqrt{2}.$$

Hence, $n_2 = 17$ and $x_2 = 12$. By substituting n_2 , we obtain a new quadratic irrational number $n\sqrt{d} = 17\sqrt{2} = \sqrt{578}$. We can confirm that the length of the period for this number is 2.

$$\sqrt{578} = 24 + \frac{1}{24 + \frac{1}{48 + \frac{1}{24 + \dots}}} = [24; \overline{24, 48}].$$

Now, let us find the solution for $m = 3$:

$$n_3 + x_3\sqrt{2} = (3 + 2\sqrt{2})^3 = 99 + 70\sqrt{2}.$$

In this case, $n_3 = 99$ and $x_3 = 70$. By substituting n_3 , we obtain a new quadratic irrational number $n\sqrt{d} = 99\sqrt{2} = \sqrt{19602}$. For this quadratic irrational number, the length of its period is also 2.

$$\sqrt{19602} = 140 + \frac{1}{140 + \frac{1}{280 + \frac{1}{140 + \dots}}} = [140; \overline{140, 280}].$$

Example 2: Let us consider $d = 3$. In this case, $D(n\sqrt{3}) = 2$, and the equation becomes

$$n^2 - 3x^2 = 1. \quad (4.2.6)$$

We use the same method as in the previous example for solving Pell's Diophantine equations. By knowing the fundamental solution (n_1, x_1) , we can easily find other solutions using the formula

$$(n_1 + x_1\sqrt{d})^m = n_m + x_m\sqrt{d},$$

where $m \geq 2$.

The fundamental solution for Equation 4.2.6 is $(7, 4)$. Therefore, for $n_1 = 7$, we have $n\sqrt{d} = 7\sqrt{3} = \sqrt{147}$. For this quadratic irrational $\sqrt{147}$, we indeed have $D(\sqrt{147}) = 2$.

$$\sqrt{147} = 12 + \frac{1}{8 + \frac{1}{24 + \frac{1}{8 + \dots}}} = [12; \overline{8, 24}].$$

Next, let us find the solution for $m = 2$:

$$n_2 + x_2\sqrt{2} = (7 + 4\sqrt{3})^2 = 97 + 56\sqrt{3}.$$

Hence, $n_2 = 97$ and $x_2 = 56$. By substituting n_2 , we obtain a new quadratic irrational number $n\sqrt{d} = 97\sqrt{3} = \sqrt{28227}$. We can confirm that the length of the period for this number is 2.

$$\sqrt{28227} = 168 + \frac{1}{112 + \frac{1}{336 + \frac{1}{112 + \dots}}} = [168; \overline{112, 336}].$$

Now, let us find the solution for $m = 3$:

$$n_3 + x_3\sqrt{3} = (7 + 4\sqrt{3})^3 = 1351 + 780\sqrt{3}.$$

In this case, $n_3 = 1351$ and $x_3 = 780$. By substituting n_3 , we obtain a new quadratic irrational number $n\sqrt{d} = 1351\sqrt{3} = \sqrt{5475603}$. For this quadratic irrational number, the length of its period is also 2.

$$\sqrt{5475603} = 2340 + \frac{1}{1560 + \frac{1}{4680 + \frac{1}{1560 + \dots}}} = [2340; \overline{1560, 4680}].$$

Hence, using the fundamental solution, we can obtain infinitely many solutions for n for any d such that $D(n\sqrt{d}) = 2$.

Chapter 5

Numerical application

We have developed a numerical algorithm capable of efficiently calculating the continued fraction expansions of large numbers and determining the lengths of their periodic parts. Our objective is to identify prominent patterns and discern consistent rules that govern the distribution of periodic part lengths across a range of quadratic irrational numbers.

To achieve this, we focused on square-free integers within the range of 1 to 1000 and computed the continued fraction expansions for their corresponding quadratic roots. Subsequently, we constructed graphical representations that illustrate the distribution of periodic part lengths.

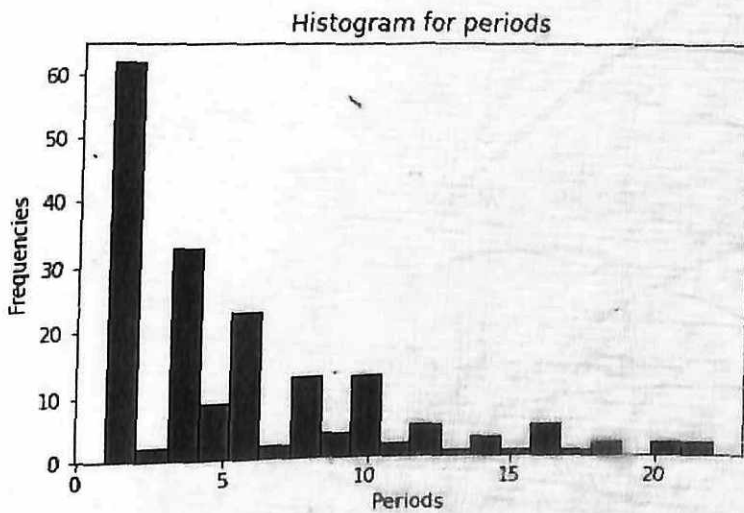


Figure 5.1:

Figure 1 presents one such visualization, depicting the diverse range of period lengths and their respective frequencies. Notably, for the initial numbers in our range, the frequency of shorter periods appears to be higher. Through the utilization of these histograms and by employing our algorithmic approach, we aim to uncover intriguing insights regarding the frequency distribution of periods. This

algorithm may help to the identification of novel observations and potentially lead to significant discoveries in the field and it makes computing the expansion of continued fractions easier.

By employing advanced algorithms and harnessing the power of computational analysis, we can systematically explore the behavior of continued fractions and uncover hidden patterns within their periodic parts. The ability to efficiently process large numbers enables us to probe a wide range of quadratic irrational numbers and obtain comprehensive data for analysis.

In summary, our algorithmic approach to studying continued fractions and their periodic parts opens up exciting possibilities for uncovering new observations and patterns within this mathematical realm. By employing advanced computational techniques and graphical representations, we strive to enhance our comprehension of the distribution of periodic part lengths and potentially make significant contributions to the field of number theory.

Algorithm 1 Part 1

```

function CONTFRAC( $x, k$ )
   $x \leftarrow \text{Decimal}(x)$ 
   $cf \leftarrow []$ 
   $q \leftarrow \lfloor x \rfloor$ 
   $cf.append(q)$ 
   $x \leftarrow x - q$ 
   $i \leftarrow 0$ 
  while  $x \neq 0$  and  $i < k$  do
     $q \leftarrow \lfloor \frac{1}{x} \rfloor$ 
     $cf.append(q)$ 
     $x \leftarrow \frac{1}{x} - q$ 
     $i \leftarrow i + 1$ 
  end while
  return  $cf$ 
end function
function IS_K_PERIODIC( $lst, k$ )
  if  $\text{len}(lst) < \frac{k}{2}$  then
    return False
  end if
  return all( $x == y$  for  $x, y$  in zip( $lst, \text{cycle}(lst[: k])$ ))
end function

```

Algorithm 1 Part 2

```
function IS_PERIODIC(lst)
  for k in range(1,  $\lfloor \frac{\text{len}(\textit{lst})}{2} \rfloor + 1$ ) do
    if IS_K_PERIODIC(lst, k) then
      return lst[ : k]
    end if
  end for
  return lst
end function
final ← []
for i in range(2, 1000) do
  if  $\lfloor \sqrt{i} + 0.5 \rfloor^2 \neq i$  then
    z ← CONTFRAC(Decimal(i).sqrt(), 45)
    w ← IS_PERIODIC(lst[1:])
    x ← lst[0]
    w.insert(0, x)
    final.append(w)
  end if
end for
print(final)
```

Chapter 6

Discussion and Conclusion

Summary

This dissertation thesis provides a comprehensive exploration of continued fractions for quadratic irrationals, with a specific focus on their distribution of periodicity. By delving into the classical results of Lagrange's theorem and Galois' findings, we established a solid foundation for understanding the continued fraction expansions of quadratic irrationals.

Building upon the recent research conducted by Gawron and Kobos [3], we undertook an investigation into the distribution of the lengths of periodic parts for quadratic irrationals in the form of $n\sqrt{d}$, where $n \geq 1$ and d is a square-free integer. Their discovery of infinitely many limit points in the sequence of these lengths prompted us to pose a novel question regarding the existence of limit points equal to specific values. Through some analysis and proof, we addressed this question by demonstrating that for every positive square-free integer d , an infinite number of solutions exist such that the length of the periodic part of the continued fraction expansion of $n\sqrt{d}$ is equal to 2.

To establish a deeper connection between the length of the periodic part and the equation representing $n\sqrt{d}$, we considered the special case when the continued fractions of $n\sqrt{d}$ have a period of 2. Our analysis and deductions led to the formulation of Theorem 1.0.2, which describes the equation governing $n\sqrt{d}$. This theorem provided crucial insights and served as a vital component in the proof of our main result.

Through our investigation, we have successfully established a profound relationship between the equation $n\sqrt{d}$, with the length $D(n\sqrt{d}) = 2$ of its continued fraction expansion, and Pell's equation (Theorem 1.0.1). By demonstrating the solvability of the equation $n^2 - dx^2 = 1$ for infinitely many positive integers n , we have unequivocally established the existence of an infinite number of solutions.

This work not only contributes to the field of number theory but also opens up new avenues for further exploration and research. The connections we have uncovered between continued fractions, quadratic irrationals, and Diophantine equa-

tions offer rich opportunities for deeper investigations into patterns, structures, and properties.

Future research directions could include exploring generalizations of the main theorems, investigating the relationship between other lengths of periodic parts and equations, and addressing open questions posed by other researchers. Through continued exploration, the field stands to benefit from enhanced theoretical understanding and potential practical applications.

Possible Future Directions

In light of the findings presented in this thesis, several avenues for future research and exploration emerge. These directions aim to deepen our understanding of the relationships between quadratic irrationals, their continued fraction representations, and Pell's equation. The following suggestions offer potential paths for further investigation:

Pattern Recognition and Formula Derivation: In Chapter 4, we have derived explicit equations for $n\sqrt{d}$ when the length of its periodic part is 2, 3, and 4. Building upon this foundation, it would be worthwhile to explore consistent patterns across different quadratic irrational numbers. By doing so, we may uncover recurrent themes and potentially construct formulas or recurrence relations that extend beyond the cases examined. This endeavor has the potential to reveal elegant mathematical structures underlying quadratic irrationals.

Generalization of Theorem 1.0.1: The main result of this thesis, Theorem 1.0.1, establishes a connection between the length of the periodic part in the continued fraction expansion of $n\sqrt{d}$ and Pell's equation for the case when the length is 2. An intriguing direction for future research involves expanding this analysis to encompass other lengths of the periodic part. Investigating the relationship between the length and the solutions of Pell's equation for various quadratic irrational numbers has the potential to unveil deeper connections and shed light on the interplay between continued fractions and Diophantine equations.

Addressing Gawron and Kobos' Open Question: The research article by Gawron and Kobos [3] poses an open question regarding the limit points in the sequence $(D(n\sqrt{d}))_{n=1}^{\infty}$. Specifically, they inquire whether, for every non-square integer $d \geq 1$ and positive integer $k \geq 1$, at least one of the numbers k or $k + 1$ appears as a limit point. Addressing this question would contribute significantly to the understanding of the distribution and properties of the lengths of periodic parts in continued fraction expansions. It invites further exploration into the patterns and behaviors exhibited by quadratic irrationals and has the potential to uncover new insights.

Analyzing Periodic Patterns through Advanced Numerical Algorithms: In Chapter 5, we introduced a comprehensive numerical algorithm designed to efficiently compute the continued fraction expansions of large numbers and accurately determine the lengths of their periodic parts. To leverage this algorithm effectively, future research endeavors may focus on refining its computational framework and exploring additional parameters that may influence the distribution of periods.

By embarking on these possible future directions, we can deepen our knowledge and uncover new facets within the realm of quadratic irrationals, continued fractions, and their connections to Diophantine equations. These research avenues hold promise for both theoretical advancements and practical applications, further enriching the field of number theory.

Bibliography

- [1] Joseph Lagrange. Recherches d'arithmétique. *In Nouveaux mémoires de l'Académie royale des sciences et belles-lettres*, 4:337–363, 1770.
- [2] Évariste Galois. Sur la théorie des nombres. *Journal de Mathématiques Pures et Appliquées*, 11:411–425, 1846.
- [3] Filip Gawron and Tomasz Kobos. On length of the period of the continued fraction of nd . *International Journal of Number Theory*, pages 1–11, 2023.
- [4] Andrew M Rockett et al. Continued fractions. 1992.
- [5] Gautam Gopa Krishnan. Continued fractions. *Cornell University*, 2016.
- [6] Kenneth H Rosen. Elementary number theory and its applications. *Addison-Wesley Publishing Company*, 1984.
- [7] Marcin Acewicz and Karol Pał. Pell's equation. *Formalized Mathematics*, 25(3):197–204, 2017.
- [8] Tuyl Van and L Adam. An introduction to the theory and applications of continued fractions. 1996.
- [9] C.D. Olds. Continued fractions. *Random House*, 1963.
- [10] Kline Morris. Mathematical thought from ancient to modern times. *Oxford University Press*, 1972.
- [11] Charles G Moore. An introduction to continued fractions. *The National Council of Teachers of Mathematics*, 1964.
- [12] P Chowla and S Chowla. Problems on periodic simple continued fractions. *Proceedings of the National Academy of Sciences*, 69(12):3745–3745, 1972.
- [13] Christian Friesen. On continued fractions of given period. *Proceedings of the American Mathematical Society*, 103(1):9–14, 1988.
- [14] Franz Halter-Koch. Continued fractions of given symmetric period. *Fibonacci Quart*, 29(4):298–303, 1991.
- [15] L'ubomíra Balková and Aranka Hrušková. Continued fractions of square roots of natural numbers. *Acta Polytechnica*, 53(4), 2013.

- [16] Hanka Řada and Štěpán Starosta. Bounds on the period of the continued fraction after a möbius transformation. *Journal of Number Theory*, 212:122–172, 2020.