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Variety of special Tortken algebras

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Kaskelen, 2026

Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Meruyert Baigali

June 2026

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I am also thankful to the faculty members of the Department of Mathematics and Natural Sciences of SDU University for the knowledge and academic support provided throughout my master's studies.

Dedication

I dedicate this thesis to my beloved family: my parents, my brother, and my sisters.

Thank you for your endless love, support, patience, and belief in me. Your encouragement has been one of my greatest sources of strength throughout this journey.

Abstract

This dissertation studies special Tortken algebras arising from the symmetrization of Novikov algebras. The main object is the space of symmetric elements in the free Novikov algebra generated by one element. Using the differential realization of free Novikov algebras, the symmetrized product is written as $a \circ b = (ab)'$, which allows us to expand elements in the $\circ$-language into differential monomials and study their linear relations by methods of linear algebra. For the homogeneous component T_n of degree n, we describe the dimensions and obtain $\dim T_1 = \dim T_2 = 1$ and $\dim T_n = p(n-2)$ for $n \geq 3$, where $p(n)$ is the partition function. Low-degree components are constructed explicitly, and special attention is given to relations in degrees 6 and 7. The computations suggest a recursive structure for the spaces T_n and contribute to the study of free special Tortken algebras.

Keywords: Novikov algebra, Tortken algebra, special Tortken algebra, free algebra, symmetrized product, symmetric elements.

Аңдатпа

Бұл диссертация Новиков алгебраларын симметриялау арқылы алынатын арнайы Торткен алгебраларын зерттеуге арналған. Жұмыстың негізгі объектісі — бір генератормен берілген еркін Новиков алгебрасындағы симметриялық элементтер кеңістігі. Еркін Новиков алгебраларының дифференциалдық жүзеге асырылуын қолдану арқылы симметрияланған көбейтінді $a \circ b = (ab)'$ түрінде жазылады. Бұл $\circ$-тіліндегі элементтерді дифференциалдық мономдарға жіктеуге және олардың арасындағы сызықтық қатынастарды сызықтық алгебра әдістерімен зерттеуге мүмкіндік береді. n-дәрежелі біртекті компонент T_n үшін өлшемдер сипатталып, $\dim T_1 = \dim T_2 = 1$ және $n \geq 3$ үшін $\dim T_n = p(n-2)$ екені алынады. Мұндағы $p(n)$ — бөлшектеу функциясы. Төмен дәрежелердегі компоненттер нақты құрылады, ал 6 және 7-дәрежелердегі қатынастарға ерекше көңіл бөлінеді.

Түйін сөздер: Новиков алгебрасы, Торткен алгебрасы, арнайы Торткен алгебрасы, еркін алгебра, симметрияланған көбейтінді, симметриялық элементтер.

Аннотация

Данная диссертация посвящена изучению специальных алгебр Торткена, возникающих при симметризации алгебр Новикова. Основным объектом работы является пространство симметрических элементов в свободной алгебре Новикова с одним порождающим. С помощью дифференциальной реализации свободных алгебр Новикова симметризованное произведение записывается в виде $a \circ b = (ab)'$. Это позволяет раскладывать элементы, записанные на языке операции $\circ$, в дифференциальные мономы и изучать линейные отношения между ними методами линейной алгебры. Для однородной компоненты T_n степени n описываются размерности и получаются равенства $\dim T_1 = \dim T_2 = 1$ и $\dim T_n = p(n-2)$ при $n \geq 3$, где $p(n)$ — функция числа разбиений. В работе явно строятся компоненты малых степеней, а также рассматриваются отношения в степенях 6 и 7.

Ключевые слова: алгебра Новикова, алгебра Торткена, специальная алгебра Торткена, свободная алгебра, симметризованное произведение, симметрические элементы.

Abbreviations

F	A field of characteristic zero
A	An algebra over a field F
X	A set of generators
Nov$\langle X\rangle$	The free Novikov algebra generated by X
Nov$\langle x\rangle$	The one-generated free Novikov algebra
$F\{X\}$	The differential polynomial algebra generated by X
D	A derivation
$x^{(k)}$	The k-th derivative of x, that is, $D^k(x)$
u'	The derivative $D(u)$
deg(u)	The standard degree of a differential monomial u
$d(u)$	The differential degree of a differential monomial u
$\cdot$	The Novikov product
$\circ$	The symmetrized product, $a \circ b = a \cdot b + b \cdot a$
(a, b, c)	The associator $(ab)c - a(bc)$
$(a, b, c)_\circ$	The associator with respect to the product $\circ$
$S\langle X\rangle$	The space of symmetric elements in Nov$\langle X\rangle$
$S\langle x\rangle$	The space of symmetric elements in Nov$\langle x\rangle$
T_n	The homogeneous component of degree n of $S\langle x\rangle$
$p(n)$	The number of partitions of n
S_n	The symmetric group on n elements
dim V	The dimension of a vector space V

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Chapter 1

Introduction

1.1 Motivation and Background

Nonassociative algebras form an important part of modern algebra. In contrast to associative algebras, the multiplication in a nonassociative algebra does not necessarily satisfy

$$(ab)c = a(bc).$$

Because of this, the order of brackets becomes essential, and the structure of the algebra may be much more flexible. Many important classes of algebras, including Lie algebras, Jordan algebras, right-symmetric algebras, and Novikov algebras, belong to the theory of nonassociative algebras. General background on nonassociative algebras and polynomial identities can be found, for example, in [3, 11, 12].

In this work, we focus on Novikov algebras and on the algebras obtained from them by symmetrization. Novikov algebras first appeared in the study of Hamiltonian operators and Poisson brackets of hydrodynamic type. The connection between Hamiltonian operators and algebraic structures was developed by Gel'fand and Dorfman [9], while Poisson brackets of hydrodynamic type were studied by Balinskii and Novikov [1]. Later, Novikov algebras became an independent object of study in nonassociative algebra.

A useful way to study identities in a class of algebras is to work with free algebras. A free algebra contains all formal consequences of the defining identities of the variety, but it does not contain additional relations coming from a particular example. Therefore, identities obtained in a free algebra are universal. This is one of the reasons why free Novikov algebras are important in the present work.

The structure of free Novikov algebras was studied by Dzhumadil'daev and Löfwall, who described them using differential polynomial algebras and differential monomials [8]. This realization is especially convenient because it allows one to translate nonassociative expressions into differential polynomials. After such a translation, questions about bases, dimensions, and identities can be studied using linear algebra.

The main construction used in this work is the symmetrization of the Novikov prod-

uct. If $(A, \cdot)$ is a Novikov algebra, then we define a new product by

$$a \circ b = a \cdot b + b \cdot a.$$

This product is commutative, but it is not necessarily associative. The algebras obtained from Novikov algebras in this way satisfy the Tortken identity and are called special Tortken algebras.

The identities satisfied by the symmetrized Novikov product were studied by Dzhumadil'daev in the context of Novikov-Jordan algebras [4, 5]. More recently, Dzhumadil'daev and Ismailov studied the symmetrization of the Novikov product and described the space of elements closed under this product in free Novikov algebras [7]. This result is the main theoretical foundation for the present dissertation.

In this work, we study the one-generated case. Although this case is simpler than the general case with several generators, it is already rich enough to show nontrivial relations and identities. In particular, it allows us to compare two descriptions of the same space: the differential description and the description in the language of the symmetrized product.

1.2 Problem Statement

Let $X = \{x\}$. We work with the free Novikov algebra on this single generator and denote it by

$$\mathrm{Nov}\langle x \rangle.$$

Inside this algebra, we consider the subspace generated from x by repeated use of the symmetrized product

$$a \circ b = ab + ba.$$

This subspace will be denoted by $S\langle x \rangle$. Equivalently, $S\langle x \rangle$ can be described as the intersection of all subspaces of $\mathrm{Nov}\langle x \rangle$ that contain x and are stable with respect to the product $\circ$ [7].

For every $n \geq 1$, we write T_n for the degree-n homogeneous part of $S\langle x \rangle$. The known differential basis for the space of Tortken elements makes it possible to compute the dimensions of these spaces. However, this basis is given in terms of differential monomials, and therefore it does not immediately explain how the same elements can be represented through the symmetrized product $\circ$.

Thus, the main problem considered in this dissertation is to describe the spaces T_n not only by differential monomials, but also in the language of the product $\circ$. This point of view is important because the product $\circ$ is the natural product appearing in special Tortken algebras. Consequently, the $\circ$-language reflects the structure of special Tortken algebras more directly.

Natural $\circ$-monomials may become linearly dependent after expansion into differential monomials. Because of this, one has to determine which $\circ$-monomials form a basis and which identities appear between them.

The main questions considered in this work are:

1. How can the dimensions of the homogeneous components T_n be described?
2. Which elements form bases of T_n in low degrees?
3. What linear relations appear between natural $\circ$-monomials?
4. How are the relations in degrees 6 and 7 connected with Tortken and special identities?
5. Is there a recursive way to construct the spaces T_n from lower-degree components?

1.3 Aim and Objectives

This dissertation aims to study the homogeneous components of the space of Tortken elements in the one-generated free Novikov algebra and to describe their structure in the language of the Tortken product.

To achieve this aim, we consider the following objectives:

1. to recall the necessary background on nonassociative algebras, polynomial identities, varieties, and free algebras;
2. to describe Novikov algebras and their realization in differential polynomial algebras;
3. to explain the Tortken product and its relation to Tortken and special Tortken algebras;
4. to define the homogeneous components T_n of the space of Tortken elements in the one-generated case;
5. to prove the dimension formula

$$\dim T_1 = \dim T_2 = 1, \qquad \dim T_n = p(n-2) \quad (n \geq 3),$$

   where $p(n)$ denotes the number of partitions of n;
6. to construct basis elements of T_n in low degrees using the $\circ$-language;
7. to find linear relations among natural $\circ$-monomials in degrees 6 and 7;
8. to formulate a recursive inclusion for the spaces T_n.

1.4 Methodology

The main method used in this dissertation is based on the differential realization of free Novikov algebras. In this realization, the Novikov product is defined by

$$a \cdot b = D(a)b,$$

where D is a derivation. Therefore, the Tortken product has the form

$$a \circ b = a \cdot b + b \cdot a = (ab)'.$$

This formula is the main computational tool of the work. It allows us to take an element written in the $\circ$-language and expand it into differential monomials. Once this is done, the element can be written in a fixed differential basis.

1.5 Main Results and Contributions

The main contribution of this dissertation is the explicit study of the first homogeneous components

$$T_n$$

of the space of Tortken elements in the free Novikov algebra with one generator element.

First, we describe the dimensions of these components. We show that

$$\dim T_1 = \dim T_2 = 1,$$

and for $n \geq 3$,

$$\dim T_n = p(n-2),$$

where $p(n)$ is the partition function.

Second, we construct the first components in the $\circ$-language. In particular, we study the spaces

$$T_n, \quad where \quad n \in \{1, 2, 3, 4, 5, 6, 7\}$$

and compare the natural $\circ$-monomials with the differential basis.

Third, we find linear relations in degrees 6 and 7. In degree 6, the relation is a consequence of the Tortken identity. In degree 7, additional relations appear and are obtained by null-space computations. These relations show that the structure in the $\circ$-language is more subtle than the dimension formula alone suggests.

Finally, the computations lead to the recursive inclusion

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2.$$

This inclusion has been verified in low degrees and suggests a possible recursive construction of the spaces T_n.

1.6 Novelty of the Research

The novelty of this research lies in the description of the one-generated space of Tortken elements in the $\circ$-language.

The differential basis of the space of symmetric elements is known from [7]. However, this basis is expressed in terms of differential monomials. In the present disserta-

tion, we focus on how these elements are represented using the Tortken product. This is important because the Tortken product is the operation defining special Tortken algebras.

The computations in low degrees show where linear dependencies appear and how identities arise among natural $\circ$-monomials. In particular, degrees 6 and 7 reveal relations connected with Tortken and special identities.

1.7 Theoretical Significance

The results of this dissertation are theoretical in nature. They contribute to the study of nonassociative algebras, especially Novikov algebras, Tortken algebras, and special identities.

The work helps to understand how special Tortken algebras arise from free Novikov algebras. It also shows how the differential realization can be used to study identities and bases in the language of the symmetrized product.

The low-degree computations may be useful for further investigation of the variety of special Tortken algebras. They also suggest a recursive structure for the homogeneous components of the one-generated free special Tortken algebra.

1.8 Structure of the Dissertation

The dissertation consists of six chapters.

Chapter 1 presents the motivation, background, problem statement, aim and objectives, methodology, main results, novelty, theoretical significance, and structure of the dissertation.

Chapter 2 recalls basic notions from the theory of nonassociative algebras, including polynomial identities, varieties of algebras, free algebras, homogeneous components, bases, and the partition function.

Chapter 3 introduces Novikov algebras and free Novikov algebras. It also describes the differential polynomial realization of free Novikov algebras and the corresponding basis in terms of differential monomials.

Chapter 4 discusses the symmetrized product of a Novikov algebra and introduces Tortken algebras and special Tortken algebras. It also defines the space of symmetric elements in a free Novikov algebra.

Chapter 5 contains the main results of the dissertation. It studies the homogeneous components T_n in the one-generated case, proves the dimension formula, describes low-degree bases, and discusses relations in degrees 6 and 7.

Chapter 6 describes the computational methodology used in the dissertation. It explains the expansion into differential monomials, coefficient matrices, and null-space computations. It also contains the discussion, limitations, conclusion, and directions for future work.

Chapter 2

Preliminaries on Nonassociative Algebras

This chapter contains the basic algebraic notions that will be used in the rest of the dissertation. We recall the definitions of algebras over a field, nonassociative products, polynomial identities, varieties of algebras, free algebras, homogeneous components, and bases. These concepts are standard in the theory of nonassociative algebras and polynomial identities; see, for example, [3, 10, 11, 12].

Throughout this dissertation, all algebras are considered over a field F of characteristic zero, unless otherwise stated.

2.1 Algebras over a Field

Let F be a field. An algebra over F is a vector space A over F together with a bilinear multiplication

$$A \times A \to A, \qquad (a, b) \mapsto ab.$$

Thus, an algebra is not only a vector space: in addition to addition and scalar multiplication, its elements can also be multiplied.

Many familiar algebraic structures are associative. However, in this dissertation we work with nonassociative algebras. This means that we do not assume that

$$(ab)c = a(bc)$$

holds for all $a, b, c \in A$. Therefore, in a nonassociative algebra the placement of brackets is important. For example, the expressions

$$(ab)c \qquad \text{and} \qquad a(bc)$$

may represent different elements.

The failure of associativity is measured by the associator [11]. For $a, b, c \in A$, the

associator is defined by

$$(a, b, c) = (ab)c - a(bc).$$

An algebra is associative if and only if

$$(a, b, c) = 0$$

for all $a, b, c \in A$.

We will also use two standard operations constructed from a given multiplication. The commutator is defined by

$$[a, b] = ab - ba,$$

while the anticommutator is defined by

$$a \circ b = ab + ba.$$

The commutator is skew-symmetric ($[a, b] = -[b, a]$), whereas the anticommutator is symmetric ($a \circ b = b \circ a$) [11]. The second operation is especially important in this work, because special Tortken algebras arise from Novikov algebras by symmetrizing the Novikov product.

2.2 Subalgebras and Generated Subalgebras

Let A be an algebra over F. A subspace $B \subseteq A$ is called a subalgebra if it is closed under the multiplication of A. This means that

$$b_1 b_2 \in B$$

for any $b_1, b_2 \in B$. In this case, B itself becomes an algebra with the product inherited from A.

If $S \subseteq A$ is a subset, then the subalgebra generated by S is the smallest subalgebra of A that contains S. It is usually denoted by

$$\langle S \rangle.$$

Equivalently, $\langle S \rangle$ consists of all elements that can be obtained from the elements of S by using multiplication, brackets, and finite linear combinations.

For example, if $S = \{x_1, \ldots, x_n\}$, then $\langle S \rangle$ consists of all nonassociative polynomials in the generators $x_1, \ldots, x_n$, interpreted inside the algebra A. This idea will be used later when we consider the free Novikov algebra generated by a set X.

2.3 Nonassociative Polynomials

Let $X = \{x_1, x_2, \ldots\}$ be a set of variables. A nonassociative monomial is a product of variables with a fixed arrangement of brackets. Unlike the associative case, different bracketings give different monomials.

For example, in degree three the expressions

$$(x_1x_2)x_3 \qquad \text{and} \qquad x_1(x_2x_3)$$

are different monomials. They contain the same variables in the same order, but their bracketings are not the same.

A nonassociative polynomial is a finite linear combination of such monomials. For example,

$$(x_1x_2)x_3 - x_1(x_2x_3)$$

is a nonassociative polynomial. It is also the associator

$$(x_1, x_2, x_3).$$

The degree of a monomial is the number of variables occurring in it, counted with multiplicity. Thus,

$$(x_1x_2)x_3$$

has degree 3, while

$$(x_1x_1)(x_2x_3)$$

has degree 4.

A polynomial is called homogeneous if all its monomials have the same degree. It is called multilinear if each variable occurs exactly once in every monomial. For example,

$$(x_1x_2)x_3 - x_1(x_2x_3)$$

is multilinear, whereas

$$(x_1x_1)x_2 - x_1(x_1x_2)$$

is homogeneous but not multilinear.

These notions are useful because many questions about polynomial identities can be studied degree by degree. In particular, the computations in this dissertation are carried out in fixed homogeneous components.

2.4 Polynomial Identities

Let A be an algebra over a field F. Consider a nonassociative polynomial $f(x_1, \ldots, x_n)$ with coefficients in F. We call $f = 0$ an identity of the algebra A if the value of f is zero after every substitution of elements of A for its variables. In other words,

$$f(a_1, \ldots, a_n) = 0$$

for all $a_1, \ldots, a_n \in A$.

For example, the polynomial

$$(x_1x_2)x_3 - x_1(x_2x_3)$$

is an identity of A exactly when A is associative. Similarly,

$$x_1x_2 - x_2x_1$$

is an identity of A exactly when A is commutative.

Polynomial identities are a natural language for describing classes of nonassociative algebras. For example, Lie algebras, Jordan algebras, right-symmetric algebras, Novikov algebras, and Tortken algebras are all defined by certain polynomial identities. In this dissertation, identities play an essential role because the passage from Novikov algebras to special Tortken algebras is controlled by identities satisfied by the symmetrized product.

2.5 Varieties of Algebras

Let $\mathcal{S}$ be a collection of polynomial identities. We denote by $\mathrm{Var}(\mathcal{S})$ the class of all algebras of the same type in which every identity from $\mathcal{S}$ holds. Such a class is called the variety defined by $\mathcal{S}$ [3, 6].

For example, associative algebras are defined by the identity

$$(x_1x_2)x_3 - x_1(x_2x_3) = 0.$$

Commutative algebras are defined by

$$x_1x_2 - x_2x_1 = 0.$$

If both identities are imposed, one obtains the variety of commutative associative algebras.

The notion of a variety is useful because it allows one to study a whole class of algebras through the identities common to all its members. This point of view is especially relevant for Novikov and Tortken algebras, since both of them are defined by polynomial identities. In this dissertation, we use this language in connection with the variety of special Tortken algebras.

If A is an algebra, the variety generated by A, denoted by

$$\mathrm{Var}(A),$$

is the class of all algebras satisfying every polynomial identity that holds in A. Therefore, describing $\mathrm{Var}(A)$ is closely related to describing the identities of A.

2.6 Free Algebras

Free algebras play an important role in the study of varieties. Informally, a free algebra is the algebra in a given variety whose elements are formed from a chosen set of variables, subject only to the identities that define the variety.

Let $\mathcal{V}$ be a variety of algebras and let X be a set. We denote by

$$F_{\mathcal{V}}(X)$$

the free algebra of the variety $\mathcal{V}$ on the set X.

The defining property of $F_{\mathcal{V}}(X)$ is the following. Whenever A is an algebra from $\mathcal{V}$, any assignment of the generators X to elements of A,

$$\varphi : X \to A,$$

can be extended in exactly one way to a homomorphism

$$\overline{\varphi} : F_{\mathcal{V}}(X) \to A.$$

This property means that once the images of the generators are chosen, the images of all elements of the free algebra are determined uniquely. In this sense, free algebras contain all formal expressions in the generators, considered modulo the identities of the variety.

In this dissertation, the most important example is the free Novikov algebra. We use it because identities and relations found in a free Novikov algebra are universal: they are consequences of the defining identities of Novikov algebras and therefore hold after substitution in every Novikov algebra.

2.7 Homogeneous Components

In many cases, free algebras are studied degree by degree. This means that the free algebra $F_{\mathcal{V}}(X)$ can be written as a direct sum of its homogeneous components:

$$F_{\mathcal{V}}(X) = \bigoplus_{n \geq 1} F_{\mathcal{V}}^{(n)}(X).$$

Here $F_{\mathcal{V}}^{(n)}(X)$ denotes the part of $F_{\mathcal{V}}(X)$ consisting of homogeneous elements of total degree n.

The component $F_{\mathcal{V}}^{(n)}(X)$ consists of linear combinations of monomials involving n variables, counted with multiplicity.

This degree-by-degree approach is used throughout the dissertation. In the one-generated case, we study the homogeneous components of the space of symmetric elements and denote them by

$$T_n.$$

Here T_n is the component of degree n. The main computations concern the structure of these spaces in low degrees.

2.8 Bases and Linear Dependence

Let V be a vector space over a field F. Consider a family of vectors

$$\{v_i\}_{i \in I} \subset V.$$

We say that this family has no nontrivial linear relation if, whenever a finite linear combination of its elements is equal to zero,

$$\alpha_1 v_{i_1} + \cdots + \alpha_m v_{i_m} = 0,$$

all coefficients must be zero:

$$\alpha_1 = \cdots = \alpha_m = 0.$$

In this case, the family is linearly independent.

A span of a set of vectors in V implies that any vector from V can be written as a finite linear combination of these vectors. Any linearly independent set that spans the vector space is called a basis for V.

If V has a basis which is finite, then the number of elements in the basis set is known as the dimension of V and is denoted by

$$\dim V.$$

In the study of free algebras, the choice of a suitable basis is important. After a basis is fixed, each element can be represented in a unique way as a linear combination of basis elements. Therefore, relations between algebraic expressions can be studied by comparing the coefficients of their expansions.

This idea is used in the later chapters of the dissertation. Elements written in the language of the symmetrized product are expanded into differential monomials. Then linear dependencies and polynomial identities are found by forming coefficient matrices and computing their null spaces.

2.9 The Partition Function

It will be helpful to use the partition function. For a non-negative integer n, the number of partitions of n is denoted by

$$p(n).$$

A partition of n is an expression of n as a sum of positive integers without considering the order of summands.

Thus, the partitions of 4 are

$$4, \qquad 3+1, \qquad 2+2, \qquad 2+1+1, \qquad 1+1+1+1.$$

Therefore,

$$p(4) = 5.$$

The first values of the partition function are

$$p(0) = 1, \quad p(1) = 1, \quad p(2) = 2, \quad p(3) = 3, \quad p(4) = 5, \quad p(5) = 7.$$

The partition function appears naturally in the one-generated case considered in this dissertation. In particular, the dimensions of the homogeneous components T_n are expressed in terms of $p(n-2)$.

In this chapter, we recalled the main algebraic notions needed for the dissertation. We discussed algebras over a field, nonassociative products, associators, polynomial identities, varieties, free algebras, homogeneous components, bases, and the partition function.

These notions provide the general framework for the next chapter, where we specialize to Novikov algebras and describe their realization in differential polynomial algebras.

Chapter 3

Novikov Algebras and Free Novikov Algebras

In this chapter, we recall the definition of Novikov algebras and describe the construction of free Novikov algebras by means of differential polynomial algebras. This realization is important for the present work, because it allows us to replace abstract nonassociative expressions by differential polynomials. As a result, questions about bases, dimensions, and identities can be studied using methods of linear algebra.

Throughout this chapter, F denotes a field of characteristic zero.

3.1 Definition of Novikov Algebras

Novikov algebras appeared in the study of Hamiltonian operators and Poisson brackets of hydrodynamic type [1, 9]. Later, they became an important object in the theory of nonassociative algebras, especially in connection with polynomial identities, free algebras, and operads [2, 6, 8].

Let A be an algebra over a field F with a bilinear product

$$(a, b) \mapsto ab.$$

The algebra A is called a right Novikov algebra if it satisfies the identities

$$(ab)c - a(bc) = (ac)b - a(cb), \tag{3.1.1}$$

and

$$a(bc) = b(ac) \tag{3.1.2}$$

for all $a, b, c \in A$.

The first identity can be written using the associator

$$(a, b, c) = (ab)c - a(bc).$$

Then (3.1.1) becomes

$$(a, b, c) = (a, c, b).$$

Thus, in a right Novikov algebra, the associator is symmetric in the second and third arguments. The second identity (3.1.2) is often called right-commutativity.

There is also a left version of Novikov algebras. The two versions are closely related by passing to the opposite multiplication. In the literature, both conventions are used. In this dissertation, we use the right Novikov convention, since it is compatible with the differential realization that will be used below.

Novikov algebras form a variety of nonassociative algebras, because they are defined by polynomial identities. This allows us to consider free Novikov algebras and study universal relations that follow only from the Novikov identities.

For this dissertation, Novikov algebras are important mainly because they provide a natural source of Tortken algebras. If A is a Novikov algebra with product ab, then one can define the symmetrized product

$$a \circ b = ab + ba.$$

This product is commutative, but in general it is not associative. Nevertheless, it satisfies the Tortken identity. Therefore, special Tortken algebras arise from Novikov algebras by symmetrization.

This connection explains why the study of special Tortken algebras naturally begins with Novikov algebras and, more specifically, with free Novikov algebras. A free Novikov algebra contains all formal consequences of the Novikov identities and does not contain additional relations coming from a particular example.

3.2 Derivations and Differential Algebras

A useful way to construct Novikov algebras is through differential algebras. Let A be a commutative associative algebra over F. A linear map

$$D : A \to A$$

is called a derivation if it satisfies the Leibniz rule

$$D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$.

A commutative associative algebra equipped with a derivation is called a differential algebra. Such algebras give a standard construction of Novikov algebras [8, 9]. Namely, define a new product by

$$a \cdot b = D(a)b. \tag{3.2.1}$$

Then A becomes a Novikov algebra with respect to this product.

Let us verify this directly. For $a, b, c \in A$, we have

$$(a \cdot b) \cdot c = D(D(a)b)c = (D^2(a)b + D(a)D(b))c,$$

while

$$a \cdot (b \cdot c) = D(a)(D(b)c) = D(a)D(b)c.$$

Therefore,

$$(a, b, c) = (a \cdot b) \cdot c - a \cdot (b \cdot c) = D^2(a)bc.$$

Similarly,

$$(a, c, b) = D^2(a)cb.$$

Since the algebra A is commutative, $bc = cb$, and hence

$$(a, b, c) = (a, c, b).$$

Thus, the first Novikov identity holds.

For the second identity, we compute

$$a \cdot (b \cdot c) = D(a)D(b)c$$

and

$$b \cdot (a \cdot c) = D(b)D(a)c.$$

Again, by commutativity,

$$D(a)D(b)c = D(b)D(a)c.$$

Hence,

$$a \cdot (b \cdot c) = b \cdot (a \cdot c).$$

Thus, the product (3.2.1) defines a Novikov algebra.

This construction is not only a source of examples. It is also the main tool for realizing free Novikov algebras inside differential polynomial algebras.

3.3 Differential Polynomial Algebra

Let

$$X = \{x_1, x_2, \ldots, x_n\}$$

be a set of variables. For each variable x_i, we introduce formal derivatives

$$x_i, \quad x_i', \quad x_i'', \quad \ldots, \quad x_i^{(k)}, \quad \ldots$$

where

$$x_i^{(k)} = D^k(x_i).$$

The differential polynomial algebra generated by X is denoted by

$$F\{X\}.$$

It is the commutative associative polynomial algebra in the variables

$$x_i^{(k)}, \qquad 1 \le i \le n, \quad k \ge 0,$$

equipped with the derivation D defined by

$$D(x_i^{(k)}) = x_i^{(k+1)}.$$

A typical differential monomial has the form

$$u = x_{i_1}^{(k_1)} x_{i_2}^{(k_2)} \cdots x_{i_m}^{(k_m)}.$$

Since $F\{X\}$ is commutative, the order of factors is not important.

For such a monomial, we use two degree functions. The standard degree is the number of factors:

$$\deg(u) = m.$$

The differential degree is the sum of the orders of derivatives:

$$d(u) = k_1 + k_2 + \cdots + k_m.$$

For example, if

$$u = x_1'' x_2' x_3,$$

then

$$\deg(u) = 3$$

and

$$d(u) = 2 + 1 + 0 = 3.$$

The set of all differential monomials forms a basis of the vector space $F\{X\}$. Therefore, every differential polynomial can be written uniquely as a finite linear combination of differential monomials.

3.4 Free Novikov Algebra in a Differential Polynomial Algebra

The differential polynomial algebra $F\{X\}$, equipped with the product

$$a \cdot b = D(a)b,$$

is a Novikov algebra. We denote by

$$\mathrm{Nov}\langle X\rangle$$

the subalgebra of this Novikov algebra generated by the set X.

Thus, $\mathrm{Nov}\langle X\rangle$ consists of all elements that can be obtained from the variables $x_1, \ldots, x_n$ by applying the Novikov product and taking finite linear combinations.

A fundamental result states that this algebra is the free Novikov algebra generated by X [8]. This means that the abstract free Novikov algebra can be studied as a concrete subalgebra of the differential polynomial algebra $F\{X\}$.

This realization is especially useful for computations. Instead of working only with bracketed nonassociative expressions, we can expand them into differential monomials. Then linear relations and identities can be found by comparing coefficients.

3.5 A Basis of a Free Novikov Algebra

The differential realization gives a simple description of a basis of the free Novikov algebra. Let $\mathcal{M}$ be the set of all differential monomials in $F\{X\}$. Then the set

$$\mathcal{N} = \{u \in \mathcal{M} \mid \deg(u) - d(u) = 1\} \tag{3.5.1}$$

forms a basis of $\mathrm{Nov}\langle X\rangle$ [7, 8].

This criterion is convenient because it allows us to check whether a differential monomial belongs to the basis of the free Novikov algebra. The condition is simply

$$\deg(u) - d(u) = 1.$$

Let us consider a few examples. For $u = x_i$, we have

$$\deg(x_i) = 1, \qquad d(x_i) = 0,$$

so

$$\deg(x_i) - d(x_i) = 1.$$

Hence, x_i is a basis element.

For

$$u = x_i' x_j,$$

we have

$$\deg(u) = 2, \qquad d(u) = 1.$$

Thus,

$$\deg(u) - d(u) = 1,$$

so $x_i' x_j$ also belongs to the basis.

Similarly, for

$$u = x_i'' x_j x_k,$$

we have

$$\deg(u) = 3, \qquad d(u) = 2,$$

and hence

$$\deg(u) - d(u) = 1.$$

Therefore, this monomial is also a basis element of the free Novikov algebra.

On the other hand, the monomial

$$v = x_i' x_j'$$

has

$$\deg(v) = 2, \qquad d(v) = 2.$$

Thus,

$$\deg(v) - d(v) = 0.$$

Therefore, v is not a basis monomial of $\mathrm{Nov}\langle X\rangle$.

3.6 The One-Generated Case

In this dissertation, we mainly work with the one-generated case

$$X = \{x\}.$$

The corresponding free Novikov algebra is denoted by

$$\mathrm{Nov}\langle x\rangle.$$

In this case, differential monomials have the form

$$u = x^{(k_1)} x^{(k_2)} \cdots x^{(k_m)}.$$

The basis condition

$$\deg(u) - d(u) = 1$$

becomes

$$m - (k_1 + k_2 + \cdots + k_m) = 1.$$

Equivalently,

$$k_1 + k_2 + \cdots + k_m = m - 1.$$

This gives a clear degree-by-degree description. In degree 1, the basis consists of

$$x.$$

In degree 2, the basis element is

$$x'x.$$

In degree 3, the basis elements are

$$x''xx$$

and

$$x'x'x.$$

Indeed, both monomials have standard degree 3 and differential degree 2, so they satisfy the condition

$$\deg(u) - d(u) = 1.$$

This simple description becomes very useful later.

3.7 Homogeneous Components of a free Novikov algebra

The free Novikov algebra is naturally graded by degree:

$$\mathrm{Nov}\langle X\rangle = \bigoplus_{m\geq 1} \mathrm{Nov}^{(m)}\langle X\rangle,$$

where $\mathrm{Nov}^{(m)}\langle X\rangle$ is the homogeneous component of degree m.

The component $\mathrm{Nov}^{(m)}\langle X\rangle$ is spanned by all basis monomials u such that

$$\deg(u) = m$$

and

$$\deg(u) - d(u) = 1.$$

Therefore, for basis monomials of degree m, the differential degree is fixed:

$$d(u) = m - 1.$$

This observation is useful in computations because it reduces the problem to the enumeration of differential monomials of a fixed standard degree and a fixed differential degree.

In the multilinear case, where each of the variables $x_1, \ldots, x_n$ appears exactly once, the dimension of the multilinear component of the free Novikov algebra is known:

$$\dim F_n^{\text{multi}} = \binom{2n-2}{n-1}$$

[6, 7]. This formula shows that the size of the components grows quickly, which explains why explicit computations are usually restricted to low degrees.

It is important to explain why the free Novikov algebra is the natural setting for this

dissertation.

If we worked with one particular Novikov algebra, then the relations we found could depend on special properties of that algebra. Such relations would not necessarily hold in all Novikov algebras. By contrast, the free Novikov algebra contains only the relations that follow from the defining Novikov identities.

Therefore, any identity found in the free Novikov algebra is universal. After substituting elements of any Novikov algebra for the generators, the same identity remains valid. This is why free algebras are the correct objects for studying identities and varieties.

In our case, the free Novikov algebra is used as the ambient algebra in which the symmetrized product is considered. The space generated by this product is the main object of the next chapters.

In this chapter, we introduced Novikov algebras and recalled their defining identities. We also discussed their construction from commutative associative differential algebras.

The main point of the chapter is the differential realization of free Novikov algebras. The free Novikov algebra $\mathrm{Nov}\langle X\rangle$ can be realized inside the differential polynomial algebra $F\{X\}$, and its basis is given by differential monomials.

This description will be used in the next chapter to study the symmetrized product and the corresponding space of symmetric elements.

Chapter 4

Symmetrization and Special Tortken Algebras

In the previous chapter, we recalled the definition of Novikov algebras and described their realization inside differential polynomial algebras. In this chapter, we pass from Novikov algebras to algebras obtained by symmetrizing the Novikov product. This construction leads naturally to Tortken algebras and, in particular, to special Tortken algebras.

The main purpose of this chapter is to explain how the symmetrized product is defined, why it satisfies the Tortken identity, and how the space of symmetric elements appears in a free Novikov algebra. These notions form the theoretical background for the computations in the following chapters.

Throughout this chapter, F is a field of characteristic zero, and A is a Novikov algebra over F.

4.1 The Symmetrized Product

Let $(A, \cdot)$ be a Novikov algebra. Starting from the Novikov product $a \cdot b$, we define a new product on the same vector space A by

$$a \circ b = a \cdot b + b \cdot a. \tag{4.1.1}$$

This product is called the symmetrized product, or the anticommutator, of the Novikov product. Also, we call it a Tortken product.

The product $\circ$ is commutative. Indeed,

$$b \circ a = b \cdot a + a \cdot b = a \cdot b + b \cdot a = a \circ b.$$

However, it is not necessarily associative. In general,

$$(a \circ b) \circ c$$

and

$$a \circ (b \circ c)$$

do not have to be equal. Therefore, after symmetrization, we obtain a commutative but still nonassociative algebra.

The associator with respect to the product $\circ$ will be denoted by

$$(a, b, c)_\circ = a \circ (b \circ c) - (a \circ b) \circ c.$$

This notation helps us to differentiate it from the associator of the original Novikov product.

Symmetrization is a common construction in nonassociative algebra. For example, Jordan algebras can be obtained by symmetrizing associative algebras. In a similar way, special Tortken algebras arise from Novikov algebras by symmetrizing the Novikov product.

4.2 Symmetrization in Differential Form

The differential realization of free Novikov algebras makes the symmetrized product especially transparent. Recall that if A is a commutative associative differential algebra with derivation D, then the product

$$a \cdot b = D(a)b$$

defines a Novikov algebra structure.

In this case, the symmetrized product becomes

$$a \circ b = a \cdot b + b \cdot a.$$

Substituting the definition of the Novikov product, we get

$$a \circ b = D(a)b + D(b)a.$$

Since the underlying algebra is commutative and associative, this expression is exactly the derivative of the product ab:

$$a \circ b = (ab)'. \tag{4.2.1}$$

This simple formula is one of the main reasons why the differential realization is useful. It allows us to translate expressions in the $\circ$-language into differential polynomials.

For example,

$$x \circ x = (x^2)' = 2x'x.$$

Similarly,

$$(x \circ x) \circ x = \big((x \circ x)x\big)'.$$

Since

$$x \circ x = 2x'x,$$

we obtain

$$(x \circ x) \circ x = (2x'x^2)' = 2x''x^2 + 4(x')^2x.$$

Thus, each expression built from the symmetrized product can be expanded into differential monomials. This is the computational idea used later in the dissertation: we write elements in the $\circ$-language, expand them using (4.2.1), and then compare coefficients.

4.3 The Tortken Identity and Besken Identity

Let $(A, \circ)$ be a commutative algebra. Dzhumadil'daev [4, 5] proved that the symmetrized product satisfies the following identities:

$$(a \circ b) \circ (c \circ d) - (a \circ d) \circ (c \circ b) = (a, b, c)_\circ \circ d - (a, d, c)_\circ \circ b, \tag{4.3.1}$$

$$\begin{aligned}
&(((a \circ a) \circ a) \circ b) \circ b + (((a \circ b) \circ b) \circ a) \circ a \\
&\quad + 2(((a \circ a) \circ b) \circ b) \circ a + 2(((a \circ b) \circ a) \circ a) \circ b \\
&\quad - 3(((a \circ a) \circ b) \circ a) \circ b - 3(((a \circ b) \circ a) \circ b) \circ a = 0
\end{aligned} \tag{4.3.2}$$

where

$$(a, b, c)_\circ = a \circ (b \circ c) - (a \circ b) \circ c.$$

The first identity of degree four is called the Tortken identity, and the second identity is the Besken identity. Algebras that satisfy (4.3.1) are called Tortken algebras.

4.4 Special Tortken Algebras

Let $(A, \cdot)$ be a Novikov algebra. Denote by $A^{(+)}$ the algebra obtained from A by replacing the original product with the symmetrized product. As explained above, $A^{(+)}$ satisfies the Tortken identity. Hence $A^{(+)}$ is a Tortken algebra.

A Tortken algebra is called special if it can be embedded into $A^{(+)}$ for some Novikov algebra A. Thus, special Tortken algebras are not arbitrary Tortken algebras; they are those that come from Novikov algebras by symmetrization.

This distinction is similar to the distinction between Jordan algebras and special Jordan algebras. A special Jordan algebra is obtained from an associative algebra by symmetrizing the associative product. In the same way, a special Tortken algebra is obtained from a Novikov algebra by symmetrizing the Novikov product.

The study of special Tortken algebras is closely connected with polynomial identities. Some identities are satisfied by all Tortken algebras, while others arise specifically from

the special construction. Special identities related to symmetrized Novikov products were studied in the context of Novikov-Jordan algebras [4, 5].

4.5 The Space of Symmetric Elements

Now let

$$\mathrm{Nov}\langle X\rangle$$

be the free Novikov algebra generated by a set X. We define the space of symmetric (Tortken) elements, denoted by

$$S\langle X\rangle,$$

as the smallest subspace of $\mathrm{Nov}\langle X\rangle$ satisfying the following two conditions:

$$X \subseteq S\langle X\rangle,$$

and

$$f, g \in S\langle X\rangle \quad \Rightarrow \quad f \circ g \in S\langle X\rangle.$$

This definition means that $S\langle X\rangle$ is generated from the variables in X using the symmetrized product.

It is important to clarify the meaning of the word "symmetric" here. It does not mean that the elements are invariant under all permutations of variables. Rather, it means that these elements are obtained by using the symmetric product(Tortken product). The space $S\langle X\rangle$ is the natural object for studying free special Tortken algebras. Since special Tortken algebras come from symmetrized Novikov algebras, the free Novikov algebra provides the universal ambient space, and $S\langle X\rangle$ describes the part generated by the symmetrized product.

4.6 Differential Basis of Symmetric Elements

The differential realization gives a concrete description of the space $S\langle X\rangle$. Let $\mathcal{M}$ be the set of all differential monomials in $F\{X\}$. In the free Novikov algebra, the basis is given by differential monomials $u \in \mathcal{M}$ satisfying

$$\deg(u) - d(u) = 1.$$

For the space of symmetric elements, the basis has a related form. Namely, the space $S\langle X\rangle$ has a basis

$$X \cup \{u' \mid u \in \mathcal{M},\ \deg(u) - d(u) = 2\}. \tag{4.6.1}$$

This description follows from the differential form of the symmetrized product and is used in the study of null Lagrangians in free Novikov algebras [7].

The condition

$$\deg(u) - d(u) = 2$$

appears naturally. If u satisfies this condition, then after applying one derivative we

obtain u', and the difference between the standard degree and the differential degree decreases by one. Hence u' belongs to the free Novikov algebra.

For example, take

$$u = x^2.$$

Then

$$\deg(x^2) = 2, \qquad d(x^2) = 0,$$

so

$$\deg(x^2) - d(x^2) = 2.$$

Therefore,

$$u' = (x^2)' = 2x'x$$

is a symmetric element. This element corresponds to

$$x \circ x.$$

As another example, let

$$u = x'x^2.$$

Then

$$\deg(u) = 3, \qquad d(u) = 1,$$

and hence

$$\deg(u) - d(u) = 2.$$

Thus,

$$u' = (x'x^2)' = x''x^2 + 2(x')^2x$$

belongs to the space of symmetric elements.

This basis is useful because it gives a clear way to count dimensions and to compare elements written in the $\circ$-language with differential monomials.

4.7 The One-Generated Case

In this dissertation, we mainly consider the one-generated case

$$X = \{x\}.$$

Then the space of symmetric elements is denoted by

$$S\langle x\rangle.$$

For each $n \geq 1$, we denote by

$$T_n$$

the homogeneous component of degree n of $S\langle x\rangle$. Thus,

$$S\langle x\rangle = \bigoplus_{n\geq 1} T_n.$$

The first components are simple:

$$T_1 = \langle x\rangle,$$

$$T_2 = \langle x \circ x\rangle,$$

and

$$T_3 = \langle (x \circ x) \circ x\rangle.$$

Starting from higher degrees, the structure becomes more interesting. There may be several natural $\circ$-monomials of the same degree, and they may satisfy nontrivial linear relations after expansion into differential monomials. For this reason, the main problem is not only to count the dimension of T_n, but also to find convenient basis elements in the $\circ$-language.

There are two complementary ways to study $S\langle X\rangle$. The first one is the differential description, which gives a basis and dimension formulas. The second one is the $\circ$-language description, which reflects the algebraic structure of the symmetrized product. The main computations in this dissertation use both viewpoints.

In this chapter, we introduced the symmetrized product on a Novikov algebra. We recalled that this product satisfies the Tortken identity, and algebras satisfying this identity are called Tortken algebras. Those Tortken algebras that arise from Novikov algebras by symmetrization are called special Tortken algebras.

We also introduced the space $S\langle X\rangle$ of symmetric elements in a free Novikov algebra. In the differential realization, the symmetrized product is given by

$$a \circ b = (ab)'.$$

This formula allows us to expand elements in the $\circ$-language into differential monomials.

In the next chapter, we will focus on the one-generated case and study the homogeneous components T_n of the space $S\langle x\rangle$.

Chapter 5

The Space of Tortken Elements in the One-Generated Case

In this chapter, we study the homogeneous components of the space of symmetric elements in the free Novikov algebra generated by one element. This is the main part of the dissertation. The previous chapters introduced the necessary background: Novikov algebras, their differential realization, the symmetrized product, and the space of symmetric elements. We now apply these tools to the one-generated case.

Let

$$X = \{x\}$$

and let

$$\mathrm{Nov}\langle x\rangle$$

be the free Novikov algebra generated by x. We denote by

$$S\langle x\rangle$$

the space of symmetric elements in $\mathrm{Nov}\langle x\rangle$. By definition, this is the smallest subspace containing x and closed under the symmetrized product

$$a \circ b = a \cdot b + b \cdot a.$$

In the differential realization, this product has the form

$$a \circ b = (ab)'.$$

For each $n \geq 1$, we denote by

$$T_n$$

the homogeneous component of degree n of $S\langle x\rangle$. Thus,

$$S\langle x\rangle = \bigoplus_{n\geq 1} T_n.$$

The aim of this chapter is to describe the spaces T_n, compute their dimensions, and study how their elements can be written in the $\circ$-language. The differential description gives a convenient basis, while the $\circ$-language shows the structure related to special Tortken algebras.

5.1 Differential Description of T_n

Recall that the space of symmetric elements in the free Novikov algebra has a basis of the form

$$X \cup \{u' \mid u \in \mathcal{M},\ \deg(u) - d(u) = 2\},$$

where $\mathcal{M}$ is the set of differential monomials [7].

In the one-generated case, a differential monomial has the form

$$u = x^{(i_1)}x^{(i_2)}\cdots x^{(i_n)}.$$

Here

$$\deg(u) = n$$

and

$$d(u) = i_1 + i_2 + \cdots + i_n.$$

Therefore, the condition

$$\deg(u) - d(u) = 2$$

is equivalent to

$$i_1 + i_2 + \cdots + i_n = n - 2.$$

Hence, for $n \geq 2$, the component T_n is spanned by elements of the form

$$\left(x^{(i_1)}x^{(i_2)}\cdots x^{(i_n)}\right)',$$

where

$$i_1 + \cdots + i_n = n - 2.$$

Since the algebra of differential polynomials is commutative, the order of the factors is not important. Therefore, such monomials are naturally indexed by partitions of the integer $n-2$. This observation leads to the dimension formula below.

5.2 Dimension Formula

Proposition 5.2.1. *Let T_n be the homogeneous component of degree n of the space of symmetric elements in the free Novikov algebra generated by one element. Then*

$$\dim T_1 = \dim T_2 = 1,$$

and for $n \geq 3$,

$$\dim T_n = p(n-2),$$

where $p(m)$ denotes the number of partitions of m.

Proof. For degree 1, the component is generated only by the element x. Hence

$$T_1 = \langle x \rangle$$

and

$$\dim T_1 = 1.$$

For degree 2, we have

$$T_2 = \langle x \circ x \rangle.$$

Using the formula $a \circ b = (ab)'$, we get

$$x \circ x = (x^2)' = 2x'x.$$

Thus, T_2 is also one-dimensional.

Now let $n \geq 3$. By the differential basis of symmetric elements, T_n has a basis consisting of elements

$$\left(x^{(i_1)}x^{(i_2)} \cdots x^{(i_n)}\right)'$$

such that

$$i_1 + i_2 + \cdots + i_n = n - 2.$$

Because the differential polynomial algebra is commutative, two monomials that differ only by a permutation of factors are the same. Therefore, the number of basis elements is equal to the number of partitions of $n - 2$. Hence

$$\dim T_n = p(n-2).$$

This proves the proposition. □

Thus, the first dimensions are

$$\dim T_1 = 1, \quad \dim T_2 = 1, \quad \dim T_3 = 1, \quad \dim T_4 = 2, \quad \dim T_5 = 3, \quad \dim T_6 = 5, \quad \dim T_7 = 7.$$

This sequence already shows that the differential description of the spaces T_n is quite simple. However, the situation becomes more subtle when we try to describe the same spaces using only the symmetrized product $\circ$.

5.3 First Components

We begin with the first low-degree components. In degree 1, there is only the generator:

$$T_1 = \langle x \rangle.$$

In degree 2, the component is generated by

$$x \circ x.$$

Thus,

$$T_2 = \langle x \circ x \rangle.$$

In degree 3, we obtain

$$T_3 = \langle (x \circ x) \circ x \rangle.$$

Since the product $\circ$ is commutative, we do not distinguish between

$$(x \circ x) \circ x$$

and

$$x \circ (x \circ x).$$

These first components are one-dimensional. The first degree where more than one basis element appears is degree 4.

5.4 Degree Four

Since $\dim T_4 = p(2) = 2$, degree 4 is the first case in which the homogeneous component has more than one basis element, so we write it out in detail. Two natural elements in the $\circ$-language are

$$w_1 = ((x \circ x) \circ x) \circ x$$

and

$$w_2 = (x \circ x) \circ (x \circ x).$$

They correspond to two different ways of forming a degree 4 expression using the symmetrized product. The differential basis of T_4 is given by

$$b_1 = (x''xxx)' = x'''xxx + 3x''x'xx$$

and

$$b_2 = (x'x'xx)' = 2x''x'xx + 2x'x'x'x.$$

Using the identity $a \circ b = (ab)'$, we obtain

$$w_1 = ((x \circ x) \circ x) \circ x = 2x'''xxx + 14x''x'xx + 8x'x'x'x$$

$$w_2 = (x \circ x) \circ (x \circ x) = 8x''x'xx + 8x'x'x'x$$

$$2x'''xxx + 14x''x'xx + 8x'x'x'x = \alpha_1 b_1 + \alpha_2 b_2 \quad \Rightarrow \quad \alpha_1 = 2, \quad \alpha_2 = 4$$

$$8x''x'xx + 8x'x'x'x = \alpha_1 b_1 + \alpha_2 b_2 \quad \Rightarrow \quad \alpha_1 = 0, \quad \alpha_2 = 4$$

That is,

$$w_1 = 2b_1 + 4b_2, \qquad w_2 = 4b_2.$$

After expansion, both elements are expressed as linear combinations of b_1 and b_2, which means that w_1 and w_2 are linearly independent and form a basis of T_4:

$$T_4 = \langle w_1, w_2 \rangle.$$

This example illustrates the general method used below: we write elements in the $\circ$-language, expand them in the differential basis, and then check linear independence by comparing coefficients.

5.5 Degree Five

For degree 5, the dimension formula gives

$$\dim T_5 = p(3) = 3.$$

The partitions of 3 are

$$3, \qquad 2+1, \qquad 1+1+1.$$

Therefore, a differential basis of T_5 can be chosen as

$$(x'''xxxx)',$$

$$(x''x'xxx)',$$

and

$$(x'x'x'xx)'.$$

In the $\circ$-language, natural elements of degree 5 are

$$w_1 = ((((x \circ x) \circ x) \circ x) \circ x),$$
$$w_2 = ((x \circ x) \circ (x \circ x)) \circ x,$$

and

$$w_3 = ((x \circ x) \circ x) \circ (x \circ x).$$

Expanding these elements using

$$a \circ b = (ab)'$$

and comparing their coordinates in the differential basis, we compute

$$w_1 = 2b_1 + 14b_2 + 8b_3,$$

$$w_2 = 8b_2 + 8b_3,$$

and

$$w_3 = 4b_2 + 8b_3.$$

Here, w_1, w_2 and w_3 are linearly independent. Therefore, these three elements form a basis of T_5:

$$T_5 = \langle w_1, w_2, w_3 \rangle.$$

In degrees 4 and 5, the number of natural elements in the $\circ$-language agrees with the dimension of the corresponding component. Starting from degree 6, linear relations begin to appear.

5.6 Degree Six

The first degree where linear dependence among natural $\circ$-words appears is degree 6. Consider the six natural commutative nonassociative words

$$\begin{aligned}
w_1 &= ((((x \circ x) \circ x) \circ x) \circ x) \circ x, \\
w_2 &= (((x \circ x) \circ (x \circ x)) \circ x) \circ x, \\
w_3 &= (((x \circ x) \circ x) \circ (x \circ x)) \circ x, \\
w_4 &= (((x \circ x) \circ x) \circ x) \circ (x \circ x), \\
w_5 &= ((x \circ x) \circ (x \circ x)) \circ (x \circ x), \\
w_6 &= (x \circ (x \circ x)) \circ (x \circ (x \circ x)).
\end{aligned}$$

Since

$$\dim T_6 = p(4) = 5,$$

these six elements cannot be linearly independent. Therefore, there must be a nontrivial relation between them.

To construct the identity in degree 6, we use *Wolfram Mathematica*. As a first step, we implement the derivation operator and the symmetrized product in the commutative differential algebra by encoding their basic properties; see Appendix A.1

Then we enumerate all six elements of degree 6 in the $\circ$-language and list all seven basis elements of the homogeneous component of degree 6 of the free Novikov algebra on one generator and assign to each element w_i a scalar coefficient λ_i and consider the linear combination

$$\sum_{i=1}^{6} \lambda_i w_i = 0.$$

The program then determines the relations among the coefficients λ_i and computes their admissible values; see Appendix A.2.

The resulting relations among the coefficients in the case $n = 6$ are

$$\begin{aligned}
\lambda_1 &= 0,\\
\lambda_3 &= -\frac{\lambda_2}{2},\\
\lambda_4 &= -\frac{\lambda_2}{2},\\
\lambda_5 &= \frac{\lambda_2}{2},\\
\lambda_6 &= -\frac{\lambda_2}{2}.
\end{aligned}$$

Taking $\lambda_2 = 2$, we obtain the following identity:

$$\begin{aligned}
&2((x \circ x) \circ (x \circ x)) \circ (x \circ x) - ((x \circ x) \circ x) \circ ((x \circ x) \circ x)\\
&\quad - (((x \circ x) \circ x) \circ x) \circ (x \circ x) + (((x \circ x) \circ (x \circ x)) \circ x) \circ x\\
&\quad - (((x \circ x) \circ x) \circ (x \circ x)) \circ x = 0
\end{aligned} \tag{5.6.1}$$

or equivalently,

$$2w_5 - w_6 - w_4 + w_2 - w_3 = 0.$$

This identity is a consequence of the Tortken identity with

$$a = d = x \circ x, \qquad b = c = x.$$

Substituting these values into the Tortken identity gives

$$((x\circ x)\circ x)\circ(x\circ(x\circ x))-((x\circ x)\circ(x\circ x))\circ(x\circ x) = (x\circ x, x, x)^{\circ}\circ(x\circ x)-(x\circ x, x\circ x, x)^{\circ}\circ x.$$

Indeed, this substitution gives

$$-2w_5 + w_6 + w_4 - w_2 + w_3 = 0,$$

or equivalently,

$$2w_5 - w_6 - w_4 + w_2 - w_3 = 0.$$

Therefore, we can eliminate the element w_6:

$$w_6 = w_2 - w_3 - w_4 + 2w_5.$$

As a result, one of the six natural degree 6 monomials can be expressed as a linear combination of the other five. Hence a basis of T_6 in the $\circ$-language consists of five independent elements.

Notice that

$$w_6 \in T_3 \circ T_3$$

and

$$w_1 \in ((((T_2 \circ T_1) \circ T_1) \circ T_1) \circ T_1)$$

where $((T_2 \circ T_1) \circ T_1) \circ T_1 \in T_5$. That is,

$$w_1 \in T_5 \circ T_1.$$

The same holds for w_2 and w_3,

$$w_2 \in T_5 \circ T_1, \qquad w_3 \in T_5 \circ T_1.$$

And for w_4 and w_5 we define

$$w_4 \in T_4 \circ T_2, \qquad w_5 \in T_4 \circ T_2.$$

From the identity $w_6 = w_2 - w_3 - w_4 + 2w_5$, it follows that

$$T_3 \circ T_3 \subseteq T_5 \circ T_1 + T_4 \circ T_2.$$

That is,

$$T_6 \subseteq T_5 \circ T_1 + T_4 \circ T_2.$$

This is the first degree in which the difference between the differential description and the $\circ$-language description becomes visible. The differential basis gives the dimension directly, while the $\circ$-language produces more elements than necessary and reveals an identity between them.

5.7 Degree Seven

For degree $n = 7$, we have 11 elements:

$$\begin{aligned}
w_1 &= (((((x \circ x) \circ x) \circ x) \circ x) \circ x) \circ x, & w_2 &= ((((x \circ x) \circ (x \circ x)) \circ x) \circ x) \circ x,\\
w_3 &= ((((x \circ x) \circ x) \circ (x \circ x)) \circ x) \circ x, & w_4 &= ((((x \circ x) \circ x) \circ x) \circ (x \circ x)) \circ x,\\
w_5 &= (((x \circ x) \circ (x \circ x)) \circ (x \circ x)) \circ x, & w_6 &= (((x \circ x) \circ x) \circ ((x \circ x) \circ x)) \circ x,\\
w_7 &= ((((x \circ x) \circ x) \circ x) \circ x) \circ (x \circ x), & w_8 &= (((x \circ x) \circ (x \circ x)) \circ x) \circ (x \circ x),\\
w_9 &= (((x \circ x) \circ x) \circ (x \circ x)) \circ (x \circ x), & w_{10} &= (((x \circ x) \circ x) \circ x) \circ ((x \circ x) \circ x),\\
w_{11} &= ((x \circ x) \circ (x \circ x)) \circ ((x \circ x) \circ x).
\end{aligned}$$

Since $p(7-2) = p(5) = 7$, these 11 elements cannot be linearly independent. Thus, four linear relations occur among these eleven candidate elements.: We eliminate w_6 since in the $n = 6$ case we concluded that

$$T_3 \circ T_3 \subseteq T_5 \circ T_1 + T_4 \circ T_2.$$

which means w_6 can be expressed as a linear combination of the other 7 monomials.

Hence, in degree 7, we obtain only three identities out of four.

For our calculations, we will use *Wolfram Mathematica* (see Appendix A.3). By this, we obtained

$$\begin{aligned}
\lambda_1 &= 0,\\
\lambda_2 &= \lambda_{10} + 2\lambda_{11},\\
\lambda_4 &= -7\lambda_{10} - 12\lambda_{11} - 3\lambda_3,\\
\lambda_5 &= 6\lambda_{10} + 10\lambda_{11} + 2\lambda_3,\\
\lambda_7 &= 5\lambda_{10} + 8\lambda_{11} + 2\lambda_3,\\
\lambda_8 &= -6\lambda_{10} - 9\lambda_{11} - 3\lambda_3,\\
\lambda_9 &= \lambda_3.
\end{aligned}$$

Taking

$$(\lambda_3, \lambda_{10}, \lambda_{11}) = (0,1,0), \quad (0,0,1), \quad (1,0,0),$$

we obtain the following three identities:

$$w_{10} + w_2 - 7w_4 + 6w_5 + 5w_7 - 6w_8 = 0, \tag{5.7.1a}$$

$$w_{11} + 2w_2 - 12w_4 + 10w_5 + 8w_7 - 9w_8 = 0, \tag{5.7.1b}$$

$$w_3 - 3w_4 + 2w_5 + 2w_7 - 3w_8 + w_9 = 0. \tag{5.7.1c}$$

The third relation (5.7.1c) coincides with the jetken polynomial, which is obtained from the Besken identity by taking $a = x$ and $b = x \circ x$ [5]. Identity (5.7.1a) allows us to eliminate w_{10}, while identity (5.7.1b) allows us to eliminate w_{11}. The third identity (5.7.1c) eliminates one more element from the remaining set. Thus, we obtain 7 linearly independent elements.

Since

$$w_{10},\ w_{11} \in T_4 \circ T_3, \qquad w_2, w_4, w_5 \in T_6 \circ T_1, \qquad w_7, w_8 \in T_5 \circ T_2,$$

and

$$w_{10} = -w_2 + 7w_4 - 6w_5 - 5w_7 + 6w_8, \qquad w_{11} = -2w_2 + 12w_4 - 10w_5 - 8w_7 + 9w_8,$$

it follows that

$$T_4 \circ T_3 \subseteq T_6 \circ T_1 + T_5 \circ T_2.$$

That is,

$$T_7 \subseteq T_6 \circ T_1 + T_5 \circ T_2.$$

Our computations show that, up to degree 7, the stronger recursive inclusion holds. These computations motivate the following conjecture in the next section.

5.8 Recursive Structure

The computations in low degrees suggest that the spaces T_n may have a recursive structure. More precisely, the following inclusion is verified in low degrees:

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2. \tag{5.8.1}$$

This means that elements of T_n can be obtained from the two previous components by multiplying them with elements of T_1 and T_2. Since

$$T_1 = \langle x \rangle$$

and

$$T_2 = \langle x \circ x \rangle,$$

the right-hand side of (5.8.1) consists of elements of the form

$$u \circ x, \qquad u \in T_{n-1},$$

and

$$v \circ (x \circ x), \qquad v \in T_{n-2}.$$

For small degrees, the inclusion can be checked directly. For example, for $n = 3$,

$$T_3 \subseteq T_2 \circ T_1,$$

because

$$T_3 = \langle (x \circ x) \circ x \rangle.$$

For $n = 4$, we have

$$T_4 \subseteq T_3 \circ T_1 + T_2 \circ T_2.$$

Indeed, the two basis elements

$$((x \circ x) \circ x) \circ x$$

and

$$(x \circ x) \circ (x \circ x)$$

belong to the right-hand side.

The same inclusion has been verified computationally for degrees up to 7:

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \qquad n \leq 7.$$

Based on these computations, we formulate the following conjecture.

Conjecture 5.8.1. Let T_n be the homogeneous component of degree n in the one-

generated case. Then,

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \qquad n \geq 3.$$

If this conjecture holds in general, it would give a useful recursive method for constructing the components T_n in the $\circ$-language.

In this chapter, we studied the homogeneous components

$$T_n$$

of the space of symmetric elements in the free Novikov algebra generated by one element.

First, we described the differential basis of T_n and proved the dimension formula

$$\dim T_1 = \dim T_2 = 1,$$

and

$$\dim T_n = p(n-2)$$

for $n \geq 3$.

Then we considered the first low-degree components in the $\circ$-language. In degrees 1, 2, and 3, the components are one-dimensional. In degree 4, the first nontrivial two-dimensional component appears. In degree 5, three independent $\circ$-monomials form a basis.

In degree 6, the first nontrivial linear relation appears among natural $\circ$-monomials. This relation is a consequence of the Tortken identity. In degree 7, there appear three identities, the third of which is obtained from the Besken identity.

Finally, the computations suggest the recursive inclusion (5.8.1). This inclusion has been verified in low degrees and may be useful for further study of one-generated free special Tortken algebras.

Chapter 6

Computational Methodology and Conclusion

In this chapter, we discuss the results obtained in the previous chapter. The main computations are based on the differential realization of the free Novikov algebra and on the formula

$$a \circ b = (ab)'.$$

This formula makes the symmetrized product suitable for explicit calculation: every expression written in the $\circ$-language can be expanded into a differential polynomial. After that, questions about linear independence, bases, and identities can be studied by standard methods of linear algebra.

The computations in this work were carried out degree by degree. This approach is natural because the space of symmetric elements is graded, and each homogeneous component T_n can be studied separately.

6.1 Discussion

The results show that the one-generated space of symmetric elements has two different but complementary descriptions.

The first description is differential. It gives the dimension formula

$$\dim T_n = p(n-2)$$

for $n \geq 3$. This formula is simple and follows from the basis of symmetric elements in terms of differential monomials.

The second description is written in the $\circ$-language. This description is more complicated, because natural $\circ$-monomials may be linearly dependent. However, it is also more meaningful for the study of special Tortken algebras, since the product $\circ$ is the operation defining the special Tortken structure.

The comparison between these two descriptions is the main idea of the dissertation.

The differential basis gives a reliable way to compute dimensions and check linear independence. The $\circ$-language reveals identities and relations that are not immediately visible from the differential basis alone.

In degrees 6 and 7, the computations show how identities appear among natural $\circ$-monomials. In degree 6, the relation is connected with the Tortken identity. In degree 7, additional relations arise, which are related to special identities of the symmetrized Novikov product.

These results suggest that the structure of T_n is not random. In low degrees, the components can be generated recursively from previous components. This leads to the inclusion

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2,$$

which was verified computationally for low degrees.

6.2 Limitations of the Work

The main limitation of this work is that explicit computations were carried out only in low degrees. As the degree increases, the number of possible $\circ$-monomials grows quickly. This makes the computations more complicated and requires more systematic algorithms.

Another limitation is that the recursive inclusion

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2$$

has been verified only in low degrees. The computations provide evidence for this inclusion, but a general proof for all n is still needed.

Finally, this dissertation focuses mainly on the one-generated case. The case of several generators is more complicated because one has to consider multidegrees, multilinear components, and the action of symmetric groups. This direction requires further investigation.

6.3 Conclusion

In this dissertation, we studied the space of symmetric elements in a free Novikov algebra and its connection with special Tortken algebras. The main focus was the one-generated case.

We first recalled the basic notions of nonassociative algebras, polynomial identities, varieties, free algebras, and homogeneous components. Then we introduced Novikov algebras and described their realization in differential polynomial algebras. This realization allowed us to work with free Novikov algebras in a concrete way.

Next, we considered the symmetrized product

$$a \circ b = a \cdot b + b \cdot a.$$

In the differential realization, this product is written as

$$a \circ b = (ab)'.$$

This formula was the main computational tool of the dissertation.

For the one-generated case, we denoted by

$$T_n$$

the homogeneous component of degree n of the space of symmetric elements. We described the differential basis of these components and obtained the dimension formula

$$\dim T_1 = \dim T_2 = 1,$$

and

$$\dim T_n = p(n-2)$$

for $n \geq 3$.

We then studied the first components explicitly in the $\circ$-language. In degrees 1 through 5, bases were constructed directly. In degree 6, a nontrivial relation appears, and this relation follows from the Tortken identity. In degree 7, additional relations were found by null-space computations.

The computations also suggest a recursive structure of the spaces T_n:

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2.$$

This inclusion was verified in low degrees and may provide a useful way to construct bases in higher degrees.

Thus, the dissertation gives a computational and structural description of the first homogeneous components of the one-generated free special Tortken algebra.

6.4 Future Work

There are several natural directions for further research.

First, it would be useful to prove the recursive inclusion

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2$$

for all $n \geq 3$. A proof of this statement would give a general recursive method for constructing the components T_n.

Second, the computations can be extended to higher degrees. This may reveal new identities of special Tortken algebras and help to understand the structure of the corresponding variety.

Third, the case of several generators should be studied. In this case, the structure

becomes richer because multilinear components and symmetric group representations appear naturally.

Fourth, it would be interesting to compare identities of special Tortken algebras with identities of general Tortken algebras. This comparison may help to clarify which identities come only from the Tortken identity and which are specific to the special construction from Novikov algebras.

Overall, the results of this dissertation can be viewed as a first step toward a more detailed description of the variety of special Tortken algebras.

Appendix A

Computational Results

This appendix contains the computational results used in Chapter 5. The computations were carried out in Wolfram Mathematica. The main purpose of the computations was to expand elements written in the ∘-language into differential monomials and to find linear relations between them.

A.1 Mathematica Code

```
Dr[C_Integer a_] := C Dr[a]

Dr[a_ + b_] := Dr[a] + Dr[b]

Dr[C_Integer a_ + K_Integer b_] := C Dr[a] + K Dr[b]

Dr[C_Integer a_ + K_Integer b_ + D_Integer d_] :=
  C Dr[a] + K Dr[b] + D Dr[d]

Dr[a_ b_] := Dr[a] b + a Dr[b]

Dr[a_^n_Integer] := Dr[a^(n - 1)] a + a^(n - 1) Dr[a]

Dr'[x_] := 0

Dr[0] := 0

t[a_, b_] := ExpandAll[Dr[a b]]
```

A.2 Degree Six Computations

Here we present the computation in degree 6. The following six candidate elements of degree six in the ∘-language were used.

```
Gen1deg6 = {
  g1d6[1] = t[t[t[t[t[x, x], x], x], x], x],
  g1d6[2] = t[t[t[x, x], t[x, x]], t[x, x]],
  g1d6[3] = t[t[t[x, x], x], t[t[x, x], x]],
  g1d6[4] = t[t[t[t[x, x], x], x], t[x, x]],
  g1d6[5] = t[t[t[t[x, x], t[x, x]], x], x],
  g1d6[6] = t[t[t[t[x, x], x], t[x, x]], x]
};
```

The program then determines the relations among the coefficients λ_i and computes their admissible values.

```
Combin1deg6 =
  Coefficient[
    Collect[
      ExpandAll[Sum[Subscript[lambda, i] g1d6[i], {i, 1, 6}]],
      {
        x Dr[x]^5,
        x^2 Dr[x]^3 Dr[Dr[x]],
        x^3 Dr[x] Dr[Dr[x]]^2,
        x^3 Dr[x]^2 Dr[Dr[Dr[x]]],
        x^4 Dr[Dr[x]]*Dr[Dr[Dr[x]]],
        x^4 Dr[x]*Dr[Dr[Dr[Dr[x]]]],
        x^5 Dr[Dr[Dr[Dr[Dr[x]]]]]
      }
    ],
    {
      x Dr[x]^5,
      x^2 Dr[x]^3 Dr[Dr[x]],
      x^3 Dr[x] Dr[Dr[x]]^2,
      x^3 Dr[x]^2 Dr[Dr[Dr[x]]],
      x^4 Dr[Dr[x]]*Dr[Dr[Dr[x]]],
      x^4 Dr[x]*Dr[Dr[Dr[Dr[x]]]],
      x^5 Dr[Dr[Dr[Dr[Dr[x]]]]]
    }
  ];

S[x_] := x == 0;

TableForm[
  First[
    Solve[
      Map[S, Combin1deg6],
      Array[Subscript[lambda, #] &, 6]
    ]
  ]
]
```

A.3 Degree Seven Computations

```
Gen1deg7 = {
  g1d7[1] = t[t[t[t[x, x], x], x], t[t[x, x], x]],
  g1d7[2] = t[t[t[x, x], t[x, x]], t[t[x, x], x]],
  g1d7[3] = t[t[t[t[t[t[x, x], x], x], x], x], x],
  g1d7[4] = t[t[t[t[x, x], t[x, x]], x], x], x],
  g1d7[5] = t[t[t[t[x, x], x], t[x, x]], x], x],
  g1d7[6] = t[t[t[t[x, x], x], x], t[x, x]], x],
  g1d7[7] = t[t[t[x, x], t[x, x]], t[x, x]], x],
  g1d7[8] = t[t[t[t[x, x], x], x], x], t[x, x]],
  g1d7[9] = t[t[t[x, x], t[x, x]], x], t[x, x]],
  g1d7[10] = t[t[t[x, x], x], t[x, x]], t[x, x]]
};

Combin1deg7 =
  Coefficient[
    Collect[
      ExpandAll[Sum[Subscript[\[Lambda], i] g1d7[i], {i, 1, 10}]],
      {
        x Dr[x]^6,
        x^2 Dr[x]^4 Dr[Dr[x]],
        x^3 Dr[x]^2 Dr[Dr[x]]^2,
        x^4 Dr[Dr[x]]^3,
        x^3 Dr[x]^3 Dr[Dr[Dr[x]]],
        x^4 Dr[x] Dr[Dr[x]] Dr[Dr[Dr[x]]],
        x^5 Dr[Dr[Dr[x]]]^2,
        x^4 Dr[x]^2 Dr[Dr[Dr[Dr[x]]]],
        x^5 Dr[Dr[x]] Dr[Dr[Dr[Dr[x]]]],
        x^5 Dr[x] Dr[Dr[Dr[Dr[Dr[x]]]]],
        x^6 Dr[Dr[Dr[Dr[Dr[Dr[x]]]]]]
      }
    ],
    {
      x Dr[x]^6,
      x^2 Dr[x]^4 Dr[Dr[x]],
      x^3 Dr[x]^2 Dr[Dr[x]]^2,
      x^4 Dr[Dr[x]]^3,
      x^3 Dr[x]^3 Dr[Dr[Dr[x]]],
      x^4 Dr[x] Dr[Dr[x]] Dr[Dr[Dr[x]]],
      x^5 Dr[Dr[Dr[x]]]^2,
      x^4 Dr[x]^2 Dr[Dr[Dr[Dr[x]]]],
      x^5 Dr[Dr[x]] Dr[Dr[Dr[Dr[x]]]],
      x^5 Dr[x] Dr[Dr[Dr[Dr[Dr[x]]]]],
      x^6 Dr[Dr[Dr[Dr[Dr[Dr[x]]]]]]
    }
  ];
```

```
S[x_] := x == 0;

TableForm[
  First[
    Solve[
      Map[S, Combin1deg7],
      Array[Subscript[\[Lambda], #] &, 10]
    ]
  ]
]
```

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Article

Space of Tortken elements in free Novikov algebra

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Abstract

In this paper, we study the one-generated case of the space of Tortken elements in a free Novikov algebra. Using the differential-monomial realization, we prove that the dimensions of the homogeneous components are given by $\dim T_1 = \dim T_2 = 1$ and $\dim T_n = p(n-2)$ for all $n \geq 3$, where $p(n-2)$ denotes the partition number. We also describe the first homogeneous components in the circle language and analyze the relations among their elements in low degrees. In particular, we show that the elements are linearly independent up to degree 5, while the first nontrivial relation appears in degree 6. We derive explicit identities in degrees 6 and 7 and prove a recursive inclusion for homogeneous components in the one-generated case for $n \leq 7$. These results give a more precise description of the low-degree structure of the space of Tortken elements and suggest a regular pattern that may persist in higher degrees. We also formulate a conjecture extending the recursive inclusion to all degrees.

Keywords: Novikov algebra, Tortken algebra, symmetric elements, free algebra

I. Introduction

An algebra $(A, \cdot)$ is called a *(right) Novikov algebra* if, for all $a, b, c \in A$, the following identities hold:

$$a \cdot (b \cdot c) = b \cdot (a \cdot c), \tag{1}$$

$$(a \cdot b) \cdot c - a \cdot (b \cdot c) = (a \cdot c) \cdot b - a \cdot (c \cdot b). \tag{2}$$

The first identity (1) is called the **left-commutativity** identity, and the second identity (2) is called the **right-symmetric** identity, and any algebra satisfying this identity is called the right-symmetric algebra. Thus, throughout the paper we use the convention that a (right) Novikov algebra is an algebra satisfying identities (1) and (2).

Example I.1. Let us consider the polynomial algebra $A = K[x]$ with multiplication

$$a \cdot b = D(a)b$$

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A direct verification shows that $(A, \cdot)$ is a Novikov algebra.

Now, let us define a new product on the Novikov algebra, which is called a symmetric product (anti-commutator). Let A be a (right) Novikov algebra with multiplication $a \cdot b$. Then the anti-commutator is defined by:

$$a \circ b := a \cdot b + b \cdot a$$

Notice that $a \circ b = b \circ a$.

Dzhumadil'daev [1], [3] proved that the symmetrized product satisfies the following identities:

$$(a \circ b) \circ (c \circ d) - (a \circ d) \circ (c \circ b) = (a, b, c)^+ \circ d - (a, d, c)^+ \circ b, \tag{3}$$

$$\begin{aligned} &(((a \circ a) \circ a) \circ b) \circ b + (((a \circ b) \circ b) \circ a) \circ a \\ &\quad + 2(((a \circ a) \circ b) \circ b) \circ a + 2(((a \circ b) \circ a) \circ a) \circ b \\ &\quad - 3(((a \circ a) \circ b) \circ a) \circ b - 3(((a \circ b) \circ a) \circ b) \circ a = 0 \end{aligned} \tag{4}$$

where

$$(a, b, c)^+ = a \circ (b \circ c) - (a \circ b) \circ c.$$

The first identity is called the Tortken identity, and the second identity is the Besken identity. Algebras that satisfy (3) are called Tortken algebras.

Let us define the free Novikov algebra generated by a set X and call it $Nov\langle X\rangle$. Then the space of symmetric elements $S\langle X\rangle$ is the smallest subspace of $Nov\langle X\rangle$ that contains X and is closed under the $\circ$ - product. Elements of $S\langle X\rangle$ are called symmetric elements or Tortken elements, otherwise.

Now, let us define $\mathcal{M}$ as a set of all differential monomials with one derivation. A basis of the free Novikov algebras was constructed by Dzhumadil'daev and Löfwall [2]:

Theorem I.2. *The set*

$$\mathcal{N} = \{\, u \in \mathcal{M} \mid \deg(u) - d(u) = 1 \,\}$$

is a basis of the free Novikov algebra $Nov\langle X\rangle$.

And recently, Dzhumadil'daev and Ismailov [4] constructed a basis of the space of Tortken elements in the free Novikov algebra in terms of differential polynomials:

Theorem I.3. *The set*

$$\overline{\mathcal{M}} := X \cup \{\, u' \mid u \in \mathcal{M},\ \deg(u) - d(u) = 2 \,\}$$

is a basis of the space of Tortken elements $S\langle X\rangle$.

In this paper, we investigate the space of Tortken elements in a free Novikov algebra in the one-generated case. While a basis in terms of differential monomials is already known [4], the corresponding description in terms of $\circ$-product remains unclear. Our aim is to describe this basis in low degrees and to

compare it with the known differential-monomial model. We show that the dimensions of the homogeneous components are governed by the partition numbers $p(n-2)$, and we construct explicit basis elements in the $\circ$-language for small degrees. The necessary identities are verified up to degree seven by means of computations carried out in *Wolfram Mathematica*. This work may be regarded as a step toward the study of free special Tortken algebras in the $\circ$-product language.

II. Methods

Proposition II.1. *Let T_n be the homogeneous component of degree n of the space of Tortken elements in the free Novikov algebra generated by one element x. Then*

$$\dim T_1 = \dim T_2 = 1, \qquad \dim T_n = p(n-2) \quad \textit{for all } n \geq 3,$$

where $p(m)$ denotes the number of partitions of m.

Proof. By Theorem I.3, a basis of $S\langle x\rangle$ is given by

$$\overline{\mathcal{M}} := X \cup \{\, u' \mid u \in \mathcal{M},\ \deg(u) - d(u) = 2 \,\},$$

where $\mathcal{M}$ is the set of differential monomials. In the one-generated case, every differential monomial can be written uniquely in the form

$$u = x^{(i_1)} x^{(i_2)} \cdots x^{(i_n)}$$

Moreover,

$$\deg(u) = n, \qquad d(u) = i_1 + \cdots + i_n.$$

Hence the condition $\deg(u) - d(u) = 2$ becomes

$$i_1 + \cdots + i_n = n - 2.$$

Since all factors involve the same generator x and the product is commutative, we may assume without loss of generality that the derivation orders are arranged in nonincreasing order:

$$i_1 \geq i_2 \geq \cdots \geq i_n \geq 0.$$

Then the number such nonincreasing sequences is equal to the number of partitions of $n-2$. Therefore,

$$\dim T_n = p(n-2), \qquad n \geq 2.$$

Finally, for $n = 1$, the homogeneous component T_1 is spanned by x. For $n = 2$, the component T_2 is spanned by $x \circ x$. Hence,

$$\dim T_1 = \dim T_2 = 1.$$

This proves the claim. □

Proposition II.2. *Let T_n be the homogeneous component of degree n in the one-generated case. Then,*

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \qquad n \leq 7.$$

Proof. In the one-generator case, bases in the $\circ$-language for the first degrees may be chosen as follows:

$$T_1 = \langle x \rangle, \quad T_2 = \langle x \circ x \rangle, \quad T_3 = \langle (x \circ x) \circ x \rangle,$$

The first three degrees are immediate, since each homogeneous component is one-dimensional.

A. *Degree* $n = 4$

Degree 4 is the first case in which the homogeneous component has more than one basis element, so we write it out in detail. Let

$$w_1 = ((x \circ x) \circ x) \circ x, \qquad w_2 = (x \circ x) \circ (x \circ x).$$

The differential basis of T_4 is given by

$$b_1 = (x''xxx)' = x'''xxx + 3x''x'xx, \qquad b_2 = (x'x'xx)' = 2x''x'xx + 2x'x'x'x.$$

Using the identity $a \circ b = (ab)'$, we obtain

$$w_1 = ((x \circ x) \circ x) \circ x = 2x'''xxx + 14x''x'xx + 8x'x'x'x$$

$$w_2 = (x \circ x) \circ (x \circ x) = 8x''x'xx + 8x'x'x'x$$

$$2x'''xxx + 14x''x'xx + 8x'x'x'x = \alpha_1 b_1 + \alpha_2 b_2 \quad \Rightarrow \quad \alpha_1 = 2, \quad \alpha_2 = 4$$

$$8x''x'xx + 8x'x'x'x = \alpha_1 b_1 + \alpha_2 b_2 \quad \Rightarrow \quad \alpha_1 = 0, \quad \alpha_2 = 4$$

That is,

$$w_1 = 2b_1 + 4b_2, \qquad w_2 = 4b_2.$$

w_1 and w_2 are linearly independent.

B. *Degree* $n = 5$

Since

$$\dim T_5 = p(3) = 3$$

in degree $n = 5$, we have three basis elements which are

$$w_1 = (((x \circ x) \circ x) \circ x) \circ x, \quad w_2 = ((x \circ x) \circ (x \circ x)) \circ x, \quad w_3 = ((x \circ x) \circ x) \circ (x \circ x).$$

The differential basis of T_5 is given by

$$b_1 = (x'''xxx)', \qquad b_2 = (x''x'xx)', \qquad b_3 = (x'x'x'xx)'.$$

Therefore, we compute

$$w_1 = 2b_1 + 14b_2 + 8b_3, \quad w_2 = 8b_2 + 8b_3, \quad w_3 = 4b_2 + 8b_3$$

Here, w_1, w_2 and w_3 are linearly independent.

C. Degree $n = 6$

The first degree where linear dependence among natural $\circ$-words appears is degree 6. Consider the six natural commutative nonassociative words

$$\begin{aligned}
w_1 &= ((((x \circ x) \circ x) \circ x) \circ x) \circ x,\\
w_2 &= (((x \circ x) \circ (x \circ x)) \circ x) \circ x,\\
w_3 &= (((x \circ x) \circ x) \circ (x \circ x)) \circ x,\\
w_4 &= (((x \circ x) \circ x) \circ x) \circ (x \circ x),\\
w_5 &= ((x \circ x) \circ (x \circ x)) \circ (x \circ x),\\
w_6 &= (x \circ (x \circ x)) \circ (x \circ (x \circ x)).
\end{aligned}$$

Since

$$\dim T_6 = p(4) = 5,$$

these six elements cannot be linearly independent.

To construct the identity in degree 6, we use *Wolfram Mathematica*. As a first step, we implement the derivation operator and the symmetrized product in the commutative differential algebra by encoding their basic properties; see Appendix A.

Then we enumerate all six elements of degree 6 in the $\circ$-language and list all seven basis elements of the homogeneous component of degree 6 of the free Novikov algebra on one generator and assign to each element w_i a scalar coefficient λ_i and consider the linear combination

$$\sum_{i=1}^{6} \lambda_i w_i = 0.$$

The program then determines the relations among the coefficients λ_i and computes their admissible values; see Appendix B.

The resulting relations among the coefficients in the case $n = 6$ are

$$\begin{aligned}
\lambda_1 &= 0,\\
\lambda_3 &= -\frac{\lambda_2}{2},\\
\lambda_4 &= -\frac{\lambda_2}{2},\\
\lambda_5 &= \frac{\lambda_2}{2},\\
\lambda_6 &= -\frac{\lambda_2}{2}.
\end{aligned}$$

Taking $\lambda_2 = 2$, we obtain the following identity:

$$\begin{aligned}
&2((x \circ x) \circ (x \circ x)) \circ (x \circ x) - ((x \circ x) \circ x) \circ ((x \circ x) \circ x)\\
&\quad - (((x \circ x) \circ x) \circ x) \circ (x \circ x) + (((x \circ x) \circ (x \circ x)) \circ x) \circ x\\
&\quad - (((x \circ x) \circ x) \circ (x \circ x)) \circ x = 0
\end{aligned} \tag{5}$$

or equivalently,

$$2w_5 - w_6 - w_4 + w_2 - w_3 = 0.$$

This identity is a consequence of the Tortken identity with

$$a = d = x \circ x, \qquad b = c = x.$$

Substituting these values into the Tortken identity gives

$$((x \circ x) \circ x) \circ (x \circ (x \circ x)) - ((x \circ x) \circ (x \circ x)) \circ (x \circ x) = (x \circ x, x, x)^{+} \circ (x \circ x) - (x \circ x, x \circ x, x)^{+} \circ x.$$

Indeed, this substitution gives

$$-2w_5 + w_6 + w_4 - w_2 + w_3 = 0,$$

or equivalently,

$$2w_5 - w_6 - w_4 + w_2 - w_3 = 0.$$

Therefore, we can eliminate the element w_6:

$$w_6 = w_2 - w_3 - w_4 + 2w_5.$$

Notice that

$$w_6 \in T_3 \circ T_3$$

and

$$w_1 \in ((((T_2 \circ T_1) \circ T_1) \circ T_1) \circ T_1)$$

where $((T_2 \circ T_1) \circ T_1) \circ T_1 \in T_5$. That is,

$$w_1 \in T_5 \circ T_1.$$

The same holds for w_2 and w_3,

$$w_2 \in T_5 \circ T_1, \qquad w_3 \in T_5 \circ T_1.$$

And for w_4 and w_5 we define

$$w_4 \in T_4 \circ T_2, \qquad w_5 \in T_4 \circ T_2.$$

From the identity $w_6 = w_2 - w_3 - w_4 + 2w_5$, it follows that

$$T_3 \circ T_3 \subseteq T_5 \circ T_1 + T_4 \circ T_2.$$

That is,

$$T_6 \subseteq T_5 \circ T_1 + T_4 \circ T_2.$$

D. Degree $n = 7$

For degree $n = 7$, we have 11 elements:

$$\begin{aligned}
w_1 &= (((((x \circ x) \circ x) \circ x) \circ x) \circ x) \circ x, & w_2 &= ((((x \circ x) \circ (x \circ x)) \circ x) \circ x) \circ x,\\
w_3 &= ((((x \circ x) \circ x) \circ (x \circ x)) \circ x) \circ x, & w_4 &= ((((x \circ x) \circ x) \circ x) \circ (x \circ x)) \circ x,\\
w_5 &= (((x \circ x) \circ (x \circ x)) \circ (x \circ x)) \circ x, & w_6 &= (((x \circ x) \circ x) \circ ((x \circ x) \circ x)) \circ x,\\
w_7 &= ((((x \circ x) \circ x) \circ x) \circ x) \circ (x \circ x), & w_8 &= (((x \circ x) \circ (x \circ x)) \circ x) \circ (x \circ x),\\
w_9 &= (((x \circ x) \circ x) \circ (x \circ x)) \circ (x \circ x), & w_{10} &= (((x \circ x) \circ x) \circ x) \circ ((x \circ x) \circ x),\\
w_{11} &= ((x \circ x) \circ (x \circ x)) \circ ((x \circ x) \circ x).
\end{aligned}$$

Since $p(7-2) = p(5) = 7$, these 11 elements cannot be linearly independent. Thus, four linear relations occur among these eleven candidate elements.: We eliminate w_6 since in the $n = 6$ case we concluded that

$$T_3 \circ T_3 \subseteq T_5 \circ T_1 + T_4 \circ T_2.$$

For our calculations, we will use *Wolfram Mathematica* (see Appendix C). By this, we obtained

$$\begin{aligned}
\lambda_1 &= 0,\\
\lambda_2 &= \lambda_{10} + 2\lambda_{11},\\
\lambda_4 &= -7\lambda_{10} - 12\lambda_{11} - 3\lambda_3,\\
\lambda_5 &= 6\lambda_{10} + 10\lambda_{11} + 2\lambda_3,\\
\lambda_7 &= 5\lambda_{10} + 8\lambda_{11} + 2\lambda_3,\\
\lambda_8 &= -6\lambda_{10} - 9\lambda_{11} - 3\lambda_3,\\
\lambda_9 &= \lambda_3.
\end{aligned}$$

Taking

$$(\lambda_3, \lambda_{10}, \lambda_{11}) = (0, 1, 0), \quad (0, 0, 1), \quad (1, 0, 0),$$

we obtain the following three identities:

$$w_{10} + w_2 - 7w_4 + 6w_5 + 5w_7 - 6w_8 = 0, \tag{6a}$$

$$w_{11} + 2w_2 - 12w_4 + 10w_5 + 8w_7 - 9w_8 = 0, \tag{6b}$$

$$w_3 - 3w_4 + 2w_5 + 2w_7 - 3w_8 + w_9 = 0. \tag{6c}$$

The third relation (6c) coincides with the jetken polynomial, which is obtained from the Besken identity by taking $a = x$ and $b = x \circ x$ [1]. Identity (6a) allows us to eliminate w_{10}, while identity (6b) allows us

to eliminate w_{11}. The third identity (6c) eliminates one more element from the remaining set. Thus, we obtain 7 linearly independent elements.

Since

$$w_{10},\, w_{11} \in T_4 \circ T_3, \qquad w_2, w_4, w_5 \in T_6 \circ T_1, \qquad w_7, w_8 \in T_5 \circ T_2,$$

and

$$w_{10} = -w_2 + 7w_4 - 6w_5 - 5w_7 + 6w_8, \qquad w_{11} = -2w_2 + 12w_4 - 10w_5 - 8w_7 + 9w_8,$$

it follows that

$$T_4 \circ T_3 \subseteq T_6 \circ T_1 + T_5 \circ T_2.$$

That is,

$$T_7 \subseteq T_6 \circ T_1 + T_5 \circ T_2.$$

□

These computations motivate the following conjecture below. Our computations show that, up to degree 7, the stronger recursive inclusion holds.

Conjecture II.3. Let T_n be the homogeneous component of degree n in the one-generated case. Then,

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \qquad n \geq 3.$$

These observations may be helpful in the study of the problem at higher degrees.

III. Results and Discussion

In this work, we studied the homogeneous components T_n in the one-generated case. Our first main result gives the exact dimensions of these components:

$$\dim T_1 = \dim T_2 = 1, \qquad \dim T_n = p(n-2) \quad \text{for all } n \geq 3.$$

Thus, the dimensions of the homogeneous components are described in combinatorial terms by the partition numbers.

We next describe the first homogeneous components in the $\circ$-language and analyze the relations among their candidate basis elements. In degrees 1 through 5, the elements under consideration are linearly independent. The first nontrivial relation appears in degree 6. More precisely, by passing from the $\circ$-language to the differential-monomial realization of the free Novikov algebra, we obtain identity (5), which is a consequence of the Tortken identity. This shows that in degree 6 the list of candidate elements is no longer independent, and one of them can be expressed through the others.

The same persists in degree 7, where further relations arise among the candidate elements. Together with the identity obtained in degree 6, these relations refine the structure of the homogeneous component T_7. Thus, beginning with degree 6, the description of T_n requires not only the enumeration of all possible $\circ$-elements, but also a careful analysis of the identities they satisfy.

Another main result is the recursive inclusion

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \qquad n \leq 7.$$

This inclusion gives a relation between consecutive homogeneous components and shows that, in low degrees, elements of T_n can be obtained from elements of smaller degree by means of the symmetrized product.

A. Discussion

The obtained results show that even the one-generated case is already nontrivial. Although the dimensions of the homogeneous components admit a simple description, the identities appearing in degrees 6 and 7 show that the space of symmetric elements cannot be understood by listing $\circ$-elements alone. One must also take into account the relations between them.

These observations indicate that the one-generated case already reflects essential features of Tortken elements. In particular, the low-degree computations reveal a regular pattern in the structure of the homogeneous components, which may remain visible in higher degrees.

The results obtained here may therefore be useful for the further study of Tortken elements in higher degrees and for the investigation of the case of several generators.

IV. Conclusion and Future Work

In this paper, we studied the one-generated case of the space of Tortken elements in a free Novikov algebra. We proved the dimension formula

$$\dim T_1 = \dim T_2 = 1, \qquad \dim T_n = p(n-2) \quad \text{for all } n \geq 3,$$

described the first homogeneous components in the $\circ$-language, obtained explicit identities in degrees 6 and 7, and established the recursive inclusion

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \qquad n \leq 7.$$

These results provide a precise description of the low-degree structure of the one-generated case. In particular, they indicate that the one-generated case already captures essential features of the structure of special Tortken algebras.

A natural problem for future work is to determine whether the recursive inclusion extends to all degrees. This leads to the conjecture

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \qquad n \geq 3.$$

If true, this conjecture would provide a systematic way to describe the higher-degree components in the one-generated case. It may also serve as a useful starting point for studying the operadic and structural properties of special Tortken algebras and their connections with related Novikov-type algebras, such as δ-Novikov algebras, Gelfand–Dorfman algebras, and Novikov–Poisson algebras.

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Appendix

A. Implementation of the derivation and the symmetrized product

The derivation was implemented by the following rules. The symmetrized product $a \circ b = (ab)'$ was implemented as `t[a,b]`.

```
Dr[C_Integer a_] := C Dr[a]

Dr[a_ + b_] := Dr[a] + Dr[b]

Dr[C_Integer a_ + K_Integer b_] := C Dr[a] + K Dr[b]

Dr[C_Integer a_ + K_Integer b_ + D_Integer d_] :=
  C Dr[a] + K Dr[b] + D Dr[d]

Dr[a_ b_] := Dr[a] b + a Dr[b]

Dr[a_^n_Integer] := Dr[a^(n - 1)] a + a^(n - 1) Dr[a]

Dr'[x_] := 0

Dr[0] := 0

t[a_, b_] := ExpandAll[Dr[a b]]
```

B. Degree 6 computation

Here we present the computation in degree 6. The following six candidate elements of degree six in the $\circ$-language were used.

```
Gen1deg6 = {
  g1d6[1] = t[t[t[t[t[x, x], x], x], x], x],
  g1d6[2] = t[t[t[x, x], t[x, x]], t[x, x]],
  g1d6[3] = t[t[t[x, x], x], t[t[x, x], x]],
  g1d6[4] = t[t[t[t[x, x], x], x], t[x, x]],
  g1d6[5] = t[t[t[t[x, x], t[x, x]], x], x],
  g1d6[6] = t[t[t[t[x, x], x], t[x, x]], x]
```

```
};
```

The program then determines the relations among the coefficients λ_i and computes their admissible values.

```
Combin1deg6 =
  Coefficient[
    Collect[
      ExpandAll[Sum[Subscript[lambda, i] g1d6[i], {i, 1, 6}]],
      {
        x Dr[x]^5,
        x^2 Dr[x]^3 Dr[Dr[x]],
        x^3 Dr[x] Dr[Dr[x]]^2,
        x^3 Dr[x]^2 Dr[Dr[Dr[x]]],
        x^4 Dr[Dr[x]]*Dr[Dr[Dr[x]]],
        x^4 Dr[x]*Dr[Dr[Dr[Dr[x]]]],
        x^5 Dr[Dr[Dr[Dr[Dr[x]]]]]
      }
    ],
    {
      x Dr[x]^5,
      x^2 Dr[x]^3 Dr[Dr[x]],
      x^3 Dr[x] Dr[Dr[x]]^2,
      x^3 Dr[x]^2 Dr[Dr[Dr[x]]],
      x^4 Dr[Dr[x]]*Dr[Dr[Dr[x]]],
      x^4 Dr[x]*Dr[Dr[Dr[Dr[x]]]],
      x^5 Dr[Dr[Dr[Dr[Dr[x]]]]]
    }
  ];

S[x_] := x == 0;

TableForm[
  First[
    Solve[
      Map[S, Combin1deg6],
      Array[Subscript[lambda, #] &, 6]
    ]
  ]
]
```

C. Degree 7 computation

Here we present the computation in degree 7.

```
Gen1deg7 = {
  g1d7[1] = t[t[t[t[x, x], x], x], t[t[x, x], x]],
  g1d7[2] = t[t[t[x, x], t[x, x]], t[t[x, x], x]],
  g1d7[3] = t[t[t[t[t[t[x, x], x], x], x], x], x],
  g1d7[4] = t[t[t[t[x, x], t[x, x]], x], x], x],
  g1d7[5] = t[t[t[t[x, x], x], t[x, x]], x], x],
  g1d7[6] = t[t[t[t[x, x], x], x], t[x, x]], x],
  g1d7[7] = t[t[t[x, x], t[x, x]], t[x, x]], x],
  g1d7[8] = t[t[t[t[x, x], x], x], x], t[x, x]],
  g1d7[9] = t[t[t[x, x], t[x, x]], x], t[x, x]],
  g1d7[10] = t[t[t[x, x], x], t[x, x]], t[x, x]]
};

Combin1deg7 =
  Coefficient[
    Collect[
      ExpandAll[Sum[Subscript[\[Lambda], i] g1d7[i], {i, 1, 10}]],
      {
        x Dr[x]^6,
        x^2 Dr[x]^4 Dr[Dr[x]],
        x^3 Dr[x]^2 Dr[Dr[x]]^2,
        x^4 Dr[Dr[x]]^3,
        x^3 Dr[x]^3 Dr[Dr[Dr[x]]],
        x^4 Dr[x] Dr[Dr[x]] Dr[Dr[Dr[x]]],
        x^5 Dr[Dr[Dr[x]]]^2,
        x^4 Dr[x]^2 Dr[Dr[Dr[Dr[x]]]],
        x^5 Dr[Dr[x]] Dr[Dr[Dr[Dr[x]]]],
        x^5 Dr[x] Dr[Dr[Dr[Dr[Dr[x]]]]],
        x^6 Dr[Dr[Dr[Dr[Dr[Dr[x]]]]]]
      }
    ],
    {
      x Dr[x]^6,
      x^2 Dr[x]^4 Dr[Dr[x]],
      x^3 Dr[x]^2 Dr[Dr[x]]^2,
      x^4 Dr[Dr[x]]^3,
```

```
      x^3 Dr[x]^3 Dr[Dr[Dr[x]]],
      x^4 Dr[x] Dr[Dr[x]] Dr[Dr[Dr[x]]],
      x^5 Dr[Dr[Dr[x]]]^2,
      x^4 Dr[x]^2 Dr[Dr[Dr[Dr[x]]]],
      x^5 Dr[Dr[x]] Dr[Dr[Dr[Dr[x]]]],
      x^5 Dr[x] Dr[Dr[Dr[Dr[Dr[x]]]]],
      x^6 Dr[Dr[Dr[Dr[Dr[Dr[x]]]]]]
    }
  ];

S[x_] := x == 0;

TableForm[
  First[
    Solve[
      Map[S, Combin1deg7],
      Array[Subscript[\[Lambda], #] &, 10]
    ]
  ]
]
```

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