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Topic of the thesis:

JORDAN ELEMENTS IN ASSOSYMMETRIC ALGEBRAS

Thesis submitted as part of the requirements for the award of the MSc in "7M05401-Mathematics", SDU, 2020-2022

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THESIS

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Abstract

We consider Jordan brackets in a free assosymmetric algebra. We investigate expansions of left-normed Jordan brackets in free assosymmetric algebra and give a conjecture. In general, we show the proposition and some examples then the proof. For associative algebras P.M. Cohn gave a criterion for Jordanian elements generated by three elements, but we further advanced to assosymetrical algebras not with three elements, but with four, and we showed five elements, but we have the degree and the elements are equal. In general, we have shown a special case, but in the end, there are assumptions that it can work on any dimension n.

Аңдатпа

Біз еркін ассимметриялық алгебрада, Джордан жақшаларында қарастырамыз. Еркін асимметриялық алгебрада сол жақ нормаланған иордан жақшаларының ыдырауын дәлелдеп, болжамды алға тартамыз. Жалпы, біз ұсынысты және кейбір мысалдарды, содан кейін дәлелді көрсетеміз. Ассоциативті алгебралар үшін П.М. Кон үш элементтен құрылған иордандық элементтердің критерийін берді, бірақ біз одан әрі ассиметриялық алгебраларға көштік үш элементі емес, төрт-еу және бес элементт алдық , бірақ бізде дәреже және элементтер саны тең. Жалпы, біз ерекше жағдайды көрсеттік, бірақ соңында ол кез келген n өлшемінде жұмыс істей алады деген болжамдар бар.

Аннотация

Мы рассматриваем жордановые скобки в свободной ассосимметричной алгебре. Мы доказываем разложения левонормированных жордановых скобок в свободной асимметричной алгебре и выдвигаем гипотезу. В общем, мы показываем предложение и примеров, а затем доказательство. Для ассоциативных алгебр П.М. Кон дал критерий жордановых элементов, порожденных тремя элементами, но мы далее перешли к ассосимметричным алгебрам не с тремя элементами, а с четырьмя, и показали пять элементов, но имеем степень и элементы равны. В общем, мы показали частный случай, но в итоге есть предположения, что он может работать на любой размерности n.

Contents

Chapter 1

Introduction

Assosymmetric rings first appeared in the paper by Kleinfeld [9] in 1957. He proved that an assosymmetric ring of characteristic different from 2 and 3 without ideals $I \neq 0$, such that $I^2 = 0$ is associative. Furthermore, assosymmetric algebras were studied in [2],[1],[10], [8],[6]. A base of free assosymmetric algebras was exhibited in [4].

Assosymmetric algebras are defined by the following left-symmetric and right-symmetric identities:

$$(x, y, z) = (x, z, y), \quad (x, y, z) = (y, x, z), \tag{1.1}$$

where $(x, y, z) = (xy)z - x(yz)$. The variety of assosymmetric algebras admits the commutative associative and associative algebras as a subvariety.

Let $A^{(+)} = (A, \{,\})$ be a subalgebra of A under Jordan product (anticommutator) i.e. multiplication is defined by the anticommutator product $\{a, b\} = ab + ba$.

Let $\mathcal{A}$ be a variety of assosymmetric algebras. We define $\mathcal{A}^{(+)}$ as a class of algebras of the form $A^{(+)}$, where $A \in \mathcal{A}$. In [1], established that any assosymmetric algebra under Jordan product satises three polynomial identities

$$\{a, b\} = \{b, a\} \textit{ (commutativity)},$$

$$\{a, \{b, b\}, c\} = 2\{b, \{a, b, c\}\} \textit{ (Lie triple)},$$

and Glennie identity of degree 8.

Let $X = \{x_1, x_2, \ldots\}$ be a set and $A(X)$ be a free assosymmetric algebra generated by X. A polynomial in $A(X)$ is called *Jordan* element of $A(X)$ if it can be expressed by elements of X in terms of anticommutators. For associative algebras P.M. Cohn gave a criterion for the Jordan elements generated by three elements [3]. In addition, he gave a criterion of homomorphic images to be special and showed that the homomorphic image of the free special Jordan algebra generated by three elements is exceptional.

In this study, we describe all Jordan elements up to degree 6 in a free assosymmetric algebra generated by proposition (3.1.1).

1.1 Motivation

We investigate expansions of left-normed Jordan brackets in free assosymmetric algebra and give a conjecture. In general, we show the proposition and some examples then the proof.

1.2 Aims and Objectives

We consider Jordan brackets in a free assosymmetric algebra. If we take Assosymetic algebra as a mountain and we study inside the cave there is a lot that we do not know, and our goal is to make our part of the contribution to the study. There are still open questions, we will not be able to answer practically the weight of the question in the mountain is very many grains. in our duty, advance the process a few steps

1.3 Thesis Outline

First chapter is Introduction chapter. It is this one part of that you are currently reading now and there is all about what's to come. Provides a basic overview of our work. In Chapter 2 we showing short history review related work.In Chapter 3 we have main result our works and formulate the problem to solve, propositions, theorems and proof. And in the Conclusion chapter's have result and we conclude our conclusion.

Chapter 2

Background and literature review

2.1 Ring

The theory of rings has an in-depth historical inquiry into some basic standard branches of mathematics and we have the beginning of the hundred years of the nineteenth century, such as which we know today and they are the theory of algebraic numbers, field theory, the theory of polynomial forms, the theory of quaternions and the theory of algebraic and of course, functions of hypercomplex numbers, etc. All these areas of unification reached its apogee in the works of the German mathematicians Emmy Noether (1882-1935) and Wolfgang Krull (1899-1970). Using these, by combining strength in place to give us the abstract notion of a ring, these two famous mathematicians published nineteenth-century systematic and comprehensive explanations of their researched theorem, especially this factorization theorem, which are common and understandable to all these domains[5].

2.2 Group

For the first time in the history of the group, the famous mathematician Lagrange used in his works in his research, he opened the topic of whether it is possible to express the roots of a polynomial in some way using the addition operations to find its coefficient, multiplication, division, subtraction and, of course, extracting the root (solution of the problem of equations in radicals). We have known since the time of ancient Greece for polynomials of the second degree such formulas in general terms are known. By the works of the famous Italian mathematicians Del Ferro and Tartaglia, and of course our common well-known Cardano, such formulas were found in their time for simple polynomials of the third degree. The beautiful Ferrari method is used to find a solution for a non-simple equation of the fourth degree, which reduces to the previously known solution of the equations of the third and the usual common known second degrees. Lagrange in modern times established a general idea for us to reduce the solution of these equations, which are given for the degree n = 2, 3, 4, to the solution of which is the so-called resolving our equation for an understandable and lesser degree. However, in this case, n = 5, the resolvent turns out to be for a sixth-degree equation. In his study of this issue in the works of Lagrange, for the first time, permutations of the group appear. For example, for the possibility that for reducing the solution of polynomial equations that are of the fourth degree and for solving the usual equation of the well-known third degree, the presence in the group of which is a substitution of the fourth degree S_4 with an invariant subgroup. All this starting time our parts, we have identities idea. An element y of an algebra is called a left (right) adverse of an element x of the algebra in case x+y=yx (x+y = xy), and is called simply an adverse in case it is both a left and a right adverse. An adverse of x is denoted x^A [11].

2.3 Fields

A very important milestone on the way to the devastating development of a structural approach to algebra was one of the interesting publications that were published [12] in 1910, the work of which the authors is one of the brightest mathematicians Ernst Steinitz on the topic of abstract field theory: Corper's algebraic theory. In this work, our Steinitz presented a very marvelous exhaustive account of the results of the theory up to that time and, at the same time, opened important new avenues for the investigation of any abstractly constructive and formulated algebraic concept. Frenkel's definition on abstract rings is a classic and accepted example of work published under the direct influence of Steinitz. In that article, Steinitz well formulated a new program and one could say directions for future research, which he fully applied for the first time to a specific topic (abstract fields), but which could also be extended to perfection, and in fact some time later was adopted, for algebra Bosch and in general. Steinitz began that article by bluntly declaring his new methodological view: he proclaimed what today would somehow be called a "program for structural research", and although only for a particular or specific algebraic field, i.e. for the study of a part fields. His subject matter would be abstract only of the fields, as they were defined with Weber in his 1893 paper. But his study of the field would slightly diverge from Weber's study on an important part of the meaning. And this is before the story for the field and the concept of where it began about him and interest[5].

2.4 Algebra

Abstract algebra is both a mathematical discipline and also structural algebra or called modern algebra, meaning, unlike classical algebra, which focuses on the formal form of manipulating abstract symbols to solve normal equations, and the new image of algebra has a special look. In we began to give representations in the Modern Algebra textbook (Waerden 1930), which "inverted the conceptual hierarchy of classical algebra" (Corry 2016), bringing its hypothetical algebraic structures to the fore and considering the properties of numbers, polynomials, etc. as corollaries abstract algebra steel slightly widely used [7].

Chapter 3

Main result

3.1 JORDAN ELEMENTS IN ASSOSYMMETRIC ALGEBRAS

Let $A(X)$ be a free assosymmetric algebra on X. Denote by $[x_{i_1}x_{i_2}\cdots x_{i_n}]$ or $x_{i_1}x_{i_2}\cdots x_{i_n}$ a *left-normed element* $(\cdots(x_{i_1}x_{i_2})x_{i_3}\cdots)x_{i_n}$ for $x_{i_1},\ldots,x_{i_n}\in X$. The element

$$\langle x_{i_1}\cdots x_{i_n}, y_{i_1}\cdots y_{i_m}\rangle =$$
$$x_{i_1}(x_{i_2}(\cdots(x_{i_n}[\ldots[(y_{i_1},y_{i_2},y_{i_3}),y_{i_4}]\cdots,y_{i_m}]\cdots))$$

is called *ordered expression*, where $x_{i_1},\ldots x_{i_n},y_{i_1},\ldots,y_{i_m}\in X$ and we have order $x_{i_1}\leq x_{i_2}\leq\cdots\leq x_{i_n}$ and $y_{i_1}\leq y_{i_2}\leq\cdots\leq y_{i_m}$. In [4] was proved that the set of left-normed and ordered expression elements forms a basis for the free assosymmetric algebra $A(X)$ over field of characteristic $\neq 2,3$. A multiplication rule of the base elements was given in [4], here we give it as proposition.

Proposition 3.1.1

$$\langle u_1,v_1\rangle\langle u_1,v_1\rangle = 0, \tag{3.1}$$

$$[x]\langle u,v\rangle = \langle xu,v\rangle, \tag{3.2}$$

$$\langle u,v\rangle[x] = \sum_{x=x_1x_2}\langle x_1u,x_2v\rangle, \tag{3.3}$$

$$[x][y] = [xy] - \sum_{\substack{x=x_1x_2\\|x_2|\geq 1\\y=y_1y_2\\|y_2|\geq 2}}(|y_2-1|)\langle x_1y_1,x_2y_2\rangle. \tag{3.4}$$

Let $F(X)$ be a free nonassociative algebra on the set X. We define *reversal* operator ρ on $F(X)$ as linear mapping $\rho: F(X)\to F(X)$ defined as follows

$$\rho(x_i) = x_i,\ \textit{for}\ x_i\in X,$$

$$\rho(uv) = \rho(v)\rho(u),\ \textit{for}\ u,v\in F(X).$$

If $u \in F(X)$, then $\rho(u)$ is called the reverse of u, and u is reversible, if $\rho(u) = u$.

Now we give our main result of this paper.

Theorem 3.1.2 *Every reversible element $u \in F(\{x_1, x_2, x_3\})$, with length ≤ 4 is Jordan element in a free assosymmetric algebra $A(\{x_1, x_2, x_3\})$.*

A free assosymmetric algebra $A(X)$ in degree 3 has a 7-dimensional polylinear part with a basis

$$\{x_1x_2x_3, x_1x_3x_2, x_2x_1x_3, x_2x_3x_1, x_3x_1x_2, x_3x_2x_1, (x_1, x_2, x_3)\}.$$

By the Propostion 3.1.1, other 6 right-bracketed elements are linear combinations of basic elements:

$$x_{i_1}(x_{i_2}x_{i_3}) = x_{i_1}x_{i_2}x_{i_3} - (x_{i_1}, x_{i_2}, x_{i_3}), \tag{3.5}$$

where $x_{i_1}, x_{i_2}, x_{i_3} \in X$.

A free assosymmetric algebra $A(X)$ in degree 4 has a 29-dimensional polylinear part with a basis: 24 elements of the left-normed form $x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}$ where $\sigma \in S_4$, and

$$\{[(x_1, x_2, x_3), x_4], x_1(x_2, x_3, x_4), x_2(x_1, x_3, x_4), x_2(x_1, x_2, x_4), x_4(x_1, x_2, x_3)\}.$$

Lemma 3.1.3 *The polynomial $x_{i_1}x_{i_2}x_{i_3} + x_{i_3}x_{i_2}x_{i_1} - (x_{i_1}, x_{i_2}, x_{i_3})$ is a Jordan element in $A(X)$, that is*

$$x_{i_1}x_{i_2}x_{i_3} + x_{i_3}x_{i_2}x_{i_1} - (x_{i_1}, x_{i_2}, x_{i_3}) =$$

$$12\{\{x_{i_1}, x_{i_2}\}, x_{i_3}\} - 12\{\{x_{i_1}, x_{i_3}\}, x_{i_2}\} + 12\{\{x_{i_2}, x_{i_3}\}, x_{i_1}\},$$

where $x_{i_1}, x_{i_2}, x_{i_3} \in X$.

Proof 3.1.4 *By definition of $\{a, b\} = ab + ba$ we have*

$$\{\{x_{i_1}, x_{i_2}\}, x_{i_3}\} = (x_{i_1}x_{i_2})x_{i_3} + (x_{i_2}x_{i_1})x_{i_3} + x_{i_3}(x_{i_2}x_{i_1}) + x_{i_3}(x_{i_1}x_{i_2}) =$$

by Proposition 3.1.1 we have

$$= x_{i_1}x_{i_2}x_{i_3} + x_{i_2}x_{i_1}x_{i_3} + x_{i_3}x_{i_2}x_{i_1} + x_{i_3}x_{i_1}x_{i_2} - 2(x_{i_1}, x_{i_2}, x_{i_3}).$$

By the same way we obtain

$$\{\{x_{i_1}, x_{i_3}\}, x_{i_2}\} = x_{i_1}x_{i_3}x_{i_2} + x_{i_3}x_{i_1}x_{i_2} + x_{i_2}x_{i_3}x_{i_1} + x_{i_2}x_{i_1}x_{i_3} - 2(x_{i_1}, x_{i_2}, x_{i_3}),$$

and

$$\{\{x_{i_2}, x_{i_3}\}, x_{i_1}\} = x_{i_1}x_{i_2}x_{i_3} + x_{i_1}x_{i_3}x_{i_2} + x_{i_2}x_{i_3}x_{i_1} + x_{i_3}x_{i_2}x_{i_1} - 2(x_{i_1}, x_{i_2}, x_{i_3}).$$

Let $SJ(X)$ be the free special Lie-triple algebra on X under anticommutator, i.e., subalgebra of $A(X)^{(+)} = (A(X), \{\ ,\ \})$ generated by X.

Lemma 3.1.5 *The following element*

$$x_{i_1}x_{i_1}x_{i_2}x_{i_3} + x_{i_3}x_{i_2}x_{i_1}x_{i_1} - 2\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - \langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - \langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle$$

is a Jordan element in $A(X)$, where $x_{i_1}, x_{i_2}, x_{i_3} \in X$.

Proof 3.1.6 *Write $a \equiv b$ if $a - b \in SJ(X)$. Then by Lemma 3.1.3 and (3.29) we have*

$$x_{i_1}x_{i_2}x_{i_3} + x_{i_1}(x_{i_2}x_{i_3}) = x_{i_1}x_{i_2}x_{i_3} + x_{i_3}x_{i_2}x_{i_1} - (x_{i_1}, x_{i_2}, x_{i_3}) \equiv 0. \qquad (3.6)$$

Since, $x_1^2 = \frac{1}{2}\{x_1, x_1\}$ is a Jordan element, we put $x_1^2 = x_1x_1$ instead of x_1 in (3.8), then by Proposition 3.1.1 we have

$$((x_1x_1)x_2)x_3 + x_3(x_2(x_1x_1)) = ((x_1x_1)x_2)x_3 + ((x_3x_2)x_1)x_1 -$$
$$2\langle x_1, x_1x_2x_3\rangle - \langle x_2, x_1x_1x_3\rangle - \langle x_3, x_1x_1x_2\rangle - 2[(x_1, x_1, x_2), x_3] \equiv 0$$

By Lemma 3.1.5 in [6] we have $[(x_1, x_1, x_2), x_3] \equiv 0$. Hence,

$$x_{i_1}x_{i_1}x_{i_2}x_{i_3} + x_{i_3}x_{i_2}x_{i_1}x_{i_1} -$$
$$2\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - \langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - \langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle \equiv 0.$$

Lemma 3.1.7 *The following element*

$$x_{i_1}x_{i_2}x_{i_3}x_{i_1} + x_{i_1}x_{i_3}x_{i_2}x_{i_1} - 2\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - \langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - \langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle$$

is a Jordan element in $A(X)$, where $x_{i_1}, x_{i_2}, x_{i_3} \in X$.

Proof 3.1.8 *Write $a \equiv b$ if $a - b \in SJ(X)$. Then by Lemma 3.1.3 and (3.29) we have*

$$\{x_{i_1}x_{i_2}x_{i_3} + x_{i_3}x_{i_2}x_{i_1} - (x_{i_1}, x_{i_2}, x_{i_3}), x_{i_1}\} \equiv 0.$$

By definition of $\{a, b\} = ab + ba$ we obtain

$$x_{i_1}x_{i_2}x_{i_3}x_{i_1} + x_{i_3}x_{i_2}x_{i_1}x_{i_1} - (x_{i_1}, x_{i_2}, x_{i_3})x_{i_1} +$$
$$x_{i_1} \cdot x_{i_1}x_{i_2}x_{i_3} + x_{i_1} \cdot x_{i_3}x_{i_2}x_{i_1} - x_{i_1}(x_{i_1}, x_{i_2}, x_{i_3}) \equiv 0.$$

Using Proposition 3.1.1 we have

$$x_{i_1}x_{i_2}x_{i_3}x_{i_1} + x_{i_3}x_{i_2}x_{i_1}x_{i_1} + x_{i_1}x_{i_1}x_{i_2}x_{i_3} + x_{i_1}x_{i_3}x_{i_2}x_{i_1} - 5[(x_{i_1}, x_{i_1}, x_{i_2}), x_{i_3}] -$$
$$4\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - 2\langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - 2\langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle \equiv 0.$$

By Lemma 3.1.7 in [6] and Lemma 3.1.5 we have

$$x_{i_1}x_{i_2}x_{i_3}x_{i_1} + x_{i_1}x_{i_3}x_{i_2}x_{i_1} -$$
$$2\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - \langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - \langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle \equiv 0.$$

Lemma 3.1.9 *The following element*

$$x_{i_2}x_{i_1}x_{i_1}x_{i_3} + x_{i_2}x_{i_1}x_{i_1}x_{i_3} - 2\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - \langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - \langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle$$

is a Jordan element in $A(X)$, where $x_{i_1}, x_{i_2}, x_{i_3} \in X$.

Proof 3.1.10 *Write $a \equiv b$ if $a - b \in SJ(X)$. Then by Lemma 3.1.3 and (3.29) we have*

$$x_{i_2}x_{i_1}x_{i_3} + x_{i_3}(x_{i_1}x_{i_2}) = x_{i_2}x_{i_1}x_{i_3} + x_{i_3}x_{i_1}x_{i_2} - (x_{i_1}, x_{i_2}, x_{i_3}) \equiv 0. \tag{3.7}$$

Since, $x_{i_1}^2 = \frac{1}{2}\{x_{i_1}, x_{i_1}\}$ is a Jordan element, we put $x_{i_1}^2 = x_{i_1}x_{i_1}$ instead of x_{i_1} in (3.8), then by Proposition 3.1.1 we have

$$(x_{i_2}(x_{i_1}x_{i_1}))x_{i_3} + x_{i_2}((x_{i_1}x_{i_1})x_{i_3}) = x_{i_2}x_{i_1}x_{i_1}x_{i_3} + x_{i_2}x_{i_1}x_{i_1}x_{i_3} -$$

$$2\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - \langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - \langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle - 3[(x_{i_1}, x_{i_1}, x_{i_2}), x_{i_3}] \equiv 0.$$

By Lemma 3.1.9 in [6] we have $[(x_1, x_1, x_2), x_3] \equiv 0$. Hence,

$$x_{i_2}x_{i_1}x_{i_1}x_{i_3} + x_{i_2}x_{i_1}x_{i_1}x_{i_3} -$$

$$2\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - \langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - \langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle \equiv 0.$$

Lemma 3.1.11 *The following element*

$$x_{i_1}x_{i_2}x_{i_1}x_{i_3} + x_{i_3}x_{i_1}x_{i_2}x_{i_1} - 2\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - \langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - \langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle$$

is a Jordan element in $A(X)$, where $x_{i_1}, x_{i_2}, x_{i_3} \in X$.

Proof 3.1.12 *Write $a \equiv b$ if $a - b \in SJ(X)$. Then by Lemma 3.1.3 and (3.29) we have*

$$x_{i_2}x_{i_1}x_{i_3} + x_{i_3}(x_{i_1}x_{i_2}) = x_{i_2}x_{i_1}x_{i_3} + x_{i_3}x_{i_1}x_{i_2} - (x_{i_1}, x_{i_2}, x_{i_3}) \equiv 0. \tag{3.8}$$

Since, we put $\{x_{i_1}, x_{i_2}\}$ instead of x_{i_2} in (3.8), then by Proposition 3.1.1 we have

$$(\{x_{i_2}x_{i_1}\}x_{i_1})x_{i_3} + x_{i_3}(x_{i_1}\{x_{i_2}, x_{i_1}\}) =$$

$$((x_{i_2}x_{i_1})x_{i_1})x_{i_3} + ((x_{i_1}x_{i_2})x_{i_1})x_{i_3} + x_{i_3}(x_{i_1}(x_{i_2}x_{i_1})) + x_{i_3}(x_{i_1}(x_{i_1}x_{i_2})) =$$

$$x_{i_2}x_{i_1}x_{i_1}x_{i_3} + x_{i_1}x_{i_2}x_{i_1}x_{i_3} + x_{i_3}x_{i_1}x_{i_2}x_{i_1} + x_{i_3}x_{i_1}x_{i_1}x_{i_2} -$$

$$4\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - 2\langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - 2\langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle - 4[(x_{i_1}, x_{i_1}, x_{i_2}), x_{i_3}] \equiv 0.$$

By Lemma 3.1.11 in [6]

$$[(x_{i_1}, x_{i_1}, x_{i_2}), x_{i_3}] \equiv 0$$

and by Lemma 3.1.9 we have

$$x_{i_2}x_{i_1}x_{i_1}x_{i_3} + x_{i_2}x_{i_1}x_{i_1}x_{i_3} -$$

$$2\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - \langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - \langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle \equiv 0.$$

Hence,

$$x_{i_1}x_{i_2}x_{i_1}x_{i_3} + x_{i_3}x_{i_1}x_{i_2}x_{i_1} -$$

$$2\langle x_{i_1}, x_{i_1}x_{i_2}x_{i_3}\rangle - \langle x_{i_2}, x_{i_1}x_{i_1}x_{i_3}\rangle - \langle x_{i_3}, x_{i_1}x_{i_1}x_{i_2}\rangle \equiv 0.$$

Proof 3.1.13 (Proof of Theorem 3.1.2) *For length of u is 2, there is only one type of reversible elements, that is*

$$x_{i_2}x_{i_2} + x_{i_2}x_{i_1} = \{x_{i_1}, x_{i_2}\}.$$

If length of u is 3, then in $F(X)$ we have one form of reversible elements

$$u = (x_{i_1}x_{i_2})x_{i_3} + x_{i_3}(x_{i_2}x_{i_1}),$$

where $x_{i_1}, x_{i_2}, x_{i_3} \in X$. The above element u in $A(X)$ by Proposition 3.1.1 is equal to

$$x_{i_1}x_{i_2}x_{i_3} + x_{i_3}x_{i_2}x_{i_1} - (x_{i_1}, x_{i_2}, x_{i_3}),$$

and by Lemma 3.1.3 it is Jordan element. Now assume that the length of u is 4, then we have the following reversible elements in $F(X)$

$$\begin{gathered}\{((x_{i_1}x_{i_2})x_{i_3})x_{i_4} + x_{i_4}(x_{i_3}(x_{i_2}x_{i_1})), \\ (x_{i_1}x_{i_2})(x_{i_3}x_{i_4}) + (x_{i_4}x_{i_3})(x_{i_2}x_{i_1}), \\ x_{i_1}((x_{i_2}x_{i_3})x_{i_4})) + (x_{i_4}(x_{i_3}x_{i_2}))x_{i_1}\},\end{gathered} \tag{3.9}$$

where $x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4} \in X$.

In $A(X)$, by the Proposition 3.1.1, elements of (3.9) can be expressed by base elements in $A(X)$ in the following form

$$\begin{gathered}x_{i_1}(x_{i_2}(x_{i_3}x_{i_4})) = x_{i_1}x_{i_2}x_{i_3}x_{i_4} - 2[(x_{i_1}, x_{i_2}, x_{i_3}), x_{i_4}] - \\ x_{i_1}(x_{i_2}, x_{i_3}, x_{i_4}) - x_{i_2}(x_{i_1}, x_{i_3}, x_{i_4}) - x_{i_3}(x_{i_1}, x_{i_2}, x_{i_4}) - x_{i_4}(x_{i_1}, x_{i_2}, x_{i_3}),\end{gathered} \tag{3.10}$$

$$\begin{gathered}(x_{i_1}x_{i_2})(x_{i_3}x_{i_4}) = x_{i_1}x_{i_2}x_{i_3}x_{i_4} - [(x_{i_1}, x_{i_2}, x_{i_3}), x_{i_4}] - \\ x_{i_1}(x_{i_2}, x_{i_3}, x_{i_4}) - x_{i_2}(x_{i_1}, x_{i_3}, x_{i_4}),\end{gathered} \tag{3.11}$$

$$\begin{gathered}x_{i_1}((x_{i_2}x_{i_3})x_{i_4})) = x_{i_1}x_{i_2}x_{i_3}x_{i_4} - 2[(x_{i_1}, x_{i_2}, x_{i_3}), x_{i_4}] - \\ x_{i_2}(x_{i_1}, x_{i_3}, x_{i_4}) - x_{i_3}(x_{i_1}, x_{i_2}, x_{i_4}) - x_{i_4}(x_{i_1}, x_{i_2}, x_{i_3}),\end{gathered} \tag{3.12}$$

$$(x_{i_1}(x_{i_2}x_{i_3}))x_{i_4} = x_{i_1}x_{i_2}x_{i_3}x_{i_4} - [(x_{i_1}, x_{i_2}, x_{i_3}), x_{i_4}] - x_{i_4}(x_{i_1}, x_{i_2}, x_{i_3}), \tag{3.13}$$

where $x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4} \in X$.

Therefore, all reversible elements (3.9) *are equal to each other in* $A(X)$:

$$\begin{gathered}((x_{i_1}x_{i_2})x_{i_3})x_{i_4} + x_{i_4}(x_{i_3}(x_{i_2}x_{i_1})) = \\ (x_{i_1}x_{i_2})(x_{i_3}x_{i_4}) + (x_{i_4}x_{i_3})(x_{i_2}x_{i_1}) = \\ x_{i_1}((x_{i_2}x_{i_3})x_{i_4})) + (x_{i_4}(x_{i_3}x_{i_2}))x_{i_1} = \\ x_{i_1}x_{i_2}x_{i_3}x_{i_4} + x_{i_4}x_{i_3}x_{i_2}x_{i_1} - 2[(x_{i_1}, x_{i_2}, x_{i_3}), x_{i_4}] - \\ x_{i_1}(x_{i_2}, x_{i_3}, x_{i_4}) - x_{i_2}(x_{i_1}, x_{i_3}, x_{i_4}) - x_{i_3}(x_{i_1}, x_{i_2}, x_{i_4}) - x_{i_4}(x_{i_1}, x_{i_2}, x_{i_3}).\end{gathered}$$

If $X = \{x_1, x_2, x_3\}$ *then by Lemma* 3.1.5, 3.1.7, 3.1.9 *and* 3.1.11 *we have they are Jordan elements. And this completes the proof.*

See Appendix **??**.

3.2 JORDAN ELEMENTS IN ASSOSYMMETRIC ALGEBRAS N 5

Assosymmetric algebras are defined by the following left-symmetric and right-symmetric identities:

$$(x, y, z) = (x, z, y), \quad (x, y, z) = (y, x, z), \tag{3.14}$$

where $(x, y, z) = (xy)z - x(yz)$. The variety of assosymmetric algebras admits the commutative associative and associative algebras as a subvariety.

Let $A^{(+)} = (A, \{,\})$ be a subalgebra of A under Jordan product (anticommutator) i.e. multiplication is defined by the anticommutator product $\{a, b\} = ab + ba$.

Let $\mathcal{A}$ be a variety of assosymmetric algebras. We define $\mathcal{A}^{(+)}$ as a class of algebras of the form $A^{(+)}$, where $A \in \mathcal{A}$. In [6], established that any assosymmetric algebra under Jordan product satises three polynomial identities

$$\{a, b\} = \{b, a\} \textit{ (commutativity)},$$

$$\{a, \{b, b\}, c\} = 2\{b, \{a, b, c\}\} \textit{ (Lie triple)},$$

and Glennie identity of degree 8.

Let $X = \{x_1, x_2, \ldots\}$ be a set and $A(X)$ be a free assosymmetric algebra generated by X. A polynomial in $A(X)$ is called *Jordan* element of $A(X)$ if it can be expressed by elements of X in terms of anticommutators. For associative algebras P.M. Cohn gave a criterion for the Jordan elements generated by three elements [3]. In addition, he gave a criterion of homomorphic images to be special and showed that the homomorphic image of the free special Jordan algebra generated by three elements is exceptional.

Let $A(X)$ be a free assosymmetric algebra on X. Denote by $[x_{i_1}x_{i_2}\cdots x_{i_n}]$ or $x_{i_1}x_{i_2}\cdots x_{i_n}$ a *left-normed element* $(\cdots(x_{i_1}x_{i_2})x_{i_3}\cdots)x_{i_n}$ for $x_{i_1}, \ldots, x_{i_n} \in X$. The element

$$\langle x_{i_1}\cdots x_{i_n}, y_{i_1}\cdots y_{i_m}\rangle =$$

$$x_{i_1}(x_{i_2}(\cdots(x_{i_n}[\ldots[(y_{i_1},y_{i_2},y_{i_3}),y_{i_4}]\cdots,y_{i_m}]\cdots))$$

is called *ordered expression*, where $x_{i_1},\ldots x_{i_n}, y_{i_1},\ldots,y_{i_m} \in X$ and we have order $x_{i_1} \le x_{i_2} \le \cdots \le x_{i_n}$ and $y_{i_1} \le y_{i_2} \le \cdots \le y_{i_m}$. In [4] was proved that the set of left-normed and ordered expression elements forms a basis for the free assosymmetric algebra $A(X)$ over field of characteristic $\neq 2,3$. A multiplication rule of the base elements was given in [4],here we give it as proposition. Some times x_{i_1} = "a", x_{i_2} = "b", x_{i_3} = "c", x_{i_4} = "d", x_{i_5} = "e", here we give it as proposition.

Proposition 3.2.1

$$\langle u_1,v_1\rangle \circ \langle u_1,v_1\rangle = 0, \tag{3.15}$$

$$[x] \circ \langle u,v\rangle = \langle xu,v\rangle, \tag{3.16}$$

$$\langle u,v\rangle \circ [x] = \sum_{x=x_1x_2} \langle x_1u, x_2v\rangle, \tag{3.17}$$

$$[x]\circ[y] = [xy] - \sum_{\substack{x=x_1x_2\\ |x_2|\ge 1\\ y=y_1y_2\\ |y_2|\ge 2}} (|y_2-1|)\langle x_1y_1, x_2y_2\rangle. \tag{3.18}$$

Every reversible element u $\in F(\{x_1,x_2,x_3\})$, with length 4 is Jordan element in a free assosymmetric algebra A$(\{x_1,x_2,x_3\})$.

A free assosymmetric algebra $A(X)$ in degree 3 has a 7-dimensional polylinear part with a basis

$$\{x_1x_2x_3, x_1x_3x_2, x_2x_1x_3, x_2x_3x_1, x_3x_1x_2, x_3x_2x_1, (x_1,x_2,x_3)\}.$$

By the Propostion (3.3.1), other 6 right-bracketed elements are linear combinations of basic elements:

$$(x_{i_1},x_{i_2},x_{i_3}) = x_{i_1}x_{i_2}x_{i_3} - x_{i_1}(x_{i_2}x_{i_3}), \tag{3.19}$$

$$x_{i_1}(x_{i_2}x_{i_3}) = x_{i_1}x_{i_2}x_{i_3} - (x_{i_1},x_{i_2},x_{i_3}),$$

Theorem 3.2.2 *From using proposition (3.3.1) and next propositions (3.30),(3.31) and (3.32) we can find {{{a, b}, c}, d},*

$$\begin{gathered}\{\{\{a,b\},c\},d\} = (((ab)c)d) + (((ba)c)d) + (((ca)b)d) + \\ + (((cb)a)d) + (((da)b)c) + (((db)a)c) + (((dc)a)b) + \\ + (((dc)b)a) - 10[(a,b,c),d] - 4[a,(b,c,d] - \\ - 4[b,(a,c,d)] - 4[c,(a,b,d)] - 4[d,(a,b,c)]\end{gathered}$$

Proposition 3.2.3 *By propostion (3.3.1), we can find ((a,b,c)d),*

$$((a,b,c)d) = <1, abc> \circ[d] =< 1, abcd > +$$
$$+ <d, abc> = [(a, b, c), d] + [d (a, b, c)].$$

There is we can take,

$$< 1, abc > \circ[d] = [(a, b, c), d] + [d(a, b, c)] \tag{3.20}$$

Proposition 3.2.4 *By propostion (3.3.1), there is our (d(a,b,c)) = [d] is equal*

$$(d(a, b, c)) = [d]\circ < 1, abc >=< d, abc >= [d(a, b, c)]$$

There is we can take [d] $\circ < 1, abc >$,

$$[d]\circ < 1, abc >= [d(a, b, c)] \tag{3.21}$$

Proposition 3.2.5 *By propostion (3.3.1), we can write (d((ab)c) =* $[d] \circ [abc]$ *and this is,*

$$(d((ab)c) = [d] \circ[abc] = [dabc] - 2 < 1, abcd > -$$
$$- <a, bcd> - <b, acd> - <c, abd> = (((da)b)c) -$$
$$- 2[(a, b, c), d] - (a(b, c, d)) - (b(a, c, d)) -$$
$$- (c(a, b, c)) = (((da)b)c) - 2[(a, b, c), d] -$$
$$- [a, (b, c, d)] - [b, (a, c, d)] - [c, (a, b, d)]$$

There is $[d] \circ [abc]$ is equal

$$[d] \circ [abc] = (((da)b)c) - 2[(a, b, c), d] - [a, (b, c, d)] - [b, (a, c, d)] - [c, (a, b, d)] \tag{3.22}$$

Proof 3.2.6 *By definition of {a, b} = ab + ba and proposition (3.3.1) we have*

$$\{\{\{a, b\}, c\}, d\} = (((ab)c)d) + (((ba)c)d) +$$
$$+ (((c(ab))d) + ((c(ba))d) + (d((ab)c)) +$$
$$+ (d((ba)c)) + (d(c(ab))) + (d(c(ba)))$$

So when we have {{{a,b},c},d} we using (3.3.1) and propositions (3.30),(3.31) and (3.32) this one gived

$$\{\{\{a,b\},c\},d\} = \{((ab)c) + ((ba)c) + ((ca)b) +$$
$$+ ((cb)a) - 2(a,b,c), d\}= [abc] + [bac] + [cab] +$$
$$+ [cba] - 2<1,(abc)>, d = [abc] \circ[d] + [bac] \circ [d] +$$
$$+ [cab] \circ[d] + [cba] \circ [d] - 2 < 1, abc > \circ[d] +$$
$$+ [d] \circ[abc] + [d] \circ [bac] + [d] \circ [cab] + [d] \circ [cba]$$
$$- 2[d] \circ < 1, abc >= [abcd] + [bacd] + [cabd] +$$
$$+ [cbad] - 2<1, abcd> - 2<d, abc> + [dabc] -$$

$$
\begin{aligned}
&- 2<1, abcd> - <a, bcd> - <b, acd> - <c, abd> + \\
&+ [dbac] - 2<1, abcd> - <a, bcd> - <b, acd> - \\
&- <c, abd> + [dcab] - 2<1, abcd> - <a, bcd> - \\
&- <b, acd> - <c, abd> + [dcba] - 2<1, abcd> - \\
&- <a, bcd> - <b, acd> - <c, abd> - 2<d, abc> = \\
&= (((ab)c)d) + (((ba)c)d) + (((ca)b)d) + (((cb)a)d) - \\
&- 2[(a, b, c), d] - 2[d (a, b, c)] - 2[(a, b, c), d] + \\
&+ (((da)b)c) - [a (b, c, d)] - [b (a, c, d)] - \\
&- [c (a, b, d)] + (((db)a)c) - 2[(a, b, c), d] - \\
&- [a (b, c, d)] - [b (a, c, d)] - [c (a, b, d)] + \\
&+ (((dc)a)b) - 2[(a, b, c), d] - [a (b, c, d)] - \\
&- [b (a, c, d)] - [c (a, b, d)] + (((dc)b)a) - \\
&- 2[(a, b, c), d] - [a (b, c, d)] - [b (a, c, d)] - \\
&- [c (a, b, d)] - 2[d (a, b, c)] = (((ab)c)d) + \\
&+ (((ba)c)d) + (((ca)b)d) + (((cb)a)d) + \\
&+ (((da)b)c) + (((db)a)c) + (((dc)a)b) + \\
&+ (((dc)b)a) - 10[(a, b, c), d] - 4[a, (b, c, d)] - \\
&- 4[b, (a, c, d)] - 4[c, (a, b, d)] - 4[d, (a, b, c)]
\end{aligned}
$$

Theorem 3.2.7 *By definition of* $\{a, b\} = ab + ba$ *and proposition (3.3.1) we have*

$$
\begin{aligned}
&\{\{a, b\}, \{c, d\}\} = (((ab)c)d) + (((cd)a)b) + \\
&+ (((ab)d)c) + (((dc)a)b) + (((ba)c)d) + (((cd)b)a) + \\
&+ (((ba)d)c) + (((dc)b)a) - 4[a, (b, c, d)] - 4[b, (a, c, d)] - \\
&- 4[c, (a, b, d)] - 4[d, (a, b, c)] - 8[(a, b, c), d]
\end{aligned}
$$

Proposition 3.2.8 *Proposition (3.3.1) we can find (ab)(cd),*

$$(ab)(cd) = [ab] \circ [cd] = [abcd] - < a, bcd > - < b, acd > - < 1, abcd > \quad (3.23)$$

Proof 3.2.9 *By definition of* $\{a, b\} = ab + ba$ *and proposition (3.3.1) we have* $\{\{a, b\}, \{c, d\}\} = ((ab)(cd)) + ((ba)(cd)) + (ab)(dc) + (ab)(cd) + + ((dc)(ab)) + ((ba)(dc)) + ((ba)(cd)) + ((dc)(ba))$

So when we know a, b, c, d we can use proposition (3.3.1) and proposition (3.3.8) to proof

$$
\begin{aligned}
&\{\{a, b\}, \{c, d\}\} = (ab)(cd) + (cd)(ab) + (ab)(dc) + (dc)(ab) + \\
&+ (ba)(cd) + (cd)(ba) + (ba)(dc) + (dc)(ba) = \\
&= [abcd] - <a, bcd> - <b, acd> - <1, abcd> + \\
&+ [cdab] - <c, abd> - <d, abc> - <1, abcd> + \\
&+ [abdc] - <a, bcd> - <b, acd> - <1, abcd> + \\
&+ [dcab] - <c, abd> - <d, abc> - <1, abcd> + \\
&+ [bacd] - <a, bcd> - <b, acd> - <1, abcd> +
\end{aligned}
$$

$$
\begin{aligned}
&+ [cdba] - <c, abd> - <d, abc> - <1, abcd> + \\
&+ [badc] - <a, bcd> - <b, acd> - <1, abcd> + \\
&+ [dcba] - <c, abd> - <d, abc> - <1, abcd> = \\
&= [abcd] + [cdab] + [abdc] + [dcab] + [bacd] + \\
&+ [cdba] + [badc] + [dcba] - 4<a, bcd> - 4<b, acd> - \\
&- 4<c, abd> - 4<d, abc> - 8<1, abcd> = \\
&= (((ab)c)d) + (((cd)a)b) + (((ab)d)c) + (((dc)a)b) + \\
&+ (((ba)c)d) + (((cd)b)a) + (((ba)d)c) + (((dc)b)a) - \\
&- 4[a, (b, c, d)] - 4[b, (a, c, d)] - 4[c, (a, b, d)] - \\
&- 4[d, (a, b, c)] - 8[(a, b, c), d]
\end{aligned}
$$

Theorem 3.2.10 *By definition of $\{a, b\} = ab + ba$ and proposition (3.3.1) we have*

$$
\begin{aligned}
\{\{\{\{a, b\}, c\}, d\}, e\} &= [abcde] + [bacde] + \\
&+ [cabde] + [cbade] + [dabce] + [dbace] + \\
&+ [dcabe] + [dcbae] + [eabcd] + [ebacd] + \\
&+ [ecabd] + [ecbad] + [edabc] + [edbac] + \\
&+ [edcab] + [edcba] - 20<a, bcde> - 20<b, acde> - \\
&- 20<c, abde> - 20<d, abce> - 20<e, abcd> - \\
&- 8<ab, cde> - 8<ac, bde> - 8<ad, bce> - 8<ae, bcd> - \\
&- 8<bc, ade> - 8<bd, ace> - 8<be, acd> - 8<cd, abe> - \\
&- 8<ce, abd> - 8<de, abc> - 34<1, abcde>
\end{aligned}
$$

when our multiply elements of on left side we use formula 5 from proposition (3.3.1)

Proposition 3.2.11 *By proposition (3.3.1) our $[e] \circ [abcd]$ is equal*

$$
\begin{aligned}
[e] \circ [abcd] &= [eabcd] - 3 < 1, abcde > - 2 < a, bcde > - \\
&- 2<b, acde> - 2<c, abde> - 2<d, abce> - <ab, cde> - \\
&- <ac, bde> - <ad, bce> - <bc, ade> - <bd, ace> - <cd, abe>
\end{aligned}
$$

Proof 3.2.12 *By definition of $\{a, b\} = ab + ba$ and proposition (3.3.1) we have*

$$
\begin{aligned}
\{\{\{\{a, b\}, c\}, d\}, e\} &= ((((ab)c)d)e) + ((((ba)c)d)e) + \\
&+ (((c(ab))d)e) + (((c(ba))d)e) + ((d((ab)c))e) + \\
&+ ((d((ba)c))e) + ((d(c(ab)))e) + ((d(c(ba)))e) + \\
&+ (e(((ab)c)d)) + (e((ba)c)d) + (e(c(ab))d) + \\
&+ (e((c(ba))d)) + (e(d((ab)c))) + (e(d((ba)c))) + \\
&+ (e(d(c(ab)))) + (e(d(c(ba))))
\end{aligned}
$$

Then By theorem (3.3.2) and propositoion (3.3.11) we can show $\{\{\{\{a, b\}, c\}, d\}, e\}$

$$\{\{\{\{a,b\},c\},d\},e\} = \{(((ab)c)d) + (((ba)c)d) +$$
$$+ (((ca)b)d) + (((cb)a)d) + (((da)b)c) + (((db)a)c) +$$
$$+ (((dc)a)b) + (((dc)b)a) - 10[(a,\ b,\ c),\ d] -$$
$$- 4[a\ (b,\ c,\ d)] - 4[b\ (a,\ c,\ d)] - 4[c\ (a,\ b,\ d)] -$$
$$- 4[d\ (a,\ b,\ c)],\ e\} = \{[abcd] + [bacd] +$$
$$+ [cabd] + [cbad] + [dabc] + [dbac] +$$
$$+ [dcab] + [dcba] - 10<1,\ abcd> - 4<a,\ bcd> -$$
$$- 4<b,\ acd> - 4<c,\ abd> - 4<d,\ abc>,\ e\} =$$
$$= [abcd] \circ[e] + [bacd] \circ [e] + [cabd] \circ [e] +$$
$$+ [cbad] \circ[e] + [dabc] \circ [e] + [dbac] \circ [e] +$$
$$+ [dcab] \circ[e] + [dcba] \circ [e] - 10 < 1, abcd > \circ[e] -$$
$$- 4<a,\ bcd> \circ[e] - 4 < b, acd > \circ[e] -$$
$$- 4<c,\ abd> \circ[e] - 4 < d, abc > \circ[e] +$$
$$+ [e] \circ[abcd] + [e] \circ [bacd] + [e] \circ [cabd] +$$
$$+ [e] \circ[cbad] + [e] \circ [dabc] + [e] \circ [dbac] +$$
$$+ [e] \circ[dcab] + [e] \circ [dcba] + 10[e]\circ < 1, abcd > -$$
$$- 4[e] \circ < a, bcd > -4[e]\circ < b, acd > -$$
$$- 4[e] \circ < c, abd > -4[e]\circ < d, abc >=$$
$$= [abcde] + [bacde] + [cabde] + [cbade] +$$
$$+ [dabce] + [dbace] + [dcabe] + [dcbae] -$$
$$- 10<1,\ abcde> - 10<e,\ abcd> - 4<a,\ bcde> -$$
$$- 4<b,\ acde> - 4<c,\ abde> - 4<d,\ abce> - 4<ae,\ bcd> +$$
$$- 4<be,\ acd> - 4<ce,\ abd> - 4<de,\ abc> +$$
$$+ [eabcd] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> -$$
$$- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> +$$
$$+ [ebacd] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> -$$
$$- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> +$$
$$+ [ecabd] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - - <ab,\ cde> - <ac,\ bde> -$$
$$- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> +$$
$$+ [ecbad] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> -$$
$$- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> +$$
$$+ [edabc] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> -$$
$$- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> +$$
$$+ [edbac] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> -$$
$$- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> +$$
$$+ [edcab] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> -$$

$$
\begin{gathered}
- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> + \\
+ [edcba] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> - \\
- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> - \\
- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> + \\
-10<e,\ abcd> - 4<ae,\ bcd> - 4<be,\ acd> - \\
- 4<ce,\ abd> - 4<de,\ abc> = \\
= [abcde] + [bacde] + [cabde] + [cbade] + \\
+ [dabce] + [dbace] + [dcabe] + [dcbae] + \\
+ [eabcd] + [ebacd] + [ecabd] + [ecbad] + \\
+ [edabc] + [edbac] + [edcab] + [edcba] + \\
- 20<a,\ bcde> - 20<b,\ acde> - 20<c,\ abde> - \\
- 20<d,\ abce> - 20<e,\ abcd> - 8<ab,\ cde> - \\
- 8<ac,\ bde> - 8<ad,\ bce> - 8<ae,\ bcd> - \\
- 8<bc,\ ade> - 8<bd,\ ace> - 8<be,\ acd> - \\
- 8<cd,\ abe> - 8<ce,\ abd> - 8<de,\ abc> - \\
- 34<1,\ abcde>
\end{gathered}
$$

Theorem 3.2.13 *By definition of $\{a,\ b\} = ab + ba$ and proposition (3.3.1) we have*

$$
\begin{gathered}
\{\{\{a,\ b\},\ c\},\ \{d,\ e\}\} = [abcde] + [bacde] + \\
+ [cabde] + [cbade] + [deabc] + [debac] + \\
+ [decab] + [decba] + [abced] + [baced] + \\
+ [cabed] + [cbaed] + [edabc] + [edbac] + \\
+ [edcab] + [edcba] - 28<1,\ abcde> - \\
- 20<d,\ abce> - 20<e,\ abcd> - 16<a,\ bcde> - \\
- 16<b,\ acde> - 16<c,\ abde> - 8<ad,\ bce> - \\
- 8<ae,\ bcd> - 8<bd,\ ace> - 8<be,\ acd> - \\
- 8<cd,\ abe> - 8<ce,\ abd> - 8\ <de,\ abc
\end{gathered}
$$

Proposition 3.2.14 *By proposition (3.3.1) we have,*

$$
\begin{gathered}
[abc] \circ [de] = [abcde] - <1,\ abcde> - \\
- <a,\ bcde> - <b,\ acde> - <c,\ abde> - \\
- <ab,\ cde> - <ac,\ bde> - <bc,\ ade>
\end{gathered}
$$

Proposition 3.2.15 *By proposition (3.3.1) we have,*

$$
\begin{gathered}
[de] \circ [abc] = [deabc] - 2<1,\ abcde> - \\
- 2<d,\ abce> - 2<e,\ abcd> - <a,\ bcde> - \\
- <b,\ acde> - <c,\ abde> - <ad,\ bce> - <ae,\ bcd> - \\
- <bd,\ ace> - <be,\ acd> - <cd,\ abe> - <ce,\ abd>
\end{gathered}
$$

Proposition 3.2.16 *By proposition (3.3.1) we have,*

$$<1, abc> \circ [de] = <1, abcde> + + <d, abce> + <e, abcd> + <de, abc>$$

Proposition 3.2.17 *By proposition (3.3.1) we have,*

$$[de] \circ <1, abc> = <de, abc>$$

Proof 3.2.18 *By definition of* $\{a, b\} = ab + ba$ *and proposition (3.3.1) we have*

$$\{\{\{a, b\}, c\}, \{d, e\}\} = (((ab)c)(de)) + (((ba)c)(de)) + + (((c(ab))(de)) + (((c(ba))(de)) + ((de)\ ((ab)c)) + + ((de)((ba)c)) + ((de)((c(ab))) + ((de)((c(ba))) + + ((ed)((ab)c)) + ((ed)((ba)c)) + ((ed)(c(ab))) + + ((ed)(c(ba))) + (((ab)c)(ed)) + (((ba)c)(ed)) + + ((c(ab))(ed)) + ((c(ba))(ed))$$

Then by proposition (3.3.14), (3.3.15) and (3.3.16) we can showing $\{\{\{a, b\}, c\}, \{d, e\}\}$,

$$\{\{\{a, b\}, c\}, \{d, e\}\} = \{((ab)c) + ((ba)c) + + ((ca)b) + ((cb)a) - 2(a,b,c),\ de+ed\} = = [abc] \circ [de] + [bac] \circ [de] + [cab] \circ [de] + + [cba] \circ [de] - 2<1, abc> \circ [de] + [de] \circ [abc] + + [de] \circ [bac] + [de] \circ [cab] + [de] \circ [cba] - - 2[de] \circ <1, abc> + [abc] \circ [ed] + [bac] \circ [ed] + + [cab] \circ [ed] + [cba] \circ [ed] - 2<1, abc> \circ [ed] + + [ed] \circ [abc] + [ed] \circ [bac] + [ed] \circ [cab] + + [ed] \circ [cba] - 2[ed] \circ <1, abc> = = [abcde] - <1, abcde> - <a, bcde> - <b, acde> - - <c, abde> - <ab, cde> - <ac, bde> - <bc, ade> + + [bacde] - <1, abcde> - <a, bcde> - <b, acde> - - <c, abde> - <ab, cde> - <ac, bde> - <bc, ade> + + [cabde] - <1, abcde> - <a, bcde> - <b, acde> - - <c, abde> - <ab, cde> - <ac, bde> - <bc, ade> + + [cbade] - <1, abcde> - <a, bcde> - <b, acde> - - <c, abde> - <ab, cde> - <ac, bde> - <bc, ade> - - 2<1, abcde> - 2<d, abce> - 2<e, abcd> - 2<de, abc> + + [deabc] - 2<1, abcde> - 2<d, abce> - - 2<e, abcd> - <a, bcde> - <b, acde> - - <c, abde> - <ad, bce> - <ae, bcd> - - <bd, ace> - <be, acd> - <cd, abe> - <ce, abd> + + [debac] - 2<1, abcde> - 2<d, abce> - - 2<e, abcd> - <a, bcde> - <b, acde> - - <c, abde> - <ad, bce> - <ae, bcd> -$$

- <bd, ace> - <be, acd> - <cd, abe> - <ce, abd> +
+ [decab] - 2<1, abcde> - 2<d, abce> -
- 2<e, abcd> - <a, bcde> - <b, acde> -
- <c, abde> - <ad, bce> - <ae, bcd> -
- <bd, ace> - <be, acd> - <cd, abe> - <ce, abd> +
+ [decba] - 2<1, abcde> - 2<d, abce> -
- 2<e, abcd> - <a, bcde> - <b, acde> -
- <c, abde> - <ad, bce> - <ae, bcd> -
- <bd, ace> - <be, acd> - <cd, abe> -
- <ce, abd> - 2 <de, abc> +
+ [abced] - <1, abcde> - <a, bcde> -
- <b, acde> -<c, abde> - <ab, cde> -
- <ac, bde> - <bc, ade> + [baced] -
- <1, abcde> - <a, bcde> - <b, acde> -
- <c, abde> - <ab, cde> - <ac, bde> -
- <bc, ade> + [cabed] - <1, abcde> -
- <a, bcde> - <b, acde> - <c, abde> -
- <ab, cde> - <ac, bde> - <bc, ade> +
+ [cbaed] - <1, abcde> - <a, bcde> -
- <b, acde> - <c, abde> - <ab, cde> -
-<ac, bde> - <bc, aed> -
- 2<1, abcde> - 2<d, abce> - 2<e, abcd> -
- 2<de, abc> + [edabc] - 2<1, abcde> -
- 2<d, abce> - 2<e, abcd> - <a, bcde> -
- <b, acde> - <c, abde> - <ad, bce> -
- <ae, bcd> - <bd, ace> - <be, acd> -
- <cd, abe> - <ce, abd> + [edbac] -
- 2<1, abcde> - 2<d, abce> - 2<e, abcd> -
- <a, bcde> - <b, acde> - <c, abde> -
- <ad, bce> - <ae, bcd> - <bd, ace> -
- <be, acd> - <cd, abe> - <ce, abd> +
+ [edcab] - 2<1, abcde> - 2<d, abce> -
- 2<e, abcd> - <a, bcde> - <b, acde> -
- <c, abde> - <ad, bce> - <ae, bcd> -
- <bd, ace> - <be, acd> - <cd, abe> -
- <ce, abd> + [edcba] - 2<1, abcde> -
- 2<d, abce> - 2<e, abcd> - <a, bcde> -
- <b, acde> - <c, abde> - <ad, bce> -
- <ae, bcd> - <bd, ace> - <be, acd> -
- <cd, abe> - <ce, abd> - 2 <de, abc> =
= [abcde] + [bacde] + [cabde] + [cbade] +
+ [deabc] + [debac] + [decab] + [decba] +
+ [abced] + [baced] + [cabed] + [cbaed] +

$$+ [edabc] + [edbac] + [edcab] + [edcba] -$$
$$- 28<1, abcde> - 20<d, abce> -$$
$$- 20<e, abcd> - 16<a, bcde> -$$
$$- 16<b, acde> - 16<c, abde> - 8<ad, bce> -$$
$$- 8<ae, bcd> - 8<bd, ace> - 8<be, acd> -$$
$$- 8<cd, abe> - 8<ce, abd> - 8 <de, abc>$$

Theorem 3.2.19 *By definition of* $\{a, b\} = ab + ba$ *and proposition (3.3.1) we have*

$$\{\{\{a, b\}, \{c, d\}\}, e\} = [abcde] +$$
$$+ [cdabe] + [abdce] + [dcabe] + [bacde] +$$
$$+ [cdbae] + [badce] + [dcbae] + [eabcd] +$$
$$+ [ecdab] + [eabdc] + [edcab] + [ebacd] +$$
$$+ [ecdba] + [ebadc] + [edcba] -$$
$$- 32<1, abcde> - 20<a, bcde> - 20<b, acde> -$$
$$- 20<c, abde> - 20<d, abce> - 16<e, abcd> -$$
$$- 8<ab, cde> - 8<ac, bde> - 8<ad, bce> -$$
$$- 8<bc, ade> - 8<bd, ace> - 8<cd, abe> -$$
$$- 8<ae, bcd> - 8<be, acd> - 8<ce, abd> -$$
$$- 8<de, abc> - 16<e, abcd>$$

Proof 3.2.20 *By definition of* $\{a, b\} = ab + ba$ *and proposition (3.3.1) we have*

$$\{\{\{a, b\}, \{c, d\}\}, e\} = (((ab)(cd))e) + (((ba)(cd))e) +$$
$$+ (((ab)(dc))e) + (((ab)(cd))e) + (((dc)(ab))e) +$$
$$+ (((ba)(dc))e) + (((ba)(cd))e) + (((dc)(ba))e) +$$
$$+ (e((ab)(cd))) + (e((ba)(cd))) + (e((ab)(dc))) +$$
$$+ (e((ab)(cd))) + (e((dc)(ab))) + (e((ba)(dc))) +$$
$$+ (e((ba)(cd))) + (e((dc)(ba)))$$

When we have $\{\{\{a, b\}, \{c, d\}\}, e\}$ *we using theorem (3.3.2) and proposition (3.3.1) we can write*

$$\{\{\{a, b\}, \{c, d\}\}, e\} = \{((((ab)c)d) + (((cd)a)b) +$$
$$+ (((ab)d)c) + (((dc)a)b) + (((ba)c)d) +$$
$$+ (((cd)b)a) + (((ba)d)c) + (((dc)b)a) -$$
$$- 4[a, (b, c, d)] - 4[b, (a, c, d)] - 4[c, (a, b, d)] -$$
$$- 4[d, (a, b, c)] - 8[(a, b, c), d], e\} =$$
$$= [abcd] \circ [e] + [cdab] \circ [e] + [abdc] \circ [e] +$$
$$+ [dcab] \circ [e] + [bacd] \circ [e] + [cdba] \circ [e] +$$
$$+ [badc] \circ [e] + [dcba] \circ [e] - 4<a, bcd> \circ [e] -$$
$$- 4<b, acd> \circ [e] - 4<c, abd> \circ [e]- 4<d, abc> \circ [e] -$$
$$- 8<1, abcd> \circ [e] + [e] \circ [abcd] + [e] \circ [cdab] +$$

$$+ [e] \circ [abdc] + [e] \circ [dcab] + [e] \circ [bacd] +$$
$$+ [e] \circ [cdba] + [e] \circ [badc] + [e] \circ [dcba] -$$
$$- 4[e] \circ \langle a, bcd\rangle - 4[e] \circ \langle b, acd\rangle - 4[e] \circ \langle c, abd\rangle -$$
$$- 4[e] \circ \langle d, abc\rangle - 8[e] \circ \langle 1, abcd\rangle =$$
$$= [abcde] + [cdabe] + [abdce] + [dcabe] +$$
$$+ [bacde] + [cdbae] + [badce] + [dcbae] +$$
$$- 4\langle a, bcde\rangle - 4\langle ae, bcd\rangle - 4\langle be, acd\rangle -$$
$$- 4\langle b, acde\rangle - 4\langle c, abde\rangle - 4\langle ce, abd\rangle -$$
$$- 4\langle d, abce\rangle - 4\langle de, abc\rangle - 8\langle 1, abcde\rangle -$$
$$- 8\langle e, abcd\rangle + [eabcd] - 3\langle 1, abcde\rangle -$$
$$- 2\langle a, bcde\rangle - 2\langle b, acde\rangle - 2\langle c, abde\rangle -$$
$$- 2\langle d, abce\rangle - \langle ab, cde\rangle - \langle ac, bde\rangle -$$
$$- \langle ad, bce\rangle - \langle bc, ade\rangle - \langle bd, ace\rangle -$$
$$- \langle cd, abe\rangle + [ecdab] - 3\langle 1, abcde\rangle -$$
$$- 2\langle a, bcde\rangle - 2\langle b, acde\rangle - 2\langle c, abde\rangle -$$
$$- 2\langle d, abce\rangle - \langle ab, cde\rangle - \langle ac, bde\rangle -$$
$$- \langle ad, bce\rangle - \langle bc, ade\rangle - \langle bd, ace\rangle -$$
$$- \langle cd, abe\rangle + [eabdc] - 3\langle 1, abcde\rangle -$$
$$- 2\langle a, bcde\rangle - 2\langle b, acde\rangle - 2\langle c, abde\rangle -$$
$$- 2\langle d, abce\rangle - \langle ab, cde\rangle - \langle ac, bde\rangle -$$
$$- \langle ad, bce\rangle - \langle bc, ade\rangle - \langle bd, ace\rangle -$$
$$- \langle cd, abe\rangle + [edcab] - 3\langle 1, abcde\rangle -$$
$$- 2\langle a, bcde\rangle - 2\langle b, acde\rangle - 2\langle c, abde\rangle -$$
$$- 2\langle d, abce\rangle - \langle ab, cde\rangle - \langle ac, bde\rangle -$$
$$- \langle ad, bce\rangle - \langle bc, ade\rangle - \langle bd, ace\rangle -$$
$$- \langle cd, abe\rangle + [ebacd] - 3\langle 1, abcde\rangle -$$
$$- 2\langle a, bcde\rangle - 2\langle b, acde\rangle - 2\langle c, abde\rangle -$$
$$- 2\langle d, abce\rangle - \langle ab, cde\rangle - \langle ac, bde\rangle -$$
$$- \langle ad, bce\rangle - \langle bc, ade\rangle - \langle bd, ace\rangle -$$
$$- \langle cd, abe\rangle + [ecdba] - 3\langle 1, abcde\rangle -$$
$$- 2\langle a, bcde\rangle - 2\langle b, acde\rangle - 2\langle c, abde\rangle -$$
$$- 2\langle d, abce\rangle - \langle ab, cde\rangle - \langle ac, bde\rangle -$$
$$- \langle ad, bce\rangle - \langle bc, ade\rangle - \langle bd, ace\rangle -$$
$$- \langle cd, abe\rangle + [ebadc] - 3\langle 1, abcde\rangle -$$
$$- 2\langle a, bcde\rangle - 2\langle b, acde\rangle - 2\langle c, abde\rangle -$$
$$- 2\langle d, abce\rangle - \langle ab, cde\rangle - \langle ac, bde\rangle -$$
$$- \langle ad, bce\rangle - \langle bc, ade\rangle - \langle bd, ace\rangle -$$
$$- \langle cd, abe\rangle + [edcba] - 3\langle 1, abcde\rangle -$$
$$- 2\langle a, bcde\rangle - 2\langle b, acde\rangle - 2\langle c, abde\rangle -$$
$$- 2\langle d, abce\rangle - \langle ab, cde\rangle - \langle ac, bde\rangle -$$
$$- \langle ad, bce\rangle - \langle bc, ade\rangle - \langle bd, ace\rangle -$$
$$- \langle cd, abe\rangle - 4\langle ae, bcd\rangle - 4\langle be, acd\rangle -$$
$$- 4\langle ce, abd\rangle - 4\langle de, abc\rangle - 8\langle e, abcd\rangle =$$

= [abcde] + [cdabe] + [abdce] + [dcabe] +
+ [bacde] + [cdbae] + [badce] + [dcbae] +
+ [eabcd] + [ecdab] + [eabdc] + [edcab] +
+ [ebacd] + [ecdba] + [ebadc] + [edcba] -
- 32<1, abcde> - 20<a, bcde> - 20<b, acde> -
- 20<c, abde> - 20<d, abce> - 16<e, abcd> -
- 8<ab, cde> - 8<ac, bde> - 8<ad, bce> -
- 8<bc, ade> - 8<bd, ace> - 8<cd, abe> -
- 8<ae, bcd> - 8<be, acd> - 8<ce, abd> -
- 8<de, abc> - 16<e, abcd>

$S'_n = \{\sigma \in S_n | \sigma(1) > \sigma(2) \ldots > \sigma(l) < \sigma(l+1) < \sigma(l+2) \ldots < \sigma(n), 1 \leq l \leq n\}$

S' is the set of ordered expression elements generated by X.

$$\{\ldots\{\{x_1, x_2\}, x_3\}\ldots x_n\} = \sum_{\sigma \in S'_n} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \ldots x_{\sigma(n)} -$$

$$-\sum_{k=3}^{n} 2^{k-2}(1 + 2^{n-k}(n-k)) \sum_{\substack{u = u_1 u_2 = x_1 x_2 \ldots x_n \\ |u_2| = n-k+3}} \langle u_1, u_2 \rangle,$$

where the length of u is equal to k.

3.3 JORDAN ELEMENTS IN ASSOSYMMETRIC ALGEBRAS N 6

Assosymmetric algebras are defined by the following left-symmetric and right-symmetric identities:

$$(x, y, z) = (x, z, y), \quad (x, y, z) = (y, x, z), \tag{3.24}$$

where $(x, y, z) = (xy)z - x(yz)$. The variety of assosymmetric algebras admits the commutative associative and associative algebras as a subvariety.

Let $A^{(+)} = (A, \{,\})$ be a subalgebra of A under Jordan product (anticommutator) i.e. multiplication is defined by the anticommutator product $\{a, b\} = ab + ba$.

Let $\mathcal{A}$ be a variety of assosymmetric algebras. We define $\mathcal{A}^{(+)}$ as a class of algebras of the form $A^{(+)}$, where $A \in \mathcal{A}$. In [**dzhumadil2018assosymmetric**], established that any assosymmetric algebra under Jordan product satises three polynomial identities

$$\{a, b\} = \{b, a\} \textit{ (commutativity)},$$

$$\{a, \{b, b\}, c\} = 2\{b, \{a, b, c\}\} \textit{ (Lie triple)},$$

and Glennie identity of degree 8.

Let $X = \{x_1, x_2, \ldots\}$ be a set and $A(X)$ be a free assosymmetric algebra generated by X. A polynomial in $A(X)$ is called *Jordan* element of $A(X)$ if it can be expressed by elements of X in terms of anticommutators. For associative algebras P.M. Cohn gave a criterion for the Jordan elements generated by three elements [3]. In addition, he gave a criterion of homomorphic images to be special and showed that the homomorphic image of the free special Jordan algebra generated by three elements is exceptional.

Let $A(X)$ be a free assosymmetric algebra on X. Denote by $[x_{i_1}x_{i_2}\cdots x_{i_n}]$ or $x_{i_1}x_{i_2}\cdots x_{i_n}$ a *left-normed element* $(\cdots(x_{i_1}x_{i_2})x_{i_3}\cdots)x_{i_n}$ for $x_{i_1}, \ldots, x_{i_n} \in X$. The element

$$\langle x_{i_1}\cdots x_{i_n}, y_{i_1}\cdots y_{i_m}\rangle =$$

$$x_{i_1}(x_{i_2}(\cdots(x_{i_n}[\ldots[(y_{i_1}, y_{i_2}, y_{i_3}), y_{i_4}]\cdots, y_{i_m}]\cdots))$$

is called *ordered expression*, where $x_{i_1}, \ldots x_{i_n}, y_{i_1}, \ldots, y_{i_m} \in X$ and we have order $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n}$ and $y_{i_1} \leq y_{i_2} \leq \cdots \leq y_{i_m}$. In [4] was proved that the set of left-normed and ordered expression elements forms a basis for the free assosymmetric algebra $A(X)$ over field of characteristic $\neq 2, 3$. A multiplication rule of the base elements was given in [4],here we give it as proposition. Some times x_{i_1} = "a", x_{i_2} = "b", x_{i_3} = "c", x_{i_4} = "d", x_{i_5} = "e", here we give it as proposition.

Proposition 3.3.1

$$\langle u_1, v_1\rangle \circ \langle u_1, v_1\rangle = 0, \tag{3.25}$$

$$[x] \circ \langle u, v\rangle = \langle xu, v\rangle, \tag{3.26}$$

$$\langle u, v\rangle \circ [x] = \sum_{x=x_1x_2} \langle x_1u, x_2v\rangle, \tag{3.27}$$

$$[x] \circ [y] = [xy] - \sum_{\substack{x = x_1x_2 \\ |x_2| \geq 1 \\ y = y_1y_2 \\ |y_2| \geq 2}} (|y_2 - 1|)\langle x_1y_1, x_2y_2\rangle. \tag{3.28}$$

Every reversible element u $\in F(\{\mathrm{x}_1, x_2, x_3\})$, with length 4 is Jordan element in a free assosymmetric algebra A($\{x_1, x_2, x_3\}$).

A free assosymmetric algebra $A(X)$ in degree 3 has a 7-dimensional polylinear part with a basis

$$\{x_1x_2x_3, x_1x_3x_2, x_2x_1x_3, x_2x_3x_1, x_3x_1x_2, x_3x_2x_1, (x_1, x_2, x_3)\}.$$

By the Propostion (3.3.1), other 6 right-bracketed elements are linear combinations of basic elements:

$$(x_{i_1}, x_{i_2}, x_{i_3}) = x_{i_1}x_{i_2}x_{i_3} - x_{i_1}(x_{i_2}x_{i_3}), \tag{3.29}$$

$$x_{i_1}(x_{i_2}x_{i_3}) = x_{i_1}x_{i_2}x_{i_3} - (x_{i_1}, x_{i_2}, x_{i_3}),$$

Theorem 3.3.2 *From using proposition (3.3.1) and next propositions (3.30),(3.31) and (3.32) we can find* $\{\{\{a,\ b\},\ c\},\ d\}$,

$$\{\{\{a,\ b\},\ c\},\ d\} = (((ab)c)d) + (((ba)c)d) + (((ca)b)d) + \\ + (((cb)a)d) + (((da)b)c) + (((db)a)c) + (((dc)a)b) + \\ + (((dc)b)a) - 10[(a,\ b,\ c),\ d] - 4[a,\ (b,\ c,\ d] - \\ - 4[b,\ (a,\ c,\ d)] - 4[c,\ (a,\ b,\ d)] - 4[d,\ (a,\ b,\ c)]$$

Proposition 3.3.3 *By propostion (3.3.1), we can find* $((a,b,c)d)$,

$$((a,b,c)d) = <1,\ abc> \circ[d] =< 1, abcd > + \\ + <d,\ abc> = [(a,\ b,\ c),\ d] + [d\ (a,\ b,\ c)].$$

There is we can take,

$$< 1, abc > \circ[d] = [(a, b, c), d] + [d(a, b, c)] \tag{3.30}$$

Proposition 3.3.4 *By propostion (3.3.1), there is our* $(d(a,b,c)) = [d]$ *is equal*

$$(d(a, b, c)) = [d]\circ < 1, abc >=< d, abc >= [d(a, b, c)]$$

There is we can take [d] $\circ < 1, abc >$,

$$[d]\circ < 1, abc >= [d(a, b, c)] \tag{3.31}$$

Proposition 3.3.5 *By propostion (3.3.1), we can write* $(d((ab)c) = [d] \circ [abc]$ *and this is,*

$$(d((ab)c) = [d] \circ[abc] = [dabc] - 2 < 1, abcd > - \\ - <a,\ bcd> - <b,\ acd> - <c,\ abd> = (((da)b)c) - \\ - 2[(a,\ b,\ c),\ d] - (a(b,\ c,\ d)) - (b(a,\ c,\ d)) - \\ - (c(a,\ b,\ c)) = (((da)b)c) - 2[(a,\ b,\ c),\ d] - \\ - [a,\ (b,\ c,\ d)] - [b,\ (a,\ c,\ d)] - [c,\ (a,\ b,\ d)]$$

There is $[d] \circ [abc]$ is equal

$$[d] \circ [abc] = (((da)b)c) - 2[(a, b, c), d] - [a, (b, c, d)] - [b, (a, c, d)] - [c, (a, b, d)] \tag{3.32}$$

Proof 3.3.6 *By definition of* $\{a,\ b\} = ab + ba$ *and proposition (3.3.1) we have*

$$\{\{\{a,\ b\},\ c\},\ d\} = (((ab)c)d) + (((ba)c)d) + \\ + (((c(ab))d) + ((c(ba))d) + (d((ab)c)) + \\ + (d((ba)c)) + (d(c(ab))) + (d(c(ba)))$$

So when we have $\{\{\{a,b\},c\},d\}$ we using (3.3.1) and propositions (3.30),(3.31) and (3.32) this one gived

$$\{\{\{a,b\},c\},d\} = \{((ab)c) + ((ba)c) + ((ca)b) + \\ + ((cb)a) - 2(a,b,c),\ d\} = [abc] + [bac] + [cab] + \\ + [cba] - 2<1,(abc)>,\ d = [abc] \circ [d] + [bac] \circ [d] + \\ + [cab] \circ [d] + [cba] \circ [d] \text{ - } 2 < 1, abc > \circ [d] + \\ + [d] \circ [abc] + [d] \circ [bac] + [d] \circ [cab] + [d] \circ [cba] \\ - 2[d] \circ < 1, abc >= [abcd] + [bacd] + [cabd] + \\ + [cbad] - 2<1,\ abcd> - 2<d,\ abc> + [dabc] - \\ - 2<1,\ abcd> - <a,\ bcd> - <b,\ acd> - <c,\ abd> + \\ + [dbac] - 2<1,\ abcd> - <a,\ bcd> - <b,\ acd> - \\ - <c,\ abd> + [dcab] - 2<1,\ abcd> - <a,\ bcd> - \\ - <b,\ acd> - <c,\ abd> + [dcba] - 2<1,\ abcd> - \\ - <a,\ bcd> - <b,\ acd> - <c,\ abd> - 2<d,\ abc> = \\ = (((ab)c)d) + (((ba)c)d) + (((ca)b)d) + (((cb)a)d) - \\ - 2[(a,\ b,\ c),\ d] - 2[d\ (a,\ b,\ c)] - 2[(a,\ b,\ c),\ d] + \\ + (((da)b)c) - [a\ (b,\ c,\ d)] - [b\ (a,\ c,\ d)] - \\ - [c\ (a,\ b,\ d)] + (((db)a)c) - 2[(a,\ b,\ c),\ d] - \\ - [a\ (b,\ c,\ d)] - [b\ (a,\ c,\ d)] - [c\ (a,\ b,\ d)] + \\ + (((dc)a)b) - 2[(a,\ b,\ c),\ d] - [a\ (b,\ c,\ d)] - \\ - [b\ (a,\ c,\ d)] - [c\ (a,\ b,\ d)] + (((dc)b)a) - \\ - 2[(a,\ b,\ c),\ d] - [a\ (b,\ c,\ d)] - [b\ (a,\ c,\ d)] - \\ - [c\ (a,\ b,\ d)] - 2[d\ (a,\ b,\ c)] = (((ab)c)d) + \\ + (((ba)c)d) + (((ca)b)d) + (((cb)a)d) + \\ + (((da)b)c) + (((db)a)c) + (((dc)a)b) + \\ + (((dc)b)a) - 10[(a,\ b,\ c),\ d] - 4[a,\ (b,\ c,\ d)] - \\ - 4[b,\ (a,\ c,\ d)] - 4[c,\ (a,\ b,\ d)] - 4[d,\ (a,\ b,\ c)]$$

Theorem 3.3.7 *By definition of $\{a,\ b\} = ab + ba$ and proposition (3.3.1) we have*

$$\{\{a,\ b\},\ \{c,\ d\}\} = (((ab)c)d) + (((cd)a)b) + \\ + (((ab)d)c) + (((dc)a)b) + (((ba)c)d) + (((cd)b)a) + \\ + (((ba)d)c) + (((dc)b)a) \text{ - } 4[a,\ (b,\ c,\ d)] \text{ - } 4[b,\ (a,\ c,\ d)] \text{ - } \\ \text{ - } 4[c,\ (a,\ b,\ d)] \text{ - } 4[d,\ (a,\ b,\ c)] \text{ - } 8[(a,\ b,\ c),\ d]$$

Proposition 3.3.8 *Proposition (3.3.1) we can find $(ab)(cd)$,*

$$(ab)(cd) = [ab] \circ [cd] = [abcd] - < a, bcd > - < b, acd > - < 1, abcd > \quad (3.33)$$

Proof 3.3.9 *By definition of $\{a,\ b\} = ab + ba$ and proposition (3.3.1) we have $\{\{a,\ b\},\ \{c,\ d\}\} = ((ab)(cd)) + ((ba)(cd)) + (ab)(dc) + (ab)(cd) + + ((dc)(ab)) + ((ba)(dc)) + ((ba)(cd)) + ((dc)(ba))$*

So when we know $a,\ b,\ c,\ d$ we can use proposition (3.3.1) and proposition (3.3.8) to proof

$$
\begin{aligned}
\{\{a, b\}, \{c, d\}\} &= (ab)(cd) + (cd)(ab) + (ab)(dc) + (dc)(ab) + \\
&+ (ba)(cd) + (cd)(ba) + (ba)(dc) + (dc)(ba) = \\
&= [abcd] - <a, bcd> - <b, acd> - <1, abcd> + \\
&+ [cdab] - <c, abd> - <d, abc> - <1, abcd> + \\
&+ [abdc] - <a, bcd> - <b, acd> - <1, abcd> + \\
&+ [dcab] - <c, abd> - <d, abc> - <1, abcd> + \\
&+ [bacd] - <a, bcd> - <b, acd> - <1, abcd> + \\
&+ [cdba] - <c, abd> - <d, abc> - <1, abcd> + \\
&+ [badc] - <a, bcd> - <b, acd> - <1, abcd> + \\
&+ [dcba] - <c, abd> - <d, abc> - <1, abcd> = \\
&= [abcd] + [cdab] + [abdc] + [dcab] + [bacd] + \\
&+ [cdba] + [badc] + [dcba] - 4<a, bcd> - 4<b, acd> - \\
&- 4<c, abd> - 4<d, abc> - 8<1, abcd> = \\
&= (((ab)c)d) + (((cd)a)b) + (((ab)d)c) + (((dc)a)b) + \\
&+ (((ba)c)d) + (((cd)b)a) + (((ba)d)c) + (((dc)b)a) - \\
&- 4[a, (b, c, d)] - 4[b, (a, c, d)] - 4[c, (a, b, d)] - \\
&- 4[d, (a, b, c)] - 8[(a, b, c), d]
\end{aligned}
$$

Theorem 3.3.10 *By definition of $\{a, b\} = ab + ba$ and proposition (3.3.1) we have*

$$
\begin{aligned}
\{\{\{\{a, b\}, c\}, d\}, e\} &= [abcde] + [bacde] + \\
&+ [cabde] + [cbade] + [dabce] + [dbace] + \\
&+ [dcabe] + [dcbae] + [eabcd] + [ebacd] + \\
&+ [ecabd] + [ecbad] + [edabc] + [edbac] + \\
&+ [edcab] + [edcba] - 20<a, bcde> - 20<b, acde> - \\
&- 20<c, abde> - 20<d, abce> - 20<e, abcd> - \\
&- 8<ab, cde> - 8<ac, bde> - 8<ad, bce> - 8<ae, bcd> - \\
&- 8<bc, ade> - 8<bd, ace> - 8<be, acd> - 8<cd, abe> - \\
&- 8<ce, abd> - 8<de, abc> - 34<1, abcde>
\end{aligned}
$$

when our multiply elements of on left side we use formula 5 from proposition (3.3.1)

Proposition 3.3.11 *By proposition (3.3.1) our $[e] \circ [abcd]$ is equal*

$$
\begin{aligned}
[e] \circ [abcd] &= [eabcd] - 3 < 1, abcde > - 2 < a, bcde > - \\
&- 2<b, acde> - 2<c, abde> - 2<d, abce> - <ab, cde> - \\
&- <ac, bde> - <ad, bce> - <bc, ade> - <bd, ace> - <cd, abe>
\end{aligned}
$$

Proof 3.3.12 *By definition of $\{a, b\} = ab + ba$ and proposition (3.3.1) we have*

$$
\begin{aligned}
\{\{\{\{a, b\}, c\}, d\}, e\} &= ((((ab)c)d)e) + ((((ba)c)d)e) + \\
&+ (((c(ab))d)e) + (((c(ba))d)e) + ((d((ab)c))e) + \\
&+ ((d((ba)c))e) + ((d(c(ab)))e) + ((d(c(ba)))e) +
\end{aligned}
$$

$$+ (e(((ab)c)d)) + (e((ba)c)d) + (e(c(ab))d) +$$
$$+ (e((c(ba))d)) + (e(d((ab)c))) + (e(d((ba)c))) +$$
$$+ (e(d(c(ab)))) + (e(d(c(ba))))$$

Then By theorem (3.3.2) and propositoion (3.3.11) we can show $\{\{\{\{a, b\}, c\}, d\}, e\}$

$$\{\{\{\{a,b\},c\},d\},e\} = \{((((ab)c)d) + (((ba)c)d) +$$
$$+ (((ca)b)d) + (((cb)a)d) + (((da)b)c) + (((db)a)c) +$$
$$+ (((dc)a)b) + (((dc)b)a) - 10[(a,\ b,\ c),\ d] -$$
$$- 4[a\ (b,\ c,\ d)] - 4[b\ (a,\ c,\ d)] - 4[c\ (a,\ b,\ d)] -$$
$$- 4[d\ (a,\ b,\ c)],\ e\} = \{[abcd] + [bacd] +$$
$$+ [cabd] + [cbad] + [dabc] + [dbac] +$$
$$+ [dcab] + [dcba] - 10<1,\ abcd> - 4<a,\ bcd> -$$
$$- 4<b,\ acd> - 4<c,\ abd> - 4<d,\ abc>,\ e\} =$$
$$= [abcd] \circ [e] + [bacd] \circ [e] + [cabd] \circ [e] +$$
$$+ [cbad] \circ [e] + [dabc] \circ [e] + [dbac] \circ [e] +$$
$$+ [dcab] \circ [e] + [dcba] \circ [e] - 10 < 1, abcd > \circ [e] -$$
$$- 4<a,\ bcd> \circ [e] - 4 < b, acd > \circ [e] -$$
$$- 4<c,\ abd> \circ [e] - 4 < d, abc > \circ [e] +$$
$$+ [e] \circ [abcd] + [e] \circ [bacd] + [e] \circ [cabd] +$$
$$+ [e] \circ [cbad] + [e] \circ [dabc] + [e] \circ [dbac] +$$
$$+ [e] \circ [dcab] + [e] \circ [dcba] + 10[e] \circ < 1, abcd > -$$
$$- 4[e] \circ < a, bcd > - 4[e] \circ < b, acd > -$$
$$- 4[e] \circ < c, abd > - 4[e] \circ < d, abc > =$$
$$= [abcde] + [bacde] + [cabde] + [cbade] +$$
$$+ [dabce] + [dbace] + [dcabe] + [dcbae] -$$
$$- 10<1,\ abcde> - 10<e,\ abcd> - 4<a,\ bcde> -$$
$$- 4<b,\ acde> - 4<c,\ abde> - 4<d,\ abce> - 4<ae,\ bcd> +$$
$$- 4<be,\ acd> - 4<ce,\ abd> - 4<de,\ abc> +$$
$$+ [eabcd] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> -$$
$$- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> +$$
$$+ [ebacd] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> -$$
$$- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> +$$
$$+ [ecabd] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - - <ab,\ cde> - <ac,\ bde> -$$
$$- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> +$$
$$+ [ecbad] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> -$$
$$- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> +$$
$$+ [edabc] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> -$$
$$- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> -$$

$$
\begin{gathered}
- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> + \\
+ [edbac] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> - \\
- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> - \\
- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> + \\
+ [edcab] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> - \\
- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> - \\
- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> + \\
+ [edcba] - 3<1,\ abcde> - 2<a,\ bcde> - 2<b,\ acde> - \\
- 2<c,\ abde> - 2<d,\ abce> - <ab,\ cde> - <ac,\ bde> - \\
- <ad,\ bce> - <bc,\ ade> - <bd,\ ace> - <cd,\ abe> + \\
-10<e,\ abcd> - 4<ae,\ bcd> - 4<be,\ acd> - \\
- 4<ce,\ abd> - 4<de,\ abc> = \\
= [abcde] + [bacde] + [cabde] + [cbade] + \\
+ [dabce] + [dbace] + [dcabe] + [dcbae] + \\
+ [eabcd] + [ebacd] + [ecabd] + [ecbad] + \\
+ [edabc] + [edbac] + [edcab] + [edcba] + \\
- 20<a,\ bcde> - 20<b,\ acde> - 20<c,\ abde> - \\
- 20<d,\ abce> - 20<e,\ abcd> - 8<ab,\ cde> - \\
- 8<ac,\ bde> - 8<ad,\ bce> - 8<ae,\ bcd> - \\
- 8<bc,\ ade> - 8<bd,\ ace> - 8<be,\ acd> - \\
- 8<cd,\ abe> - 8<ce,\ abd> - 8<de,\ abc> - \\
- 34<1,\ abcde>
\end{gathered}
$$

Theorem 3.3.13 *By definition of* $\{a,\ b\} = ab + ba$ *and proposition (3.3.1) we have*

$$
\begin{gathered}
\{\{\{a,\ b\},\ c\},\ \{d,\ e\}\} = [abcde] + [bacde] + \\
+ [cabde] + [cbade] + [deabc] + [debac] + \\
+ [decab] + [decba] + [abced] + [baced] + \\
+ [cabed] + [cbaed] + [edabc] + [edbac] + \\
+ [edcab] + [edcba] - 28<1,\ abcde> - \\
- 20<d,\ abce> - 20<e,\ abcd> - 16<a,\ bcde> - \\
- 16<b,\ acde> - 16<c,\ abde> - 8<ad,\ bce> - \\
- 8<ae,\ bcd> - 8<bd,\ ace> - 8<be,\ acd> - \\
- 8<cd,\ abe> - 8<ce,\ abd> - 8\ <de,\ abc
\end{gathered}
$$

Proposition 3.3.14 *By proposition (3.3.1) we have,*

$$
\begin{gathered}
[abc] \circ [de] = [abcde] - <1,\ abcde> - \\
- <a,\ bcde> - <b,\ acde> - <c,\ abde> - \\
- <ab,\ cde> - <ac,\ bde> - <bc,\ ade>
\end{gathered}
$$

Proposition 3.3.15 *By proposition (3.3.1) we have,*

$$\begin{aligned} [de] \circ [abc] = [deabc] - 2<1,\ abcde> - \\ - 2<d,\ abce> - 2<e,\ abcd> - <a,\ bcde> - \\ - <b,\ acde> - <c,\ abde> - <ad,\ bce> - <ae,\ bcd> - \\ - <bd,\ ace> - <be,\ acd> - <cd,\ abe> - <ce,\ abd> \end{aligned}$$

Proposition 3.3.16 *By proposition (3.3.1) we have,*

$$\begin{aligned} <1,\ abc> \circ [de] = <1,\ abcde> + \\ + <d,\ abce> + <e,\ abcd> + <de,\ abc> \end{aligned}$$

Proposition 3.3.17 *By proposition (3.3.1) we have,*

$$[de] \circ <1,\ abc> = <de,\ abc>$$

Proof 3.3.18 *By definition of* $\{a,\ b\} = ab + ba$ *and proposition (3.3.1) we have*

$$\begin{aligned} \{\{\{a,\ b\},\ c\},\ \{d,\ e\}\} = (((ab)c)(de)) + (((ba)c)(de)) + \\ + (((c(ab))(de)) + (((c(ba))(de)) + ((de)\ ((ab)c)) + \\ + ((de)((ba)c)) + ((de)((c(ab))) + ((de)((c(ba))) + \\ + ((ed)((ab)c)) + ((ed)((ba)c)) + ((ed)(c(ab))) + \\ + ((ed)(c(ba))) + (((ab)c)(ed)) + (((ba)c)(ed)) + \\ + ((c(ab))(ed)) + ((c(ba))(ed)) \end{aligned}$$

Then by proposition (3.3.14), (3.3.15) and (3.3.16) we can showing $\{\{\{a,\ b\},\ c\},$ $\{d,\ e\}\}$,

$$\begin{aligned} \{\{\{a,\ b\},\ c\},\ \{d,\ e\}\} = \{((ab)c) + ((ba)c) + \\ + ((ca)b) + ((cb)a) - 2(a,b,c),\ de+ed\} = \\ = [abc] \circ [de] + [bac] \circ [de] + [cab] \circ [de] + \\ + [cba] \circ [de] - 2<1,\ abc> \circ [de] + [de] \circ [abc] + \\ + [de] \circ [bac] + [de] \circ [cab] + [de] \circ [cba] - \\ - 2[de] \circ <1,\ abc> + [abc] \circ [ed] + [bac] \circ [ed] + \\ + [cab] \circ [ed] + [cba] \circ [ed] - 2<1,\ abc> \circ [ed] + \\ + [ed] \circ [abc] + [ed] \circ [bac] + [ed] \circ [cab] + \\ + [ed] \circ [cba] - 2[ed] \circ <1,\ abc> = \\ = [abcde] - <1,\ abcde> - <a,\ bcde> - <b,\ acde> - \\ - <c,\ abde> - <ab,\ cde> - <ac,\ bde> - <bc,\ ade> + \\ + [bacde] - <1,\ abcde> - <a,\ bcde> - <b,\ acde> - \\ - <c,\ abde> - <ab,\ cde> - <ac,\ bde> - <bc,\ ade> + \\ + [cabde] - <1,\ abcde> - <a,\ bcde> - <b,\ acde> - \\ - <c,\ abde> - <ab,\ cde> - <ac,\ bde> - <bc,\ ade> + \\ + [cbade] - <1,\ abcde> - <a,\ bcde> - <b,\ acde> - \\ - <c,\ abde> - <ab,\ cde> - <ac,\ bde> - <bc,\ ade> - \\ - 2<1,\ abcde> - 2<d,\ abce> - 2<e,\ abcd> - 2<de,\ abc> + \\ + [deabc] - 2<1,\ abcde> - 2<d,\ abce> - \end{aligned}$$

- 2<e, abcd> - <a, bcde> - <b, acde> -
- <c, abde> - <ad, bce> - <ae, bcd> -
- <bd, ace> - <be, acd> - <cd, abe> - <ce, abd> +
+ [debac] - 2<1, abcde> - 2<d, abce> -
- 2<e, abcd> - <a, bcde> - <b, acde> -
- <c, abde> - <ad, bce> - <ae, bcd> -
- <bd, ace> - <be, acd> - <cd, abe> - <ce, abd> +
+ [decab] - 2<1, abcde> - 2<d, abce> -
- 2<e, abcd> - <a, bcde> - <b, acde> -
- <c, abde> - <ad, bce> - <ae, bcd> -
- <bd, ace> - <be, acd> - <cd, abe> - <ce, abd> +
+ [decba] - 2<1, abcde> - 2<d, abce> -
- 2<e, abcd> - <a, bcde> - <b, acde> -
- <c, abde> - <ad, bce> - <ae, bcd> -
- <bd, ace> - <be, acd> - <cd, abe> -
- <ce, abd> - 2 <de, abc> +
+ [abced] - <1, abcde> - <a, bcde> -
- <b, acde> -<c, abde> - <ab, cde> -
- <ac, bde> - <bc, ade> + [baced] -
- <1, abcde> - <a, bcde> - <b, acde> -
- <c, abde> - <ab, cde> - <ac, bde> -
- <bc, ade> + [cabed] - <1, abcde> -
- <a, bcde> - <b, acde> - <c, abde> -
- <ab, cde> - <ac, bde> - <bc, ade> +
+ [cbaed] - <1, abcde> - <a, bcde> -
- <b, acde> - <c, abde> - <ab, cde> -
-<ac, bde> - <bc, aed> -
- 2<1, abcde> - 2<d, abce> - 2<e, abcd> -
- 2<de, abc> + [edabc] - 2<1, abcde> -
- 2<d, abce> - 2<e, abcd> - <a, bcde> -
- <b, acde> - <c, abde> - <ad, bce> -
- <ae, bcd> - <bd, ace> - <be, acd> -
- <cd, abe> - <ce, abd> + [edbac] -
- 2<1, abcde> - 2<d, abce> - 2<e, abcd> -
- <a, bcde> - <b, acde> - <c, abde> -
- <ad, bce> - <ae, bcd> - <bd, ace> -
- <be, acd> - <cd, abe> - <ce, abd> +
+ [edcab] - 2<1, abcde> - 2<d, abce> -
- 2<e, abcd> - <a, bcde> - <b, acde> -
- <c, abde> - <ad, bce> - <ae, bcd> -
- <bd, ace> - <be, acd> - <cd, abe> -
- <ce, abd> + [edcba] - 2<1, abcde> -
- 2<d, abce> - 2<e, abcd> - <a, bcde> -

$$
\begin{aligned}
&- <b, acde> - <c, abde> - <ad, bce> - \\
&- <ae, bcd> - <bd, ace> - <be, acd> - \\
&- <cd, abe> - <ce, abd> - 2 <de, abc> = \\
&= [abcde] + [bacde] + [cabde] + [cbade] + \\
&+ [deabc] + [debac] + [decab] + [decba] + \\
&+ [abced] + [baced] + [cabed] + [cbaed] + \\
&+ [edabc] + [edbac] + [edcab] + [edcba] - \\
&- 28<1, abcde> - 20<d, abce> - \\
&- 20<e, abcd> - 16<a, bcde> - \\
&- 16<b, acde> - 16<c, abde> - 8<ad, bce> - \\
&- 8<ae, bcd> - 8<bd, ace> - 8<be, acd> - \\
&- 8<cd, abe> - 8<ce, abd> - 8 <de, abc>
\end{aligned}
$$

Theorem 3.3.19 *By definition of $\{a, b\} = ab + ba$ and proposition (3.3.1) we have*

$$
\begin{aligned}
&\{\{\{a, b\}, \{c, d\}\}, e\} = [abcde] + \\
&+ [cdabe] + [abdce] + [dcabe] + [bacde] + \\
&+ [cdbae] + [badce] + [dcbae] + [eabcd] + \\
&+ [ecdab] + [eabdc] + [edcab] + [ebacd] + \\
&+ [ecdba] + [ebadc] + [edcba] - \\
&- 32<1, abcde> - 20<a, bcde> - 20<b, acde> - \\
&- 20<c, abde> - 20<d, abce> - 16<e, abcd> - \\
&- 8<ab, cde> - 8<ac, bde> - 8<ad, bce> - \\
&- 8<bc, ade> - 8<bd, ace> - 8<cd, abe> - \\
&- 8<ae, bcd> - 8<be, acd> - 8<ce, abd> - \\
&- 8<de, abc> - 16<e, abcd>
\end{aligned}
$$

Proof 3.3.20 *By definition of $\{a, b\} = ab + ba$ and proposition (3.3.1) we have*

$$
\begin{aligned}
&\{\{\{a, b\}, \{c, d\}\}, e\} = (((ab)(cd))e) + (((ba)(cd))e) + \\
&+ (((ab)(dc))e) + (((ab)(cd))e) + (((dc)(ab))e) + \\
&+ (((ba)(dc))e) + (((ba)(cd))e) + (((dc)(ba))e) + \\
&+ (e((ab)(cd))) + (e((ba)(cd))) + (e((ab)(dc))) + \\
&+ (e((ab)(cd))) + (e((dc)(ab))) + (e((ba)(dc))) + \\
&+ (e((ba)(cd))) + (e((dc)(ba)))
\end{aligned}
$$

When we have $\{\{\{a, b\}, \{c, d\}\}, e\}$ we using theorem (3.3.2) and proposition (3.3.1) we can write

$$
\begin{aligned}
&\{\{\{a, b\}, \{c, d\}\}, e\} = \{(((ab)c)d) + (((cd)a)b) + \\
&+ (((ab)d)c) + (((dc)a)b) + (((ba)c)d) + \\
&+ (((cd)b)a) + (((ba)d)c) + (((dc)b)a) - \\
&- 4[a, (b, c, d)] - 4[b, (a, c, d)] - 4[c, (a, b, d)] -
\end{aligned}
$$

$$
\begin{gathered}
- 4[d, (a, b, c)] - 8[(a, b, c), d], e\} = \\
= [abcd] \circ [e] + [cdab] \circ [e] + [abdc] \circ [e] + \\
+ [dcab] \circ [e] + [bacd] \circ [e] + [cdba] \circ [e] + \\
+ [badc] \circ [e] + [dcba] \circ [e] - 4<a, bcd> \circ [e] - \\
- 4<b, acd> \circ [e] - 4<c, abd> \circ [e] - 4<d, abc> \circ [e] - \\
- 8<1, abcd> \circ [e] + [e] \circ [abcd] + [e] \circ [cdab] + \\
+ [e] \circ [abdc] + [e] \circ [dcab] + [e] \circ [bacd] + \\
+ [e] \circ [cdba] + [e] \circ [badc] + [e] \circ [dcba] - \\
- 4[e] \circ <a, bcd> - 4[e] \circ <b, acd> - 4[e] \circ <c, abd> - \\
- 4[e] \circ <d, abc> - 8[e] \circ <1, abcd> = \\
= [abcde] + [cdabe] + [abdce] + [dcabe] + \\
+ [bacde] + [cdbae] + [badce] + [dcbae] + \\
- 4<a, bcde> - 4<ae, bcd> - 4<be, acd> - \\
- 4<b, acde> - 4<c, abde> - 4<ce, abd> - \\
- 4<d, abce> - 4<de, abc> - 8<1, abcde> - \\
- 8<e, abcd> + [eabcd] - 3<1, abcde> - \\
- 2<a, bcde> - 2<b, acde> - 2<c, abde> - \\
- 2<d, abce> - <ab, cde> - <ac, bde> - \\
- <ad, bce> - <bc, ade> - <bd, ace> - \\
- <cd, abe> + [ecdab] - 3<1, abcde> - \\
- 2<a, bcde> - 2<b, acde> - 2<c, abde> - \\
- 2<d, abce> - <ab, cde> - <ac, bde> - \\
- <ad, bce> - <bc, ade> - <bd, ace> - \\
- <cd, abe> + [eabdc] - 3<1, abcde> - \\
- 2<a, bcde> - 2<b, acde> - 2<c, abde> - \\
- 2<d, abce> - <ab, cde> - <ac, bde> - \\
- <ad, bce> - <bc, ade> - <bd, ace> - \\
- <cd, abe> + [edcab] - 3<1, abcde> - \\
- 2<a, bcde> - 2<b, acde> - 2<c, abde> - \\
- 2<d, abce> - <ab, cde> - <ac, bde> - \\
- <ad, bce> - <bc, ade> - <bd, ace> - \\
- <cd, abe> + [ebacd] - 3<1, abcde> - \\
- 2<a, bcde> - 2<b, acde> - 2<c, abde> - \\
- 2<d, abce> - <ab, cde> - <ac, bde> - \\
- <ad, bce> - <bc, ade> - <bd, ace> - \\
- <cd, abe> + [ecdba] - 3<1, abcde> - \\
- 2<a, bcde> - 2<b, acde> - 2<c, abde> - \\
- 2<d, abce> - <ab, cde> - <ac, bde> - \\
- <ad, bce> - <bc, ade> - <bd, ace> - \\
- <cd, abe> + [ebadc] - 3<1, abcde> - \\
- 2<a, bcde> - 2<b, acde> - 2<c, abde> - \\
- 2<d, abce> - <ab, cde> - <ac, bde> - \\
- <ad, bce> - <bc, ade> - <bd, ace> -
\end{gathered}
$$

$$
\begin{gathered}
- <cd, abe> + [edcba] - 3<1, abcde> - \\
- 2<a, bcde> - 2<b, acde> - 2<c, abde> - \\
- 2<d, abce> - <ab, cde> - <ac, bde> - \\
- <ad, bce> - <bc, ade> - <bd, ace> - \\
- <cd, abe> - 4<ae, bcd> - 4<be, acd> - \\
- 4<ce, abd> - 4<de, abc> - 8<e, abcd> = \\
= [abcde] + [cdabe] + [abdce] + [dcabe] + \\
+ [bacde] + [cdbae] + [badce] + [dcbae] + \\
+ [eabcd] + [ecdab] + [eabdc] + [edcab] + \\
+ [ebacd] + [ecdba] + [ebadc] + [edcba] - \\
- 32<1, abcde> - 20<a, bcde> - 20<b, acde> - \\
- 20<c, abde> - 20<d, abce> - 16<e, abcd> - \\
- 8<ab, cde> - 8<ac, bde> - 8<ad, bce> - \\
- 8<bc, ade> - 8<bd, ace> - 8<cd, abe> - \\
- 8<ae, bcd> - 8<be, acd> - 8<ce, abd> - \\
- 8<de, abc> - 16<e, abcd>
\end{gathered}
$$

Theorem 3.3.21 *By definition of $\{a, b\} = ab + ba$ and proposition (3.3.1) we have*

$$
\begin{gathered}
\{\{\{\{\{a, b\}, c\}, d\}, e\}, f\} = [abcdef] + [bacdef] + \\
+ [cabdef] + [cbadef] + [dabcef] + \\
+ [dbacef] + [dcabef] + [dcbaef] + \\
+ [eabcdf] + [ebacdf] + [ecabdf] + \\
+ [ecbadf] + [edabcf] + [edbacf] + \\
+ [edcabf] + [edcbaf] + [fabcde] + \\
+ [fbacde] + [fcabde] + [fcbade] + \\
+ [fdabce] + [fdbace] + [fdcabe] + \\
+ [fdcbae] + [feabcd] + [febacd] + \\
+ [fecabd] + [fecbad] + [fedabc] + \\
+ [fedbac] + [fedcab] + [edcbaf] - \\
- 50<1, abcdef> - 36<a, bcdef> - \\
- 36<b, acdef> - 36<c, abdef> - 36<d, abcef> - \\
- 36<e, abcdf> - 24<ab, cdef> - 24<ac, bdef> - \\
- 24<ad, bcef> - 24<ae, bcdf> - 24<bc, adef> - \\
- 24<bd, acef> - 24<be, acdf> - 24<cd, abef> - \\
- 24<ce, abdf> - 24<de, abcf> - 16<abc, def> - \\
- 16<abd, cef> - 16<abe, cdf> - 16<acd, bdf> - \\
- 16<ace, bdf> - 16<ade, bcf> - 16<bcd, aef> - \\
- 16<bce, adf> - 16<bde, acf> - 16<cde, abf>
\end{gathered}
$$

Proposition 3.3.22 *By proposition (3.3.1) we have,*

$$[abcde] \circ [f] = [abcdef]$$

Proposition 3.3.23 *By proposition (3.3.1) we have,*

$$\begin{gathered}
[f] \circ [abcde] = <1, abcdef> - <a, bcdef> - \\
- <b, acdef> - <c, abdef> - <d, abcef> - \\
- <e, abcdf> - <ab, cdef> - <ac, bdef> - \\
- <ad, bcef> - <ae, bcdf> - <bc, adef> - \\
- <bd, acef> - <be, acdf> - <cd, abef> - \\
- <ce, abdf> - <de, abcf> - <abc, def> - \\
- <abd, cef> - <abe, cdf> - <acd,bdf> - \\
- <ace, bdf> - <ade, bcf> - <bcd, aef> - \\
- <bce, adf> - <bde, acf> - <cde, abf>
\end{gathered}$$

Proposition 3.3.24 *By proposition (3.3.1) we have,*

$$<1, abcde> \circ [f] = <f, abcde> + <1, abcde>$$

Proposition 3.3.25 *By proposition (3.3.1) we have,*

$$[f] \circ <1, abcde> = <f, abcde>$$

Proposition 3.3.26 *By proposition (3.3.1) we have,*

$$<e, abcd> \circ [f] = <fe, abce> + <e, abcdf>$$

Proposition 3.3.27 *By proposition (3.3.1) we have,*

$$[f] \circ <d, abce> = <fd, abce>$$

Proposition 3.3.28 *By proposition (3.3.1) we have,*

$$<d, abce> \circ [f] = <fd, abce> + <d, abcef>$$

Proposition 3.3.29 *By proposition (3.3.1) we have,*

$$[f] \circ <ab, cde> = <fab, cde>$$

Proposition 3.3.30 *By proposition (3.3.1) we have,*

$$<ab, cde> \circ [f] = <fab, cde> + <ab, cdef>$$

Proposition 3.3.31 *By proposition (3.3.1) we have,*

$$[f] \circ <ab, cde> = <fab, cde>$$

Proof. By definition of $\{a, b\} = ab + ba$ we have

$$
\begin{aligned}
\{\{\{\{\{a, b\}, c\}, d\}, e\}, f\} &= ((((ab)c)d)e)f) + ((((ba)c)d)e)f) + \\
&+ ((((c(ab))d)e)f) + ((((c(ba))d)e)f) + (((d((ab)c))e)f) + \\
&+ (((d((ba)c))e)f) + (((d(c(ab)))e)f) + (((d(c(ba)))e)f) + \\
&+ ((e(((ab)c)d))f) + ((e((ba)c)d)f) + ((e(c(ab))d)f) + \\
&+ ((e((c(ba))d))f) + ((e(d((ab)c)))f) + ((e(d((ba)c)))f) + \\
&+ ((e(d(c(ab))))f) + ((e(d(c(ba))))f) + (f((((ab)c)d)e)) + \\
&+ (f((((ba)c)d)e)) + (f(((c(ab))d)e)) + (f(((c(ba))d)e)) + \\
&+ (f((d((ab)c))e)) + (f((d((ba)c))e)) + (f((d(c(ab)))e)) + \\
&+ (f((d(c(ba)))e)) + (f(e(((ab)c)d))) + (f(e((ba)c)d)) + \\
&+ (f(e(c(ab))d)) + (f(e((c(ba))d))) + (f(e(d((ab)c)))) + \\
&+ (f(e(d((ba)c)))) + (f(e(d(c(ab))))) + (e(d(c(ba)))))
\end{aligned}
$$

Than by proposition 3.3.22-3.3.31 we showing $\{\{\{\{\{a, b\}, c\}, d\}, e\}, f\}$

$$
\begin{aligned}
\{\{\{\{\{a, b\}, c\}, d\}, e\}, f\} &= [abcde] + [bacde] + \\
&+ [cabde] + [cbade] + [dabce] + [dbace] + \\
&+ [dcabe] + [dcbae] + [eabcd] + [ebacd] + \\
&+ [ecabd] + [ecbad] + [edabc] + [edbac] + \\
&+ [edcab] + [edcba] - 20<a, bcde> - \\
&- 20<b, acde> - 20<c, abde> - 20<d, abce> - \\
&- 20<e, abcd> - 8<ab, cde> - 8<ac, bde> - \\
&- 8<ad, bce> - 8<ae, bcd> - 8<bc, ade> - \\
&- 8<bd, ace> - 8<be, acd> - 8<cd, abe> - \\
&- 8<ce, abd> - 8<de, abc> - 34<1, abcde>, f = \\
&= [abcde] \circ [f] + [bacde] \circ [f] + [cabde] \circ [f] + \\
&+ [cbade] \circ [f] + [dabce] \circ [f] + [dbace] \circ [f] + \\
&+ [dcabe] \circ [f] + [dcbae] \circ [f] + [eabcd] \circ [f] + \\
&+ [ebacd] \circ [f] + [ecabd] \circ [f] + [ecbad] \circ [f] + \\
&+ [edabc] \circ [f] + [edbac] \circ [f] + [edcab] \circ [f] + \\
&+ [edcba] \circ [f] - 20<a, bcde> \circ [f] - 20<b, acde> \circ [f] - \\
&- 20<c, abde> \circ [f] - 20<d, abce> \circ [f] - 20<e, abcd> \circ [f] - \\
&- 8<ab, cde> \circ [f] - 8<ac, bde> \circ [f] - 8<ad, bce> \circ [f] - \\
&- 8<ae, bcd> \circ [f] - 8<bc, ade> \circ [f] - 8<bd, ace> \circ [f] - \\
&- 8<be, acd> \circ [f] - 8<cd, abe> \circ [f] - 8<ce, abd> \circ [f] - \\
&- 8<de, abc> \circ [f] - 34<1, abcde> \circ [f] + \\
&+ [f] \circ [abcde] + [f] \circ [bacde] + [f] \circ [cabde] + \\
&+ [f] \circ [cbade] + [f] \circ [dabce] + [f] \circ [dbace] + \\
&+ [f] \circ [dcabe] + [f] \circ [dcbae] + [f] \circ [eabcd] + \\
&+ [f] \circ [ebacd] + [f] \circ [ecabd] + [f] \circ [ecbad] + \\
&+ [f] \circ [edabc] + [f] \circ [edbac] + [f] \circ [edcab] + \\
&+ [f] \circ [edcba] - 20[f] \circ <a, bcde> - 20[f] \circ <b, acde> - \\
&- 20[f] \circ <c, abde> - 20[f] \circ <d, abce> - 20[f] \circ <e, abcd> - \\
&- 8[f] \circ <ab, cde> - 8[f] \circ <ac, bde> - 8[f] \circ <ad, bce> - \\
&- 8[f] \circ <ae, bcd> - 8[f] \circ <bc, ade> - 8[f] \circ <bd, ace> -
\end{aligned}
$$

– 8[f] ∘ <be, acd> – 8[f] ∘ <cd, abe> – 8[f] ∘ <ce, abd> -
– 8[f] ∘ <de, abc> - 34[f] ∘ <1, abcde> =
= [abcdef] + [bacdef] + [cabdef] +
+ [cbadef] + [dabcef] + [dbacef] +
+ [dcabef] + [dcbaef] + [eabcdf] +
+ [ebacdf] + [ecabdf] + [ecbadf] +
+ [edabcf] + [edbacf] + [edcabf] +
+ [edcbaf] – 20<a, bcdef> – 20<b, acdef> -
– 20<c, abdef> – 20<d, abcef> – 20<e, abcdf> -
– 8<ab, cdef> – 8<ac, bdef> – 8<ad, bcef> -
– 8<ae, bcdf> – 8<bc, adef> – 8<bd, acef> –
- 8<be, acdf> – 8<cd, abef> – 8<ce, abdf> –
- 8<de, abcf> - 34<1, abcdef> +
20<af, bcde> – 20<bf, acde> -
– 20<cf, abde> – 20<df, abce> – 20<ef, abcd> -
– 8<abf, cde> – 8<acf, bde> – 8<adf, bce> -
– 8<aef, bcd> – 8<bcf, ade> – 8<bdf, ace> –
- 8<bef, acd> – 8<cdf, abe> – 8<cef, abd> –
- 8<def, abc> - 34<f, abcde> +
+ [fabcde] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [fbacde] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [fcabde] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -

- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [fcbade] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [fdabce] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [fdbace] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
[fdcabe] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [fdcbae] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -

- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
[feabcd] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [febacd] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [fecabd] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [fecbad] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [fedabc] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -

- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [fedbac] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [fedcab] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> -
+ [edcbaf] - <1, abcdef> - <a, bcdef> -
- <b, acdef> - <c, abdef> - <d, abcef> -
- <e, abcdf> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <ae, bcdf> - <bc, adef> -
- <bd, acef> - <be, acdf> - <cd, abef> -
- <ce, abdf> - <de, abcf> - <abc, def> -
- <abd, cef> - <abe, cdf> - <acd,bdf> -
- <ace, bdf> - <ade, bcf> - <bcd, aef> -
- <bce, adf> - <bde, acf> - <cde, abf> =
= [abcdef] + [bacdef] + [cabdef] +
+ [cbadef] + [dabcef] + [dbacef] +
+ [dcabef] + [dcbaef] + [eabcdf] +
+ [ebacdf] + [ecabdf] + [ecbadf] +
+ [edabcf] + [edbacf] + [edcabf] +
+ [edcbaf] +[fabcde] + [fbacde] +
+ [fcabde] + [fcbade] + [fdabce] +
+[fdbace] + [fdcabe] + [fdcbae] +
+ [feabcd] + [febacd] + [fecabd] +

$$
\begin{gathered}
+ [fecbad] + [fedabc] + [fedbac] + \\
+ [fedcab] + [edcbaf] - \\
- 50<1, abcdef> - 36<a, bcdef> - \\
- 36<b, acdef> - 36<c, abdef> - 36<d, abcef> - \\
- 36<e, abcdf> - 24<ab, cdef> - 24<ac, bdef> - \\
- 24<ad, bcef> - 24<ae, bcdf> - 24<bc, adef> - \\
- 24<bd, acef> - 24<be, acdf> - 24<cd, abef> - \\
- 24<ce, abdf> - 24<de, abcf> - 16<abc, def> - \\
- 16<abd, cef> - 16<abe, cdf> - 16<acd,bdf> - \\
- 16<ace, bdf> - 16<ade, bcf> - 16<bcd, aef> - \\
- 16<bce, adf> - 16<bde, acf> - 16<cde, abf>
\end{gathered}
$$

Theorem 3.3.32 *By definition of $\{a, b\} = ab + ba$ and proposition (3.3.1) we have $\{\{\{\{a, b\}, c\}, d\}, \{e\ f\}\}$and when we have $\{\{\{\{a, b\}, c\}, d\}, \{e\ f\}\}$ we using theorem (3.3.2) and and proposition (3.3.1) and we can write*

$$
\begin{gathered}
\{\{\{\{a, b\}, c\}, d\}, \{e\ f\}\} = (((ab)c)d) + (((ba)c)d) + \\
+ (((c(ab))d) + ((c(ba))d) + (d((ab)c)) + \\
+ (d((ba)c)) + (d(c(ab))) + (d(c(ba))), (ef) + (fe) = \\
= ((((ab)c)d)(ef)) + ((((ba)c)d)(ef)) + ((((c(ab))d)(ef)) + \\
+ (((c(ba))d)(ef)) + ((d((ab)c))(ef)) + ((d((ba)c))(ef)) + \\
+ ((d(c(ab)))\ (ef)) + ((d(c(ba)))(ef)) + \\
+ ((ef)(((ab)c)d)) + ((ef)(((ba)c)d)) + ((ef)(((c(ab))d)) + \\
+ ((ef)((c(ba))d)) + ((ef)(d((ab)c))) + ((ef)(d((ba)c))) + \\
+ ((ef)(d(c(ab)))) + ((ef)(d(c(ba)))) + \\
+ ((((ab)c)d)(fe)) + ((((ba)c)d)(fe)) + (((c(ab))d)(fe)) + \\
+ (((c(ba))d)(fe)) + ((d((ab)c))(fe)) + ((d((ba)c))(fe)) + \\
+ ((d(c(ab)))(fe)) + ((d(c(ba)))(fe)) + \\
+ ((fe)(((ab)c)d)) + ((fe)(((ba)c)d)) + ((fe)(((c(ab))d)) + \\
+ ((fe)((c(ba))d)) + ((fe)(d((ab)c))) + ((fe)(d((ba)c))) + \\
+ ((fe)(d(c(ab)))) + ((fe)(d(c(ba)))) = \\
= [abcdef] + [bacdef] + [cabdef] + [cbadef] + \\
+ [dabcef] + [dbacef] + [dcabef] + [dcbaef] + \\
+ [efabcd] + [efbacd] + [efcabd] + [efcbad] + \\
+ [efdabc] + [efdbac] + [efdcab] + [efdcba] + \\
= [abcdfe] + [bacdfe] + [cabdfe] + [cbadfe] + \\
+ [dabcfe] + [dbacfe] + [dcabfe] + [dcbafe] + \\
+ [feabcd] + [febacd] + [fecabd] + [fecbad] + \\
+ [fedabc] + [fedbac] + [fedcab] + [fedcba] - \\
- 60<1, abcdef> - 20<e, abcdf> - 20<f, abcde> - 40<ef, abcd> - \\
- 40<a, bcdef> - 24<ae, bcdf> - 8<af, bcde> - 16<aef, bcd> - \\
- 40<b, acdef> - 24<be, acdf> - 8<bf, acde> - 16<bef, acd> - \\
- 40<c, abdef> - 24<ce, abdf> - 8<cf, abde> - 16<cef, abd> - \\
- 40<d, abcef> - 24<de, abcf> - 8<df, abce> - 16<def, abc>
\end{gathered}
$$

Proposition 3.3.33 *By proposition (3.3.1) we have,*

$$\begin{aligned}&((((ab)c)d)(ef)) = [abcd] \circ [ef] = [abcdef] \text{ - } \\ &\text{ - } <1,\ abcdef> \text{ - } <a,\ bcdef> \text{ - } <b,\ acdef> \text{ - } \\ &\text{ - } <c,\ abdef> \text{ - } <d,\ abcef> \text{ - } <ab,\ cdef> \text{ - } \\ &\text{ - } <ac,\ bdef> \text{ - } <ad,\ bcef> \text{ - } <bc,\ adef> \text{ - } \\ &\text{ - } <bd,\ acef> \text{ - } <cd,\ abef> \text{ - } <abc,\ def> \text{ - } \\ &\text{ - } <abd,\ cef> \text{ - } <acd,\ bef> \text{ - } <bcd,\ aef>\end{aligned}$$

Proposition 3.3.34 *By proposition (3.3.1) we have,*

$$\begin{aligned}&((ef)(((ab)c)d)) = [ef] \circ [abcd] = [efabcd] \text{ - } 2<1,\ abcdef> \text{ - } \\ &\text{ - } <a,\ bcdef> \text{ - } <b,\ acdef> \text{ - } <c,\ abdef> \text{ - } \\ &\text{ - } <d,\ abcef> \text{ - } <ab,\ cdef> \text{ - } <ac,\ bdef> \text{ - } \\ &\text{ - } <ad,\ bcef> \text{ - } <bc,\ adef> \text{ - } <bd,\ acef> \text{ - } \\ &\text{ - } <cd,\ abef> \text{ - } <ae,\ bcdf> \text{ - } <be,\ acdf> \text{ - } \\ &\text{ - } <ce,\ abdf> \text{ - } <de,\ abcf> \text{ - } <af,\ bcdef> \text{ - } \\ &\text{ - } <bf,\ acdef> \text{ - } <cf,\ abdef> \text{ - } <df,\ abcef> =\end{aligned}$$

By proposition (3.3.1) we have,

Proposition 3.3.35 $<1,\ abcd> \circ [ef] = <1,\ abcdef> +$
$+ <e,\ abcdf> + <f,\ abcde> + <ef,\ abcd>$

Proposition 3.3.36 *By proposition (3.3.1) we have,*

$$[ef] \circ <1,\ abcd> = <ef,\ abcd>$$

Proposition 3.3.37 *By proposition (3.3.1) we have,*

$$<a,\ bcd> \circ [ef] = <a,\ bcdef> + <ae,\ bcdf> + <af,\ bcde> + <aef,\ bcd>$$

Proposition 3.3.38 *By proposition (3.3.1) we have,*

$$[ef] \circ <a,\ bcd> = <aef,\ bcd>$$

Then by proposition (3.3.33) - (3.3.38) we can start proof part

$$\begin{aligned}&\{\{\{\{a, b\}, c\}, d\}, \{e, f\}\} = \{(((ab)c)d) + (((ba)c)d) + \\ &+ (((ca)b)d) + (((cb)a)d) + (((da)b)c) + (((db)a)c) + \\ &+ (((dc)a)b) + (((dc)b)a) - 10[(a, b, c), d] - 4[a, (b, c, d)] \text{ - } \\ &- 4[b, (a, c, d)] - 4[c, (a, b, d)] - 4[d, (a, b, c)], \{e, f\}\} = \\ &= \{(((ab)c)d) + (((ba)c)d) + (((ca)b)d) + (((cb)a)d) + \\ &+ (((da)b)c) + (((db)a)c) + (((dc)a)b) + (((dc)b)a) - \\ &\text{ - } 10[(a, b, c), d] - 4[a, (b, c, d)] - 4[b, (a, c, d)] - \\ &\text{ - } 4[c, (a, b, d)] - 4[d, (a, b, c)], (ef)+(fe)\} =\end{aligned}$$

={[abcd] + [bacd] + [cabd] + [cbad] +
+ [dabc] + [dbac] + [dcab] + [dcba] -
- 10<1, abcdef> - 4<a, bcdef> - 4<b, acdef> -
- 4<c, abdef> - 4<d, abcef> -
- 10[(a, b, c), d] – 4[a, (b, c, d)] – 4[b, (a, c, d)] –
- 4[c, (a, b, d)] – 4[d, (a, b, c)], [ef]+[fe]} =
= [abcd] ∘ [ef] + [bacd] ∘ [ef] + [cabd] ∘ [ef] +
+ [cbad] ∘ [ef] + [dabc] ∘ [ef] + [dbac] ∘ [ef] +
+ [dcab] ∘ [ef] + [dcba] ∘ [ef] - 10<1, abcd> ∘ [ef] -
– 4<a, bcd> ∘ [ef] – 4<b, acd> ∘ [ef] –
- 4<c, abd> ∘ [ef] – 4<d, abc> ∘ [ef] +
+ [ef] ∘ [abcd] + [ef] ∘ [bacd] + [ef] ∘ [cabd] +
+ [ef] ∘ [cbad] + [ef] ∘ [dabc] + [ef] ∘ [dbac] +
+ [ef] ∘ [dcab] +[ef] ∘ [dcba] - 10[ef] ∘ <1, abcd> -
– 4[ef] ∘ <a, bcd> – 4[ef] ∘ <b, acd> –
- 4[ef] ∘ <c, abd> – 4[ef] ∘ <d, abc> +
+ [abcd] ∘ [fe] + [bacd] ∘ [fe] + [cabd] ∘ [fe] +
+ [cbad] ∘ [fe] + [dabc] ∘ [fe] + [dbac] ∘ [fe] +
+ [dcab] ∘ [fe] + [dcba] ∘ [fe] - 10<1, bcd> ∘ [fe] -
– 4<a, bcd> ∘ [fe] – 4<b, acd> ∘ [fe] –
- 4<c, abd> ∘ [fe] – 4<d, abc> ∘ [fe] +
+ [fe] ∘ [abcd] + [fe] ∘ [bacd] + [fe] ∘ [cabd] +
+[fe] ∘ [cbad] + [fe] ∘ [dabc] + [fe] ∘ [dbac] +
+ [fe] ∘ [dcab] + [fe] ∘ [dcba] - 10[fe] ∘ <1, bcd> –
- 4[fe] ∘ <a, bcd> – 4[fe] ∘ <b, acd> –
- 4[fe] ∘ <c, abd> – 4[fe] ∘ <d, abc> =
= [abcdef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [bacdef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [cabdef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [cbadef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -

- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [dabcef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [dbacef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [dcabef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [dcbaef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> -
- 10<1, abcdef> - 10<e, abcdf> - 10<f, abcde> - 10<ef, abcd> -
- 4<a, bcdef> - 4<ae, bcdf> - 4<af, bcde> - 4<aef, bcd> -
- 4<b, acdef> - 4<be, acdf> - 4<bf, acde> - 4<bef, acd> -
- 4<c, abdef> - 4<ce, abdf> - 4<cf, abde> - 4<cef, abd> -
- 4<d, abcef> - 4<de, abcf> - 4<df, abce> - 4<def, abc> +
+ [efabcd] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efbacd] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efcabd] - 2<1, abcdef> -

- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efcbad] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efdabc] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efdbac] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efdcab] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efdcba] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> =
- 10<ef, abcd> - 4<aef, bcd> – 4 <bef, acd> –
- 4<cef, abd> – 4<def, abc> +

+ [abcdfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [bacdfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [cabdfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [cbadfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [dabcfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [dbacfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [dcabfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [dcbafe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> -
- 10<1, abcdef> - 10<e, abcdf> - 10<f, abcde> - 10<ef, abcd> -
- 4<a, bcdef> - 4<ae, bcdf> - 4<af, bcde> - 4<aef, bcd> -
- 4<b, acdef> - 4<be, acdf> - 4<bf, acde> - 4<bef, acd> -

- 4<c, abdef> - 4<ce, abdf> - 4<cf, abde> - 4<cef, abd> -
- 4<d, abcef> - 4<de, abcf> - 4<df, abce> - 4<def, abc> +
+ [feabcd] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [febacd] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [fecabd] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [fecbad] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [fedabc] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [fedbac] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -

- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [fedcab] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [fedcba] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> =
- 10<ef, abcd> - 4<aef, bcd> – 4 <bef, acd> –
- 4<cef, abd> – 4<def, abc> =
= [abcdef] + [bacdef] + [cabdef] + [cbadef] +
+ [dabcef] + [dbacef] + [dcabef] + [dcbaef] +
+ [efabcd] + [efbacd] + [efcabd] + [efcbad] +
+ [efdabc] + [efdbac] + [efdcab] + [efdcba] +
= [abcdfe] + [bacdfe] + [cabdfe] + [cbadfe] +
+ [dabcfe] + [dbacfe] + [dcabfe] + [dcbafe] +
+ [feabcd] + [febacd] + [fecabd] + [fecbad] +
+ [fedabc] + [fedbac] + [fedcab] + [fedcba] -
- 60<1, abcdef> - 20<e, abcdf> - 20<f, abcde> - 40<ef, abcd> -
- 40<a, bcdef> - 24<ae, bcdf> - 8<af, bcde> - 16<aef, bcd> -
- 40<b, acdef> - 24<be, acdf> - 8<bf, acde> - 16<bef, acd> -
- 40<c, abdef> - 24<ce, abdf> - 8<cf, abde> - 16<cef, abd> -
- 40<d, abcef> - 24<de, abcf> - 8<df, abce> - 16<def, abc>

Teorem is proved

Theorem 3.3.39 *We have {{{a, b}, {c, d}}, {e f}} by definition of {a, b} = ab + ba and proposition (3.3.1) when we have {{{{a, b}, c}, d}, {e f}} we can using theorem (3.3.7) and and proposition (3.3.1) ,*

{{{a, b}, {c, d}}, {e f}} = {((ab)(cd)) + ((ba)(cd)) +
+ (ab)(dc) + (ab)(cd) + ((dc)(ab)) +
+ ((ba)(dc)) + ((ba)(cd)) + ((dc)(ba)), (ef) + (fe)} =
= (((ab)(cd))(ef)) + (((ba)(cd))(ef)) +
+ (((ab)(dc))(ef)) + (((ab)(cd)(ef)) + (((dc)(ab))(ef)) +
+ (((ba)(dc)))(ef)) + (((ba)(cd))(ef)) + (((dc)(ba))(ef)) +
+ ((ef)((ab)(cd))) + ((ef)((ba)(cd))) +

+ ((ef)((ab)(dc))) + ((ef)((ab)(cd))) + ((ef)((dc)(ab))) +
+ ((ef)((ba)(dc))) + ((ef)((ba)(cd))) + ((ef)((dc)(ba))) +
+ (((ab)(cd))(fe)) + (((ba)(cd))(fe)) +
+ (((ab)(dc))(fe)) + (((ab)(cd)(fe)) + (((dc)(ab))(fe)) +
+ (((ba)(dc)))(fe)) + (((ba)(cd))(fe)) + (((dc)(ba))(fe))
+ ((fe)((ab)(cd))) + ((fe)((ba)(cd))) +
+ ((fe)((ab)(dc))) + ((fe)((ab)(cd))) + ((fe)((dc)(ab))) +
+ ((fe)((ba)(dc))) + ((fe)((ba)(cd))) + ((fe)((dc)(ba))) =
= [abcdef] + [cdabef] + [abdcef] + [dcabef] +
+ [bacdef] + [cdbaef] + [badcef] + [dcbaef] +
+ [efabcd] + [efcdab] + [efabdc] + [efdcab] +
+ [efbacd] + [efcdba] + [efbadc] + [efdcba] +
+ [abcdfe] + [cdabfe] + [abdcfe] + [dcabfe] +
+ [bacdfe] + [cdbafe] + [badcfe] + [dcbafe] +
+ [feabcd] + [fecdab] + [feabdc] + [fedcab] +
+ [febacd] + [fecdba] + [febadc] + [fedcba] -
- 40<a, bcdef> - 24<ae, bcdf> - 8<af, bcde> - 24<aef, bcd> -
- 40<b, acdef> - 24<be, acdf> - 8<bf, acde> - 24<bef, acd> -
- 40<c, abdef> - 24<ce, abdf> - 8<cf, abde> - 24<cef, abd> -
- 40<d, abcef> - 24<de, abcf> - 8<df, abce> - 24<def, abc> -
- 56<1, abcdef> - 16<e, abcdf> - 16<f, abcde> - 32<ef, abcd> +

Proposition 3.3.40 *By theorem 3.3.7 and by proposition (3.3.1) we can find [abcd] ∘ [ef]*

[abcd] ∘ [ef] = [abcdef] -
- <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef>

Proposition 3.3.41 *[ef] ∘ [abcd] = [efabcd] - 2<1, abcdef> -*
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> =

Proposition 3.3.42 *<1, abcd> ∘ [ef] = <1, abcdef> +*
+ <e, abcdf> + <f, abcde> + <ef, abcd>

Proposition 3.3.43 *[ef] ∘ <1, abcd> = <ef, abcd>*

Proposition 3.3.44 $\langle a, bcd\rangle \circ [ef] = \langle a, bcdef\rangle +$
$+ \langle ae, bcdf\rangle + \langle af, bcde\rangle + \langle aef, bcd\rangle$

Proposition 3.3.45 $[ef] \circ \langle a, bcd\rangle = \langle aef, bcd\rangle$

Then by proposition (3.3.40) - (3.3.45) we can proof (3.3.39),

$$
\begin{gathered}
\{\{\{a, b\}, \{c, d\}\}, \{e\ f\}\} = \{((ab)(cd)) + ((ba)(cd)) + \\
+ (ab)(dc) + (ab)(cd) + ((dc)(ab)) + \\
+ ((ba)(dc)) + ((ba)(cd)) + ((dc)(ba)), (ef) + (fe)\} = \\
= [abcd] \circ [ef] + [cdab] \circ [ef] + [abdc] \circ [ef] + [dcab] \circ [ef] + \\
+ [bacd] \circ [ef] + [cdba] \circ [ef] + [badc] \circ [ef] + [dcba] \circ [ef] - \\
- 4\langle a, bcd\rangle \circ [ef] - 4\langle b, acd\rangle \circ [ef] - 4\langle c, abd\rangle \circ [ef] - \\
- 4\langle d, abc\rangle \circ [ef] - 8\langle 1, abcd\rangle \circ [ef] + \\
+ [ef] \circ [abcd] + [ef] \circ [cdab] + [ef] \circ [abdc] + [ef] \circ [dcab] + \\
+ [ef] \circ [bacd] + [ef] \circ [cdba] + [ef] \circ [badc] + [ef] \circ [dcba] - \\
- 4[ef] \circ \langle a, bcd\rangle - 4[ef] \circ \langle b, acd\rangle - 4[ef] \circ \langle c, abd\rangle - \\
- 4[ef] \circ \langle d, abc\rangle - 8[ef] \circ \langle 1, abcd\rangle + \\
+ [abcd] \circ [fe] + [cdab] \circ [fe] + [abdc] \circ [fe] + [dcab] \circ [fe] + \\
+ [bacd] \circ [fe] + [cdba] \circ [fe] + [badc] \circ [fe] + [dcba] \circ [fe] - \\
- 4\langle a, bcd\rangle \circ [fe] - 4\langle b, acd\rangle \circ [fe] - 4\langle c, abd\rangle \circ [fe] - \\
- 4\langle d, abc\rangle \circ [fe] - 8\langle 1, abcd\rangle \circ [fe] + \\
+ [fe] \circ [abcd] + [fe] \circ [cdab] + [fe] \circ [abdc] + [fe] \circ [dcab] + \\
+ [fe] \circ [bacd] + [fe] \circ [cdba] + [fe] \circ [badc] + [fe] \circ [dcba] - \\
- 4[fe] \circ \langle a, bcd\rangle - 4[fe] \circ \langle b, acd\rangle - 4[fe] \circ \langle c, abd\rangle - \\
- 4[fe] \circ \langle d, abc\rangle - 8[fe] \circ \langle 1, abcd\rangle = \\
= [abcdef] - \langle 1, abcdef\rangle - \langle a, bcdef\rangle - \langle b, acdef\rangle - \\
- \langle c, abdef\rangle - \langle d, abcef\rangle - \langle ab, cdef\rangle - \\
- \langle ac, bdef\rangle - \langle ad, bcef\rangle - \langle bc, adef\rangle - \\
- \langle bd, acef\rangle - \langle cd, abef\rangle - \langle abc, def\rangle - \\
- \langle abd, cef\rangle - \langle acd, bef\rangle - \langle bcd, aef\rangle + \\
+ [cdabef] - \langle 1, abcdef\rangle - \langle a, bcdef\rangle - \langle b, acdef\rangle - \\
- \langle c, abdef\rangle - \langle d, abcef\rangle - \langle ab, cdef\rangle - \\
- \langle ac, bdef\rangle - \langle ad, bcef\rangle - \langle bc, adef\rangle - \\
- \langle bd, acef\rangle - \langle cd, abef\rangle - \langle abc, def\rangle - \\
- \langle abd, cef\rangle - \langle acd, bef\rangle - \langle bcd, aef\rangle + \\
+ [abdcef] - \langle 1, abcdef\rangle - \langle a, bcdef\rangle - \langle b, acdef\rangle - \\
- \langle c, abdef\rangle - \langle d, abcef\rangle - \langle ab, cdef\rangle - \\
- \langle ac, bdef\rangle - \langle ad, bcef\rangle - \langle bc, adef\rangle - \\
- \langle bd, acef\rangle - \langle cd, abef\rangle - \langle abc, def\rangle - \\
- \langle abd, cef\rangle - \langle acd, bef\rangle - \langle bcd, aef\rangle \\
+ [dcabef] - \langle 1, abcdef\rangle - \langle a, bcdef\rangle - \langle b, acdef\rangle - \\
- \langle c, abdef\rangle - \langle d, abcef\rangle - \langle ab, cdef\rangle - \\
- \langle ac, bdef\rangle - \langle ad, bcef\rangle - \langle bc, adef\rangle -
\end{gathered}
$$

- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [bacdef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [cdbaef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [badcef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [dcbaef] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> -
- 4<a, bcdef> - 4<ae, bcdf> - 4<af, bcde> - 4<aef, bcd> -
- 4<b, acdef> - 4<be, acdf> - 4<bf, acde> - 4<bef, acd> -
- 4<c, abdef> - 4<ce, abdf> - 4<cf, abde> - 4<cef, abd> -
- 4<d, abcef> - 4<de, abcf> - 4<df, abce> - 4<def, abc> -
- 8<1, abcdef> - 8<e, abcdf> - 8<f, abcde> - 8<ef, abcd> +
+ [efabcd] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efcdab] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efabdc] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -

- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efdcab] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efbacd] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efcdba] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efbadc] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [efdcba] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
- 4<aef, bcd> − 4<bef, acd> − 4<cef, abd> −
- 4<def, abc> − 8<ef, abcd> +
+ [abcdfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -

- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [cdabfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [abdcfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef>
+ [dcabfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [bacdfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [cdbafe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [badcfe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> +
+ [dcbafe] - <1, abcdef> - <a, bcdef> - <b, acdef> -
- <c, abdef> - <d, abcef> - <ab, cdef> -
- <ac, bdef> - <ad, bcef> - <bc, adef> -
- <bd, acef> - <cd, abef> - <abc, def> -
- <abd, cef> - <acd, bef> - <bcd, aef> -
- 4<a, bcdef> - 4<ae, bcdf> - 4<af, bcde> - 4<aef, bcd> -
- 4<b, acdef> - 4<be, acdf> - 4<bf, acde> - 4<bef, acd> -
- 4<c, abdef> - 4<ce, abdf> - 4<cf, abde> - 4<cef, abd> -
- 4<d, abcef> - 4<de, abcf> - 4<df, abce> - 4<def, abc> -

- 8<1, abcdef> - 8<e, abcdf> - 8<f, abcde> - 8<ef, abcd> +
+ [feabcd] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [fecdab] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
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- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [feabdc] - 2<1, abcdef> -
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- <d, abcef> - <ab, cdef> - <ac, bdef> -
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- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [fedcab] - 2<1, abcdef> -
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- <ad, bcef> - <bc, adef> - <bd, acef> -
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- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [febacd] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
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- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [fecdba] - 2<1, abcdef> -
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- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +

+ [febadc] - 2<1, abcdef> -
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- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
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- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
+ [fedcba] - 2<1, abcdef> -
- <a, bcdef> - <b, acdef> - <c, abdef> -
- <d, abcef> - <ab, cdef> - <ac, bdef> -
- <ad, bcef> - <bc, adef> - <bd, acef> -
- <cd, abef> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <de, abcf> - <af, bcdef> -
- <bf, acdef> - <cf, abdef> - <df, abcef> +
- 4<aef, bcd> − 4<bef, acd> − 4<cef, abd> −
- 4<def, abc> − 8<ef, abcd> =
= [abcdef] + [cdabef] + [abdcef] + [dcabef] +
+ [bacdef] + [cdbaef] + [badcef] + [dcbaef] +
+ [efabcd] + [efcdab] + [efabdc] + [efdcab] +
+ [efbacd] + [efcdba] + [efbadc] + [efdcba] +
+ [abcdfe] + [cdabfe] + [abdcfe] + [dcabfe] +
+ [bacdfe] + [cdbafe] + [badcfe] + [dcbafe] +
+ [feabcd] + [fecdab] + [feabdc] + [fedcab] +
+ [febacd] + [fecdba] + [febadc] + [fedcba] -
- 40<a, bcdef> - 24<ae, bcdf> - 8<af, bcde> - 24<aef, bcd> -
- 40<b, acdef> - 24<be, acdf> - 8<bf, acde> - 24<bef, acd> -
- 40<c, abdef> - 24<ce, abdf> - 8<cf, abde> - 24<cef, abd> -
- 40<d, abcef> - 24<de, abcf> - 8<df, abce> - 24<def, abc> -
- 56<1, abcdef> - 16<e, abcdf> - 16<f, abcde> - 32<ef, abcd> +

Theorem 3.3.46 *We have {{{a, b}, c}, {{d, e}, f}} by definition of {a, b} = ab + ba and proposition (3.3.1) when we have {{{a, b}, c}, {{d, e}, f}} we can using theorem (3.3.7) and and proposition (3.3.1) ,*

{{{a, b}, c}, {{d, e}, f}} = [abc] ∘ [def] +
+ [bac] ∘ [def] + [cab] ∘ [def] + [cba] ∘ [def] +
+ [abc] ∘ [edf] + [bac] ∘ [edf] + [cab] ∘ [edf] + [cba] ∘ [edf] +
+ [abc] ∘ [fde] + [bac] ∘ [fde] + [cab] ∘ [fde] + [cba] ∘ [fde] +
+ [abc] ∘ [efd] + [bac] ∘ [efd] + [cab] ∘ [efd] + [cba] ∘ [efd] +
+ [def] ∘ [abc] + [def] ∘ [bac] + [def] ∘ [cab] + [def] ∘ [cba] +
+ [edf] ∘ [abc] + [edf] ∘ [bac] + [edf] ∘ [cab] + [edf] ∘ [cba] +
+[fde] ∘ [abc] + [fde] ∘ [bac] + [fde] ∘ [cab] + [fde] ∘ [cba] +
+ [efd] ∘ [abc] + [efd] ∘ [bac] + [efd] ∘ [cab] + [efd] ∘ [cba] +

+ [abc] ∘ 2<1,def> + [bac] ∘ 2<1,def> + [cab] ∘ 2<1,def> + [cba] ∘ 2<1,def> +
+ [def] ∘ 2<1, abc> + [edf] ∘ 2<1, abc> + [fde] ∘ 2<1, abc> + [efd] ∘ 2<1, abc> -
- 2<1,def> ∘ [abc] - 2<1,def> ∘ [bac] - 2<1,def> ∘ [cab] -
- 2<1,def> ∘ [cba] - 2<1,def> ∘ 2<1, abc> -
- 2<1, abc> ∘ [def] - 2<1, abc> ∘ [edf]- 2<1, abc> ∘ [fde] -
- 2<1, abc> ∘ [efd] - 2<1, abc> ∘ - 2<1,def> =
[abcdef] + [bacdef] + [cabdef] + [cbadef] +
+[abcedf] + [bacedf] + [cabedf] + [cbaedf] +
+ [abcfde] + [bacfde] + [cabfde] + [cbafde] +
+ [abcefd] + [bacefd] + [cabefd] + [cbaefd] +
+ [defabc] + [defbac] + [defcab] + [defcba] +
+ [edfabc] + [edfbac] + [edfcab] + [edfcba] +
+ [fdeabc] + [fdebac] + [fdecab] + [fdecba] +
+ [efdabc] + [efdbac] + [efdcab] + [efdcba] +
- 72<a, bcdef> - 72<b, acdef> - 72<c, abdef> -
- 72<ab, cdef> - 72<ac, bdef> - 72<bc, adef> -
- 40<d, abcef> - 32<ad, bcef> - 32<bd, acef> -
- 32<cd, abef> - 32<abd, cef> - 32<acd, bef> - 32<bcd, aef> -
- 40<e, abcdf> - 32<ae, bcdf> - 32<be, acdf> -
- 32<ce, abdf> - 32<abe, cdf> - 32<ace, bdf> - 32<bce, adf> -
- 40<f, abcde> - 32<af, bcde> - 32<bf, acde> -
- 32<cf, abde> - 32<abf, cde> - 32<acf, bde> - 32<bcf, ade> -
- 10<abc, def> - 10<def, abc> - 8<de, abcf> -
- 8<df, abce> - 8<ef, abcd> -80<1, abcdef>

Proposition 3.3.47 *By proposition (3.3.1) we have*

{{a,b}, c} = ((ab)c) + ((ba)c) + ((ca)b) +
+ ((cb)a) − 2(a,b,c) = [abc] +[bac] +
+ [cab] + [cba] - 2<1, abc>

Proposition 3.3.48 *By proposition (3.3.1) we have*

[abc] ∘ [def] = [abcdef] - 2<1, abcdef> -
- 2<a, bcdef> - 2<b, acdef> - 2<c, abdef> -
- 2<ab, cdef> - 2<ac, bdef> - 2<bc, adef> -
- <d, abcef> - <ad, bcef> - <bd, acef> -
- <cd, abef> - <abd, cef> - <acd, bef> - <bcd, aef> -
- <e, abcdf> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <abe, cdf> - <ace, bdf> - <bce, adf> -
- <f, abcde> - <af, bcde> - <bf, acde> -
- <cf, abde> - <abf, cde> - <acf, bde> - <bcf, ade> -

Proposition 3.3.49 *By proposition (3.3.1) we have*

$$[def] \circ <1, abc> = < def, abc>$$

Proposition 3.3.50 *By proposition (3.3.1) we have*

$$\begin{gathered}<1, abc> \circ [def] = <1, abcdef> + <d, abcef> + \\ + <e, abcdf> + <f, abcde> + <de, abcf> + \\ + <df, abce> + <ef, abcd> + <def, abc>\end{gathered}$$

And we have

$$\begin{gathered}\{\{a,b\},c\} = \{(ab) + (ba), c\}= \\ = ((ab)c) + (c(ab)) + ((ba)c) + (c(ba)) = \\ = ((ab)c) + ((ba)c) + ((ca)b) - (a,b,c) + \\ + ((cb)a) - (a,b,c) = ((ab)c) + ((ba)c) + \\ + ((ca)b) + ((cb)a) - 2(a,b,c) = \\ = [abc] +[bac] + [cab] + [cba] - 2<1, abc>\end{gathered}$$

Then by proposition (3.3.47) - (3.3.50) we can proof (3.3.46),

$$\begin{gathered}\{\{\{a, b\}, c\}, \{\{d, e\}, f\}\} =\{[abc] +[bac] + \\ + [cab] + [cba] - 2<1, abc>, [def] + \\ + [edf] + [fde] + [efd] - 2<1,def>\} = \\ = [abc] \circ [def] + [bac] \circ [def] + [cab] \circ [def] + [cba] \circ [def] + \\ + [abc] \circ [edf] + [bac] \circ [edf] + [cab] \circ [edf] + [cba] \circ [edf] + \\ + [abc] \circ [fde] + [bac] \circ [fde] + [cab] \circ [fde] + [cba] \circ [fde] + \\ + [abc] \circ [efd] + [bac] \circ [efd] + [cab] \circ [efd] + [cba] \circ [efd] + \\ + [def] \circ [abc] + [def] \circ [bac] + [def] \circ [cab] + [def] \circ [cba] + \\ + [edf] \circ [abc] + [edf] \circ [bac] + [edf] \circ [cab] + [edf] \circ [cba] + \\ +[fde] \circ [abc] + [fde] \circ [bac] + [fde] \circ [cab] + [fde] \circ [cba] + \\ + [efd] \circ [abc] + [efd] \circ [bac] + [efd] \circ [cab] + [efd] \circ [cba] + \\ + [abc] \circ 2<1,def> + [bac] \circ 2<1,def> + [cab] \circ 2<1,def> + [cba] \circ 2<1,def> \\ + \\ + [def] \circ 2<1, abc> + [edf] \circ 2<1, abc> + [fde] \circ 2<1, abc> + [efd] \circ 2<1, \\ abc> - \\ - 2<1,def> \circ [abc] - 2<1,def> \circ [bac] - 2<1,def> \circ [cab] - \\ - 2<1,def> \circ [cba] - 2<1,def> \circ 2<1, abc> - \\ - 2<1, abc> \circ [def] - 2<1, abc> \circ [edf]- 2<1, abc> \circ [fde] - \\ - 2<1, abc> \circ [efd] - 2<1, abc> \circ - 2<1,def> = \\ = [abcdef] - 2<1, abcdef> - \\ - 2<a, bcdef> - 2<b, acdef> - 2<c, abdef> - \\ - 2<ab, cdef> - 2<ac, bdef> - 2<bc, adef> - \\ - <d, abcef> - <ad, bcef> - <bd, acef> - \\ - <cd, abef> - <abd, cef> - <acd, bef> - <bcd, aef> -\end{gathered}$$

- <e, abcdf> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <abe, cdf> - <ace, bdf> - <bce, adf> -
- <f, abcde> - <af, bcde> - <bf, acde> -
- <cf, abde> - <abf, cde> - <acf, bde> - <bcf, ade> +
+ [bacdef] - 2<1, abcdef> -
- 2<a, bcdef> - 2<b, acdef> - 2<c, abdef> -
- 2<ab, cdef> - 2<ac, bdef> - 2<bc, adef> -
- <d, abcef> - <ad, bcef> - <bd, acef> -
- <cd, abef> - <abd, cef> - <acd, bef> - <bcd, aef> -
- <e, abcdf> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <abe, cdf> - <ace, bdf> - <bce, adf> -
- <f, abcde> - <af, bcde> - <bf, acde> -
- <cf, abde> - <abf, cde> - <acf, bde> - <bcf, ade> +
+ [cabdef] - 2<1, abcdef> -
- 2<a, bcdef> - 2<b, acdef> - 2<c, abdef> -
- 2<ab, cdef> - 2<ac, bdef> - 2<bc, adef> -
- <d, abcef> - <ad, bcef> - <bd, acef> -
- <cd, abef> - <abd, cef> - <acd, bef> - <bcd, aef> -
- <e, abcdf> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <abe, cdf> - <ace, bdf> - <bce, adf> -
- <f, abcde> - <af, bcde> - <bf, acde> -
- <cf, abde> - <abf, cde> - <acf, bde> - <bcf, ade> +
+ [cbadef] - 2<1, abcdef> -
- 2<a, bcdef> - 2<b, acdef> - 2<c, abdef> -
- 2<ab, cdef> - 2<ac, bdef> - 2<bc, adef> -
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- <ce, abdf> - <abe, cdf> - <ace, bdf> - <bce, adf> -
- <f, abcde> - <af, bcde> - <bf, acde> -
- <cf, abde> - <abf, cde> - <acf, bde> - <bcf, ade> +
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- 2<a, bcdef> - 2<b, acdef> - 2<c, abdef> -
- 2<ab, cdef> - 2<ac, bdef> - 2<bc, adef> -
- <d, abcef> - <ad, bcef> - <bd, acef> -
- <cd, abef> - <abd, cef> - <acd, bef> - <bcd, aef> -
- <e, abcdf> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <abe, cdf> - <ace, bdf> - <bce, adf> -
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- 2<ab, cdef> - 2<ac, bdef> - 2<bc, adef> -

- <d, abcef> - <ad, bcef> - <bd, acef> -
- <cd, abef> - <abd, cef> - <acd, bef> - <bcd, aef> -
- <e, abcdf> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <abe, cdf> - <ace, bdf> - <bce, adf> -
- <f, abcde> - <af, bcde> - <bf, acde> -
- <cf, abde> - <abf, cde> - <acf, bde> - <bcf, ade> +
+ [cabedf] - 2<1, abcdef> -
- 2<a, bcdef> - 2<b, acdef> - 2<c, abdef> -
- 2<ab, cdef> - 2<ac, bdef> - 2<bc, adef> -
- <d, abcef> - <ad, bcef> - <bd, acef> -
- <cd, abef> - <abd, cef> - <acd, bef> - <bcd, aef> -
- <e, abcdf> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <abe, cdf> - <ace, bdf> - <bce, adf> -
- <f, abcde> - <af, bcde> - <bf, acde> -
- <cf, abde> - <abf, cde> - <acf, bde> - <bcf, ade> +
+ [cbaedf] - 2<1, abcdef> -
- 2<a, bcdef> - 2<b, acdef> - 2<c, abdef> -
- 2<ab, cdef> - 2<ac, bdef> - 2<bc, adef> -
- <d, abcef> - <ad, bcef> - <bd, acef> -
- <cd, abef> - <abd, cef> - <acd, bef> - <bcd, aef> -
- <e, abcdf> - <ae, bcdf> - <be, acdf> -
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- <f, abcde> - <af, bcde> - <bf, acde> -
- <cf, abde> - <abf, cde> - <acf, bde> - <bcf, ade> +
+ [bacfde] - 2<1, abcdef> -
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- <e, abcdf> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <abe, cdf> - <ace, bdf> - <bce, adf> -
- <f, abcde> - <af, bcde> - <bf, acde> -
- <cf, abde> - <abf, cde> - <acf, bde> - <bcf, ade> +
+ [cabfde] - 2<1, abcdef> -

- 2<a, bcdef> - 2<b, acdef> - 2<c, abdef> -
- 2<ab, cdef> - 2<ac, bdef> - 2<bc, adef> -
- <d, abcef> - <ad, bcef> - <bd, acef> -
- <cd, abef> - <abd, cef> - <acd, bef> - <bcd, aef> -
- <e, abcdf> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <abe, cdf> - <ace, bdf> - <bce, adf> -
- <f, abcde> - <af, bcde> - <bf, acde> -
- <cf, abde> - <abf, cde> - <acf, bde> - <bcf, ade> +
+ [cbafde] - 2<1, abcdef> -
- 2<a, bcdef> - 2<b, acdef> - 2<c, abdef> -
- 2<ab, cdef> - 2<ac, bdef> - 2<bc, adef> -
- <d, abcef> - <ad, bcef> - <bd, acef> -
- <cd, abef> - <abd, cef> - <acd, bef> - <bcd, aef> -
- <e, abcdf> - <ae, bcdf> - <be, acdf> -
- <ce, abdf> - <abe, cdf> - <ace, bdf> - <bce, adf> -
- <f, abcde> - <af, bcde> - <bf, acde> -
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3.3.1 Conjecture

$S'_n = \{\sigma \in S_n | \sigma(1) > \sigma(2) \ldots > \sigma(l) < \sigma(l+1) < \sigma(l+2) \ldots < \sigma(n), 1 \leq l \leq n\}$

S' is the set of ordered expression elements generated by X.

$$\{\ldots\{\{x_1, x_2\}, x_3\} \ldots x_n\} = \sum_{\sigma \in S'_n} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \ldots x_{\sigma(n)} -$$

$$-\sum_{k=3}^{n} 2^{k-2}(1 + 2^{n-k}(n-k)) \sum_{\substack{u = u_1 u_2 = x_1 x_2 \ldots x_n \\ |u_2| = n-k+3}} \langle u_1, u_2 \rangle,$$

where the length of u is equal to k.

Chapter 4

Conclusion

We have shown the definition of the Lie and Jordan criteria for a free definite algebra - this is one of the most popular questions in algebra. This issue has been resolved, or largely resolved, for some manifolds, but is still open for others. It is well known that manifold associative algebras are a classic illustration of this approach. For associative algebras, there is a well-known Dynkin-Specht-Wever criterion for Lie elements, but for Jordan elements the question remains open. P.M. Kohn gave a criterion for Jordan elements generated by three elements. In this note, we consider one of the generalizations of associative algebras as asosymmetric algebras and describe Jordan elements up to degree 5 in the free assymmetric algebra generated by three elements.

In the second part, we consider Jordan brackets in a free asymmetrical algebra. We have shown decompositions of left-normed Jordan brackets in a free asymmetrical algebra and made a conjecture that is still not wrong, but not perfect. In general, we have shown a proposal and some examples, and then a proof. For associative algebras P.M. Cohn gave the criterion for the Jordanian elements. generated by three elements, it was in the first part and then we switched to asymmetrical algebra not with three elements, but with four, and we showed five elements, the end stopped at the sixth power but we have manual work up to seven elements but this is not complete and what we have shown there the degree and the number of elements are equal. In general, we have shown a special case, but, after all, there are suggestions that it can work in any dimension n.

Appendix A

Appendix

A.1 conjecture

1) "Bulletin" EURASIAN MATHEMATICAL JOURNAL (EMJ) first Publication "JORDAN ELEMENTS IN ASSOSYMMETRIC ALGEBRAS"
2) "Conference speaker" AMLG Conference IMMM (2022) topic is "JORDAN ELEMENTS IN ASSOSYMMETRIC ALGEBRAS"
3) "Bulletin" Suleyman Demirel University Bulletin second pulication "JORDAN ELEMENTS IN ASSOSYMMETRIC ALGEBRAS N 5"
4) "Conference speaker" Young Scholars Conference (IIYSC 2022) "JORDAN ELEMENTS IN ASSOSYMMETRIC ALGEBRAS"

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