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RELATION BETWEEN STATISTICS OF PERMUTATIONS.

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Contents

Abbreviation

r[min, i]- right minimal index

r[min, v]- right minimal value

r[max, i]-right maximal index

r[max, v]- right maximal value

l[min, i]-left minimal index

l[min, v]- left minimal value

l[max, i]-left maximal index

l[max, v]-left maximal value

cyc- cycle

des-descent

INTRODUCTION

Permutations have a remarkably rich combinatorial structure. Part of the reason for this is that a permutation of a finite set can be represented in many equivalent ways, including as a word (sequence), a function, a collection of disjoint cycles, a matrix, etc. Each of these representations suggests a host of natural invariants (or "statistics"), operations, transformations, structures, etc., that can be applied to or placed on permutations. The fundamental statistics, operations, and structures on permutations include descent set (with numerous specializations), excedance set, cycle type, records, subsequences, composition (product), partial orders, simplicial complexes, probability distributions, etc. This paper contains topics that relates to the main part and definitions to understand the main part of this work. In the main part, we consider statistics of permutations, equidistributions of them.

1 PERMUTATIONS

In mathematics, the notion of permutation is used with several slightly different meanings, all related to the act of permuting (rearranging) objects or values. Informally, a permutation of a set of objects is an arrangement of those objects into a particular order. For example, there are six permutations of the set {1,2,3}, namely (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), and (3,2,1). For instance, an anagram of a word is a permutation of its letters. The study of permutations in this sense generally belongs to the field of combinatorics.

The number of permutations of n distinct objects is $n\times(n-1)\times(n-2)\times\cdots\times2\times1$, which is commonly denoted as "n factorial" and written "$n!$".

Permutations occur, in more or less prominent ways, in almost every domain of mathematics. They often arise when different orderings on certain finite sets are considered, possibly only because one wants to ignore such orderings and needs to know how many configurations are thus identified. For similar reasons permutations arise in the study of sorting algorithms in computer science.

In algebra and particularly in group theory, a permutation of a set S is defined as a bijection from S to itself (i.e., a map $S \to S$ for which every element of S occurs exactly once as image value). This is related to the rearrangement of S in which each element s takes the place of the corresponding $f(s)$. The collection of such permutations form a symmetric group. The key to its structure is the possibility to compose permutations: performing two given rearrangements in succession defines a third rearrangement, the composition. Permutations may *act* on composite objects by rearranging their components, or by certain replacements (substitutions) of symbols.

1.1 In combinatorics

In combinatorics, a permutation is usually understood to be a sequence containing each element from a finite set once, and only once. The concept of *sequence* is distinct from that of a *set*, in that the elements of a sequence appear in some order: the sequence has a first element (unless it is empty), a second element (unless its length is less than 2), and so on. In contrast, the elements in a set have no order; {1, 2, 3} and {3, 2, 1} are different ways to denote the same set. In this sense a permutation of a finite set S of n elements is equivalent to a bijection from $\{1, 2, ..., n\}$ to S (in which any i is mapped to the i-th element of the sequence), or to a choice of a total ordering on S (for which $x < y$ if x comes before y in the sequence). There are $n!$ permutations of S.

There is also a weaker meaning of the term "permutation" that is sometimes used in elementary combinatorics texts, designating those sequences in which no element occurs more than once, but without the requirement to use all elements from a given set. Indeed this use often involves considering sequences of a fixed length k of elements taken from a given set of size n. These objects are also known as partial permutations or as sequences without repetition, terms that avoids confusion with the other, more common, meanings of "permutation". The number of such k-permutations of n is denoted variously by such symbols as ${}_nP_k$, ${}^nP_k$, $P_{n,k}$, or $P(n,k)$, and its value is given by the product

$$n\cdot(n\text{-}1)\cdot(n\text{-}2)\cdots(n\text{-}3)$$

which is 0 when $k > n$, and otherwise is equal to

$$\frac{n!}{(n-k)!}.$$

The product is well defined without the assumption that n is a non-negative integer and is of importance outside combinatorics as well; it is known as the Pochhammer symbol $(n)_k$ or as the k-th falling factorial power $n^{\underline{k}}$ of n.

If M is a finite multiset, then a multiset permutation is a sequence of elements of M in which each element appears exactly as often as is its multiplicity in M. If the multiplicities of the elements of M (taken in some order) are m_1, m_2, ..., m_l and their sum (i.e., the size of M) is n, then the number of multiset permutations of M is given by the multinomial coefficient

$$\binom{n}{m_1,\ m_2,\ldots,\ m_l} = \frac{n!}{m_1!\,m_2!\ldots\ m_l!}.$$

1.2 Notation

There are three main notations for permutations of a finite set S.

In Cauchy's *two-line notation*, one lists the elements of S in the first row, and for each one its image under the permutation below it in the second row. For instance, a particular permutation of the set {1,2,3,4,5} can be written as:

$$\sigma = \binom{12345}{35412}$$

this means that σ satisfies $\sigma(1)=3$, $\sigma(2)=5$, $\sigma(3)=4$, $\sigma(4)=1$, and $\sigma(5)=2$.

In *one-line notation*, one gives only the second row of this array, so the one-line notation for the permutation above is 32514. (It is typical to use commas to separate these entries only if some have two or more digits.)

Cycle notation, the third method of notation, focuses on the effect of successively applying the permutation. It expresses the permutation as a product of cycles corresponding to the orbits (with at least two elements) of the permutation; since distinct orbits are disjoint, this is loosely referred to as "the decomposition into disjoint cycles" of the permutation. It works as follows: starting from some element x of S with $\sigma(x) \neq x$, one writes the sequence $(x\ \sigma(x)\ \sigma(\sigma(x))\ \ldots)$ of successive images under σ, until the image would be x, at which point one instead closes the parenthesis. The set of values written down forms the orbit (under σ) of x, and the parenthesized expression gives the corresponding cycle of σ. One then continues choosing an element y of S that is not in the orbit already written down, and such that $\sigma(y) \neq y$, and writes down the corresponding cycle, and so on until all elements of S either belong to a cycle written down or are fixed points of σ. Since for every new cycle the starting point can be chosen in different ways, there are in general many different cycle notations for the same permutation; for the example above one has for instance

$$\begin{pmatrix} 12345 \\ 35412 \end{pmatrix} = (134)(25) = (25)(134) = (413)(52).$$

Each cycle $(x_1\, x_2 \ldots x_l)$ of σ denotes a permutation in its own right, namely the one that takes the same values as σ on this orbit (so it maps x_i to x_{i+1} for $i < l$, and x_l to x_1), while mapping all other elements of S to themselves. The size l of the orbit is called the length of the cycle. Distinct orbits of σ are by definition disjoint, so the corresponding cycles are easily seen to commute, and σ is the product of its cycles (taken in any order). Therefore the concatenation of cycles in the cycle notation can be interpreted as denoting composition of permutations, whence the name "decomposition" of the permutation. This decomposition is essentially unique: apart from the reordering the cycles in the product, there are no other ways to write σ as a product of cycles (possibly unrelated to the cycles of σ) that have disjoint orbits. The cycle notation is less unique, since each individual cycle can be written in different ways, as in the example above where (413) denotes the same cycle as (134) (but (431) would denote a different permutation).

An orbit of size 1 (a fixed point x in S) has no corresponding cycle, since that permutation would fix x as well as every other element of S, in other words it would be the identity, independently of x. It is possible to include (x) in the cycle notation for σ to stress that σ fixes x (and this is even standard in combinatorics, as described in cycles and fixed points), but this does not correspond to a factor in the (group theoretic) decomposition of σ. If the notion of "cycle" were taken to include the identity permutation, then this would spoil the uniqueness (up to order) of the decomposition of a permutation into disjoint cycles. The decomposition into disjoint cycles of the identity permutation is an empty product; its cycle notation would be empty, so some other notation like e is usually used instead.

Cycles of length two are called transpositions; such permutations merely exchange the place of two elements.

In combinatorics a permutation of a set S with n elements is a listing of the elements of S in some order (each element occurring exactly once). This can be defined formally as a bijection from the set { 1, 2, ..., n } to S. Note that if S equals { 1, 2, ..., n }, then this definition coincides with the definition in group theory. More generally one could use instead of { 1, 2, ..., n } any set equipped with a total ordering of its elements.

One combinatorial property that is related to the group theoretic interpretation of permutations, and can be defined without using a total ordering of S, is the cycle structure of a permutation σ. It is the partition of n describing the lengths of the cycles of σ. Here there is a part "1" in the partition for every fixed point of σ. A permutation that has no fixed point is called a derangement.

Other combinatorial properties however are directly related to the ordering of S, and to the way the permutation relates to it. Here are a number of such properties.

2 NUMBERING PERMUTATIONS

One way to represent permutations of n is by an integer N with $0 \leq N < n!$, provided convenient methods are given to convert between the number and the usual representation of a permutation as a sequence. This gives the most compact representation of arbitrary permutations, and in computing is particularly attractive when n is small enough that N can be held in a machine word; for 32-bit words this means $n \leq 12$, and for 64-bit words this means $n \leq 20$. The conversion can be done via the intermediate form of a sequence of numbers $d_n, d_{n-1}, \ldots, d_2, d_1$, where d_i is a non-negative integer less than i (one may omit d_1, as it is always 0, but its presence makes the subsequent conversion to a permutation easier to describe). The first step then is simply expression of N in the factorial number system, which is just a particular mixed radix representation, where for numbers up to $n!$ the bases for successive digits are $n, n-1, \ldots, 2, 1$. The second step interprets this sequence as a Lehmer code or (almost equivalently) as an inversion table.

In the Lehmer code for a permutation σ, the number d_n represents the choice made for the first term σ_1, the number d_{n-1} represents the choice made for the second term σ_2 among the remaining $n-1$ elements of the set, and so forth. More precisely, each d_{n+1-i} gives the number of *remaining* elements strictly less than the term σ_i. Since those remaining elements are bound to turn up as some later term σ_j, the digitd_{n+1-i} counts the *inversions* (i,j) involving i as smaller index (the number of values j for which $i < j$ and$\sigma_i > \sigma_j$). The inversion table for σ is quite similar, but here d_{n+1-k} counts the number of inversions (i,j) where $k = \sigma_j$ occurs as the smaller of the two values appearing in inverted order. Both encodings can be visualized by an n by n Rothe diagram (named after Heinrich August Rothe) in which dots at (i,σ_i) mark the entries of the permutation, and a cross at (i,σ_j) marks the inversion (i,j); by the definition of inversions a cross appears in any square that comes both before the dot (j,σ_j) in its column, and before the dot (i,σ_i) in its row. The Lehmer code lists the numbers of crosses in successive rows, while the inversion table lists the numbers of crosses in successive columns; it is just the Lehmer code for the inverse permutation, and vice versa.

To effectively convert a Lehmer code $d_n, d_{n-1}, \ldots, d_2, d_1$ into a permutation of an ordered set S, one can start with a list of the elements of S in increasing order, and for i increasing from 1 to n set σ_i to the element in the list that is preceded by d_{n+1-i} other ones, and remove that element from the list. To convert an inversion table $d_n, d_{n-1}, \ldots, d_2, d_1$ into the corresponding permutation, one can traverse the numbers from d_1 to d_n while inserting the elements of S from largest to smallest into an initially empty sequence; at the step using the number d from the inversion table, the element from S inserted into the sequence at the point where it is preceded by d elements already present. Alternatively one could process the numbers from the inversion table and the elements of S both in the opposite order, starting with a row of n empty slots, and at each step place the element from S into the empty slot that is preceded by d other empty slots.

Converting successive natural numbers to the factorial number system produces those sequences in lexicographic order (as is the case with any mixed radix number system), and further converting them to permutations preserves the lexicographic ordering, provided the Lehmer code interpretation is used (using inversion tables, one gets a different ordering, where one starts by comparing permutations by the *place* of their entries 1 rather than by the value of their first entries). The sum of the numbers in the factorial number system representation gives the number of inversions of the permutation, and the parity of that sum gives the signature of the permutation. Moreover the positions of the zeroes in the inversion table give the values of left-to-right maxima of the permutation (in the example 6, 8, 9) while the positions of the zeroes in the Lehmer code are the positions of the right-to-left minima (in the example positions the 4, 8, 9 of the values 1, 2, 5); this allows computing the distribution of such extrema among all permutations. A permutation with Lehmer code $d_n, d_{n-1}, \ldots, d_2, d_1$ has an ascent $n - i$ if and only if $d_i \geq d_{i+1}$.

2.1 Counting sequences without repetition

In this section, a *k*-permutation of a set *S* is an ordered sequence of *k* distinct elements of *S*. For example, given the set of letters {`C`, `E`, `G`, `I`, `N`, `R`}, the sequence `ICE` is a 3-permutation, `RING` and `RICE` are 4-permutations, `NICER` and `REIGN` are 5-permutations, and `CRINGE` is a 6-permutation; since the latter uses all letters, it is a permutation of the given set in the ordinary combinatorial sense. `ENGINE` on the other hand is not a permutation, because of the repetitions: it uses the elements `E` and `N` twice.

Let *n* be the size of *S*, the number of elements available for selection. In constructing a *k*-permutation, there are *n* possible choices for the first element of the sequence, and this is then number of 1-permutations. Once it has been chosen, there are $n - 1$ elements of *S* left to choose from, so a second element can be chosen in $n - 1$ ways, giving a total $n \times (n - 1)$ possible 2-permutations. For each successive element of the sequence, the number of possibilities decreases by 1 which leads to the number of

$n \times (n - 1) \times (n - 2) \ldots \times (n - k + 1)$ possible *k*-permutations.

This gives in particular the number of *n*-permutations (which contain all elements of *S* once, and are therefore simply permutations of *S*):

$n \times (n - 1) \times (n - 2) \times \ldots \times 2 \times 1,$

a number that occurs so frequently in mathematics that it is given a compact notation "*n*!", and is called "*n* factorial". These *n*-permutations are the longest sequences without repetition of elements of *S*, which is reflected by the fact that the above formula for the number of *k*-permutations gives zero whenever $k > n$.

The number of *k*-permutations of a set of *n* elements is sometimes denoted by *P(n,k)* or a similar notation (usually accompanied by a notation for the number of *k*-combinations of a set of *n* elements in which the "*P*" is replaced by "*C*"). That notation is rarely used in other contexts than that of counting k-permutations, but the expression for the number does arise in many other situations. Being a product of k factors starting at n and decreasing by unit steps, it is called the k-th falling factorial power of n:

$$^{k}n = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$$

though many other names and notations are in use, as detailed at Pochhammer symbol. When $k \leq n$ the factorial power can be completed by additional factors: $^{k}n \times (n-k)! = n!$, which allows writing

$$^{k}n = \frac{n!}{(n-k)!}.$$

The right hand side is often given as expression for the number of k-permutations, but its main merit is using the compact factorial notation. Expressing a product of k factors as a quotient of potentially much larger products, where all factors in the denominator are also explicitly present in the numerator, is not particularly efficient; as a method of computation there is the additional danger of overflow or rounding errors. It should also be noted that the expression is undefined when $k > n$, whereas in those cases the number n^k of k-permutations is just 0.

2.2 Lehmer code

The Lehmer code makes evident the fact that there are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

permutations of a sequence of n numbers. If a permutation σ is specified by the sequence $(\sigma_1, \ldots, \sigma_n)$ of its images of $1, \ldots, n$, then it is encoded by a sequence of n numbers, but not all such sequences are valid since every number must be used only once. By contrast the encodings considered here choose the first number from a set of n values, the next number from a fixed set of $n-1$ values, and so forth decreasing the number of possibilities until the last number for which only a single fixed value is allowed; *every* sequence of numbers chosen from these sets encodes a single permutation. While several encodings can be defined, the Lehmer code has several additional useful properties; it is the sequence

$$L(\sigma) = (L(\sigma)_1, \ldots, L(\sigma)_n) \text{ where } L(\sigma)_i = \{j > i: \sigma_j < \sigma_i\}$$

in other words the term $L(\sigma)_i$ counts the number of terms in $(\sigma_1, \ldots, \sigma_n)$ to the right of σ_i that are smaller than it, a number between 0 and $n-i$, allowing for $n+1-i$ different values.

A pair of indices (i,j) with $i < j$ and $\sigma_i > \sigma_j$ is called an inversion of σ, and $L(\sigma)_i$ counts the number of inversions (i,j) with i fixed and varying j. It follows that $L(\sigma)_1 + L(\sigma)_2 + \ldots + L(\sigma)_n$ is the total number of inversions of σ, which is also the number of adjacent transpositions that are needed to transform the permutation into the identity permutation. Other properties of the Lehmer code include that the lexicographical order of the encodings of two permutations is the same as that of their sequences ($\sigma_1, \ldots, \sigma_n$), that any value 0 in the code represents a right-to-left minimum in the permutation (i.e., a σ_i smaller than any σ_j to its right), and a value $n - i$ at position i similarly signifies a right-to-left maximum, and that the Lehmer code of σ coincides with the factorial number system representation of its position in the list of permutations of n in lexicographical order (numbering the positions starting from 0).

Variations of this encoding can be obtained by counting inversions (i,j) for fixed j rather than fixed i, by counting inversions with a fixed smaller *value* σ_j rather than smaller index i, or by counting non-inversions rather than inversions; while this does not produce a fundamentally different type of encoding, some properties of the encoding will change correspondingly. In particular counting inversions with a fixed smaller value σ_j gives the inversion table of σ, which can be seen to be the Lehmer code of the inverse permutation.

2.3 Encoding and decoding

The usual way to prove that there are $n!$ different permutations of n objects is to observe that the first object can be chosen in n different ways, the next object in $n - 1$ different ways (because choosing the same number as the first is forbidden), the next in $n - 2$ different ways (because there are now 2 forbidden values), and so forth. Translating this freedom of choice at each step into a number, one obtains an encoding algorithm, one that finds the Lehmer code of a given permutation. One need not suppose the objects permuted to be numbers, but one needs a total ordering of the set of objects. Since the code numbers are to start from 0, the appropriate number to encode each object σ_i by is the number of objects that were available at that point (so they do not occur before position i), but which are smaller than the object σ_i actually chosen. (Inevitably such objects must appear at some position $j > i$, and (i,j) will be an inversion, which shows that this number is indeed $L(\sigma)_i$.)

This number to encode each object can be found by direct counting, in several ways (directly counting inversions, or correcting the total number of objects smaller than a given one, which is its sequence number starting from 0 in the set, by those that are unavailable at its position). Another method which is in-place, but not really more efficient, is to start with the permutation of $\{0, 1, \ldots n - 1\}$ obtained by representing each object by its mentioned sequence number, and then for each entry x, in order from left to right, correct the items to its right by subtracting 1 from all entries (still) greater than x (to reflect the fact that the object corresponding to x is no longer available). Concretely a Lehmer code for the permutation B,F,A,G,D,E,C of letters, ordered alphabetically, would first give the list of sequence numbers 1,5,0,6,3,4,2, which is successively transformed

1 5 0 6 3 4 2

1 4 0 5 2 3 1

1 4 0 4 2 3 1

1 4 0 3 1 2 0
1 4 0 3 1 2 0
1 4 0 3 1 1 0
1 4 0 3 1 1 0

where the final line is the Lehmer code (at each line one subtracts 1 from the larger entries to the right of the boldface element to form the next line).

For decoding a Lehmer code into a permutation of a given set, the latter procedure may be reversed: for each entry x, in order from right to left, correct the items to its right by adding 1 to all those (currently) greater than or equal to x; finally interpret the resulting permutation of $\{0, 1, \dots n-1\}$ as sequence numbers (which amounts to adding 1 to each entry if a permutation of $\{1, 2, \dots n\}$ is sought). Alternatively the entries of the Lehmer code can be processed from left to right, and interpreted as a number determining the next choice of an element as indicated above; this requires maintaining a list of available elements, from which each chosen element is removed. In the example this would mean choosing element 1 from {A,B,C,D,E,F,G} (which is B) then element 4 from {A,C,D,E,F,G} (which is F), then element 0 from {A,C,D,E,G} (giving A) and so on, reconstructing the sequence B,F,A,G,D,E,C.

2.4 Independence of relative ranks

The Lehmer code defines a bijection from the symmetric group S_n to the Cartesian product $[n]\times[n-1]\times[n-2]\times\cdots\times[2]\times[1]$, where $[k]$ designates the k-element set $\{0,1,\dots,k-1\}$. As a consequence, under the uniform law on the S_n, the component $L(\sigma)_i$ defines a uniformly distributed random variable on $[n+1-i]$, and these random variables are mutually independent, because they are projections on different factors of a Cartesian product.

Here are the 24 elements of the symmetric group on {1,2,3,4} expressed using the cycle notation, and grouped according to their conjugacy classes:

()

(12), (13), (14), (23), (24), (34) (transpositions)

(123), (132), (124), (142), (134), (143), (234), (243)

(12)(34), (13)(24), (14)(23)

(1234), (1243), (1324), (1342), (1423), (1432)

Another method for determining whether a given permutation is even or odd is to construct the corresponding Permutation matrix and compute its determinant. The value of the determinant is same as the parity of the permutation.

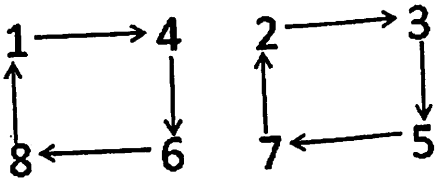
Every permutation of odd order must be even. The permutation (12)(34) in A_4 shows that the converse is not true in general.

2.5 Algorithms to generate permutations

In computing it may be required to generate permutations of a given sequence of values. The methods best adapted to do this depend on whether one wants some randomly chosen permutations, or all permutations, and in the latter case if a specific ordering is required. Another question is whether possible equality among entries in the given sequence is to be taken into account; if so, one should only generate distinct multiset permutations of the sequence.

An obvious way to generate permutations of n is to generate values for the Lehmer code (possibly using the factorial number system representation of integers up to $n!$), and convert those into the corresponding permutations. However, the latter step, while straightforward, is hard to implement efficiently, because it requires n operations each of selection from a sequence and deletion from it, at an arbitrary position; of the obvious representations of the sequence as an array or a linked list, both require (for different reasons) about $n^2/4$ operations to perform the conversion. With n likely to be rather small (especially if generation of all permutations is needed) that is not too much of a problem, but it turns out that both for random and for systematic generation there are simple alternatives that do considerably better. For this reason it does not seem useful, although certainly possible, to employ a special data structure that would allow performing the conversion from Lehmer code to permutation in $O(n \log n)$ time.

Definition 1



A permutation P over a set S with k elements is called a cyclic permutation with offset t if and only if

the elements of S may be ordered ($c[1] < c[2] < \ldots < c[k]$) and the mapping of P may be written as:

$p(c[i]) = c[i + t]$ for $i = 1, 2, \ldots, k - t$, and

$p(c[i]) = c[i + t - k]$ for $i = k - t + 1, k - t + 2, \ldots, k$.

Note: Every cyclic permutation of definition type 1 will be constructed with exactly gcd (k, t) disjoint cycles of equal length; see cycles and fixed points.

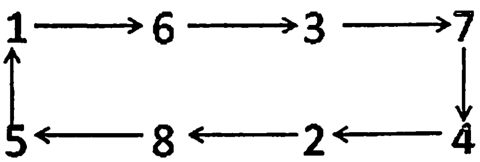
Cyclic permutations of definition type 1 are also called *rotations*, or *circular shifts*.

Example:

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 4\,3\,5\,6\,7\,8\,2\,1 \end{pmatrix} = \begin{pmatrix} 1\,2\,3\,4\,5\,7\,6\,8 \\ 4\,3\,5\,6\,7\,2\,8\,1 \end{pmatrix} = (1468)(2357)$$

is a cyclic permutation with offset 2. It may be constructed with gcd(8, 2) = 2 cycles; see image. The used order is: $c[6] := 7$, $c[7] := 6$, $c[i] = i$ else.

Definition 2



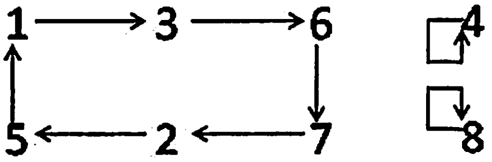
A permutation is called a cyclic permutation if and only if it will be constructed with exactly 1 cycle.

Note: Every permutation over a set with k elements is a cyclic permutation of definition type 2 if and only if it is a cyclic permutation of definition type 1 with gcd(k, offset) = 1

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 7 & 2 & 1 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 3 & 7 & 4 & 2 & 8 & 5 \\ 6 & 3 & 7 & 4 & 2 & 8 & 5 & 1 \end{pmatrix} = (16374285)$$

Definition 3



A permutation is called a cyclic permutation if and only if only one of the constructing cycles will have length > 1.

Note: Every cyclic permutation of definition type 3 may be seen as an union of a cyclic permutation of definition type 2 and some fixed points.

Every cyclic permutation of definition type 2 may be seen "as a cyclic permutation of definition type 3 with zero fixed points.

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 6 & 4 & 1 & 7 & 2 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 & 7 & 2 & 5 & 4 & 8 \\ 3 & 6 & 7 & 2 & 5 & 1 & 4 & 8 \end{pmatrix} = (136725)(4)(8)$$

2.6 Cycle notation

In combinatorial mathematics, the cycle notation is a useful convention for writing down a permutation in terms of its constituent cycles. This is also called circular notation and the permutation called a cyclic or circular permutation.

Definition

Let S be a finite set, and $a_1, \ldots, a_k, k \geq 2$ be distinct elements of S. The expression $(a_1 \ldots a_k)$ denotes the cycle σ whose action is

$$a_1 \longmapsto a_2 \longmapsto a_3 \longmapsto \cdots \longmapsto a_k \longmapsto a_1.$$

For each index i, $\sigma(a_i)= a_{i+1}$ where a_{k+1} is taken to mean a_1.

There are k different expressions for the same cycle; the following all represent the same cycle:

$$(a_1 a_2 a_3 \dots a_k)= (a_2 a_3 \dots a_k a_1)= \dots = (a_k a_1 a_2 \dots a_{k-1})$$

A 1-element cycle is the identity permutation. The identity permutation can also be written as an empty cycle, "()".

2.7 Permutation as product of cycles

Let π be a permutation of S, and let $S_1, \dots S_k \subset S, k \in \mathbb{N}$ be the orbits of π with more than 1 element. Consider an element $S_j, j = 1, \dots, k$, let n_j denote the cardinality of $S_j, |S_j| =n_j$. Also, choose an $a_{1,j} \in S_j$, and define $a_{i+1,j}= \pi(a_{i,j})$, for $1\leq i \leq n_j$; then also $\pi(a_{n_j,j})= a_{1,j}$.

We can now express π as a product of disjoint cycles, namely

$$\pi = (a_{1,1} \cdots a_{n_1,1})(a_{1,2} \cdots a_{n_2,2}) \cdots (a_{1,k} \cdots a_{n_k,k}).$$

Since disjoint cycles commute with each other, the meaning of this expression is independent of the convention used for the order in products of permutations, namely whether the factors in such a product operate rightmost-first (as is usual more generally for function composition), or leftmost-first as some authors prefer. The meaning of individual cycles is also independent of this convention, namely always as described above.

Example

Here are the 24 elements of the symmetric group on {1,2,3,4} expressed using the cycle notation, and grouped according to their conjugacy classes:

()

(12), (13), (14), (23), (24), (34) (transpositions)

(123), (132), (124), (142), (134), (143), (234), (243)

(12)(34), (13)(24), (14)(23)

(1234), (1243), (1324), (1342), (1423), (1432)

3 ASCENTS, DESCENTS AND RUNS

An *ascent* of a permutation σ of n is any position $i < n$ where the following value is bigger than the current one. That is, if $\sigma = \sigma_1\sigma_2...\sigma_n$, then i is an ascent if $\sigma_i < \sigma_{i+1}$.

For example, the permutation 3246517 has ascents (at positions) 3,4,6,7.

Similarly, a *descent* is a position $i < n$ with $\sigma_i > \sigma_{i+1}$, so every i with $1 \leq i < n$ either is an ascent or is a descent of σ.

For example, the permutation 3246517 has decscents (at positions) 3,2,1.

The number of permutations of n with k ascents is the Eulerian number $\left\langle {n \atop k} \right\rangle$; this is also the number of permutations of n with k descents.

An *ascending run* of a permutation is a nonempty increasing contiguous subsequence of the permutation that cannot be extended at either end; it corresponds to a maximal sequence of successive ascents (the latter may be empty: between two successive descents there is still an ascending run of length 1). By contrast an *increasing subsequence* of a permutation is not necessarily contiguous: it is an increasing sequence of elements obtained from the permutation by omitting the values at some positions. For example, the permutation 2453167 has the ascending runs 245, 3, and 167, while it has an increasing subsequence 2367.

If a permutation has $k - 1$ descents, then it must be the union of k ascending runs. Hence, the number of permutations of n with k ascending runs is the same as the number $\left\langle {n \atop k-1} \right\rangle$ of permutations with $k - 1$ descents.

3.1 Inversions

An *inversion* of a permutation σ is a pair (i,j) of positions where the entries of a permutation are in the opposite order: $i < j$ and $\sigma_i > \sigma_j$. So a descent is just an inversion at two adjacent positions. For example, the permutation $\sigma = 23154$ has three inversions: (1,3), (2,3), (4,5), for the pairs of entries (2,1), (3,1), (5,4).

Sometimes an inversion is defined as the pair of values (σ_i,σ_j) itself whose order is reversed; this makes no difference for the *number* of inversions, and this pair (reversed) is also an inversion in the above sense for the inverse permutation σ^{-1}. The number of inversions is an important measure for the degree to which the entries of a permutation are out of order; it is the same for σ and forσ^{-1}. To bring a permutation with k inversions into order (i.e., transform it into the identity permutation), by successively applying (right-multiplication by) adjacent transpositions, is always possible and requires a sequence of k such operations. Moreover any reasonable choice for the adjacent transpositions will work: it suffices to choose at each step a transposition of i and $i + 1$ where i is a descent of the permutation as modified so far (so that the transposition will remove this particular descent, although it might create other descents). This is so because applying such a transposition reduces the number of inversions by 1; also note that as long as this number is not zero, the permutation is not the identity, so it has at least one descent. Bubble sort and insertion sort can be interpreted as particular instances of this procedure to put a sequence into order. Incidentally this procedure proves that any permutation σ can be written as a product of adjacent transpositions; for this one may simply reverse any sequence of such transpositions that transforms σ into the identity. In fact, by enumerating all sequences of adjacent transpositions that would transform σ into the identity, one obtains (after reversal) a *complete* list of all expressions of minimal length writing σ as a product of adjacent transpositions.

The number of permutations of n with k inversions is expressed by a Mahonian number, it is the coefficient of X^k in the expansion of the product

$$\prod_{m=1}^{n}\sum_{i=0}^{m-1} X^i = 1(1+X)(1+X+X^2)\dots(1+X+X^2+\cdots+X^{n-1}),$$

which is also known (with q substituted for X) as the q-factorial $[n]_q!$. The expansion of the product appears in Necklace (combinatorics).

3.2 Descents

The "most orderly" of all n-permutations is obviously the increasing permutation $123\cdots n$. All other permutations have at least some "disorder" in them, for instance, it happens that an entry is immediately followed by a *smaller* entry in them.

The definition of descents

Definition Let $\sigma = \sigma_1\sigma_2 \dots \sigma_n$ be a permutation. We say that i is a descent of σ if $\sigma_i > \sigma_i+1$. Similarly, we say that i is an ascent of σ if $\sigma_i < \sigma_i+1$.

Lemma 3.2.2

Let $S = \{s_1, s_2, \dots, s_k\} \subseteq [n-1]$, and let α (S) be the number of n-permutations whose descent set is contained in S. Then we have

$$\alpha\,(S) = \binom{n}{s_1}\binom{n-s_1}{s_2-s_1}\binom{n-s_2}{s_3-s_2}\dots\binom{n-s_k}{n-s_k}$$

PROOF

The crucial idea of the proof is the following. We arrange our n entries into k+1 segments so that the first i segments together have s_i entries for each i. Then, within each segment, we put our entries in increasing order. Then the only places where the resulting permutation has a chance to have a descent is where two segments meet, that is, at $s_1, s_2, \dots, s_k$. Therefore, the descent set of the resulting permutation is contained in S. How many ways are there to arrange our entries in these segments? The first segment has to have length s_1, and therefore can be chosen in ways. The second segment has to be of length $s_2 - s_1$, and has to be disjoint from the first one. Therefore, it can be chosen in ways. In general, segment i must have length $s_i - s_i - 1$ if $i<k+1$, and has to be chosen from the remaining $n - s_i - 1$ entries, in ways. There is only one choice for the last segment as all remaining $n - s_k$ entries have to go there. This completes the proof.

Now we are in a position to state and prove the formula for the number of n-permutations with a given descent set.

3.3 Eulerian numbers

Let $A(n, k)$ be the number of n-permutations with k-1 descents. You may be wondering what the reason for this shift in the parameter k is. If σ has k-1 descents, then σ is the union of k increasing subsequences of consecutive entries. These are called the *ascending runs* of σ. (Some authors call them just "runs," some others call something else "runs." This is why we add the adjective "ascending" to avoid confusion.) Also note that in some papers, $A(n, k)$ is used to denote the number of permutations with k descents.

Example 1

The three ascending runs of p=2415367 are 24, 15, and 367.

THEOREM 3.3.1

For all positive integers k and n satisfying $k \leq n$, *we have*

$$A(n, k+1)=(k+1)A(n-1, k+1)+(n-k)A(n-1, k).$$

PROOF

There are two ways we can get an n-permutation σ with k descents from an (n-1)-permutation σ' by inserting the entry n into σ'. Either σ' has k descents, and the insertion of n does not form a new descent, or σ' has k-1 descents, and the insertion of n does form a new descent.

In the first case, we have to put the entry n at the end of σ', or we have to insert n between two entries that form one of the k descents of σ'. This means we have k+1 choices for the position of n. As we have $A(n-1, k+1)$ choices for σ', the first term of the right-hand side is explained.

In the second case, we have to put the entry n at the front of σ', or we have to insert n between two entries that form one of the (n-2)-(k-1) ascents of σ'. This means that we have n-k choices for the position of n. As we have $A(n-1, k)$ choices for σ', the second part of the right-hand side is explained, and the theorem is proved.

We note that $A(n, k+1)=A(n, n-k)$; in other words, the Eulerian numbers are symmetric. Indeed, if $\sigma=\sigma_1\sigma_2 \dots \sigma_n$ has k descents, then its reverse $\sigma^r=\sigma_n\sigma_{n-1} \dots \sigma_1$ has n-k-1 descents.

The following theorem shows some additional significance of the Eulerian numbers. In fact, the Eulerian numbers are sometimes *defined* using this relation.

3.4 Stirling numbers and Eulerian numbers

A *partition* of the set *[n]* into r blocks is a distribution of the elements of *[n]* into r sets $B_1, B_2, \cdots, B_r$ so that each element is placed into exactly one block.

DEFINITION 3.4.1 The number of partitions of [n] into r blocks is denoted by S(n, k) and is called a *Stirling number of the second kind*. By convention, we set $S(n, 0)=0$ if $n>0$, and $S(0, 0)=1$.

LEMMA 3.4.2

For all positive integers n and r, we have

$$S(n,r) = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i \binom{r}{i}(r-i)^n.$$

PROOF Note that an ordered partition of *n* into *r* blocks is just the same as a surjection from *[n]* to *[r]*, To enumerate all such surjections, let A_i be the set of functions from *[n]* into *[r]* whose image does not contain *i*. The function *f: [n]→[r]* is a surjection if and only if it is not contained in $A_1 \cup A_2 \cup A_r$, and our claim follows by a standard application of the Principle of Inclusion- Exclusion.

Stirling numbers of the second kind and Eulerian numbers are closely related, as shown by the following theorem.

THEOREM 3.4.3

For all positive integers n and r, we have

$$S(n,r) = \frac{1}{r!}\sum_{i=0}^{r} A(n,k)\binom{n-k}{r-k}. \qquad (3.9)$$

PROOF

Multiplying both sides by *r*! we get

$$r!\,S(n,r) = \sum_{i=0}^{r} A(n,k)\binom{n-k}{r-k}.$$

Here the left-hand side is obviously the number of ordered partitions, (that is, partitions whose set of blocks is totally ordered), of *[n]* into *r* blocks. We will now show that the right-hand side counts the same objects. Take a permutation *p* counted by *A(n, k)*. The *k* ascending runs of *p* then naturally define an ordered partition of *[n]* into *k* parts. If *k*=*r*, then there is nothing left to do. If *k*<*r*, then we will split up some of the ascending runs into several blocks of consecutive elements, in order to get an ordered partition of *r* blocks. As we currently have *k* blocks, we have to increase the number of blocks by *r-k*. This can be achieved by choosing *r-k* of the *n-k* "gap positions", (gaps between two consecutive entries within the same block).

This shows that we can obtain $\sum_{k=0}^{r} A(n,k)\binom{n-k}{r-k}$ ordered partitions of *[n]* into *r* blocks by the above procedure. It is straightforward to show that each such partition will be obtained exactly once. Indeed, if we write the elements within each block of the partition in increasing order, we can just read the entries of the ordered partition left to right and get the unique permutation having at most *r* ascending runs that led to it. We can then recover the gap positions used. This completes the proof.

Inverting this result leads to a formula expressing the Eulerian numbers by the Stirling numbers of the second kind.

COROLLARY

For all positive integers n and k, we have

$$A(n,k) = \sum_{r=1}^{k} S(n,r)r!\binom{n-r}{k-r}(-1)^{k-r}. \qquad (3.10)$$

PROOF Let us consider formula (1.9) for each $r \leq k$, and multiply each by $r!$. We get the equations

$$1! \cdot S(n,1) = A(n,1)\binom{n-1}{0},$$

$$2! \cdot S(n,2) = A(n,1)\binom{n-1}{1} + A(n,2)\binom{n-2}{0},$$

the equation for general r being

$$r! \cdot S(n,r) = \sum_{i=1}^{r} A(n,i)\binom{n-i}{r-i}, \qquad (3.11)$$

and the last equation being

$$k! \cdot S(n,k) = \sum_{i=1}^{k} A(n,i)\binom{n-i}{r-i}. \qquad (3.12)$$

Our goal is to eliminate each term from the right-hand side of (3.12), except for the term $A(n,k)\binom{n-k}{k-k}=A(n,k)$. We claim that this can be achieved by multiplying (3.11) by $(-1)^{k-r}\binom{n-r}{k-r}$, doing this for all $r \in [k-1]$, then adding these equations to (1.12).

To verify our claim, look at the obtained equation

$$\sum_{r=1}^{k} S(n,r)r!\,(-1)^{k-r}\binom{n-r}{k-r} = \sum_{r=1}^{k}(-1)^{k-r}\binom{n-r}{k-r}\sum_{i=1}^{r} A(n,i)\binom{n-i}{r-i}, \qquad (3.13)$$

or, after changing the order of summation,

$$\sum_{r=1}^{k} S(n,r)r!\,(-1)^{k-r}\binom{n-r}{k-r} = \sum_{i=1}^{r} A(n,i)\binom{n-i}{r-i}\sum_{r=1}^{k}(-1)^{k-r}\binom{n-r}{k-r} \qquad (3.14)$$

whose left-hand side is identical to the right-hand side of (3.10).

It is obvious that the coefficient of $A(n,k)$ on the right-hand side is $\binom{n-k}{k-k}=1$. Therefore, our statement will be proved if we can show that the coefficient $t(n,i)$ of $A(n,i)$ in the last expression is equal to zero if $i<k$.

Note that $\binom{n-i}{r-i} = 0$ if $r<i$. Therefore, for any fixed $i<k$, we have

$$t(n,i) = \sum_{r=i}^{k}\binom{n-i}{r-i}\binom{n-r}{k-r}(-1)^{k-r} = \sum_{r=i}^{k}\binom{n-i}{r-i}\binom{k-n-1}{k-r} = \binom{k-i-1}{k-i}$$
$$= 0$$

We used Cauchy's convolution formula (Lemma 3.12) in the last step. This proves that if $i<k$, then $A(n,i)$ vanishes on the right-hand side of (3.14). We have discussed that $A(n,k)$ will have coefficient 1 there, (and this can be seen again by setting $k=i$ in the last expression, leading to $t(n,i) = \binom{-1}{0} = 1$), so (3.14) implies the claim of this Corollary.

3.5 Generating functions and Eulerian numbers

The various generating functions of the Eulerian numbers have several interesting properties. Let us start with a finite version.

DEFINITION For all nonnegative integers n, the polynomial

$$A_n(x) = \sum_{k=1}^{n} A(n,k)x^k$$

is called the nth Eulerian polynomial.

The Eulerian polynomials have several interesting properties that can be proved by purely combinatorial means. We postpone the study of those properties until the next Subsection. For now, we will explore the connection between these polynomials and some infinite generating functions.

THEOREM 3.5.1

For all positive integers n, the nth Eulerian polynomial has the alternative description

$$A_n(x) = (1-x)^{n+1} \sum_{i\geq 0} i^n x^i.$$

Note that Euler first defined the polynomials $A_n(x)$ in the above form.

Example

For n=1, we have

$$A_1(x) = (1-x)^2 \sum_{i\geq 0} ix^i = (1-x)^2 \frac{x}{(1-x)^2} = x,$$

and for *n*=2, we have

$$A_2(x) = (1-x)^3 \sum_{i\geq 0} i^2 x^i = (1-x)^3 \left(\frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \right) = x + x^2.$$

PROOF Let us use formula (1.3) to write the Eulerian polynomials as

$$A(n,k)x^k = \sum_{k=1}^{n} \sum_{0\leq i\leq k} (-1)^i \binom{n+1}{i} (k-i)^n x^k$$

$$= \sum_{k=1}^{n} (\sum_{0\leq i\leq k} (-1)^{k-i} \binom{n+1}{k-i} i^n x^k).$$

Changing the order of summation, and noting that the sum in parentheses, being equal to *A(n, k),* vanishes for *k*>*n,* we get

$$\sum_{0\leq i\leq k} i^n x^k \cdot \sum_{k\geq 1} \binom{n+1}{k-i} (-x)^{k-i} = (1-x)^{n+1} \sum_{i\geq 0} i^n x^i.$$

It is often useful to collect all Eulerian numbers *A(n, k)* for all *n* and all *k* in a master generating function. This function turns out to have the following simple form.

THEOREM 3.5.2

Let $r(x,u) = \sum_{n\geq 0} \sum_{k\geq 0} A(n,k)x^k \frac{u^n}{n!}$. Then we have

$$r(x,u) = \frac{1-t}{1-te^{u(1-t)}}.$$

PROOF

Using the result of Theorem 3.5.1, we see that

$$r(x,u)=\sum_{n\geq 0}((1-x)^{n+1}\sum_{i\geq 0} i^n x^i)\frac{u^n}{n!}=(1-x)\sum_{i\geq 0} x^i \sum_{n\geq 0}\frac{(iu(1-x))^n}{n!}$$

$$=(1-x)\sum_{i\geq 0} x^i e^{iu(1-x)}=\frac{1-t}{1-te^{u(1-t)}}.$$

n=1	1					
n=2	1	1				
n=3	1	4	1			
n=4	1	11	11	1		
n=5	1	26	66	26	1	
n=6	1	57	302	302	57	1

FIGURE 1.2

Eulerian numbers for $n\leq 6$. Again, the NE-SW diagonals contain the values of $A(n, k)$ for fixed k. Row n starts with $A(n, 1)$.

3.6 The sequence of Eulerian numbers

Let us take a look at the numerical values of the Eulerian numbers for small n, and k=0, 1, ⋯, n-1. The nth row of Figure 1.2 contains the values of $A(n, k)$, for $1\leq k\leq n$, up to n=6. We notice several interesting properties. As we pointed out before, the sequence $A(n, k)$ is symmetric for any fixed n. Moreover, it seems that these sequences first increase steadily, then decrease steadily. This property is so important in combinatorics that it has its own name.

DEFINITION 3.6.1 *We say that the sequence of positive real numbers* $a_1, a_2, \cdots, a_n$ *is* unimodal *if there exists an index k such that* $1\leq k\leq n$, *and* $a_1 \leq a_2 \cdots \leq a_k \geq a_{k+1} \geq a_n$.

The sequences $A(n,k)_{\{1\leq k\leq n\}}$ seem to be unimodal for any fixed n. In fact, they seem to have a stronger property.

DEFINITION 3.6.2 *We say that the sequence of positive real numbers* $a_1, a_2, \cdots, a_n$ is log-concave *if* $a_{k-1}a_{k+1} \leq a_k^2$ *holds for all indices k.*

PROPOSITION 3.6.3

If the sequence $a_1, a_2, \cdots, a_n$ *of positive real numbers is log-concave, then it is also unimodal.*

The conjecture suggested by our observations is in fact correct. This is the content of the following theorem.

THEOREM 3.6.4

For any positive integer n, the sequence $\boldsymbol{A(n,k)\{1 \leq k \leq n\}}$ *of Eulerian number is log-concave.*

Direct combinatorial proofs of this fact are more recent. The proof we present here was obtained by Bóna and Ehrenborg who used an idea of Gasharov.

If a path on a square grid uses steps (1, 0) and (0, 1) only, we will call it a *northeastern lattice path.*

Before proving the theorem, we need to set up some tools, which will be useful in the next section as well. We will construct a bijection from the set *A(n,k)* of *n*-permutations with *k* descents onto that of labeled northeastern lattice paths with *n* edges, exactly *k* of which are vertical. (Note the shift in parameters: $|A(n,k)|=A(n, k+1)$, but this will not cause any confusion.)

Let *P(n)* be the set of labeled northeastern lattice paths that have edges $a_1, a_2, \cdots, a_n$ and that corresponding positive integers $e_1, e_2, \cdots, e_n$ as labels, so that the following hold:

(i) the edge a_i is horizontal and $e_1=1$,

(ii) if the edges a_i and a_{i+1} are both vertical, or both horizontal, then $e_i \geq e_i + 1$,

(iii) if a_i and a_{i+1} are perpendicular to each other, then $e_i + e_{i+1} \leq i + 1$.

At this point, the starting point of a path in *P*(*n*) has no additional significance. Let *P*(*n*,*k*) be the set of all lattice paths in *P*(*n*) which have *k* vertical edges, and let *P*(*n*,*k*)=$|P(n,k)|$.

PROPOSITION 3.6.5

The following two properties of paths in P(n) are immediate from the definitions.

- *For all* $i \geq 2$, *we have* $e_i \leq i - 1$.
- *Fix the label* e_i. *If* e_{i+1} *can take value* v, *then it can take all positive integer values* $w \leq v$.

Also note that all restrictions on e_{i+1} are given by e_i, independently of preceding e_j, *j*<*i*. Now we are going to explain how we will encode our permutations by these labeled lattice paths.

LEMMA 3.6.6

The following description defines a bijection from *A(n)* onto *P(n)*, where *A(n)* is the set of all *n*-permutations. Let $p \in A(n)$. To obtain the edge a_i and the label e_i for $2 \leq i \leq n$, restrict the permutation p to the i first entries and relabel the entries to obtain a permutation $q = q_1 \cdots q_i$ of $[i]$.

- *If the position i-1 is a descent of the permutation p (equivalently, of the permutation q), let the edge* a_i *be vertical and the label* e_i *be equal to* q_i.
- *If the position i-1 is an ascent of the permutation p, let the edge* a_i *be horizontal and the label* e_i *be i+1-*q_i.

Moreover, this bijection restricts naturally to a bijection between A(n, k) and P(n,k) for $0 \leq k \leq n - 1$.

PROOF

The described map is clearly injective. Assume that i-1 and i are both descents of the permutation p. Let q, respectively r, be the permutation when restricted to the i, respectively i+1, first elements. Observe that q_i is either r_i or r_i-1. Since $r_i > r_i + 1$ we have $q_i \geq r_i + 1$ and condition (ii) is satisfied in this case. By similar reasoning the three remaining cases (based on i-1 and i being ascents or descents) are shown, hence the map is into the set $P(n)$.

To see that this is a bijection, we show that we can recover the permutation p from its image. To that end, it is sufficient to show that we can recover p_n, and then use induction on n for the rest of p. To recover p_n from its image, simply recall that p_n is equal to the label l of the last edge if that edge is vertical, and to n+1-l if that edge is horizontal. Conditions (ii) and (iii) assure that this way we always get a number between 1 and n for p_n.

PROOF (of Theorem 3.6.4). We construct an *injection*

$$\phi : P(n,k-1) \times P(n,k+1) \rightarrow P(n,k) \times P(n,k).$$

This injection ϕ will be defined differently on different parts of the domain.

Let $(P, Q) \in P(n,k-1) \times P(n,k+1)$. Place the initial points of P and Q at (0, 0) and (1, -1), respectively. Then the endpoints of P and Q are (n-k+1, k-1) and (n-k, k), respectively, so P and Q intersect. Let X be their *first* intersection point (we order intersection points from southwest to northeast), and decompose $P = P_1 \cup P_2$ and $Q = Q_1 \cup Q_2$, where P_1 is a path from (0, 0) to X, P_2 is a path from X to (n-k, k), Q_1 is a path from (1, -1) to X, and Q_2 is a path from X to (n-k+1, k-1). Let a, b, c, d be the labels of the four edges adjacent to X as shown in Figure 1.5, the edges AX and XB originally belonging to P and the edges CX and XD originally belonging to Q. Then by condition (ii) we have $a \geq b$ and $c \geq d$. (It is possible that these four edges are not all distinct; A and C are always distinct as X is the first intersection point, but it could be, that B=D and so BX=DX; this singular case can be treated very similarly to the generic case we describe below and is hence omitted).

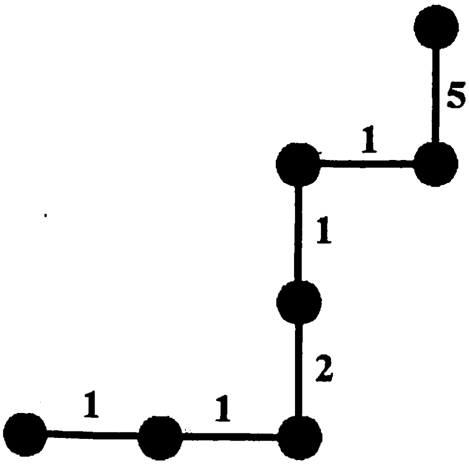


FIGURE 1.3
The image of the permutation 243165.

Let $\boldsymbol{P'} = \boldsymbol{P_1} \cup \boldsymbol{Q_2}$ and let $\boldsymbol{Q'} = \boldsymbol{Q_1} \cup \boldsymbol{P_2}$.

- If P' and Q' are valid paths, that is, if their labeling fulfills conditions (i)–(iii), then we set

ϕ (*P*, *Q*) =(*P'*, *Q'*). See Figure 1.4 for this construction. This way we have defined ϕ for pairs
$(P,Q) \in P(n,k) \times P(n,k)$in which $a + d \leq i$ and $b + c \leq i$, where *i*-1 is the sum of the two coordinates of *X*. We also point out that we have not changed any labels, therefore in (*P'*, *Q'*) we still have
$a \geq b$ and $c \geq d$, though it is no longer required as the edges in question are no longer parts of the same path.
It is clear that ϕ (*P*, *Q*) =(*P'*, *Q'*) ∈ $P(n,k) \times P(n,k)$, (in particular, (*P'*, *Q'*) belongs to the subset of $P(n,k) \times P(n,k)$ consisting of *intersecting* pairs of paths), and that ϕ is one-to-one.

- What remains to be done is to define ϕ (*P*, *Q*) for those $(P,Q) \in P(n,k-1) \times P(n,k+1)$ for which it cannot be defined this way, that is, when either $a+d>i$ or $b+c>i$ holds.
  Change the label of the edge *AX* to *i-c* and change the label of the edge *CX* to *i-a* as seen in Figure 1.6, then proceed as in the previous case to get ϕ (*P*, *Q*) =(*P'*, *Q'*), where $P' = P_1 \cup Q_2$ and $Q' = Q_1 \cup P_2$. We claim that *P'* and *Q'* are valid paths. Indeed we had at least one of $a+d>i$ and $b+c>i$, so we must have $a+c>i$ as $a \geq b$ and $c \geq d$.
  Therefore, $i-a<c$ and $i-c<a$, so we have decreased the values of the labels of edges *AX* and *CX*, and that is always possible as shown in figure 1.3

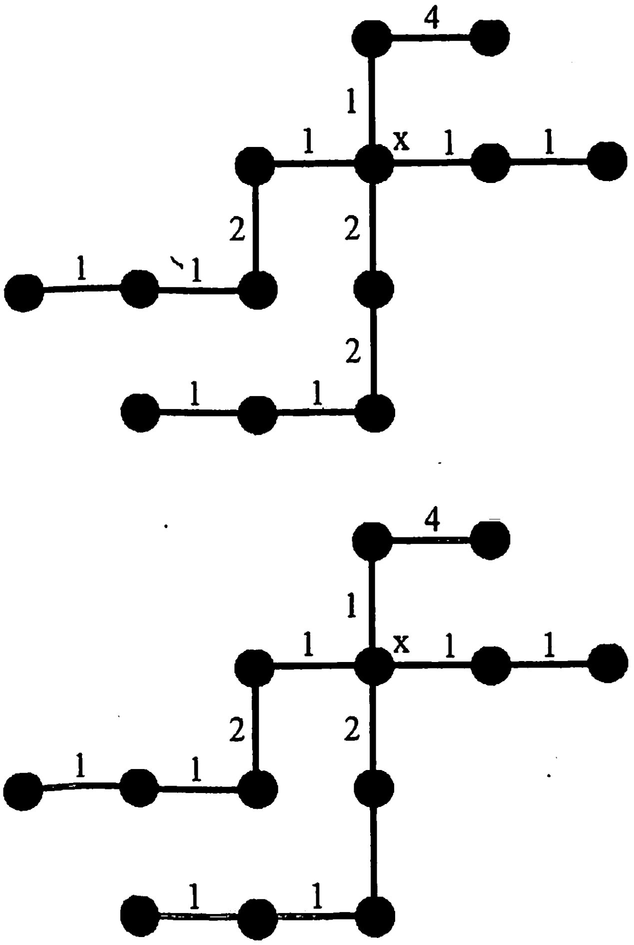


FIGURE 1.4
The new pair of paths.

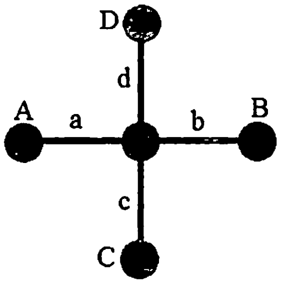


FIGURE 1.5
Labels around the point *X*.

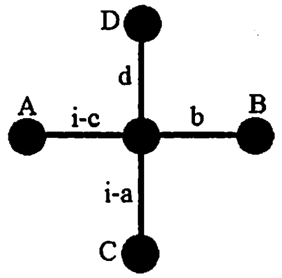


FIGURE 1.6
New labels around the point *X*.

Proposition 3.6.7. Moreover, no constraints are violated in *P'* and *Q'* by the edges adjacent *to X as* $i-c+d\leq i$ and $i-a+b\leq i$. It is also clear that Φ is one-to-one on this part of the domain, too. Finally, we have to show that the image of this part of the domain is disjoint from that of the previous part. This is true because in this part of the domain we have at least one of $a+d>i$ and $b+c>i$, that is, at least one of $i-c<b$ and $i-a<d$, so in the image, at least one of the pairs of edges *AX, XB* and *CX, XD* does not have the property that the label of the first edge is at least as large as that of the second one. And, as pointed out in the previous case, all elements of the image of the previous part of the domain do have that property.
The map ϕ we created is an injection. This shows that $A(n, k-1)A(n, k+1)\leq A(n,k)2$, so our theorem is proved.

4 RECORDS

Let $\sigma = a_1 a_2 \ldots a_n$ be any permutation of length n. An element a_i in σ is a record if $a_i > a_j$ for all $j = 1, 2, \ldots, i - 1$. Furthermore, the position of this record is i. The number of records in permutations was first studied by R′enyi. Recently, Myers and Wilf extended the study of records to multiset permutations and words. In the literature records are also referred to as left–to–right maxima or outstanding elements. In particular the study of records has applications to observations of extreme weather problems, test of randomness, determination of minimal failure, and stresses of electronic components.

Example

Here is another interesting probability problem that involves permutations. We give an example of weather in Almaty measured on a scale of Celsius for 10 days in Table1. Suppose we have started keeping records in first day. Then our first day's temperature could be considered a record temperature starting from this day. A new record was established thursday, the temperature was 14°C; the next record was established next monday, the temperature was 17°C; and there were no new records established after this day. Thus, in this ten-day period, there were three records established: first day, fourth day, and eighth day. The question that we ask is: How many records should we expect to be established in such a ten-day period? In fact, the thermometer indicators are not necessary, we will use them only for sorting from smallest to biggest. We can, therefore, change the numbers measuring temperature to numbers 1 to 10 by replacing the smallest number by 1, the next smallest by 2, and so forth. For our example, we obtain the data shown in Table 2.

Days	Temperature
1 Monday	13°C
2 Tuesday	6°C
3 Wednesday	12°C
4 Thursday	14°C
5 Friday	7°C
6 Saturday	8°C
7 Sunday	11°C
8 Monday	17°C
9 Tuesday	15°C
10 Wednesday	16°C

Table 1 Temperature in Almaty.

Days	Ranking
1 Monday	6
2 Tuesday	1
3 Wednesday	5

4 Thursday	7
5 Friday	2
6 Saturday	3
7 Sunday	4
8 Monday	10
9 Tuesday	8
10 Wednesday	9

Table 2

This gives us a permutation of the numbers from 1 to 10 and, from this permutation, we can read off the records; they are in days 1, 4, and 8. Thus we can define records for a permutation as follows:

Definition Let σ be a permutation of the set $\{1, 2, \ldots, n\}$. Then i is a record of σ if either $i = 1$ or $\sigma(j) < \sigma(i)$ for every $j = 1, 2, \ldots, i\text{-}1$.

4.1 Types of records

We have kinds of permutation records that depend on three parameters: direction (right-to-left or left-to-right), extremum (maximum or minimum) and place (index or value). Write down a permutation record briefly as f[g, h], where f = r, l; g = max,min; and h = i, v. Here “r,l” corresponds to “right-to-left, left-to-right”; “max,min” to “maximum, minimum”; and “i,v” to “index, value”. From this we get eight kind of permutation records:

r[min, i], r[min, v], r[max, i], r[max, v], l[min, i], l[min, v], l[max, i], l[max, v].

L[min,i]

Here is an example of the historical museum's attendance during the week:

Days	number of visitors
1 Monday	302
2 Tuesday	335
3 Wednesday	297
4 Thursday	420
5 Friday	218
6 Saturday	726
7 Sunday	2135

Table 3

In the table, l[min, i] - the days of the week. For records number of days is indices, and number of visitors is values. We need to find the days in which the number of visitors is less than the number of visitors of days until this day. The first will be Monday. Following Wednesday, Friday. Than l[min, i]=135.

4. $xy^3 + x^2y + x^2y+ x^2y^2 + xy^2 + x^3y^2 \ncong x^3y + xy^2 + x^2y^2 + x^2y^2 + x^2y + x^2y^3$

For n=4

1. $xy^4 + 3xy^3 + 2xy^2 + 4x^2y + 6x^2y^2 + x^2y^3 + 2x^2y + 2x^3y^2 + 2x^3y^3 + x^4y^2 \ncong 2xy^3 + 4xy^2 + 2x^2y^3 + 2x^3y^3 + 2x^2y + 6x^2y^2 + x^2y^4 + x^3y^2 + 3x^3y + x^4y$
2. $x^4y^4 + 3x^3y^3 + 2x^3y^2 + x^3y + 2x^2y^3 + 6x^2y^2 + 3x^2y + xy^3 + 3xy^2 + 2xy \cong 2y^3x^2 + y^3x + 2y^2x^3 + 2y^2x^3 + 3y^2x + yx^3 + 3yx^2+ 6x^2y^2 + x^4y^4 + 3x^3y^3$
3. $x^4y^4 + 3x^3y^3 + 2x^3y^2 + x^3y + 2x^2y^3 + 6x^2y^2 + 3x^2y + xy^3 + 3xy^2 + 2xy \cong 2x^2y^3 + xy^3 + 2x^3y^2 + 3xy^2 + x^3y + 3x^2y + x^4y^4 + 3x^3y^3+ 2xy+ 6x^2y^2$
4. $xy^4 + 2x^3y^3 + 2x^3y^2 + 2x^3y + x^2y^3 + 6x^2y^2 + 4x^2y + 3xy^3 + 2xy^2 + x^4y^2 \ncong x^4y+ 2x^3y^3+ x^3y^2+ 3x^3y + x^2y^4 + 2x^2y^3+ 2xy^3+ 6x^2y^2 + 4xy^2 + 2x^2y$

For n=5 we will check only for l$[max, v]$ and r$[min, v]$

1. $x^5y^5+ 4x^4y^4 + 3x^4y^3 + 2x^4y^2 + x^4y+ 3x^3y^4 + 12x^3y^3 + 14x^3y^2 + 6x^3y + 2x^2y^4 + 14x^2y^3 + 23x^2y^2 + 11x^2y + xy^4 + 6xy^3 + 11xy^2 + 6xy \cong x^5y^5+ 4x^4y^4 + 3x^4y^3 + 2x^4y^2 + x^4y+ 3x^3y^4 + 12x^3y^3 + 14x^3y^2 + 6x^3y + 2x^2y^4 + 14x^2y^3 + 23x^2y^2 + 11x^2y + xy^4 + 6xy^3 + 11xy^2 + 6xy$
2. $x^5y^5+ 4x^4y^4 + 3x^4y^3 + x^4y^2 + x^4y+ 3x^3y^4 + 12x^3y^3 + 15x^3y^2 + 6x^3y + 2x^2y^4 + 14x^2y^3 + 23x^2y^2 + 11x^2y + xy^4 + 6xy^3 + 11xy^2 + 6xy \ncong x^5y^5+ 4x^4y^4 + 3x^4y^3 + 2x^4y^2 + x^4y+ 3x^3y^4 + 12x^3y^3 + 14x^3y^2 + 6x^3y + x^2y^4 + 15x^2y^3 + 23x^2y^2 + 11x^2y + xy^4 + 6xy^3 + 11xy^2 + 6xy$.

Definition2: two triple statistics (f,g,h) and (f_1,g_1,h_1) are equidistributed, and write $(f,g,h) \sim (f_1,g_1,h_1)$, if their multi-variable generating functions are equal,

$$\sum_{\sigma \in S_n} x_{f(\sigma)} y_{g(\sigma)} z_{h(\sigma)} = \sum_{\sigma \in S_n} x_{f_1(\sigma)} y_{g_1(\sigma)} z_{h_1(\sigma)}$$

Example (des, l$[min, v]$, cyc) and (dec, cyc, l$[min, v]$)

For n=2 $x^0yz^2+xy^2z \ncong x^0y^2z+xyz^2$. We define that triple-statistics (des, l$[min, v]$, cyc) and (dec, cyc, l$[min, v]$) not equidistributed for n=2. Now check for all kinds of records:

For n=2

	des	l$[min, v]$	l$[max, v]$	r$[min, v]$	r$[max, v]$	cycle
12	0	1	12	21	2	2
21	1	21	2	1	12	1

There is x uses for descents, y for records and z for cycle.

1. $x^0yz^2+xy^2z \ncong x^0y^2z+xyz^2$ (record is l$[min, v]$)

2. $x^0y^2z^2+xyz \cong x^0y^2z^2+xyz$ (record is l[max, v])
3. $x^0y^2z^2+xyz \cong x^0y^2z^2+xyz$ (record is r[min, v])
4. $x^0yz^2+xy^2z \ncong x^0y^2z+xyz^2$ (record is r[max, v])

For n=3

	des	l[min, v]	l[max, v]	r[min, v]	r[max, v]	cycle
123	0	1	123	321	3	3
132	2	1	13	21	23	2
213	1	21	23	31	3	2
231	2	21	23	1	13	1
312	1	31	3	21	23	1
321	12	321	3	1	123	2

1. $x^0yz^3+xyz^2+xy^2z^2+2xy^2z+x^2y^3z^2 \ncong x^0y^3z+xy^2z+xy^2z^2+2xyz^2+x^2y^2z^3$ (record is l[min, v])
2. $x^0y^3z^3+2xy^2z^2+xy^2z+xyz+x^2yz^2 \ncong x^0y^3z^3+2xy^2z^2+xyz^2+xyz+x^2y^2z$ (record is l[max, v])
3. $x^0y^3z^3+2xy^2z^2+xyz+xy^2z+x^2yz^2 \ncong x^0y^3z^3+2xy^2z^2+xyz+xyz^2+x^2y^2z$ (record is r[min, v])
4. $x^0yz^3+xy^2z^2+xyz^2+2xy^2z+ x^2y^3z^2 \ncong x^0y^3z+xy^2z^2+xy^2z+2xyz^2+x^2y^2z^3$ (record is r[max, v])

Conclusion

If at n=k we will define that some type of bistatistics or triple statistics are not equidistributed, it is not important to check following steps. Because if triple statistics are not equidistributed at least one of this steps, they are not equidistributed at all.

We have checked all equidistributions of (rec,cyc) and (cyc,rec) bistatistics and (des, rec, cyc) and (dec, cyc, rec) triple statistics instead of rec putting all kinds of records and received some results. We see that bistatistics (rec,cyc) and (cyc,rec) are equidistributed for n=5 if and only if rec is $\mathrm{l}[max, v]$. Triple statistics (des, rec, cyc) and (dec, cyc, rec) are not equidistributed already for n=3.

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