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Mutation of nonassociative algebras

THESIS

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Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

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Dedication

This thesis is dedicated to my parents and many other for their support, help, sense of humour and useful comments for improving this project.

Abstract

We consider bicommutative algebras under mutation products. We obtain that any bicommutative algebra under the mutation product satisfies Lie-admissible identity, which follows from two independent identities of degree three. Moreover, we obtain all identities of degree four.

Аңдатпа

Біз бикоммутативті алгебраларды мутация көбейтіндісінде қарастырдық. Біз мутация көбейтіндісі бойынша кез-келген бикоммутативті алгебраның үшінші дәрежелі екі тәуелсіз тепе-теңдігінен шығатын Ли-адмиссибл тепе-теңдігін қанағаттандыратынын көрсеттік. Оған қоса, біз төртінші дәрежедегі барлық тепе-теңдіктерді таптық.

Аннотация

Мы рассматривали бикоммутативные алгебры относительно произведений мутаций. Мы получили, что любая бикоммутативная алгебра относительно произведения мутации удовлетворяет Ли-допустимому тождество, которое следует из двух независимых тождеств третьей степени. Более того, мы получили все тождества четвертой степени.

Contents

1. Introduction

An algebra $\mathcal{B}$ that satisfying the following identities:

$$a(bc) = b(ac)$$

$$(ab)c = (ac)b$$

is called bicommutative algebra [1],[2]. D. Burde and co-authors also call bicommutative algebras LR - (left and right) algebras in their work [3]. A. Dzhumadildaev, N. Ismailov and K. Tulenbayev found the basis of free bicommutative algebras [2]. Moreover, A. Zhumadildaev and N. Ismailov considered bicommutative algebras in commutator product [1].

The mutation product was first considered in associative algebras [4]. The mutation product appeared in theoretical physics in the 1980's [5], [6]. For the first time, F. Montaner studied the identity satisfied by mutations of associative algebras [4]. In [7] the mutations of non-associative algebras are considered. A. Elduque and co-authors comprehensively explained the structural theory of mutation algebras [8].

For fixed elements p and q of the algebra, the mutation product in the algebra is defined as follows:

$$\langle a, b\rangle = (ap)b - (bq)a.$$

The algebra $\mathcal{B}_{p,q} = (\mathcal{B}, \langle\cdot,\cdot\rangle)$ is called a mutation of the given algebra $\mathcal{B}$.

Let us denote the class of bicommutative algebras by $\mathcal{B}icom$. Let $\mathcal{B}icom_{p,q}$ be the class of mutations of bicommutative algebras for fixed $p, q \in \mathcal{B}$. That is, $\mathcal{B}icom_{p,q}$ elements are algebras of the form $\mathcal{B}_{p,q}$, where $\mathcal{B} \in \mathcal{B}icom$.

The aim of this work is to find identities for $\mathcal{B}icom_{p,q}$. If $p = \emptyset$, $q = \emptyset$, then the mutation product becomes the commutator product:

$$\langle a, b\rangle = ab - ba = [a, b] \qquad (\textit{the commutator}).$$

In this work, we prove that any bicommutative algebra under mutation product satisfies two identities of degree 3, and we find all identities of degree 4.

2. Bicommutative algebras under mutation product

2.1 Mutation product and identities in degree three

Let

$$f_1(a,b,c) = \langle\langle b,a\rangle,c\rangle - \langle\langle b,c\rangle,a\rangle - \langle c,\langle a,b\rangle\rangle + \langle a,\langle c,b\rangle\rangle,$$

$$f_2(a,b,c) = \langle\langle a,b\rangle,c\rangle + \langle\langle c,a\rangle,b\rangle - \langle\langle b,a\rangle,c\rangle - \langle\langle a,c\rangle,b\rangle - \langle\langle c,b\rangle,a\rangle + \langle\langle b,c\rangle,a\rangle,$$

$$f_3(a,b,c) = \langle\langle c,a\rangle,b\rangle - \langle\langle b,a\rangle,c\rangle - \langle\langle c,b\rangle,a\rangle + \langle\langle b,c\rangle,a\rangle - \langle b,\langle c,a\rangle\rangle + \langle c,\langle b,a\rangle\rangle,$$

$$f_4(a,b,c) = \langle a,\langle c,b\rangle\rangle - \langle a,\langle b,c\rangle\rangle + \langle b,\langle a,c\rangle\rangle - \langle b,\langle c,a\rangle\rangle - \langle c,\langle a,b\rangle\rangle + \langle c,\langle b,a\rangle\rangle$$

where

$$\langle a,b\rangle = (ap)b - (bq)a.$$

Lemma 2.1.1. *Let $(\mathcal{B},\cdot)$ be a bicommutative algebra. Then $(\mathcal{B},\langle\ ,\ \rangle)$ satisfies the identity $f_1(a,b,c)=0$.*

Proof. We will prove that

$$f_1(a,b,c) = \langle\langle b,a\rangle,c\rangle - \langle\langle b,c\rangle,a\rangle - \langle c,\langle a,b\rangle\rangle + \langle a,\langle c,b\rangle\rangle = 0.$$

By direct calculations, we have

$$\langle\langle b,a\rangle,c\rangle = (((ba)c)p)p - (((ab)c)p)q - c(((ba)p)q) + c(((ab)q)q),$$

$$-\langle\langle b,c\rangle,a\rangle = -(((ba)c)p)p + (((ca)b)p)q + b(((ac)p)q) - c(((ab)q)q),$$

$$-\langle c,\langle a,b\rangle\rangle = -c(((ab)p)p) + c(((ba)p)q) + (((ab)c)p)q - (((ba)c)q)q,$$

$$\langle a,\langle c,b\rangle\rangle = c(((ab)p)p) - b(((ac)p)q) - (((ca)b)p)q + (((ba)c)q)q.$$

The sum of the above elements gives us the desired result.

□

Now, we prove that the bicommutative algebra under mutation product satisfies additional identities except $f_1(a,b,c)$.

Lemma 2.1.2. *Let $(\mathcal{B},\cdot)$ be a bicommutative algebra. Then $(\mathcal{B},\langle\ ,\ \rangle)$ satisfies the identities $f_2(a,b,c)=0$, $f_3(a,b,c)=0$. and $f_4(a,b,c)=0$.*

Proof. Firstly, we will prove that

$$f_2(a,b,c)=\langle\langle a,b\rangle,c\rangle+\langle\langle c,a\rangle,b\rangle-\langle\langle b,a\rangle,c\rangle-\langle\langle a,c\rangle,b\rangle-\langle\langle c,b\rangle,a\rangle+\langle\langle b,c\rangle,a\rangle=0.$$

By direct calculations, we obtain

$$\begin{aligned}
\langle\langle a,b\rangle,c\rangle &= (((ab)c)p)p-(((ba)c)p)q-c(((ab)p)q)+c(((ba)q)q),\\
\langle\langle c,a\rangle,b\rangle &= (((ca)b)p)p-(((ab)c)p)q-c(((ba)p)q)+b(((ac)q)q),\\
-\langle\langle b,a\rangle,c\rangle &= -(((ba)c)p)p+(((ab)c)p)q+c(((ba)p)q)-c(((ab)q)q),\\
-\langle\langle a,c\rangle,b\rangle &= -(((ab)c)p)p+(((ca)b)p)q+b(((ac)p)q)-c(((ba)q)q),\\
-\langle\langle c,b\rangle,a\rangle &= -(((ab)c)p)p+(((ca)b)p)q+b(((ac)p)q)-c(((ba)q)q),\\
\langle\langle b,c\rangle,a\rangle &= (((ba)c)p)p-(((ca)b)p)q-b(((ac)p)q)+c(((ab)q)q).
\end{aligned}$$

The addition of the above elements yields the expected outcome. Now, we prove the following identity:

$$f_3(a,b,c)=\langle\langle c,a\rangle,b\rangle-\langle\langle b,a\rangle,c\rangle-\langle\langle c,b\rangle,a\rangle+\langle\langle b,c\rangle,a\rangle-\langle b,\langle c,a\rangle\rangle+\langle c,\langle b,a\rangle\rangle=0.$$

By direct calculations, we have

$$\begin{aligned}
\langle\langle c,a\rangle,b\rangle &= (((ca)b)p)p-(((ab)c)p)q-c(((ba)p)q)+b(((ac)q)q),\\
-\langle\langle b,a\rangle,c\rangle &= -(((ba)c)p)p+(((ab)c)p)q+c(((ba)p)q)-c(((ab)q)q),\\
-\langle\langle c,b\rangle,a\rangle &= -(((ab)c)p)p+(((ca)b)p)q+b(((ac)p)q)-c(((ba)q)q),\\
\langle\langle b,c\rangle,a\rangle &= (((ba)c)p)p-(((ca)b)p)q-b(((ac)p)q)+c(((ab)q)q),\\
-\langle b,\langle c,a\rangle\rangle &= -c(((ba)p)p)+b(((ac)p)q)+(((ca)b)p)q-(((ab)c)q)q,\\
\langle c,\langle b,a\rangle\rangle &= c(((ba)p)p)-c(((ab)p)q)-(((ba)c)p)q+(((ab)c)q)q.
\end{aligned}$$

The desired result can be achieved by adding the elements mentioned above. Now, we prove the following identity:

$$f_4(a,b,c)=\langle a,\langle c,b\rangle\rangle-\langle a,\langle b,c\rangle\rangle+\langle b,\langle a,c\rangle\rangle-\langle b,\langle c,a\rangle\rangle-\langle c,\langle a,b\rangle\rangle+\langle c,\langle b,a\rangle\rangle=0.$$

By direct calculations, we have

$$\begin{aligned}
\langle a,\langle c,b\rangle\rangle &= c(((ab)p)p)-b(((ac)p)q)-(((ca)b)p)q+(((ba)c)q)q,\\
-\langle a,\langle b,c\rangle\rangle &= -b(((ac)p)p)+c(((ab)p)q)+(((ba)c)p)q-(((ca)b)q)q,
\end{aligned}$$

$$\langle b, \langle a, c\rangle\rangle = b(((ac)p)p) - c(((ba)p)q) - (((ab)c)p)q + (((ca)b)q)q,$$
$$-\langle b, \langle c, a\rangle\rangle = -c(((ba)p)p) + b(((ac)p)q) + (((ca)b)p)q - (((ab)c)q)q,$$
$$-\langle c, \langle a, b\rangle\rangle = -c(((ab)p)p) + c(((ba)p)q) + (((ab)c)p)q - (((ba)c)q)q,$$
$$\langle c, \langle b, a\rangle\rangle = c(((ba)p)p) - c(((ab)p)q) - (((ba)c)p)q + (((ab)c)q)q.$$

The desired result can be achieved by adding the elements mentioned above. □

Lemma 2.1.3. *The identity $f_2(a, b, c)$ and $f_4(a, b, c)$ follows from the identities $f_1(a, b, c)$ and $f_3(a, b, c)$.*

Proof. There are 12 nonassociative monomials of degree 3, and we present them in the following order:

$$\{\langle\langle a, b\rangle, c\rangle, \langle\langle a, c\rangle, b\rangle, \langle\langle b, a\rangle, c\rangle, \langle\langle b, c\rangle, a\rangle, \langle\langle c, a\rangle, b\rangle, \langle\langle c, b\rangle, a\rangle,$$
$$\langle a, \langle b, c\rangle\rangle, \langle a, \langle c, b\rangle\rangle, \langle b, \langle a, c\rangle\rangle, \langle b, \langle c, a\rangle\rangle, \langle c, \langle a, b\rangle\rangle, \langle c, \langle b, a\rangle\rangle\}.$$

We select the coefficients of the monomials relative to the order of the above monomials. In other words, the columns correspond to the monomials and the rows represent each polynomial with all possible permutations of f_1 and f_3 in the variables a, b, c.

Then we have the following matrix, which first 6 rows are all permutations of f_1 and the next 6 rows of f_3:

$$A_{f_1,f_3} = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

Then we see that $rank(A_1) = 4$. Now, we create a matrix of size 18×12 without changing columns, as in the previous matrix. Here, columns represent monomials of degree three, while the rows represent every polynomial resulting from all possible permutations of f_1, f_2 and f_3.

$$A_{f_1,f_3,f_2} = \begin{pmatrix}
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 1 \\
0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0
\end{pmatrix}.$$

Then we see that $rank(A_2) = 4$. Therefore, the identity f_2 follows from f_1 and f_3. We form the matrix whose rows are the polynomials with all possible permutations of f_i, where $i \in \{1, 2, 3, 4\}$. Also, write the columns as a previous matrices.

$$A_{f_1,f_3,f_4} = \begin{pmatrix}
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 1 \\
0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 \\
-1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Then $rank(A_3) = 4$. Hence, the identities $f_2(a,b,c)$ and $f_4(a,b,c)$ follows from the identities $f_1(a,b,c)$ and $f_3(a,b,c)$.

□

Let

$$L(x_1,x_2,x_3) = \sum_{\sigma\in S_3} sgn(\sigma)\left(\langle\langle x_{\sigma(1)}, x_{\sigma(2)}\rangle, x_{\sigma(3)}\rangle - \langle x_{\sigma(1)}, \langle x_{\sigma(2)}, x_{\sigma(3)}\rangle\rangle\right) = 0.$$

Corollary 2.1.4. *Any bicommutative algebra under the mutation product is Lie-admissible algebra.*

Proof. Since, $f_2(a,b,c) = 0$ and $f_4(a,b,c) = 0$ we have $L(a,b,c) = 0$. □

Theorem 2.1.5. *Every identity of degree no more than 3 satisfied by the mutation products in every bicommutative algebra is a consequence of identities:*

$$f_1(a,b,c) = \langle\langle b,a\rangle,c\rangle - \langle\langle b,c\rangle,a\rangle - \langle c,\langle a,b\rangle\rangle + \langle a,\langle c,b\rangle\rangle = 0 \tag{2.1.1}$$

$$f_3(a,b,c) = \langle\langle c,a\rangle,b\rangle - \langle\langle b,a\rangle,c\rangle - \langle\langle c,b\rangle,a\rangle + \langle\langle b,c\rangle,a\rangle - \langle b,\langle c,a\rangle\rangle + \langle c,\langle b,a\rangle\rangle = 0 \tag{2.1.2}$$

Proof. Let $F(a,b,c) =$

$$\lambda_1\langle\langle a,b\rangle,c\rangle + \lambda_2\langle\langle c,a\rangle,b\rangle + \lambda_3\langle\langle b,a\rangle,c\rangle + \lambda_4\langle\langle a,c\rangle,b\rangle$$
$$+\lambda_5\langle\langle c,b\rangle,a\rangle + \lambda_6\langle\langle b,c\rangle,a\rangle + \lambda_7\langle a,\langle b,c\rangle\rangle + \lambda_8\langle b,\langle c,a\rangle\rangle$$
$$+\lambda_9\langle c,\langle a,b\rangle\rangle + \lambda_{10}\langle b,\langle a,c\rangle\rangle + \lambda_{11}\langle c,\langle b,a\rangle\rangle + \lambda_{12}\langle a,\langle c,b\rangle\rangle$$

be a nonassociative polynomial, i.e., an element of the free nonassociative algebra of degree 3.

Let $\mathcal{B}$ be the free bicommutative algebra with generators a,b,c with multiplication $(a,b) \longmapsto ab$. We calculate $F(a,b,c) \in \mathcal{B}$ in terms of the mutation product $\langle a,b\rangle = (ap)b - (bq)a$.

We have $F(a,b,c) =$

$$\lambda_1\langle\langle a,b\rangle,c\rangle + \lambda_2\langle\langle c,a\rangle,b\rangle + \lambda_3\langle\langle b,a\rangle,c\rangle + \lambda_4\langle\langle a,c\rangle,b\rangle$$
$$+\lambda_5\langle\langle c,b\rangle,a\rangle + \lambda_6\langle\langle b,c\rangle,a\rangle + \lambda_7\langle a,\langle b,c\rangle\rangle + \lambda_8\langle b,\langle c,a\rangle\rangle$$
$$+\lambda_9\langle c,\langle a,b\rangle\rangle + \lambda_{10}\langle b,\langle a,c\rangle\rangle + \lambda_{11}\langle c,\langle b,a\rangle\rangle + \lambda_{12}\langle a,\langle c,b\rangle\rangle$$

$$= (\lambda_1+\lambda_4)(((ab)c)p)p$$
$$+(\lambda_1+\lambda_4)c(((ba)q)q)$$
$$+(\lambda_2+\lambda_5)(((ca)b)p)p$$
$$+(\lambda_2+\lambda_5)b(((ac)q)q)$$

$$
\begin{aligned}
&+(\lambda_3+\lambda_6)(((ba)c)p)p\\
&+(\lambda_3+\lambda_6)c(((ab)q)q)\\
&+(\lambda_7+\lambda_{10})b(((ac)p)p)\\
&+(\lambda_7+\lambda_{10})(((ca)b)q)q\\
&+(\lambda_8+\lambda_{11})c(((ba)p)p)\\
&+(\lambda_8+\lambda_{11})(((ab)c)q)q\\
&+(\lambda_9+\lambda_{12})c(((ab)p)p)\\
&+(\lambda_9+\lambda_{12})(((ba)c)q)q\\
&+(-\lambda_1-\lambda_5-\lambda_7-\lambda_{11})(((ba)c)p)q\\
&+(-\lambda_1-\lambda_5-\lambda_7-\lambda_{11})c(((ab)p)q)\\
&+(-\lambda_2-\lambda_3-\lambda_9-\lambda_{10})(((ab)c)p)q\\
&+(-\lambda_2-\lambda_3-\lambda_9-\lambda_{10})c(((ba)p)q)\\
&+(-\lambda_4-\lambda_6-\lambda_8-\lambda_{12})(((ca)b)p)q\\
&+(-\lambda_4-\lambda_6-\lambda_8-\lambda_{12})b(((ac)p)q).
\end{aligned}
$$

Since, the mutation products of $\{a,b,c\}$ are expressed with the set of elements

$$
\begin{aligned}
&\{(((ab)c)p)p, (((ba)c)p)q, c(((ab)p)q), c(((ba)q)q), (((ca)b)p)p, (((ab)c)p)p,\\
&c(((ba)p)q), b(((ac)q)q), (((ba)c)p)p, c(((ab)q)q), (((ca)b)p)q, b(((ac)p)q),\\
&b(((ac)p)p), (((ca)b)q)q, c(((ba)p)p), (((ab)c)q)q, c(((ab)p)p), (((ba)c)q)q\}
\end{aligned}
$$

in the free bicommutative algebra in degree 3, we see that $F(a,b,c)=0$ gives us system of 18 linear equations with 12 unknowns λ_i, where $i=\{1,\dots,12\}$. We see that this system has rank 8, and we can take $\lambda_1, \lambda_{10}, \lambda_{11}$ and λ_{12} as free parameters and express the other parameters in the following way:

$$
\begin{aligned}
&\lambda_2 = \lambda_1-\lambda_{10}+\lambda_{11},\\
&\lambda_3 = -\lambda_1-\lambda_{11}+\lambda_{12},\\
&\lambda_4 = -\lambda_1,\\
&\lambda_5 = -\lambda_1+\lambda_{10}-\lambda_{11},\\
&\lambda_6 = \lambda_1+\lambda_{11}-\lambda_{12},\\
&\lambda_7 = -\lambda_{10},\\
&\lambda_8 = -\lambda_{11},\\
&\lambda_9 = -\lambda_{12}.
\end{aligned}
$$

Therefore,

$$f = \lambda_{12}g_1 + \lambda_{10}g_2 + \lambda_1 g_3 + \lambda_{11}g_4 = 0$$

where

$$\begin{aligned}
g_1 &= \langle\langle b, a\rangle, c\rangle - \langle\langle b, c\rangle, a\rangle - \langle c, \langle a, b\rangle\rangle + \langle a, \langle c, b\rangle\rangle,\\
g_2 &= \langle\langle c, b\rangle, a\rangle - \langle a, \langle b, c\rangle\rangle - \langle\langle c, a\rangle, b\rangle + \langle b, \langle a, c\rangle\rangle,\\
g_3 &= \langle\langle a, b\rangle, c\rangle + \langle\langle c, a\rangle, b\rangle - \langle\langle b, a\rangle, c\rangle - \langle\langle a, c\rangle, b\rangle - \langle\langle c, b\rangle, a\rangle + \langle\langle b, c\rangle, a\rangle,\\
g_4 &= \langle\langle c, a\rangle, b\rangle - \langle\langle b, a\rangle, c\rangle - \langle\langle c, b\rangle, a\rangle + \langle\langle b, c\rangle, a\rangle - \langle b, \langle c, a\rangle\rangle + \langle c, \langle b, a\rangle\rangle.
\end{aligned}$$

We see that the following equalities hold:

$$g_1(a, b, c) = \langle\langle b, a\rangle, c\rangle - \langle\langle b, c\rangle, a\rangle - \langle c, \langle a, b\rangle + \langle a, \langle c, b\rangle = f_1(a, b, c),$$

$$\begin{aligned}
g_2(a, b, c) &= \langle\langle c, b\rangle, a\rangle - \langle a, \langle b, c\rangle\rangle - \langle\langle c, a\rangle, b\rangle + \langle b, \langle a, c\rangle\rangle = -f_1(a, c, b),\\
g_3(a, b, c) &= \langle\langle a, b\rangle, c\rangle + \langle\langle c, a\rangle, b\rangle - \langle\langle b, a\rangle, c\rangle - \langle\langle a, c\rangle, b\rangle - \langle\langle c, b\rangle, a\rangle\\
&\quad + \langle\langle b, c\rangle, a\rangle = f_2(a, b, c),\\
g_4(a, b, c) &= \langle\langle c, a\rangle, b\rangle - \langle\langle b, a\rangle, c\rangle - \langle\langle c, b\rangle, a\rangle + \langle\langle b, c\rangle, a\rangle - \langle b, \langle c, a\rangle\rangle\\
&\quad + \langle c, \langle b, a\rangle\rangle = f_3(a, b, c).
\end{aligned}$$

This means that any identity of degree 3 of the class $\mathcal{B}icom_{p,q}$ follows from the identities $\{f_1, f_3\}$. And by Lemma 2.1.3, the proof is complete. □

Now, we prove that a bicommutative algebra, under mutation product, satisfies identities of degree four that do not follow from $f_1(a, b, c)$ and $f_3(a, b, c)$.

2.2 Mutation product and identities in degree four

Let

$$f_1(a, b, c, d) = \langle\langle\langle a, b\rangle, c\rangle, d\rangle - \langle\langle\langle a, d\rangle, b\rangle, c\rangle + \langle b, \langle\langle d, a\rangle, c\rangle\rangle - \langle d, \langle\langle b, a\rangle, c\rangle\rangle,$$

$$\begin{aligned}
f_2(a, b, c, d) &= \langle\langle\langle b, c\rangle, a\rangle, d\rangle - \langle\langle\langle b, d\rangle, a\rangle c\rangle + \langle c, \langle\langle d, a\rangle, b\rangle\rangle - \langle d, \langle\langle c, a\rangle, b\rangle\rangle\\
&\quad + \langle a, \langle c, \langle b, d\rangle\rangle\rangle - \langle a, \langle d, \langle b, c\rangle\rangle\rangle - \langle b, \langle c, \langle a, d\rangle\rangle\rangle + \langle b, \langle d, \langle a, c\rangle\rangle\rangle,\\
f_3(a, b, c, d) &= \langle\langle\langle b, a\rangle, c\rangle, d\rangle - \langle\langle\langle b, d\rangle, a\rangle, c\rangle + \langle a, \langle\langle d, c\rangle, b\rangle\rangle - \langle d, \langle\langle a, c\rangle, b\rangle\rangle\\
&\quad - \langle d, \langle\langle b, a\rangle, c\rangle\rangle + \langle d, \langle\langle b, c\rangle, a\rangle\rangle - \langle a, \langle b, \langle c, d\rangle\rangle\rangle + \langle a, \langle c, \langle b, d\rangle\rangle\rangle\\
&\quad - \langle a, \langle d, \langle b, c\rangle\rangle\rangle + \langle b, \langle d, \langle a, c\rangle\rangle\rangle,\\
f_4(a, b, c, d) &= \langle\langle a, b\rangle, \langle c, d\rangle\rangle - \langle\langle c, a\rangle, \langle d, b\rangle\rangle - \langle a, \langle\langle d, c\rangle, b\rangle\rangle - \langle c, \langle\langle a, d\rangle, b\rangle\rangle\\
&\quad - \langle c, \langle\langle b, a\rangle, d\rangle\rangle + \langle c, \langle\langle b, d\rangle, a\rangle\rangle + \langle c, \langle\langle d, a\rangle, b\rangle\rangle + \langle d, \langle\langle a, c\rangle, b\rangle\rangle,
\end{aligned}$$

$$
\begin{aligned}
f_5(a,b,c,d) &= \langle\langle a,d\rangle,\langle b,c\rangle\rangle - \langle\langle c,a\rangle,\langle d,b\rangle\rangle - \langle a,\langle\langle d,c\rangle,b\rangle\rangle - \langle b,\langle\langle a,d\rangle,c\rangle\rangle \\
&\quad - \langle c,\langle\langle b,a\rangle,d\rangle\rangle + \langle c,\langle\langle b,d\rangle,a\rangle\rangle + \langle c,\langle\langle d,a\rangle,b\rangle\rangle + \langle d,\langle\langle a,c\rangle,b\rangle\rangle \\
&\quad + \langle d,\langle\langle b,a\rangle,c\rangle\rangle - \langle d,\langle\langle b,c\rangle,a\rangle\rangle + \langle a,\langle b,\langle c,d\rangle\rangle\rangle - \langle b,\langle c,\langle a,d\rangle\rangle\rangle, \\
f_6(a,b,c,d) &= \langle\langle a,\langle b,c\rangle\rangle,d\rangle - \langle a,\langle\langle d,c\rangle,b\rangle\rangle - \langle b,\langle\langle a,d\rangle,c\rangle\rangle + \langle b,\langle\langle d,a\rangle,c\rangle\rangle \\
&\quad - \langle c,\langle\langle b,a\rangle,d\rangle\rangle + \langle c,\langle\langle b,d\rangle,a\rangle\rangle + \langle d,\langle\langle a,c\rangle,b\rangle\rangle + \langle d,\langle\langle b,a\rangle,c\rangle\rangle \\
&\quad - \langle d,\langle\langle b,c\rangle,a\rangle\rangle - \langle d,\langle\langle b,c\rangle,a\rangle\rangle + \langle a,\langle b,\langle c,d\rangle\rangle\rangle - \langle a,\langle c,\langle b,d\rangle\rangle\rangle \\
&\quad + \langle a,\langle d,\langle b,c\rangle\rangle\rangle - \langle b,\langle d,\langle a,c\rangle\rangle\rangle
\end{aligned}
$$

where

$$\langle a,b\rangle = (ap)b - (bq)a.$$

Lemma 2.2.1. *Let $(\mathcal{B},\cdot)$ be a bicommutative algebra. Then $(\mathcal{B},\langle\ ,\ \rangle)$ satisfies the identities $f_i(a,b,c,d)=0$, where $i\in\{1,2,3,4,5,6\}$.*

Proof. We will prove that

$$f_1(a,b,c,d) = \langle\langle\langle a,b\rangle,c\rangle,d\rangle - \langle\langle\langle a,d\rangle,b\rangle,c\rangle + \langle b,\langle\langle d,a\rangle,c\rangle\rangle - \langle d,\langle\langle b,a\rangle,c\rangle\rangle = 0.$$

By direct calculations, we have

$$
\begin{aligned}
\langle\langle\langle a,b\rangle,c\rangle,d\rangle &= (((((ab)c)d)p)p)p - (((((ba)c)b)p)p)q - c(((((ab)d)p)p)q) \\
&\quad + c(((((ba)d)p)q)q) - d(((((ab)c)p)p)q) + d(((((ba)c)p)q)q) \\
&\quad + d(c((((ab)p)q)q)) - d(c((((ba)q)q)q)),
\end{aligned}
$$

$$
\begin{aligned}
-\langle\langle\langle a,d\rangle,b\rangle,c\rangle &= -(((((ab)c)d)p)p)p + (((((da)b)c)p)p)q + b(((((ac)d)p)p)q) \\
&\quad - d(((((ba)c)p)q)q) + c(((((ab)d)p)p)q) - d(((((ca)b)p)q)q) \\
&\quad - c(b((((ad)p)q)q)) + d(c((((ba)q)q)q)),
\end{aligned}
$$

$$
\begin{aligned}
\langle b,\langle\langle d,a\rangle,c\rangle\rangle &= d(((((ba)c)p)p)p) - b(((((ac)d)p)p)q) - d(c((((ba)p)p)q)) \\
&\quad + c(b((((ad)p)q)q)) - (((((da)b)c)p)p)q + (((((ab)c)d)p)q)q \\
&\quad + d(((((ca)b)p)q)q) - c(((((ab)d)q)q)q),
\end{aligned}
$$

$$
\begin{aligned}
-\langle d,\langle\langle b,a\rangle,c\rangle\rangle &= -d(((((ba)c)p)p)p) + d(((((ab)c)p)p)q) + d(c((((ba)p)p)q)) \\
&\quad - d(c((((ab)p)q)q)) + (((((ba)c)d)p)p)q - (((((ab)c)d)p)q)q \\
&\quad - c(((((ba)d)p)q)q) + c(((((ab)d)q)q)q).
\end{aligned}
$$

The sum of the above elements gives us the desired result. The remaining five identities $f_i(a,b,c,d)=0$, where $i\in\{2,3,4,5,6\}$ are also prove by the same calculations. □

Now, we proved that the bicommutative algebra under mutation product satisfies additional identities except $f_i(a,b,c,d)$, where $i\in\{1,2,3,4,5,6\}$.

Lemma 2.2.2. *Let $(\mathcal{B}, \cdot)$ be a bicommutative algebra. Then $(\mathcal{B}, \langle\ ,\ \rangle)$ satisfies the identity $f(a,b,c,d) = 0$, where $\langle a, b\rangle = (ap)b - b(qa)$.*

$$\langle\langle\langle a, b\rangle, c\rangle, d\rangle = \langle\langle\langle a, b\rangle, d\rangle, c\rangle,$$

$$\langle\langle a, b\rangle, \langle c, d\rangle\rangle = \langle\langle c, d\rangle, \langle a, b\rangle\rangle,$$

$$\langle a, \langle b, \langle c, d\rangle\rangle\rangle = \langle b, \langle a, \langle c, d\rangle\rangle\rangle.$$

Proof. By direct calculations, we obtain

$$\begin{aligned}\langle\langle\langle a, b\rangle, c\rangle, d\rangle &= (((((ab)c)d)p)p)p - (((((ba)c)d)p)p)q - c(((((ab)d)p)p)q)\\ &+ c(((((ba)d)p)q)q) - d(((((ab)c)p)p)q) + d(((((ba)c)p)q)q)\\ &+ d(c((((ab)p)q)q)) - d(c((((ba)q)q)q)) = \langle\langle\langle a, b\rangle, d\rangle, c\rangle,\end{aligned}$$

$$\begin{aligned}\langle\langle a, b\rangle, \langle c, d\rangle\rangle &= c(((((ab)d)p)p)p) - d(((((ab)c)p)p)q) - c(((((ba)d)p)p)q)\\ &+ d(((((ba)c)p)q)q) - c(((((ab)d)p)p)q) + c(((((ba)d)p)q)q)\\ &+ d(((((ab)c)p)q)q) - d(((((ba)c)q)q)q) = \langle\langle c, d\rangle, \langle a, b\rangle\rangle,\end{aligned}$$

$$\begin{aligned}\langle a, \langle b, \langle c, d\rangle\rangle\rangle &= c(b((((ad)p)p)p) - d(b((((ac)p)p)q) - c(((((ab)d)p)p)q)\\ &+ d(((((ab)c)p)q)q) - c(((((ba)d)p)p)q) + d(((((ba)c)p)q)q)\\ &+ (((((ca)b)d)p)q)q - (((((da)b)c)q)q)q = \langle b, \langle a, \langle c, d\rangle\rangle\rangle.\end{aligned}$$

□

Now, we have proved that the above identities are equal to each other.

Theorem 2.2.3. *Every identity of degree no more than 4 satisfied by the mutation products in every bicommutative algebra is a consequence of identities:*

$$f_1(a,b,c,d) = \langle\langle\langle a, b\rangle, c\rangle, d\rangle - \langle\langle\langle a, d\rangle, b\rangle, c\rangle + \langle b, \langle\langle d, a\rangle, c\rangle\rangle - \langle d, \langle\langle b, a\rangle, c\rangle\rangle = 0. \tag{2.2.1}$$

$$\begin{aligned}f_2(a,b,c,d) &= \langle\langle\langle b, c\rangle, a\rangle, d\rangle - \langle\langle\langle b, d\rangle, a\rangle, c\rangle + \langle c, \langle\langle d, a\rangle, b\rangle\rangle - \langle d, \langle\langle c, a\rangle, b\rangle\rangle\\ &+ \langle a, \langle c, \langle b, d\rangle\rangle\rangle - \langle a, \langle d, \langle b, c\rangle\rangle\rangle - \langle b, \langle c, \langle a, d\rangle\rangle\rangle + \langle b, \langle d, \langle a, c\rangle\rangle\rangle = 0.\end{aligned} \tag{2.2.2}$$

$$\begin{aligned}f_3(a,b,c,d) &= \langle\langle\langle b, a\rangle, c\rangle, d\rangle - \langle\langle\langle b, d\rangle, a\rangle, c\rangle + \langle a, \langle\langle d, c\rangle, b\rangle\rangle - \langle d, \langle\langle a, c\rangle, b\rangle\rangle\\ &- \langle d, \langle\langle b, a\rangle, c\rangle\rangle + \langle d, \langle\langle b, c\rangle, a\rangle\rangle - \langle a, \langle b, \langle c, d\rangle\rangle\rangle + \langle a, \langle c, \langle b, d\rangle\rangle\rangle\\ &- \langle a, \langle d, \langle b, c\rangle\rangle\rangle + \langle b, \langle d, \langle a, c\rangle\rangle\rangle = 0.\end{aligned} \tag{2.2.3}$$

$$f_4(a,b,c,d) = \langle\langle a,b\rangle,\langle c,d\rangle\rangle - \langle\langle c,a\rangle,\langle d,b\rangle\rangle - \langle a,\langle\langle d,c\rangle,b\rangle\rangle - \langle c,\langle\langle a,d\rangle,b\rangle\rangle$$
$$- \langle c,\langle\langle b,a\rangle,d\rangle\rangle + \langle c,\langle\langle b,d\rangle,a\rangle\rangle + \langle c,\langle\langle d,a\rangle,b\rangle\rangle + \langle d,\langle\langle a,c\rangle,b\rangle\rangle = 0. \tag{2.2.4}$$

$$f_5(a,b,c,d) = \langle\langle a,d\rangle,\langle b,c\rangle\rangle - \langle\langle c,a\rangle,\langle d,b\rangle\rangle - \langle a,\langle\langle d,c\rangle,b\rangle\rangle - \langle b,\langle\langle a,d\rangle,c\rangle\rangle$$
$$- \langle c,\langle\langle b,a\rangle,d\rangle\rangle + \langle c,\langle\langle b,d\rangle,a\rangle\rangle + \langle c,\langle\langle d,a\rangle,b\rangle\rangle + \langle d,\langle\langle a,c\rangle,b\rangle\rangle$$
$$+ \langle d,\langle\langle b,a\rangle,c\rangle\rangle - \langle d,\langle\langle b,c\rangle,a\rangle\rangle + \langle a,\langle b,\langle c,d\rangle\rangle\rangle - \langle b,\langle c,\langle a,d\rangle\rangle\rangle = 0. \tag{2.2.5}$$

$$f_6(a,b,c,d) = \langle\langle a,\langle b,c\rangle\rangle,d\rangle - \langle a,\langle\langle d,c\rangle,b\rangle\rangle - \langle b,\langle\langle a,d\rangle,c\rangle\rangle + \langle b,\langle\langle d,a\rangle,c\rangle\rangle$$
$$- \langle c,\langle\langle b,a\rangle,d\rangle\rangle + \langle c,\langle\langle b,d\rangle,a\rangle\rangle + \langle d,\langle\langle a,c\rangle,b\rangle\rangle + \langle d,\langle\langle b,a\rangle,c\rangle\rangle$$
$$- \langle d,\langle\langle b,c\rangle,a\rangle\rangle - \langle d,\langle\langle b,c\rangle,a\rangle\rangle + \langle a,\langle b,\langle c,d\rangle\rangle\rangle - \langle a,\langle c,\langle b,d\rangle\rangle\rangle$$
$$+ \langle a,\langle d,\langle b,c\rangle\rangle\rangle - \langle b,\langle d,\langle a,c\rangle\rangle\rangle = 0. \tag{2.2.6}$$

Proof. By using the program Albert, we obtain a basis of degree 4 of algebra defined by identities $f_1(a,b,c) = 0$ and $f_3(a,b,c) = 0$. The basis of this algebra contains 84 nonassociative monomials of degree 4, and we present them in the following order:

$$\{\langle\langle\langle a,b\rangle,c\rangle,d\rangle,\langle\langle\langle b,c\rangle,a\rangle,d\rangle,\langle\langle\langle a,c\rangle,b\rangle,d\rangle,\langle\langle\langle a,d\rangle,b\rangle,c\rangle,\langle\langle\langle b,a\rangle,c\rangle,d\rangle,$$
$$\langle\langle\langle b,d\rangle,a\rangle,c\rangle,\langle\langle\langle c,a\rangle,b\rangle,d\rangle,\langle\langle\langle c,b\rangle,a\rangle,d\rangle,\langle\langle\langle c,d\rangle,a\rangle,b\rangle,\langle\langle\langle d,a\rangle,b\rangle,c\rangle,$$
$$\langle\langle\langle d,b\rangle,a\rangle,c\rangle,\langle\langle\langle d,c\rangle,a\rangle,b\rangle,\langle\langle a,b\rangle,\langle c,d\rangle\rangle,\langle\langle a,c\rangle,\langle b,d\rangle\rangle,\langle\langle a,d\rangle,\langle b,c\rangle\rangle,$$
$$\langle\langle b,a\rangle,\langle c,d\rangle\rangle,\langle\langle c,a\rangle,\langle b,d\rangle\rangle,\langle\langle c,b\rangle,\langle a,d\rangle\rangle,\langle\langle d,a\rangle,\langle b,c\rangle\rangle,\langle\langle d,b\rangle,\langle a,c\rangle\rangle,$$
$$\langle\langle d,c\rangle,\langle a,b\rangle\rangle,\langle\langle b,a\rangle,\langle d,c\rangle\rangle,\langle\langle b,d\rangle,\langle a,c\rangle\rangle,\langle\langle c,a\rangle,\langle d,b\rangle\rangle,\langle\langle a,\langle b,c\rangle\rangle,d\rangle,$$
$$\langle\langle a,\langle b,d\rangle\rangle,c\rangle,\langle\langle a,\langle c,b\rangle\rangle,d\rangle,\langle\langle a,\langle c,d\rangle\rangle,b\rangle,\langle\langle a,\langle d,b\rangle\rangle,c\rangle,\langle\langle a,\langle d,c\rangle\rangle,b\rangle,$$
$$\langle\langle b,\langle a,c\rangle\rangle,d\rangle,\langle\langle b,\langle a,d\rangle\rangle,c\rangle,\langle\langle b,\langle d,a\rangle\rangle,c\rangle,\langle\langle b,\langle d,c\rangle\rangle,a\rangle,\langle\langle b,\langle c,a\rangle\rangle,d\rangle,$$
$$\langle\langle b,\langle c,d\rangle\rangle,a\rangle,\langle\langle c,\langle a,b\rangle\rangle,d\rangle,\langle\langle c,\langle a,d\rangle\rangle,b\rangle,\langle\langle c,\langle b,a\rangle\rangle,b\rangle,\langle\langle c,\langle b,d\rangle\rangle,a\rangle,$$
$$\langle\langle c,\langle d,a\rangle\rangle,b\rangle,\langle\langle c,\langle d,b\rangle\rangle,a\rangle,\langle\langle d,\langle a,b\rangle\rangle,c\rangle,\langle\langle d,\langle a,c\rangle\rangle,b\rangle,\langle\langle d,\langle b,a\rangle\rangle,c\rangle,$$
$$\langle\langle d,\langle b,c\rangle\rangle,a\rangle,\langle\langle d,\langle c,a\rangle\rangle,b\rangle,\langle\langle d,\langle c,b\rangle\rangle,a\rangle,\langle a,\langle\langle b,c\rangle,d\rangle\rangle,\langle a,\langle\langle b,d\rangle,c\rangle\rangle,$$
$$\langle a,\langle\langle c,b\rangle,d\rangle\rangle,\langle a,\langle\langle c,d\rangle,b\rangle\rangle,\langle a,\langle\langle d,b\rangle,c\rangle\rangle,\langle a,\langle\langle d,c\rangle,b\rangle\rangle,\langle b,\langle\langle a,c\rangle,d\rangle\rangle,$$
$$\langle b,\langle\langle a,d\rangle,c\rangle\rangle,\langle b,\langle\langle c,a\rangle,d\rangle\rangle,\langle b,\langle\langle c,d\rangle,a\rangle\rangle,\langle b,\langle\langle d,a\rangle,c\rangle\rangle,\langle b,\langle\langle d,c\rangle,a\rangle\rangle,$$
$$\langle c,\langle\langle a,b\rangle,d\rangle\rangle,\langle c,\langle\langle a,d\rangle,b\rangle\rangle,\langle c,\langle\langle b,a\rangle,d\rangle\rangle,\langle c,\langle\langle b,d\rangle,a\rangle\rangle,\langle c,\langle\langle d,a\rangle,b\rangle\rangle,$$
$$\langle c,\langle\langle d,b\rangle,a\rangle\rangle,\langle d,\langle\langle a,b\rangle,c\rangle\rangle,\langle d,\langle\langle a,c\rangle,b\rangle\rangle,\langle d,\langle\langle b,a\rangle,c\rangle\rangle,\langle d,\langle\langle b,c\rangle,a\rangle\rangle,$$
$$\langle d,\langle\langle c,a\rangle,b\rangle\rangle,\langle d,\langle\langle c,b\rangle,a\rangle\rangle,\langle a,\langle b,\langle c,d\rangle\rangle\rangle,\langle a,\langle b,\langle d,c\rangle\rangle\rangle,\langle a,\langle c,\langle b,d\rangle\rangle\rangle,$$
$$\langle a,\langle c,\langle d,b\rangle\rangle\rangle,\langle a,\langle d,\langle b,c\rangle\rangle\rangle,\langle a,\langle d,\langle c,b\rangle\rangle\rangle,\langle b,\langle c,\langle a,d\rangle\rangle\rangle,\langle b,\langle c,\langle d,a\rangle\rangle\rangle,$$
$$\langle b,\langle d,\langle a,c\rangle\rangle\rangle,\langle b,\langle d,\langle c,a\rangle\rangle\rangle,\langle c,\langle d,\langle a,b\rangle\rangle\rangle,\langle c,\langle d,\langle b,a\rangle\rangle\rangle\}. \tag{2.2.7}$$

Let

$$F(a,b,c,d) =$$

$$\lambda_1\langle\langle\langle a,b\rangle,c\rangle,d\rangle + \lambda_2\langle\langle\langle b,c\rangle,a\rangle,d\rangle + \lambda_3\langle\langle\langle a,c\rangle,b\rangle,d\rangle + \lambda_4\langle\langle\langle a,d\rangle,b\rangle,c\rangle$$
$$+ \lambda_5\langle\langle\langle b,a\rangle,c\rangle,d\rangle + \lambda_6\langle\langle\langle b,d\rangle,a\rangle,c\rangle + \lambda_7\langle\langle\langle c,a\rangle,b\rangle,d\rangle + \lambda_8\langle\langle\langle c,b\rangle,a\rangle,d\rangle$$
$$+ \lambda_9\langle\langle\langle c,d\rangle,a\rangle,b\rangle + \lambda_{10}\langle\langle\langle d,a\rangle,b\rangle,c\rangle + \lambda_{11}\langle\langle\langle d,b\rangle,a\rangle,c\rangle + \lambda_{12}\langle\langle\langle d,c\rangle,a\rangle,b\rangle$$
$$+ \lambda_{13}\langle\langle a,b\rangle,\langle c,d\rangle\rangle + \lambda_{14}\langle\langle a,c\rangle,\langle b,d\rangle\rangle + \lambda_{15}\langle\langle a,d\rangle,\langle b,c\rangle\rangle + \lambda_{16}\langle\langle b,a\rangle,\langle c,d\rangle\rangle$$
$$+ \lambda_{17}\langle\langle c,a\rangle,\langle b,d\rangle\rangle + \lambda_{18}\langle\langle c,b\rangle,\langle a,d\rangle\rangle + \lambda_{19}\langle\langle d,a\rangle,\langle b,c\rangle\rangle + \lambda_{20}\langle\langle d,b\rangle,\langle a,c\rangle\rangle$$
$$+ \lambda_{21}\langle\langle d,c\rangle,\langle a,b\rangle\rangle + \lambda_{22}\langle\langle b,a\rangle,\langle d,c\rangle\rangle + \lambda_{23}\langle\langle b,d\rangle,\langle a,c\rangle\rangle + \lambda_{24}\langle\langle c,a\rangle,\langle d,b\rangle\rangle$$
$$+ \lambda_{25}\langle\langle a,\langle b,c\rangle\rangle,d\rangle + \lambda_{26}\langle\langle a,\langle b,d\rangle\rangle,c\rangle + \lambda_{27}\langle\langle a,\langle c,b\rangle\rangle,d\rangle + \lambda_{28}\langle\langle a,\langle c,d\rangle\rangle,b\rangle$$
$$+ \lambda_{29}\langle\langle a,\langle d,b\rangle\rangle,c\rangle + \lambda_{30}\langle\langle a,\langle d,c\rangle\rangle,b\rangle + \lambda_{31}\langle\langle b,\langle a,c\rangle\rangle,d\rangle + \lambda_{32}\langle\langle b,\langle a,d\rangle\rangle,c\rangle$$
$$+ \lambda_{33}\langle\langle b,\langle d,a\rangle\rangle,c\rangle + \lambda_{34}\langle\langle b,\langle d,c\rangle\rangle,a\rangle + \lambda_{35}\langle\langle b,\langle c,a\rangle\rangle,d\rangle + \lambda_{36}\langle\langle b,\langle c,d\rangle\rangle,a\rangle$$
$$+ \lambda_{37}\langle\langle c,\langle a,b\rangle\rangle,d\rangle + \lambda_{38}\langle\langle c,\langle a,d\rangle\rangle,b\rangle + \lambda_{39}\langle\langle c,\langle b,a\rangle\rangle,b\rangle + \lambda_{40}\langle\langle c,\langle b,d\rangle\rangle,a\rangle$$
$$+ \lambda_{41}\langle\langle c,\langle d,a\rangle\rangle,b\rangle + \lambda_{42}\langle\langle c,\langle d,b\rangle\rangle,a\rangle + \lambda_{43}\langle\langle d,\langle a,b\rangle\rangle,c\rangle + \lambda_{44}\langle\langle d,\langle a,c\rangle\rangle,b\rangle$$
$$+ \lambda_{45}\langle\langle d,\langle b,a\rangle\rangle,c\rangle + \lambda_{46}\langle\langle d,\langle b,c\rangle\rangle,a\rangle + \lambda_{47}\langle\langle d,\langle c,a\rangle\rangle,b\rangle + \lambda_{48}\langle\langle d,\langle c,b\rangle\rangle,a\rangle$$
$$+ \lambda_{49}\langle a,\langle\langle b,c\rangle,d\rangle\rangle + \lambda_{50}\langle a,\langle\langle b,d\rangle,c\rangle\rangle + \lambda_{51}\langle a,\langle\langle c,b\rangle,d\rangle\rangle + \lambda_{52}\langle a,\langle\langle c,d\rangle,b\rangle\rangle$$
$$+ \lambda_{53}\langle a,\langle\langle d,b\rangle,c\rangle\rangle + \lambda_{54}\langle a,\langle\langle d,c\rangle,b\rangle\rangle + \lambda_{55}\langle b,\langle\langle a,c\rangle,d\rangle\rangle + \lambda_{56}\langle b,\langle\langle a,d\rangle,c\rangle\rangle$$
$$+ \lambda_{57}\langle b,\langle\langle c,a\rangle,d\rangle\rangle + \lambda_{58}\langle b,\langle\langle c,d\rangle,a\rangle\rangle + \lambda_{59}\langle b,\langle\langle d,a\rangle,c\rangle\rangle + \lambda_{60}\langle b,\langle\langle d,c\rangle,a\rangle\rangle$$
$$+ \lambda_{61}\langle c,\langle\langle a,b\rangle,d\rangle\rangle + \lambda_{62}\langle c,\langle\langle a,d\rangle,b\rangle\rangle + \lambda_{63}\langle c,\langle\langle b,a\rangle,d\rangle\rangle + \lambda_{64}\langle c,\langle\langle b,d\rangle,a\rangle\rangle$$
$$+ \lambda_{65}\langle c,\langle\langle d,a\rangle,b\rangle\rangle + \lambda_{66}\langle c,\langle\langle d,b\rangle,a\rangle\rangle + \lambda_{67}\langle d,\langle\langle a,b\rangle,c\rangle\rangle + \lambda_{68}\langle d,\langle\langle a,c\rangle,b\rangle\rangle$$
$$+ \lambda_{69}\langle d,\langle\langle b,a\rangle,c\rangle\rangle + \lambda_{70}\langle d,\langle\langle b,c\rangle,a\rangle\rangle + \lambda_{71}\langle d,\langle\langle c,a\rangle,b\rangle\rangle + \lambda_{72}\langle d,\langle\langle c,b\rangle,a\rangle\rangle$$
$$+ \lambda_{73}\langle a,\langle b,\langle c,d\rangle\rangle\rangle + \lambda_{74}\langle a,\langle b,\langle d,c\rangle\rangle\rangle + \lambda_{75}\langle a,\langle c,\langle b,d\rangle\rangle\rangle + \lambda_{76}\langle a,\langle c,\langle d,b\rangle\rangle\rangle$$
$$+ \lambda_{77}\langle a,\langle d,\langle b,c\rangle\rangle\rangle + \lambda_{78}\langle a,\langle d,\langle c,b\rangle\rangle\rangle + \lambda_{79}\langle b,\langle c,\langle a,d\rangle\rangle\rangle + \lambda_{80}\langle b,\langle c,\langle d,a\rangle\rangle\rangle$$
$$+ \lambda_{81}\langle b,\langle d,\langle a,c\rangle\rangle\rangle + \lambda_{82}\langle b,\langle d,\langle c,a\rangle\rangle\rangle + \lambda_{83}\langle c,\langle d,\langle a,b\rangle\rangle\rangle + \lambda_{84}\langle c,\langle d,\langle b,a\rangle\rangle\rangle$$

be a nonassociative polynomial, i.e., an element of the free nonassociative algebra of degree 4.

Let $\mathcal{B}$ be the free bicommutative algebra with multiplication $(a,b) \longmapsto ab$. We calculate $F(a,b,c,d) \in \mathcal{B}$ in terms of the mutation product $\langle a,b\rangle = (ap)b - (bq)a$.

We have

$$\begin{aligned}
&F(a,b,c,d) = \\
&(((((ab)c)d)p)p)p(\lambda_1+\lambda_3+\lambda_4) \\
&-(((((ba)c)d)p)p)q(\lambda_1+\lambda_8+\lambda_{11}+\lambda_{25}+\lambda_{26}+\lambda_{39}+\lambda_{40}+\lambda_{45}+\lambda_{46}+\lambda_{49} \\
&\qquad +\lambda_{50}+\lambda_{63}+\lambda_{64}+\lambda_{69}+\lambda_{70}) \\
&-c(((((ab)d)p)p)q)(\lambda_1+\lambda_4+\lambda_8+\lambda_9+\lambda_{13}+\lambda_{15}+\lambda_{16}+\lambda_{18}+\lambda_{21}+\lambda_{25}+\lambda_{39} \\
&\qquad +\lambda_{41}+\lambda_{49}+\lambda_{54}+\lambda_{63}+\lambda_{65}+\lambda_{73}+\lambda_{79}+\lambda_{83}) \\
&+c(((((ba)d)p)q)q)(\lambda_1+\lambda_3+\lambda_{11}+\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{20}+\lambda_{21}+\lambda_{22}+\lambda_{23} \\
&\qquad +\lambda_{24}+\lambda_{26}+\lambda_{28}+\lambda_{45}+\lambda_{47}+\lambda_{50}+\lambda_{52}+\lambda_{69}+\lambda_{71}+\lambda_{81} \\
&\qquad +\lambda_{83})
\end{aligned}$$

$$
\begin{aligned}
&-d(((((ab)c)p)p)q)(\lambda_1+\lambda_3+\lambda_{11}+\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{20}+\lambda_{21}+\lambda_{22}+\lambda_{23}\\
&\qquad+\lambda_{24}+\lambda_{26}+\lambda_{28}+\lambda_{45}+\lambda_{47}+\lambda_{50}+\lambda_{52}+\lambda_{69}+\lambda_{71}+\lambda_{81}\\
&\qquad+\lambda_{83})\\
&+d(((((ba)c)p)q)q)(\lambda_1+\lambda_4+\lambda_8+\lambda_9+\lambda_{13}+\lambda_{15}+\lambda_{16}+\lambda_{18}+\lambda_{21}+\lambda_{25}+\lambda_{39}\\
&\qquad+\lambda_{41}+\lambda_{49}+\lambda_{54}+\lambda_{63}+\lambda_{65}+\lambda_{73}+\lambda_{79}+\lambda_{83})\\
&+d(c((((ab)p)q)q))(\lambda_1+\lambda_8+\lambda_{11}+\lambda_{25}+\lambda_{26}+\lambda_{39}+\lambda_{40}+\lambda_{45}+\lambda_{46}+\lambda_{49}\\
&\qquad+\lambda_{50}+\lambda_{63}+\lambda_{64}+\lambda_{69}+\lambda_{70})\\
&-d(c((((ab)p)q)q))(\lambda_1+\lambda_3+\lambda_4)\\
&+(((((ba)c)d)p)p)p(\lambda_2+\lambda_5+\lambda_6)\\
&-(((((ca)b)d)p)p)q(\lambda_2+\lambda_3+\lambda_{12}+\lambda_{28}+\lambda_{35}+\lambda_{36}+\lambda_{47}+\lambda_{48}+\lambda_{51}+\lambda_{52}\\
&\qquad+\lambda_{57}+\lambda_{58}+\lambda_{71}+\lambda_{72})\\
&-b(((((ac)d)p)p)q)(\lambda_2+\lambda_3+\lambda_4+\lambda_6+\lambda_{14}+\lambda_{15}+\lambda_{17}+\lambda_{18}+\lambda_{19}+\lambda_{20}+\lambda_{23}\\
&\qquad+\lambda_{33}+\lambda_{35}+\lambda_{51}+\lambda_{53}+\lambda_{57}+\lambda_{59}+\lambda_{75}+\lambda_{77}+\lambda_{79}+\lambda_{81})\\
&+c(((((ab)d)p)q)q)(\lambda_2+\lambda_5+\lambda_{10}+\lambda_{12}+\lambda_{15}+\lambda_{16}+\lambda_{19}+\lambda_{21}+\lambda_{22}+\lambda_{32}\\
&\qquad+\lambda_{36}+\lambda_{43}+\lambda_{48}+\lambda_{56}+\lambda_{58}+\lambda_{67}+\lambda_{72}+\lambda_{77}+\lambda_{80}+\lambda_{84})\\
&-d(((((ba)c)p)p)q)(\lambda_2+\lambda_5+\lambda_{10}+\lambda_{12}+\lambda_{15}+\lambda_{16}+\lambda_{19}+\lambda_{21}+\lambda_{22}+\lambda_{32}\\
&\qquad+\lambda_{36}+\lambda_{43}+\lambda_{48}+\lambda_{56}+\lambda_{58}+\lambda_{67}+\lambda_{72}+\lambda_{77}+\lambda_{80}+\lambda_{84})\\
&+d(((((ca)b)p)q)q)(\lambda_2+\lambda_3+\lambda_4+\lambda_6+\lambda_{14}+\lambda_{15}+\lambda_{17}+\lambda_{18}+\lambda_{19}+\lambda_{20}+\lambda_{23}\\
&\qquad+\lambda_{33}+\lambda_{35}+\lambda_{51}+\lambda_{53}+\lambda_{57}+\lambda_{59}+\lambda_{75}+\lambda_{77}+\lambda_{79}+\lambda_{81})\\
&+d(b((((ac)p)q)q))(\lambda_2+\lambda_3+\lambda_{12}+\lambda_{28}+\lambda_{35}+\lambda_{36}+\lambda_{47}+\lambda_{48}+\lambda_{51}+\lambda_{52}\\
&\qquad+\lambda_{57}+\lambda_{58}+\lambda_{71}+\lambda_{72})\\
&-d(c((((ab)q)q)q))(\lambda_2+\lambda_5+\lambda_6)\\
&-(((((da)b)c)p)p)q(\lambda_4+\lambda_6+\lambda_9+\lambda_{29}+\lambda_{30}+\lambda_{33}+\lambda_{34}+\lambda_{41}+\lambda_{42}+\lambda_{53}+\lambda_{54}\\
&\qquad+\lambda_{59}+\lambda_{60}+\lambda_{65}+\lambda_{66})\\
&+c(b((((ad)p)q)q))(\lambda_4+\lambda_6+\lambda_9+\lambda_{29}+\lambda_{30}+\lambda_{33}+\lambda_{34}+\lambda_{41}+\lambda_{42}+\lambda_{53}+\lambda_{54}\\
&\qquad+\lambda_{59}+\lambda_{60}+\lambda_{65}+\lambda_{66})\\
&-(((((ab)c)d)p)p)q(\lambda_5+\lambda_7+\lambda_{10}+\lambda_{31}+\lambda_{32}+\lambda_{37}+\lambda_{38}+\lambda_{43}+\lambda_{44}+\lambda_{55}\\
&\qquad+\lambda_{56}+\lambda_{61}+\lambda_{62}+\lambda_{67}+\lambda_{68})\\
&-c(((((ba)d)p)q)q)(\lambda_5+\lambda_6+\lambda_7+\lambda_9+\lambda_{13}+\lambda_{14}+\lambda_{16}+\lambda_{17}+\lambda_{22}+\lambda_{23}+\lambda_{24}\\
&\qquad+\lambda_{31}+\lambda_{37}+\lambda_{42}+\lambda_{55}+\lambda_{60}+\lambda_{61}+\lambda_{66}+\lambda_{73}+\lambda_{75}+\lambda_{82}\\
&\qquad+\lambda_{84})\\
&+d(((((ab)c)p)q)q)(\lambda_5+\lambda_6+\lambda_7+\lambda_9+\lambda_{13}+\lambda_{14}+\lambda_{16}+\lambda_{17}+\lambda_{22}+\lambda_{23}+\lambda_{24}\\
&\qquad+\lambda_{31}+\lambda_{37}+\lambda_{42}+\lambda_{55}+\lambda_{60}+\lambda_{61}+\lambda_{66}+\lambda_{73}+\lambda_{75}+\lambda_{82}\\
&\qquad+\lambda_{84})\\
&+d(c((((ba)p)q)q))(\lambda_5+\lambda_7+\lambda_{10}+\lambda_{31}+\lambda_{32}+\lambda_{37}+\lambda_{38}+\lambda_{43}+\lambda_{44}+\lambda_{55}\\
&\qquad+\lambda_{56}+\lambda_{61}+\lambda_{62}+\lambda_{67}+\lambda_{68})
\end{aligned}
$$

$$
\begin{aligned}
&+(((((ca)b)d)p)p)p(\lambda_7+\lambda_8+\lambda_9)\\
&+b(((((ac)d)p)q)q)(\lambda_7+\lambda_8+\lambda_{10}+\lambda_{11}+\lambda_{17}+\lambda_{18}+\lambda_{19}+\lambda_{20}+\lambda_{24}+\lambda_{38}\\
&\qquad+\lambda_{40}+\lambda_{44}+\lambda_{46}+\lambda_{62}+\lambda_{64}+\lambda_{68}+\lambda_{70}+\lambda_{80}+\lambda_{82})\\
&-d(((((ca)b)p)p)q)(\lambda_7+\lambda_8+\lambda_{10}+\lambda_{11}+\lambda_{17}+\lambda_{18}+\lambda_{19}+\lambda_{20}+\lambda_{24}+\lambda_{38}\\
&\qquad+\lambda_{40}+\lambda_{44}+\lambda_{46}+\lambda_{62}+\lambda_{64}+\lambda_{68}+\lambda_{70}+\lambda_{80}+\lambda_{82})\\
&-d(b((((ac)q)q)q))(\lambda_7+\lambda_8+\lambda_9)\\
&+(((((da)b)c)p)p)p(\lambda_{10}+\lambda_{11}+\lambda_{12})\\
&-c(b((((ad)q)q)q))(\lambda_{10}+\lambda_{11}+\lambda_{12})\\
&+c(((((ab)d)p)p)p)(\lambda_{13}+\lambda_{18}+\lambda_{28}+\lambda_{37}+\lambda_{38}+\lambda_{51}+\lambda_{52}+\lambda_{61}+\lambda_{62})\\
&-d(((((ba)c)q)q)q)(\lambda_{13}+\lambda_{18}+\lambda_{28}+\lambda_{37}+\lambda_{38}+\lambda_{51}+\lambda_{52}+\lambda_{61}+\lambda_{62})\\
&+b(((((ac)d)p)p)p)(\lambda_{14}+\lambda_{15}+\lambda_{23}+\lambda_{25}+\lambda_{26}+\lambda_{31}+\lambda_{32}+\lambda_{49}+\lambda_{50}+\lambda_{55}\\
&\qquad+\lambda_{56})\\
&-d(((((ca)b)q)q)q)(\lambda_{14}+\lambda_{15}+\lambda_{23}+\lambda_{25}+\lambda_{26}+\lambda_{31}+\lambda_{32}+\lambda_{49}+\lambda_{50}+\lambda_{55}\\
&\qquad+\lambda_{56})\\
&+c(((((ba)d)p)p)p)(\lambda_{16}+\lambda_{17}+\lambda_{35}+\lambda_{36}+\lambda_{39}+\lambda_{40}+\lambda_{57}+\lambda_{58}+\lambda_{63}+\lambda_{64})\\
&-d(((((ab)c)q)q)q)(\lambda_{16}+\lambda_{17}+\lambda_{35}+\lambda_{36}+\lambda_{39}+\lambda_{40}+\lambda_{57}+\lambda_{58}+\lambda_{63}+\lambda_{64})\\
&+d(((((ba)c)p)p)p)(\lambda_{19}+\lambda_{22}+\lambda_{33}+\lambda_{45}+\lambda_{46}+\lambda_{59}+\lambda_{60}+\lambda_{69}+\lambda_{70})\\
&-c(((((ab)d)q)q)q)(\lambda_{19}+\lambda_{22}+\lambda_{33}+\lambda_{45}+\lambda_{46}+\lambda_{59}+\lambda_{60}+\lambda_{69}+\lambda_{70})\\
&+d(((((ab)c)p)p)p)(\lambda_{20}+\lambda_{21}+\lambda_{43}+\lambda_{44}+\lambda_{53}+\lambda_{54}+\lambda_{67}+\lambda_{68})\\
&-c(((((ba)d)q)q)q)(\lambda_{20}+\lambda_{21}+\lambda_{43}+\lambda_{44}+\lambda_{53}+\lambda_{54}+\lambda_{67}+\lambda_{68})\\
&+d(((((ca)b)p)p)p)(\lambda_{24}+\lambda_{41}+\lambda_{42}+\lambda_{47}+\lambda_{48}+\lambda_{65}+\lambda_{66}+\lambda_{71}+\lambda_{72})\\
&-b(((((ac)d)q)q)q)(\lambda_{24}+\lambda_{41}+\lambda_{42}+\lambda_{47}+\lambda_{48}+\lambda_{65}+\lambda_{66}+\lambda_{71}+\lambda_{72})\\
&+(((((ca)b)d)p)q)q(\lambda_{25}+\lambda_{31}+\lambda_{44}+\lambda_{46}+\lambda_{49}+\lambda_{54}+\lambda_{55}+\lambda_{60}+\lambda_{68}+\lambda_{70}\\
&\qquad+\lambda_{73}+\lambda_{82})\\
&-d(b((((ac)p)p)q))(\lambda_{25}+\lambda_{31}+\lambda_{44}+\lambda_{46}+\lambda_{49}+\lambda_{54}+\lambda_{55}+\lambda_{60}+\lambda_{68}+\lambda_{70}\\
&\qquad+\lambda_{73}+\lambda_{82})\\
&+(((((da)b)c)p)q)q(\lambda_{26}+\lambda_{28}+\lambda_{32}+\lambda_{36}+\lambda_{38}+\lambda_{40}+\lambda_{50}+\lambda_{52}+\lambda_{56}+\lambda_{58}\\
&\qquad+\lambda_{62}+\lambda_{64}+\lambda_{80})\\
&-c(b((((ad)p)p)q))(\lambda_{26}+\lambda_{28}+\lambda_{32}+\lambda_{36}+\lambda_{38}+\lambda_{40}+\lambda_{50}+\lambda_{52}+\lambda_{56}+\lambda_{58}\\
&\qquad+\lambda_{62}+\lambda_{64}+\lambda_{80})\\
&+(((((ba)c)d)p)q)q(\lambda_{37}+\lambda_{42}+\lambda_{43}+\lambda_{48}+\lambda_{51}+\lambda_{53}+\lambda_{61}+\lambda_{66}+\lambda_{67}+\lambda_{72}\\
&\qquad+\lambda_{75}+\lambda_{77}+\lambda_{84})\\
&-d(c((((ab)p)p)q))(\lambda_{37}+\lambda_{42}+\lambda_{43}+\lambda_{48}+\lambda_{51}+\lambda_{53}+\lambda_{61}+\lambda_{66}+\lambda_{67}+\lambda_{72}\\
&\qquad+\lambda_{75}+\lambda_{77}+\lambda_{84})\\
&+(((((ab)c)d)p)q)q)(\lambda_{33}+\lambda_{35}+\lambda_{39}+\lambda_{41}+\lambda_{45}+\lambda_{47}+\lambda_{57}+\lambda_{59}+\lambda_{63}+\lambda_{65}\\
&\qquad+\lambda_{69}+\lambda_{71}+\lambda_{79}+\lambda_{81}+\lambda_{83})
\end{aligned}
$$

$$
\begin{aligned}
&+d(c((((ba)p)p)q))(\lambda_{33}+\lambda_{35}+\lambda_{39}+\lambda_{41}+\lambda_{45}+\lambda_{47}+\lambda_{57}+\lambda_{59}+\lambda_{63}+\lambda_{65}\\
&\qquad\qquad+\lambda_{69}+\lambda_{71}+\lambda_{79}+\lambda_{81}+\lambda_{83})\\
&+c(b((((ad)p)p)p))(\lambda_{73}+\lambda_{75}+\lambda_{79})\\
&-(((((da)b)c)q)q)q(\lambda_{73}+\lambda_{75}+\lambda_{79})\\
&+d(b((((ac)p)p)p))(\lambda_{77}+\lambda_{81})\\
&+(((((ca)b)d)q)q)q)(\lambda_{77}+\lambda_{81})\\
&+d(c((((ab)p)p)p))(\lambda_{83})\\
&+(((((ba)c)d)q)q)q(\lambda_{83})\\
&+d(c((((ba)p)p)p))(\lambda_{80}+\lambda_{82}+\lambda_{84})\\
&+(((((ab)c)d)q)q)q(\lambda_{80}+\lambda_{82}+\lambda_{84})
\end{aligned}
$$

Since, the mutation products of $\{a,b,c,d\}$ are expressed with the set of elements

$$
\begin{aligned}
&\{(((((ab)c)d)p)p)p,(((((ba)c)d)p)p)q,c(((((ab)d)p)p)q),c(((((ba)d)p)q)q),\\
&\ d(((((ab)c)p)p)q),d(((((ba)c)p)q)q),d(c((((ab)p)q)q)),d(c((((ab)p)q)q)),\\
&\ (((((ba)c)d)p)p)p,(((((ca)b)d)p)p)q,b(((((ac)d)p)p)q),c(((((ab)d)p)q)q),\\
&\ d(((((ba)c)p)p)q),d(((((ca)b)p)q)q),d(b((((ac)p)q)q)),d(c((((ab)q)q)q)),\\
&\ (((((da)b)c)p)p)q,c(b((((ad)p)q)q)),(((((ab)c)d)p)p)q,c(((((ba)d)p)q)q),\\
&\ d(((((ab)c)p)q)q),d(c((((ba)p)q)q)),(((((ca)b)d)p)p)p,b(((((ac)d)p)q)q),\\
&\ d(((((ca)b)p)p)q),d(b((((ac)q)q)q)),(((((da)b)c)p)p)p,c(b((((ad)q)q)q)),\\
&\ c(((((ab)d)p)p)p),d(((((ba)c)q)q)q),b(((((ac)d)p)p)p),d(((((ca)b)q)q)q),\\
&\ c(((((ba)d)p)p)p),d(((((ab)c)q)q)q),d(((((ba)c)p)p)p),c(((((ab)d)q)q)q),\\
&\ d(((((ab)c)p)p)p),c(((((ba)d)q)q)q),d(((((ca)b)p)p)p),b(((((ac)d)q)q)q),\\
&\ (((((ca)b)d)p)q)q,d(b((((ac)p)p)q)),(((((da)b)c)p)q)q,c(b((((ad)p)p)q)),\\
&\ (((((ba)c)d)p)q)q,d(c((((ab)p)p)q)),(((((ab)c)d)p)q)q,d(c((((ba)p)p)q)),\\
&\ c(b((((ad)p)p)p)),(((((da)b)c)q)q)q,d(b((((ac)p)p)p)),(((((ca)b)d)q)q)q,\\
&\ d(c((((ab)p)p)p)),(((((ba)c)d)q)q)q,d(c((((ba)p)p)p)),(((((ab)c)d)q)q)q\}
\end{aligned}
$$

in the free bicommutative algebra in degree 4, we see that $F(a,b,c,d)=0$ gives us system of 61 linear equations with 84 unknowns λ_i, where $i\in\{1,2,\ldots,84\}$. We see that this system has rank 23, and we can express the parameters in the following way:

$$
\begin{aligned}
&\lambda_4=-\lambda_1-\lambda_3,\\
&\lambda_6=-\lambda_2-\lambda_5,\\
&\lambda_9=-\lambda_7-\lambda_8,\\
&\lambda_{12}=-\lambda_{10}-\lambda_{11},\\
&\lambda_{24}=-\lambda_{13}-\lambda_{14}-\lambda_{15}-\lambda_{16}-\lambda_{17}-\lambda_{18}-\lambda_{19}-\lambda_{20}-\lambda_{21}-\lambda_{22}-\lambda_{23},
\end{aligned}
$$

$$\lambda_{54} = \lambda_{10} - \lambda_{13} - \lambda_{14} - \lambda_{15} - \lambda_{18} - \lambda_{20} - \lambda_{21} - \lambda_{23} - \lambda_{25} - \lambda_{26} - \lambda_{28} - \lambda_{49} + \lambda_5 - \lambda_{50} - \lambda_{51} - \lambda_{52} - \lambda_{53} + \lambda_7,$$

$$\lambda_{56} = -\lambda_{14} - \lambda_{15} - \lambda_{23} - \lambda_{25} - \lambda_{26} - \lambda_{31} - \lambda_{32} - \lambda_{49} - \lambda_{50} - \lambda_{55},$$

$$\lambda_{59} = \lambda_1 + \lambda_{11} - \lambda_{16} - \lambda_{17} - \lambda_{19} - \lambda_{22} + \lambda_{25} + \lambda_{26} - \lambda_{33} - \lambda_{35} - \lambda_{36} + \lambda_{49} + \lambda_{50} - \lambda_{57} - \lambda_{58} - \lambda_{60} + \lambda_8,$$

$$\lambda_{62} = -\lambda_{13} - \lambda_{18} - \lambda_{28} - \lambda_{37} - \lambda_{38} - \lambda_{51} - \lambda_{52} - \lambda_{61},$$

$$\lambda_{63} = -\lambda_{13} - \lambda_{14} - \lambda_{15} - \lambda_{16} - \lambda_{17} - \lambda_{18} - \lambda_{23} - \lambda_{25} - \lambda_{31} - \lambda_{35} - \lambda_{37} - \lambda_{39} - \lambda_{49} - \lambda_{51} - \lambda_{55} - \lambda_{57} - \lambda_{61} + \lambda_{80},$$

$$\lambda_{64} = \lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{18} + \lambda_{23} + \lambda_{25} + \lambda_{31} - \lambda_{36} + \lambda_{37} - \lambda_{40} + \lambda_{49} + \lambda_{51} + \lambda_{55} - \lambda_{58} + \lambda_{61} - \lambda_{80},$$

$$\lambda_{65} = -\lambda_{10} - \lambda_{11} + \lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{16} + \lambda_{17} + \lambda_{18} + \lambda_{19} + \lambda_2 + \lambda_{20} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{28} + \lambda_3 + \lambda_{35} + \lambda_{36} - \lambda_{41} - \lambda_{42} + \lambda_{51} + \lambda_{52} + \lambda_{57} + \lambda_{58} - \lambda_{66},$$

$$\lambda_{68} = -\lambda_{10} + \lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{18} + \lambda_{23} + \lambda_{25} + \lambda_{26} + \lambda_{28} - \lambda_{43} - \lambda_{44} + \lambda_{49} - \lambda_5 + \lambda_{50} + \lambda_{51} + \lambda_{52} - \lambda_{67} - \lambda_7,$$

$$\lambda_{69} = -\lambda_1 + \lambda_{14} + \lambda_{15} + \lambda_{17} + \lambda_{18} - \lambda_{21} - \lambda_{22} + \lambda_{23} + \lambda_{25} + \lambda_{31} + \lambda_{35} - \lambda_{43} - \lambda_{45} + \lambda_{49} - \lambda_5 + \lambda_{51} + \lambda_{55} + \lambda_{57} - \lambda_{67} + \lambda_{82},$$

$$\lambda_{70} = -\lambda_{11} - \lambda_{14} - \lambda_{15} + \lambda_{16} - \lambda_{18} + \lambda_{21} + \lambda_{22} - \lambda_{23} - 2\lambda_{25} - \lambda_{26} - \lambda_{31} + \lambda_{36} + \lambda_{43} - \lambda_{46} - 2\lambda_{49} + \lambda_5 - \lambda_{50} - \lambda_{51} - \lambda_{55} + \lambda_{58} + \lambda_{67} - \lambda_8 - \lambda_{82},$$

$$\lambda_{71} = \lambda_{10} + \lambda_{11} - \lambda_2 - \lambda_{28} - \lambda_3 - \lambda_{35} - \lambda_{36} - \lambda_{47} - \lambda_{48} - \lambda_{51} - \lambda_{52} - \lambda_{57} - \lambda_{58} - \lambda_{72},$$

$$\lambda_{73} = \lambda_{11} + \lambda_{14} + \lambda_{15} - \lambda_{16} + \lambda_{18} + \lambda_{20} - \lambda_{22} + \lambda_{23} + \lambda_{25} + \lambda_{26} - \lambda_{36} + \lambda_{49} - \lambda_5 + \lambda_{50} + \lambda_{51} + \lambda_{53} - \lambda_{58} - \lambda_{60} + \lambda_8,$$

$$\lambda_{75} = -\lambda_{11} - \lambda_{14} + \lambda_{16} + \lambda_{19} + \lambda_2 + \lambda_{21} + \lambda_{22} - \lambda_{23} - \lambda_{25} - \lambda_{26} - \lambda_{31} + \lambda_{36} - \lambda_{37} - \lambda_{42} - \lambda_{49} + \lambda_5 - \lambda_{50} - \lambda_{51} - \lambda_{53} - \lambda_{55} + \lambda_{58} - \lambda_{61} - \lambda_{66} + \lambda_{80},$$

$$\lambda_{77} = \lambda_{11} + \lambda_{14} - \lambda_{16} - \lambda_{19} - \lambda_2 - \lambda_{21} - \lambda_{22} + \lambda_{23} + \lambda_{25} + \lambda_{26} + \lambda_{31} - \lambda_{36} - \lambda_{43} - \lambda_{48} + \lambda_{49} - \lambda_5 + \lambda_{50} + \lambda_{55} - \lambda_{58} - \lambda_{67} - \lambda_{72} + \lambda_{82},$$

$$\lambda_{79} = -\lambda_{15} - \lambda_{18} - \lambda_{19} - \lambda_2 - \lambda_{20} - \lambda_{21} + \lambda_{31} + \lambda_{37} + \lambda_{42} + \lambda_{55} + \lambda_{60} + \lambda_{61} + \lambda_{66} - \lambda_8 - \lambda_{80},$$

$$\lambda_{81} = -\lambda_{11} - \lambda_{14} + \lambda_{16} + \lambda_{19} + \lambda_2 + \lambda_{21} + \lambda_{22} - \lambda_{23} - \lambda_{25} - \lambda_{26} - \lambda_{31} + \lambda_{36} + \lambda_{43} + \lambda_{48} - \lambda_{49} + \lambda_5 - \lambda_{50} - \lambda_{55} + \lambda_{58} + \lambda_{67} + \lambda_{72} - \lambda_{82},$$

$$\lambda_{83} = 0,$$

$$\lambda_{84} = -\lambda_{80} - \lambda_{82}$$

We write the 61 free parameters generated by us with the 84 nonassociative monomials of degree 4 (2.2.7), substituting them with the remaining pivots.

$$
\begin{aligned}
&F(a,b,c,d) = \\
&\lambda_1\langle\langle\langle a,b\rangle,c\rangle,d\rangle + \lambda_2\langle\langle\langle b,c\rangle,a\rangle,d\rangle + \lambda_3\langle\langle\langle a,c\rangle,b\rangle,d\rangle + (-\lambda_1-\lambda_3) \\
&\langle\langle\langle a,d\rangle,b\rangle,c\rangle + \lambda_5\langle\langle\langle b,a\rangle,c\rangle,d\rangle + (-\lambda_2-\lambda_5)\langle\langle\langle b,d\rangle,a\rangle,c\rangle + \lambda_7\langle\langle\langle c,a\rangle,b\rangle,d\rangle \\
&+ \lambda_8\langle\langle\langle c,b\rangle,a\rangle,d\rangle + (-\lambda_7-\lambda_8)\langle\langle\langle c,d\rangle,a\rangle,b\rangle + \lambda_{10}\langle\langle\langle d,a\rangle,b\rangle,c\rangle \\
&+ \lambda_{11}\langle\langle\langle d,b\rangle,a\rangle,c\rangle + (-\lambda_{10}-\lambda_{11})\langle\langle\langle d,c\rangle,a\rangle,b\rangle + \lambda_{13}\langle\langle a,b\rangle,\langle c,d\rangle\rangle \\
&+ \lambda_{14}\langle\langle a,c\rangle,\langle b,d\rangle\rangle + \lambda_{15}\langle\langle a,d\rangle,\langle b,c\rangle\rangle + \lambda_{16}\langle\langle b,a\rangle,\langle c,d\rangle\rangle + \lambda_{17}\langle\langle c,a\rangle,\langle b,d\rangle\rangle \\
&+ \lambda_{18}\langle\langle c,b\rangle,\langle a,d\rangle\rangle + \lambda_{19}\langle\langle d,a\rangle,\langle b,c\rangle\rangle + \lambda_{20}\langle\langle d,b\rangle,\langle a,c\rangle\rangle + \lambda_{21}\langle\langle d,c\rangle,\langle a,b\rangle\rangle \\
&+ \lambda_{22}\langle\langle b,a\rangle,\langle d,c\rangle\rangle + \lambda_{23}\langle\langle b,d\rangle,\langle a,c\rangle\rangle + (-\lambda_{13}-\lambda_{14}-\lambda_{15}-\lambda_{16}-\lambda_{17}-\lambda_{18} \\
&- \lambda_{19}-\lambda_{20}-\lambda_{21}-\lambda_{22}-\lambda_{23})\langle\langle c,a\rangle,\langle d,b\rangle\rangle + \lambda_{25}\langle\langle a,\langle b,c\rangle\rangle,d\rangle \\
&+ \lambda_{26}\langle\langle a,\langle b,d\rangle\rangle,c\rangle + \lambda_{27}\langle\langle a,\langle c,b\rangle\rangle,d\rangle + \lambda_{28}\langle\langle a,\langle c,d\rangle\rangle,b\rangle + \lambda_{29}\langle\langle a,\langle d,b\rangle\rangle,c\rangle \\
&+ \lambda_{30}\langle\langle a,\langle d,c\rangle\rangle,b\rangle + \lambda_{31}\langle\langle b,\langle a,c\rangle\rangle,d\rangle + \lambda_{32}\langle\langle b,\langle a,d\rangle\rangle,c\rangle + \lambda_{33}\langle\langle b,\langle d,a\rangle\rangle,c\rangle \\
&+ \lambda_{34}\langle\langle b,\langle d,c\rangle\rangle,a\rangle + \lambda_{35}\langle\langle b,\langle c,a\rangle\rangle,d\rangle + \lambda_{36}\langle\langle b,\langle c,d\rangle\rangle,a\rangle + \lambda_{37}\langle\langle c,\langle a,b\rangle\rangle,d\rangle \\
&+ \lambda_{38}\langle\langle c,\langle a,d\rangle\rangle,b\rangle + \lambda_{39}\langle\langle c,\langle b,a\rangle\rangle,b\rangle + \lambda_{40}\langle\langle c,\langle b,d\rangle\rangle,a\rangle + \lambda_{41}\langle\langle c,\langle d,a\rangle\rangle,b\rangle \\
&+ \lambda_{42}\langle\langle c,\langle d,b\rangle\rangle,a\rangle + \lambda_{43}\langle\langle d,\langle a,b\rangle\rangle,c\rangle + \lambda_{44}\langle\langle d,\langle a,c\rangle\rangle,b\rangle + \lambda_{45}\langle\langle d,\langle b,a\rangle\rangle,c\rangle \\
&+ \lambda_{46}\langle\langle d,\langle b,c\rangle\rangle,a\rangle + \lambda_{47}\langle\langle d,\langle c,a\rangle\rangle,b\rangle + \lambda_{48}\langle\langle d,\langle c,b\rangle\rangle,a\rangle + \lambda_{49}\langle a,\langle\langle b,c\rangle,d\rangle\rangle \\
&+ \lambda_{50}\langle a,\langle\langle b,d\rangle,c\rangle\rangle + \lambda_{51}\langle a,\langle\langle c,b\rangle,d\rangle\rangle + \lambda_{52}\langle a,\langle\langle c,d\rangle,b\rangle\rangle + \lambda_{53}\langle a,\langle\langle d,b\rangle,c\rangle\rangle \\
&+ (\lambda_{10}-\lambda_{13}-\lambda_{14}-\lambda_{15}-\lambda_{18}-\lambda_{20}-\lambda_{21}-\lambda_{23}-\lambda_{25}-\lambda_{26}-\lambda_{28}-\lambda_{49} \\
&+ \lambda_5-\lambda_{50}-\lambda_{51}-\lambda_{52}-\lambda_{53}+\lambda_7)\langle a,\langle\langle d,c\rangle,b\rangle\rangle + \lambda_{55}\langle b,\langle\langle a,c\rangle,d\rangle\rangle \\
&+ (-\lambda_{14}-\lambda_{15}-\lambda_{23}-\lambda_{25}-\lambda_{26}-\lambda_{31}-\lambda_{32}-\lambda_{49}-\lambda_{50}-\lambda_{55})\langle b,\langle\langle a,d\rangle,c\rangle\rangle \\
&+ \lambda_{57}\langle b,\langle\langle c,a\rangle,d\rangle\rangle + \lambda_{58}\langle b,\langle\langle c,d\rangle,a\rangle\rangle + (\lambda_1+\lambda_{11}-\lambda_{16}-\lambda_{17}-\lambda_{19}-\lambda_{22} \\
&+ \lambda_{25}+\lambda_{26}-\lambda_{33}-\lambda_{35}-\lambda_{36}+\lambda_{49}+\lambda_{50}-\lambda_{57}-\lambda_{58}-\lambda_{60} \\
&+ \lambda_8)\langle b,\langle\langle d,a\rangle,c\rangle\rangle + \lambda_{60}\langle b,\langle\langle d,c\rangle,a\rangle\rangle + \lambda_{61}\langle c,\langle\langle a,b\rangle,d\rangle\rangle + (-\lambda_{13}-\lambda_{18} \\
&- \lambda_{28}-\lambda_{37}-\lambda_{38}-\lambda_{51}-\lambda_{52}-\lambda_{61})\langle c,\langle\langle a,d\rangle,b\rangle\rangle + (-\lambda_{13}-\lambda_{14}-\lambda_{15}-\lambda_{16} \\
&- \lambda_{17}-\lambda_{18}-\lambda_{23}-\lambda_{25}-\lambda_{31}-\lambda_{35}-\lambda_{37}-\lambda_{39}-\lambda_{49}-\lambda_{51}-\lambda_{55}-\lambda_{57}-\lambda_{61} \\
&+ \lambda_{80})\langle c,\langle\langle b,a\rangle,d\rangle\rangle + (\lambda_{13}+\lambda_{14}+\lambda_{15}+\lambda_{18}+\lambda_{23}+\lambda_{25}+\lambda_{31}-\lambda_{36}+\lambda_{37} \\
&- \lambda_{40}+\lambda_{49}+\lambda_{51}+\lambda_{55}-\lambda_{58}+\lambda_{61}-\lambda_{80})\langle c,\langle\langle b,d\rangle,a\rangle\rangle + (-\lambda_{10}-\lambda_{11}+\lambda_{13} \\
&+ \lambda_{14}+\lambda_{15}+\lambda_{16}+\lambda_{17}+\lambda_{18}+\lambda_{19}+\lambda_2+\lambda_{20}+\lambda_{21}+\lambda_{22}+\lambda_{23}+\lambda_{28}+\lambda_3 \\
&+ \lambda_{35}+\lambda_{36}-\lambda_{41}-\lambda_{42}+\lambda_{51}+\lambda_{52}+\lambda_{57}+\lambda_{58}-\lambda_{66})\langle c,\langle\langle d,a\rangle,b\rangle\rangle \\
&+ \lambda_{66}\langle c,\langle\langle d,b\rangle,a\rangle\rangle + \lambda_{67}\langle d,\langle\langle a,b\rangle,c\rangle\rangle + (-\lambda_{10}+\lambda_{13}+\lambda_{14}+\lambda_{15}+\lambda_{18}+\lambda_{23} \\
&+ \lambda_{25}+\lambda_{26}+\lambda_{28}-\lambda_{43}-\lambda_{44}+\lambda_{49}-\lambda_5+\lambda_{50}+\lambda_{51}+\lambda_{52}-\lambda_{67} \\
&- \lambda_7)\langle d,\langle\langle a,c\rangle,b\rangle\rangle + (-\lambda_1+\lambda_{14}+\lambda_{15}+\lambda_{17}+\lambda_{18}-\lambda_{21}-\lambda_{22}+\lambda_{23}+\lambda_{25} \\
&+ \lambda_{31}+\lambda_{35}-\lambda_{43}-\lambda_{45}+\lambda_{49}-\lambda_5+\lambda_{51}+\lambda_{55}+\lambda_{57}-\lambda_{67} \\
&+ \lambda_{82})\langle d,\langle\langle b,a\rangle,c\rangle\rangle + (-\lambda_{11}-\lambda_{14}-\lambda_{15}+\lambda_{16}-\lambda_{18}+\lambda_{21}+\lambda_{22}-\lambda_{23}-2\lambda_{25} \\
&- \lambda_{26}-\lambda_{31}+\lambda_{36}+\lambda_{43}-\lambda_{46}-2\lambda_{49}+\lambda_5-\lambda_{50}-\lambda_{51}-\lambda_{55}+\lambda_{58}+\lambda_{67}-\lambda_8 \\
&- \lambda_{82})\langle d,\langle\langle b,c\rangle,a\rangle\rangle + (\lambda_{10}+\lambda_{11}-\lambda_2-\lambda_{28}-\lambda_3-\lambda_{35}-\lambda_{36}-\lambda_{47}-\lambda_{48}
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda_{51} - \lambda_{52} - \lambda_{57} - \lambda_{58} - \lambda_{72})\langle d, \langle\langle c, a\rangle, b\rangle\rangle + \lambda_{72}\langle d, \langle\langle c, b\rangle, a\rangle\rangle + (\lambda_{11} + \lambda_{14} \\
&+ \lambda_{15} - \lambda_{16} + \lambda_{18} + \lambda_{20} - \lambda_{22} + \lambda_{23} + \lambda_{25} + \lambda_{26} - \lambda_{36} + \lambda_{49} - \lambda_5 + \lambda_{50} + \lambda_{51} \\
&+ \lambda_{53} - \lambda_{58} - \lambda_{60} + \lambda_8)\langle a, \langle b, \langle c, d\rangle\rangle\rangle + \lambda_{74}\langle a, \langle b, \langle d, c\rangle\rangle\rangle + (-\lambda_{11} - \lambda_{14} + \lambda_{16} \\
&+ \lambda_{19} + \lambda_2 + \lambda_{21} + \lambda_{22} - \lambda_{23} - \lambda_{25} - \lambda_{26} - \lambda_{31} + \lambda_{36} - \lambda_{37} - \lambda_{42} - \lambda_{49} + \lambda_5 \\
&- \lambda_{50} - \lambda_{51} - \lambda_{53} - \lambda_{55} + \lambda_{58} - \lambda_{61} - \lambda_{66} + \lambda_{80})\langle a, \langle c, \langle b, d\rangle\rangle\rangle \\
&+ \lambda_{76}\langle a, \langle c, \langle d, b\rangle\rangle\rangle + (\lambda_{11} + \lambda_{14} - \lambda_{16} - \lambda_{19} - \lambda_2 - \lambda_{21} - \lambda_{22} + \lambda_{23} + \lambda_{25} \\
&+ \lambda_{26} + \lambda_{31} - \lambda_{36} - \lambda_{43} - \lambda_{48} + \lambda_{49} - \lambda_5 + \lambda_{50} + \lambda_{55} - \lambda_{58} - \lambda_{67} - \lambda_{72} \\
&+ \lambda_{82})\langle a, \langle d, \langle b, c\rangle\rangle\rangle + \lambda_{78}\langle a, \langle d, \langle c, b\rangle\rangle\rangle + (-\lambda_{15} - \lambda_{18} - \lambda_{19} - \lambda_2 - \lambda_{20} - \lambda_{21} \\
&+ \lambda_{31} + \lambda_{37} + \lambda_{42} + \lambda_{55} + \lambda_{60} + \lambda_{61} + \lambda_{66} - \lambda_8 - \lambda_{80})\langle b, \langle c, \langle a, d\rangle\rangle\rangle \\
&+ \lambda_{80}\langle b, \langle c, \langle d, a\rangle\rangle\rangle + (-\lambda_{11} - \lambda_{14} + \lambda_{16} + \lambda_{19} + \lambda_2 + \lambda_{21} + \lambda_{22} - \lambda_{23} - \lambda_{25} \\
&- \lambda_{26} - \lambda_{31} + \lambda_{36} + \lambda_{43} + \lambda_{48} - \lambda_{49} + \lambda_5 - \lambda_{50} - \lambda_{55} + \lambda_{58} + \lambda_{67} + \lambda_{72} \\
&- \lambda_{82})\langle b, \langle d, \langle a, c\rangle\rangle\rangle + \lambda_{82}\langle b, \langle d, \langle c, a\rangle\rangle\rangle + (-\lambda_{80} - \lambda_{82})\langle c, \langle d, \langle b, a\rangle\rangle\rangle.
\end{aligned}
$$

Now, let us denote the 84 nonassociative monomials of degree 4 in 2.2.7 as e_i where $i \in \{1, 2, 3, ..., 84\}$. Therefore, we decompose each monomial by λ_i.

$$
\begin{aligned}
&\lambda_1(e_1 - e_4 + e_{59} - e_{69}) + \lambda_2(e_2 - e_6 + e_{65} - e_{71} + e_{75} - e_{77} - e_{79} + e_{81}) \\
&+ \lambda_3(e_3 - e_4 + e_{65} - e_{71}) + \lambda_5(e_5 - e_6 + e_{54} - e_{68} - e_{69} + e_{70} - e_{73} + e_{75} - e_{77} \\
&+ e_{81}) + \lambda_7(e_7 - e_9 + e_{54} - e_{68}) + \lambda_8(e_8 - e_9 + e_{59} - e_{70} + e_{73} - e_{79}) + \lambda_{10}(e_{10} \\
&- e_{12} + e_{54} - e_{65} - e_{68} + e_{71}) + \lambda_{11}(e_{11} - e_{12} + e_{59} - e_{65} - e_{70} + e_{71} + e_{73} - e_{75} \\
&+ e_{77} - e_{81}) + \lambda_{13}(e_{13} - e_{24} - e_{54} - e_{62} - e_{63} + e_{64} + e_{65} + e_{68}) + \lambda_{14}(e_{14} - e_{24} \\
&- e_{54} - e_{56} - e_{63} + e_{64} + e_{65} + e_{68} + e_{69} - e_{70} + e_{73} - e_{75} + e_{77} - e_{81}) + \lambda_{15}(e_{15} \\
&- e_{24} - e_{54} - e_{56} - e_{63} + e_{64} + e_{65} + e_{68} + e_{69} - e_{70} + e_{73} - e_{79}) + \lambda_{16}(e_{16} - e_{24} \\
&- e_{59} - e_{63} + e_{65} + e_{70} - e_{73} + e_{75} - e_{77} + e_{81}) + \lambda_{17}(e_{17} - e_{24} - e_{59} - e_{63} + e_{65} \\
&+ e_{69}) + \lambda_{18}(e_{18} - e_{24} - e_{54} - e_{62} - e_{63} + e_{64} + e_{65} + e_{68} + e_{69} - e_{70} + e_{73} - e_{79}) \\
&+ \lambda_{19}(e_{19} - e_{24} - e_{59} + e_{65} + e_{75} - e_{77} - e_{79} + e_{81}) + \lambda_{20}(e_{20} - e_{24} - e_{54} + e_{65} \\
&+ e_{73} - e_{79}) + \lambda_{21}(e_{21} - e_{24} - e_{54} + e_{65} - e_{69} + e_{70} + e_{75} - e_{77} - e_{79} + e_{81}) \\
&+ \lambda_{22}(e_{22} - e_{24} - e_{59} + e_{65} - e_{69} + e_{70} - e_{73} + e_{75} - e_{77} + e_{81}) + \lambda_{23}(e_{23} - e_{24} \\
&- e_{54} - e_{56} - e_{63} + e_{64} + e_{65} + e_{68} + e_{69} - e_{70} + e_{73} - e_{75} + e_{77} - e_{81}) + \lambda_{25}(e_{25} \\
&- e_{54} - e_{56} + e_{59} - e_{63} + e_{64} + e_{68} + e_{69} - 2e_{70} + e_{73} - e_{75} + e_{77} - e_{81}) \\
&+ \lambda_{26}(e_{26} - e_{54} - e_{56} + e_{59} + e_{68} - e_{70} + e_{73} - e_{75} + e_{77} - e_{81}) + \lambda_{28}(e_{28} - e_{54} \\
&- e_{62} + e_{65} + e_{68} - e_{71}) + \lambda_{31}(e_{31} - e_{56} - e_{63} + e_{64} + e_{69} - e_{70} - e_{75} + e_{77} + e_{79} \\
&- e_{81}) + \lambda_{32}(e_{32} - e_{56}) + \lambda_{33}(e_{33} - e_{59}) + \lambda_{35}(e_{35} - e_{59} - e_{63} + e_{65} + e_{69} - e_{71}) \\
&+ \lambda_{36}(e_{36} - e_{59} - e_{64} + e_{65} + e_{70} - e_{71} - e_{73} + e_{75} - e_{77} + e_{81}) + \lambda_{37}(e_{37} - e_{62} \\
&- e_{63} + e_{64} - e_{75} + e_{79}) + \lambda_{38}(e_{38} - e_{62}) + \lambda_{39}(e_{39} - e_{63}) + \lambda_{40}(e_{40} - e_{64}) \\
&+ \lambda_{41}(e_{41} - e_{65}) + \lambda_{42}(e_{42} - e_{65} - e_{75} + e_{79}) + \lambda_{43}(e_{43} - e_{68} - e_{69} + e_{70} - e_{77} \\
&+ e_{81}) + \lambda_{44}(e_{44} - e_{68}) + \lambda_{45}(e_{45} - e_{69}) + \lambda_{46}(e_{46} - e_{70}) + \lambda_{47}(e_{47} - e_{71}) \\
&+ \lambda_{48}(e_{48} - e_{71} - e_{77} + e_{81}) + \lambda_{49}(e_{49} - e_{54} - e_{56} + e_{59} - e_{63} + e_{64} + e_{68} + e_{69}
\end{aligned}
$$

$- e_{70} - e_{70} + e_{73} - e_{75} + e_{77} - e_{81}) + \lambda_{50}(e_{50} - e_{54} - e_{56} + e_{59} + e_{68} - e_{70} + e_{73}$
$- e_{75} + e_{77} - e_{81}) + \lambda_{51}(e_{51} - e_{54} - e_{62} - e_{63} + e_{64} + e_{65} + e_{68} + e_{69} - e_{70} - e_{71}$
$+ e_{73} - e_{75}) + \lambda_{52}(e_{52} - e_{54} - e_{62} + e_{65} + e_{68} - e_{71}) + \lambda_{53}(e_{53} - e_{54} + e_{73} - e_{75})$
$+ \lambda_{55}(e_{55} - e_{56} - e_{63} + e_{64} + e_{69} - e_{70} - e_{75} + e_{77} + e_{79} - e_{81}) + \lambda_{57}(e_{57} - e_{59}$
$- e_{63} + e_{65} + e_{69} - e_{71}) + \lambda_{58}(e_{58} - e_{59} - e_{64} + e_{65} + e_{70} - e_{71} - e_{73} + e_{75} - e_{77}$
$+ e_{81}) + \lambda_{60}(e_{60} - e_{59} - e_{73} + e_{79}) + \lambda_{61}(e_{61} - e_{62} - e_{63} + e_{64} - e_{75} + e_{79})$
$+ \lambda_{66}(e_{66} - e_{65} - e_{75} + e_{79}) + \lambda_{67}(e_{67} - e_{68} - e_{69} + e_{70} - e_{77} + e_{81}) + \lambda_{72}(e_{72}$
$- e_{71} - e_{77} + e_{81}) + \lambda_{80}(e_{80} + e_{63} - e_{64} + e_{75} - e_{79} - e_{84}) + \lambda_{82}(e_{82} + e_{69} - e_{70}$
$+ e_{77} - e_{81} - e_{84}) = 0.$

So, we have obtained 61 identities. Now, we need to show that all of them follows from the Theorem 2.2.3.

We select the coefficients of the monomials based on their order in the preceding list 2.2.7. In other words, the columns of a matrix correspond to the monomials and the rows of a matrix represent each polynomial with all possible permutations of f_i where $i \in \{1, 2, 3, 4, 5, 6\}$ in the variables a, b, c, d. The resulting matrix achieves a rank of 106.

After that, we construct a matrix without changing its columns; however, the rows represent every polynomial resulting from all possible permutations of $f_i(a, b, c, d)$ where $i \in \{1, 2, 3, 4, 5, 6\}$ and other 61 identities. The rank of this matrix is also 106. Hence, the 61 identities follows from the identities in the Theorem 2.2.3.

□

3. Base elements under mutation product

3.1 List of base elements of degree three and four

Below is the list of base elements of degree 3:

$$\langle\langle a,b\rangle,c\rangle = (((ab)c)p)p - (((ba)c)p)q - c(((ab)p)q) + c(((ba)q)q) \tag{3.1.1}$$

$$\langle\langle c,a\rangle,b\rangle = (((ca)b)p)p - (((ab)c)p)q - c(((ba)p)q) + b(((ac)q)q) \tag{3.1.2}$$

$$\langle\langle b,a\rangle,c\rangle = (((ba)c)p)p - (((ab)c)p)q - c(((ba)p)q) + c(((ab)q)q) \tag{3.1.3}$$

$$\langle\langle a,c\rangle,b\rangle = (((ab)c)p)p - (((ca)b)p)q - b(((ac)p)q) + c(((ba)q)q) \tag{3.1.4}$$

$$\langle\langle c,b\rangle,a\rangle = (((ab)c)p)p - (((ca)b)p)q - b(((ac)p)q) + c(((ba)q)q) \tag{3.1.5}$$

$$\langle\langle b,c\rangle,a\rangle = (((ba)c)p)p - (((ca)b)p)q - b(((ac)p)q) + c(((ab)q)q) \tag{3.1.6}$$

$$\langle a,\langle b,c\rangle\rangle = b(((ac)p)p) - c(((ab)p)q) - (((ba)c)p)q + (((ca)b)q)q \tag{3.1.7}$$

$$\langle b,\langle c,a\rangle\rangle = c(((ba)p)p) - b(((ac)p)q) - (((ca)b)p)q + (((ab)c)q)q \tag{3.1.8}$$

$$\langle c,\langle a,b\rangle\rangle = c(((ab)p)p) - c(((ba)p)q) - (((ab)c)p)q + (((ba)c)q)q \tag{3.1.9}$$

$$\langle b,\langle a,c\rangle\rangle = b(((ac)p)p) - c(((ba)p)q) - (((ab)c)p)q + (((ca)b)q)q \tag{3.1.10}$$

$$\langle c,\langle b,a\rangle\rangle = c(((ba)p)p) - c(((ab)p)q) - (((ba)c)p)q + (((ab)c)q)q \tag{3.1.11}$$

$$\langle a,\langle c,b\rangle\rangle = c(((ab)p)p) - b(((ac)p)q) - (((ca)b)p)q + (((ba)c)q)q \tag{3.1.12}$$

Below is the list of base elements of degree 4:

$$\begin{aligned}\langle\langle\langle a,b\rangle,c\rangle,d\rangle = {} & (((((ab)c)d)p)p)p - (((((ba)c)d)p)p)q - c(((((ab)d)p)p)q) \\ & + c(((((ba)d)p)q)q) - d(((((ab)c)p)p)q) + d(((((ba)c)p)q)q) \\ & + d(c((((ab)p)q)q)) - d(c((((ba)q)q)q))\end{aligned} \tag{3.1.13}$$

$$\begin{aligned}\langle\langle\langle b,c\rangle,a\rangle,d\rangle &= (((((ba)c)d)p)p)p - (((((ca)b)d)p)p)q - b(((((ac)d)p)p)q \\ &+c(((((ab)d)p)q)q) - d(((((ba)c)p)p)q) + d(((((ca)b)p)q)q) \\ &+d(b((((ac)p)q)q)) - d(c((((ab)q)q)q))\end{aligned} \tag{3.1.14}$$

$$\begin{aligned}\langle\langle\langle a,c\rangle,b\rangle,d\rangle &= (((((ab)c)d)p)p)p - (((((ca)b)d)p)p)q - b(((((ac)d)p)p)q \\ &+c(((((ba)d)p)q)q) - d(((((ab)c)p)p)q) + d(((((ca)b)p)q)q) \\ &+d(b((((ac)p)q)q)) - d(c((((ba)q)q)q))\end{aligned} \tag{3.1.15}$$

$$\begin{aligned}\langle\langle\langle a,d\rangle,b\rangle,c\rangle &= (((((ab)c)d)p)p)p - (((((da)b)c)p)p)q - b(((((ac)d)p)p)q \\ &+d(((((ba)c)p)q)q) - c(((((ab)d)p)p)q) + d(((((ca)b)p)q)q) \\ &+c(b((((ad)p)q)q)) - d(c((((ba)q)q)q))\end{aligned} \tag{3.1.16}$$

$$\begin{aligned}\langle\langle\langle b,a\rangle,c\rangle,d\rangle &= (((((ba)c)d)p)p)p - (((((ab)c)d)p)p)q - c(((((ba)d)p)p)q \\ &+ c(((((ab)d)p)q)q) - d(((((ba)c)p)p)q) + d(((((ab)c)p)q)q) \\ &+ d(c((((ba)p)q)q)) - d(c((((ab)q)q)q))\end{aligned} \tag{3.1.17}$$

$$\begin{aligned}\langle\langle\langle b,d\rangle,a\rangle,c\rangle &= (((((ba)c)d)p)p)p - (((((da)b)c)p)p)q - b(((((ac)d)p)p)q \\ &+d(((((ab)c)p)q)q) - c(((((ba)d)p)p)q) + d(((((ca)b)p)q)q) \\ &+c(b((((ad)p)q)q)) - d(c((((ab)q)q)q))\end{aligned} \tag{3.1.18}$$

$$\begin{aligned}\langle\langle\langle c,a\rangle,b\rangle,d\rangle &= (((((ca)b)d)p)p)p - (((((ab)c)d)p)p)q - c(((((ba)d)p)p)q \\ &+b(((((ac)d)p)q)q) - d(((((ca)b)p)p)q) + d(((((ab)c)p)q)q) \\ &+d(c((((ba)p)q)q)) - d(b((((ac)q)q)q))\end{aligned} \tag{3.1.19}$$

$$\begin{aligned}\langle\langle\langle c,b\rangle,a\rangle,d\rangle &= (((((ca)b)d)p)p)p - (((((ba)c)d)p)p)q - c(((((ab)d)p)p)q \\ &+b(((((ac)d)p)q)q) - d(((((ca)b)p)p)q) + d(((((ba)c)p)q)q) \\ &+d(c((((ab)p)q)q)) - d(b((((ac)q)q)q))\end{aligned} \tag{3.1.20}$$

$$\langle\langle\langle c,d\rangle,a\rangle,b\rangle = (((((ca)b)d)p)p)p - (((((da)b)c)p)p)q - c(((((ab)d)p)p)q \\ +d(((((ab)c)p)q)q) - c(((((ba)d)p)p)q) + d(((((ba)c)p)q)q) \\ +c(b((((ad)p)q)q)) - d(b((((ac)q)q)q)) \tag{3.1.21}$$

$$\langle\langle\langle d,a\rangle,b\rangle,c\rangle = (((((da)b)c)p)p)p - (((((ab)c)d)p)p)q - d(((((ba)c)p)p)q \\ +b(((((ac)b)p)q)q) - d(((((ca)b)p)p)q) + c(((((ab)d)p)q)q) \\ +d(c((((ba)p)q)q)) - c(b((((ad)q)q)q)) \tag{3.1.22}$$

$$\langle\langle\langle d,b\rangle,a\rangle,c\rangle = (((((da)b)c)p)p)p - (((((ba)c)d)p)p)q - d(((((ab)c)p)p)q \\ +b(((((ac)d)p)q)q) - d(((((ca)b)p)p)q) + c(((((ba)d)p)q)q) \\ +d(c((((ab)p)q)q)) - c(b((((ad)q)q)q)) \tag{3.1.23}$$

$$\langle\langle\langle d,c\rangle,a\rangle,b\rangle = (((((da)b)c)p)p)p - (((((ca)b)d)p)p)q - d(((((ab)c)p)p)q \\ +c(((((ab)d)p)q)q) - d(((((ba)c)p)p)q) + c(((((ba)d)p)q)q) \\ +d(b((((ac)p)q)q)) - c(b((((ad)q)q)q)) \tag{3.1.24}$$

$$\langle\langle a,b\rangle,\langle c,d\rangle\rangle = c(((((ab)d)p)p)p) - d(((((ab)c)p)p)q) - c(((((ba)d)p)p)q) \\ +d(((((ba)c)p)q)q) - c(((((ab)d)p)p)q) + c(((((ba)d)p)q)q) \\ +d(((((ab)c)p)q)q) - d(((((ba)c)q)q)q) \tag{3.1.25}$$

$$\langle\langle a,c\rangle,\langle b,d\rangle\rangle = b(((((ac)d)p)p)p) - d(((((ab)c)p)p)q) - c(((((ba)d)p)p)q) \\ +d(((((ca)b)p)q)q) - b(((((ac)d)p)p)q) + c(((((ba)d)p)q)q) \\ +d(((((ab)c)p)q)q) - d(((((ca)b)q)q)q) \tag{3.1.26}$$

$$\langle\langle a,d\rangle,\langle b,c\rangle\rangle = b(((((ac)d)p)p)p) - c(((((ab)d)p)p)q) - d(((((ba)c)p)p)q) \\ +d(((((ca)b)p)q)q) - b(((((ac)d)p)p)q) + d(((((ba)c)p)q)q) \\ +c(((((ab)d)p)q)q) - d(((((ca)b)q)q)q) \tag{3.1.27}$$

$$\langle\langle b,a\rangle,\langle c,d\rangle\rangle = c(((((ba)d)p)p)p) - d(((((ba)c)p)p)q) - c(((((ab)d)p)p)q) \\ +d(((((ab)c)p)q)q) - c(((((ba)d)p)p)q) + c(((((ab)d)p)q)q) \\ +d(((((ba)c)p)q)q) - d(((((ab)c)q)q)q) \tag{3.1.28}$$

$$\langle\langle c,a\rangle,\langle b,d\rangle\rangle = c(((((ba)d)p)p)p) - d(((((ca)b)p)p)q) - b(((((ac)d)p)p)q) \\ +d(((((ab)c)p)q)q) - c(((((ba)d)p)p)q) + b(((((ac)d)p)q)q) \\ +d(((((ca)b)p)q)q) - d(((((ab)c)q)q)q) \tag{3.1.29}$$

$$\langle\langle c,b\rangle,\langle a,d\rangle\rangle = c(((((ab)d)p)p)p) - d(((((ca)b)p)p)q) - b(((((ac)d)p)p)q) \\ +d(((((ba)c)p)q)q) - c(((((ab)d)p)p)q) + b(((((ac)d)p)q)q) \\ +d(((((ca)b)p)q)q) - d(((((ba)c)q)q)q) \tag{3.1.30}$$

$$\langle\langle d,a\rangle,\langle b,c\rangle\rangle = d(((((ba)c)p)p)p) - d(((((ca)b)p)p)q) - b(((((ac)d)p)p)q) \\ +c(((((ab)d)p)q)q) - d(((((ba)c)p)p)q) + b(((((ac)d)p)q)q) \\ +d(((((ca)b)p)q)q) - c(((((ab)d)q)q)q) \tag{3.1.31}$$

$$\langle\langle d,b\rangle,\langle a,c\rangle\rangle = d(((((ab)c)p)p)p) - d(((((ca)b)p)p)q) - b(((((ac)d)p)p)q) \\ +c(((((ba)d)p)q)q) - d(((((ab)c)p)p)q) + b(((((ac)d)p)q)q) \\ +d(((((ca)b)p)q)q) - c(((((ba)d)q)q)q) \tag{3.1.32}$$

$$\langle\langle d,c\rangle,\langle a,b\rangle\rangle = d(((((ab)c)p)p)p) - d(((((ba)c)p)p)q) - c(((((ab)d)p)p)q) \\ +c(((((ba)d)p)q)q) - d(((((ab)c)p)p)q) + c(((((ab)d)p)q)q) \\ +d(((((ba)c)p)q)q) - c(((((ba)d)q)q)q) \tag{3.1.33}$$

$$\langle\langle b,a\rangle,\langle d,c\rangle\rangle = d(((((ba)c)p)p)p) - d(((((ab)c)p)p)q) - c(((((ba)d)p)p)q) \\ +c(((((ab)d)p)q)q) - d(((((ba)c)p)p)q) + d(((((ab)c)p)q)q) \\ +c(((((ba)d)p)q)q) - c(((((ab)d)q)q)q) \tag{3.1.34}$$

$$\langle\langle b,d\rangle,\langle a,c\rangle\rangle = b(((((ac)d)p)p)p) - c(((((ba)d)p)p)q) - d(((((ab)c)p)p)q) \\ +d(((((ca)b)p)q)q) - b(((((ac)d)p)p)q) + c(((((ba)d)p)q)q) \\ +d(((((ab)c)p)q)q) - d(((((ca)b)q)q)q) \tag{3.1.35}$$

$$\langle\langle c,a\rangle,\langle d,b\rangle\rangle = d(((((ca)b)p)p)p) - c(((((ba)d)p)p)q) - d(((((ab)c)p)p)q) \\ +b(((((ac)d)p)q)q) - d(((((ca)b)p)p)q) + d(((((ab)c)p)q)q) \\ +c(((((ba)d)p)q)q) - b(((((ac)d)q)q)q) \tag{3.1.36}$$

$$\langle\langle a,\langle b,c\rangle\rangle,d\rangle = b(((((ac)d)p)p)p) - c(((((ab)d)p)p)q) - (((((ba)c)d)p)p)q \\ +(((((ca)b)d)p)q)q - d(b((((ac)p)p)q) + d(c((((ab)p)q)q) \\ +d(((((ba)c)p)q)q) - d(((((ca)b)q)q)q) \tag{3.1.37}$$

$$\langle\langle a,\langle b,d\rangle\rangle,c\rangle = b(((((ac)d)p)p)p) - d(((((ab)c)p)p)q) - (((((ba)c)d)p)p)q \\ +(((((da)b)c)p)q)q - c(b((((ad)p)p)q) + d(c((((ab)p)q)q) \\ +c(((((ba)d)p)q)q) - d(((((ca)b)q)q)q) \tag{3.1.38}$$

$$\langle\langle a,\langle c,b\rangle\rangle,d\rangle = c(((((ab)d)p)p)p) - b(((((ac)d)p)p)q) - (((((ca)b)d)p)p)q \\ +(((((ba)c)d)p)q)q - d(c((((ab)p)p)q) + d(b((((ac)p)q)q) \\ +d(((((ca)b)p)q)q) - d(((((ba)c)q)q)q) \tag{3.1.39}$$

$$\langle\langle a,\langle c,d\rangle\rangle,b\rangle = c(((((ab)d)p)p)p) - d(((((ab)c)p)p)q) - (((((ca)b)d)p)p)q \\ +(((((da)b)c)p)q)q - c(b((((ad)p)p)q) + d(b((((ac)p)q)q) \\ +c(((((ba)d)p)q)q) - d(((((ba)c)q)q)q) \tag{3.1.40}$$

$$\langle\langle a,\langle d,b\rangle\rangle,c\rangle = d(((((ab)c)p)p)p) - b(((((ac)d)p)p)q) - (((((da)b)c)p)p)q \\ +(((((ba)c)d)p)q)q - d(c((((ab)p)p)q) + c(b((((ad)p)q)q) \\ +d(((((ca)b)p)q)q) - c(((((ba)d)q)q)q) \tag{3.1.41}$$

$$\langle\langle a,\langle d,c\rangle\rangle,b\rangle = d(((((ab)c)p)p)p) - c(((((ab)d)p)p)q) - (((((da)b)c)p)p)q \\ +(((((ca)b)d)p)q)q - d(b((((ac)p)p)q) + c(b((((ad)p)q)q) \\ +d(((((ba)c)p)q)q) - c(((((ba)d)q)q)q) \tag{3.1.42}$$

$$\langle\langle b,\langle a,c\rangle\rangle,d\rangle = b(((((ac)d)p)p)p) - c(((((ba)d)p)p)q) - (((((ab)c)d)p)p)q \\ +(((((ca)b)d)p)q)q - d(b((((ac)p)p)q) + d(c((((ba)p)q)q) \\ +d(((((ab)c)p)q)q) - d(((((ca)b)q)q)q) \tag{3.1.43}$$

$$\langle\langle b,\langle a,d\rangle\rangle,c\rangle = b(((((ac)d)p)p)p) - d(((((ba)c)p)p)q) - (((((ab)c)d)p)p)q \\ +(((((da)b)c)p)q)q - c(b((((ad)p)p)q) + d(c((((ba)p)q)q) \\ +c(((((ab)d)p)q)q) - d(((((ca)b)q)q)q) \tag{3.1.44}$$

$$\langle\langle b,\langle d,a\rangle\rangle,c\rangle = d(((((ba)c)p)p)p) - b(((((ac)d)p)p)q) - (((((da)b)c)p)p)q \\ +(((((ab)c)d)p)q)q - d(c((((ba)p)p)q) + c(b((((ad)p)q)q) \\ +d(((((ca)b)p)q)q) - c(((((ab)d)q)q)q) \tag{3.1.45}$$

$$\langle\langle b,\langle d,c\rangle\rangle,a\rangle = d(((((ba)c)p)p)p) - c(((((ba)d)p)p)q) - (((((da)b)c)p)p)q \\ +(((((ca)b)d)p)q)q - d(b((((ac)p)p)q) + c(b((((ad)p)q)q) \\ +d(((((ab)c)p)q)q) - c(((((ab)d)q)q)q) \tag{3.1.46}$$

$$\langle\langle b,\langle c,a\rangle\rangle,d\rangle = c(((((ba)d)p)p)p) - b(((((ac)d)p)p)q) - (((((ca)b)d)p)p)q \\ +(((((ab)c)d)p)q)q - d(c((((ba)p)p)q) + d(b((((ac)p)q)q) \\ +d(((((ca)b)p)q)q) - d(((((ab)c)q)q)q) \tag{3.1.47}$$

$$\langle\langle b,\langle c,d\rangle\rangle,a\rangle = c(((((ba)d)p)p)p) - d(((((ba)c)p)p)q) - (((((ca)b)d)p)p)q \\ +(((((da)b)c)p)q)q - c(b((((ad)p)p)q) + d(b((((ac)p)q)q) \\ +c(((((ab)d)p)q)q) - d(((((ab)c)q)q)q) \tag{3.1.48}$$

$$\langle\langle c,\langle a,b\rangle\rangle,d\rangle = c(((((ab)d)p)p)p) - c(((((ba)d)p)p)q) - (((((ab)c)d)p)p)q \\ +(((((ba)c)d)p)q)q - d(c((((ab)p)p)q) + d(c((((ba)p)q)q) \\ +d(((((ab)c)p)q)q) - d(((((ba)c)q)q)q) \tag{3.1.49}$$

$$\langle\langle c,\langle a,d\rangle\rangle,b\rangle = c(((((ab)d)p)p)p) - d(((((ca)b)p)p)q) - (((((ab)c)d)p)p)q \\ +(((((da)b)c)p)q)q - c(b((((ad)p)p)q) + d(b((((ca)p)q)q) \\ +b(((((ac)d)p)q)q) - b(((((da)c)q)q)q) \tag{3.1.50}$$

$$\langle\langle c,\langle b,a\rangle\rangle,b\rangle = c(((((ba)d)p)p)p) - c(((((ab)d)p)p)q) - (((((ba)c)d)p)p)q \\ +(((((ab)c)d)p)q)q - d(c((((ba)p)p)q) + d(c((((ab)p)q)q) \\ +d(((((ba)c)p)q)q) - d(((((ab)c)q)q)q) \tag{3.1.51}$$

$$\langle\langle c,\langle b,d\rangle\rangle,a\rangle = c(((((ba)d)p)p)p) - d(((((ca)b)p)p)q) - (((((ba)c)d)p)p)q \\ +(((((da)b)c)p)q)q - c(b((((ad)p)p)q) + d(c((((ab)p)q)q) \\ +b(((((ac)d)p)q)q) - d(((((ab)c)q)q)q) \tag{3.1.52}$$

$$\langle\langle c,\langle d,a\rangle\rangle,b\rangle = d(((((ca)b)p)p)p) - c(((((ab)d)p)p)q) - (((((da)b)c)p)p)q \\ +(((((ab)c)d)p)q)q - d(c((((ba)p)p)q) + c(b((((ad)p)q)q) \\ +d(((((ba)c)p)q)q) - b(((((ac)d)q)q)q) \tag{3.1.53}$$

$$\langle\langle c,\langle d,b\rangle\rangle,a\rangle = d(((((ca)b)p)p)p) - c(((((ba)d)p)p)q) - (((((da)b)c)p)p)q \\ +(((((ba)c)d)p)q)q - d(c((((ab)p)p)q) + c(b((((ad)p)q)q) \\ +d(((((ab)c)p)q)q) - b(((((ac)d)q)q)q) \tag{3.1.54}$$

$$\langle\langle d,\langle a,b\rangle\rangle,c\rangle = d(((((ab)c)p)p)p) - d(((((ba)c)p)p)q) - (((((ab)c)d)p)p)q \\ +(((((ba)c)d)p)q)q - d(c((((ab)p)p)q) + d(c((((ba)p)q)q) \\ +c(((((ab)d)p)q)q) - c(((((ba)d)q)q)q) \tag{3.1.55}$$

$$\langle\langle d,\langle a,c\rangle\rangle,b\rangle = d(((((ab)c)p)p)p) - d(((((ca)b)p)p)q) - (((((ab)c)d)p)p)q \\ +(((((ca)b)d)p)q)q - d(b((((ac)p)p)q) + d(c((((ba)p)q)q) \\ +b(((((ac)d)p)q)q) - c(((((ba)d)q)q)q) \tag{3.1.56}$$

$$\langle\langle d,\langle b,a\rangle\rangle,c\rangle = d(((((ba)c)p)p)p) - d(((((ab)c)p)p)q) - (((((ba)c)d)p)p)q \\ +(((((ab)c)d)p)q)q - d(c((((ba)p)p)q) + d(c((((ab)p)q)q) \\ +c(((((ba)d)p)q)q) - a(((((bc)d)q)q)q) \tag{3.1.57}$$

$$\langle\langle d,\langle b,c\rangle\rangle,a\rangle = d(((((ba)c)p)p)p) - d(((((ca)b)p)p)q) - (((((ba)c)d)p)p)q \\ +(((((ca)b)d)p)q)q - d(b((((ac)p)p)q) + d(c((((ab)p)q)q) \\ +b(((((ac)d)p)q)q) - c(((((ab)d)q)q)q) \tag{3.1.58}$$

$$\langle\langle d,\langle c,a\rangle\rangle,b\rangle = d(((((ca)b)p)p)p) - d(((((ab)c)p)p)q) - (((((ca)b)d)p)p)q \\ +(((((ab)c)d)p)q)q - d(c((((ba)p)p)q) + d(b((((ac)p)q)q) \\ +c(((((ba)d)p)q)q) - b(((((ac)d)q)q)q) \tag{3.1.59}$$

$$\langle\langle d,\langle c,b\rangle\rangle,a\rangle = d(((((ca)b)p)p)p) - d(((((ba)c)p)p)q) - (((((ca)b)d)p)p)q \\ +(((((ba)c)d)p)q)q - d(c((((ab)p)p)q) + d(b((((ac)p)q)q) \\ +c(((((ab)d)p)q)q) - b(((((ac)d)q)q)q) \tag{3.1.60}$$

$$\langle a,\langle\langle b,c\rangle,d\rangle\rangle = b(((((ac)d)p)p)p) - c(((((ab)d)p)p)q) - d(b((((ac)p)p)q) \\ +d(c((((ab)p)q)q) - (((((ba)c)d)p)p)q + (((((ca)b)d)p)q)q \\ +d(((((ba)c)p)q)q) - d(((((ca)b)q)q)q) \tag{3.1.61}$$

$$\langle a,\langle\langle b,d\rangle,c\rangle\rangle = b(((((ac)d)p)p)p) - d(((((ab)c)p)p)q) - c(b((((ad)p)p)q) \\ +d(c((((ab)p)q)q) - (((((ba)c)d)p)p)q + (((((da)b)c)p)q)q \\ +c(((((ba)d)p)q)q) - d(((((ca)b)q)q)q) \tag{3.1.62}$$

$$\begin{aligned}\langle a, \langle\langle c, b\rangle, d\rangle\rangle = c(((((ab)d)p)p)p) - b(((((ac)d)p)p)q) - d(c((((ab)p)p)q)\\ +d(b((((ac)p)q)q) - (((((ca)b)d)p)p)q + (((((ba)c)d)p)q)q\\ +d(((((ca)b)p)q)q) - d(((((ba)c)q)q)q)\end{aligned} \tag{3.1.63}$$

$$\begin{aligned}\langle a, \langle\langle c, d\rangle, b\rangle\rangle = c(((((ab)d)p)p)p) - d(((((ab)c)p)p)q) - c(b((((ad)p)p)q)\\ +d(b((((ac)p)q)q) - (((((ca)b)d)p)p)q + (((((da)b)c)p)q)q\\ +c(((((ba)d)p)q)q) - d(((((ba)c)q)q)q)\end{aligned} \tag{3.1.64}$$

$$\begin{aligned}\langle a, \langle\langle d, b\rangle, c\rangle\rangle = d(((((ab)c)p)p)p) - b(((((ac)d)p)p)q) - d(c((((ab)p)p)q)\\ +c(b((((ad)p)q)q) - (((((da)b)c)p)p)q + (((((ba)c)d)p)q)q\\ +d(((((ca)b)p)q)q) - c(((((ba)d)q)q)q)\end{aligned} \tag{3.1.65}$$

$$\begin{aligned}\langle a, \langle\langle d, c\rangle, b\rangle\rangle = d(((((ab)c)p)p)p) - c(((((ab)d)p)p)q) - d(b((((ac)p)p)q)\\ +c(b((((ad)p)q)q) - (((((da)b)c)p)p)q + (((((ca)b)d)p)q)q\\ +d(((((ba)c)p)q)q) - c(((((ba)b)q)q)q)\end{aligned} \tag{3.1.66}$$

$$\begin{aligned}\langle b, \langle\langle a, c\rangle, d\rangle\rangle = b(((((ac)d)p)p)p) - c(((((ba)d)p)p)q) - d(b((((ac)p)p)q)\\ +d(b((((ca)p)q)q) - (((((ab)c)d)p)p)q + (((((ca)b)d)p)q)q\\ +d(((((ab)c)p)q)q) - d(((((ca)b)q)q)q)\end{aligned} \tag{3.1.67}$$

$$\begin{aligned}\langle b, \langle\langle a, d\rangle, c\rangle\rangle = b(((((ac)d)p)p)p) - d(((((ca)b)p)p)q) - c(b((((ad)p)p)q)\\ +d(c((((ba)p)q)q) - (((((ab)c)d)p)p)q + (((((da)b)c)p)q)q\\ +c(((((ab)d)p)q)q) - d(((((ca)b)q)q)q)\end{aligned} \tag{3.1.68}$$

$$\begin{aligned}\langle b, \langle\langle c, a\rangle, d\rangle\rangle = c(((((ba)d)p)p)p) - b(((((ac)d)p)p)q) - d(c((((ba)p)p)q)\\ +d(b((((ac)p)q)q) - (((((ca)b)d)p)p)q + (((((ab)c)d)p)q)q\\ +d(((((ca)b)p)q)q) - d(((((ab)c)q)q)q)\end{aligned} \tag{3.1.69}$$

$$\langle b, \langle\langle c, d\rangle, a\rangle\rangle = c(((((ba)d)p)p)p) - d(((((ba)c)p)p)q) - c(b((((ad)p)p)q) + d(b((((ac)p)q)q) - (((((ca)b)d)p)p)q + (((((da)b)c)p)q)q + c(((((ab)d)p)q)q) - d(((((ab)c)q)q)q) \tag{3.1.70}$$

$$\langle b, \langle\langle d, a\rangle, c\rangle\rangle = d(((((ba)c)p)p)p) - b(((((ac)d)p)p)q) - d(c((((ba)p)p)q) + c(b((((ad)p)q)q) - (((((da)b)c)p)p)q + (((((ab)c)d)p)q)q + d(((((ca)b)p)q)q) - c(((((ab)d)q)q)q) \tag{3.1.71}$$

$$\langle b, \langle\langle d, c\rangle, a\rangle\rangle = d(((((ba)c)p)p)p) - c(((((ba)d)p)p)q) - d(b((((ac)p)p)q) + c(b((((ad)p)q)q) - (((((da)b)c)p)p)q + (((((ca)b)d)p)q)q + d(((((ab)c)p)q)q) - c(((((ab)d)q)q)q) \tag{3.1.72}$$

$$\langle c, \langle\langle a, b\rangle, d\rangle\rangle = c(((((ab)d)p)p)p) - c(((((ba)d)p)p)q) - d(c((((ab)p)p)q) + d(c((((ba)p)q)q) - (((((ab)c)d)p)p)q + (((((ba)c)d)p)q)q + d(((((ab)c)p)q)q) - d(((((ba)c)q)q)q) \tag{3.1.73}$$

$$\langle c, \langle\langle a, d\rangle, b\rangle\rangle = c(((((ab)d)p)p)p) - d(((((ca)b)p)p)q) - c(b((((ad)p)p)q) + d(c((((ba)p)q)q) - (((((ab)c)d)p)p)q + (((((da)b)c)p)q)q + b(((((ac)d)p)q)q) - d(((((ba)c)q)q)q) \tag{3.1.74}$$

$$\langle c, \langle\langle b, a\rangle, d\rangle\rangle = c(((((ba)d)p)p)p) - c(((((ab)d)p)p)q) - d(c((((ba)p)p)q) + d(c((((ab)p)q)q) - (((((ba)c)d)p)p)q + (((((ab)c)d)p)q)q + d(((((ba)c)p)q)q) - d(((((ab)c)q)q)q) \tag{3.1.75}$$

$$\langle c, \langle\langle b, d\rangle, a\rangle\rangle = c(((((ba)d)p)p)p) - d(((((ca)b)p)p)q) - c(b((((ad)p)p)q) + d(c((((ab)p)q)q) - (((((ba)c)d)p)p)q + (((((da)b)c)p)q)q + b(((((ac)d)p)q)q) - d(((((ab)c)q)q)q) \tag{3.1.76}$$

$$\langle c, \langle\langle d, a\rangle, b\rangle\rangle = d(((((ca)b)p)p)p) - c(((((ab)d)p)p)q) - d(c((((ba)p)p)q) + c(b((((ad)p)q)q) - (((((da)b)c)p)p)q + (((((ab)c)d)p)q)q + d(((((ba)c)p)q)q) - b(((((ac)b)q)q)q) \tag{3.1.77}$$

$$\langle c, \langle\langle d, b\rangle, a\rangle\rangle = d(((((ca)b)p)p)p) - c(((((ba)d)p)p)q) - d(c((((ab)p)p)q) + c(b((((ad)p)q)q) - (((((da)b)c)p)p)q + (((((ba)c)d)p)q)q + d(((((ab)c)p)q)q) - b(((((ac)d)q)q)q) \tag{3.1.78}$$

$$\langle d, \langle\langle a, b\rangle, c\rangle\rangle = d(((((ab)c)p)p)p) - d(((((ba)c)p)p)q) - d(c((((ab)p)p)q) + d(c((((ba)p)q)q) - (((((ab)c)d)p)p)q + (((((ba)c)d)p)q)q + c(((((ab)d)p)q)q) - c(((((ba)d)q)q)q) \tag{3.1.79}$$

$$\langle d, \langle\langle a, c\rangle, b\rangle\rangle = d(((((ab)c)p)p)p) - d(((((ca)b)p)p)q) - d(b((((ac)p)p)q) + d(c((((ba)p)q)q) - (((((ab)c)d)p)p)q + (((((ca)b)d)p)q)q + b(((((ac)d)p)q)q) - c(((((ba)d)q)q)q) \tag{3.1.80}$$

$$\langle d, \langle\langle b, a\rangle, c\rangle\rangle = d(((((ba)c)p)p)p) - d(((((ab)c)p)p)q) - d(c((((ba)p)p)q) + d(c((((ab)p)q)q) - (((((ba)c)d)p)p)q + (((((ab)c)d)p)q)q + c(((((ba)d)p)q)q) - c(((((ab)d)q)q)q) \tag{3.1.81}$$

$$\langle d, \langle\langle b, c\rangle, a\rangle\rangle = d(((((ba)c)p)p)p) - d(((((ca)b)p)p)q) - d(b((((ac)p)p)q) + d(c((((ab)p)q)q) - (((((ba)c)d)p)p)q + (((((ca)b)d)p)q)q + b(((((ac)d)p)q)q) - c(((((ab)d)q)q)q) \tag{3.1.82}$$

$$\langle d, \langle\langle c, a\rangle, b\rangle\rangle = d(((((ca)b)p)p)p) - d(((((ab)d)p)p)q) - d(c((((ba)p)p)q) + d(b((((ac)p)q)q) - (((((ca)b)d)p)p)q + (((((ab)c)d)p)q)q + c(((((ba)d)p)q)q) - b(((((ac)d)q)q)q) \tag{3.1.83}$$

$$\langle d, \langle\langle c, b\rangle, a\rangle\rangle = d(((((ca)b)p)p)p) - d(((((ba)c)p)p)q) - d(c((((ab)p)p)q) \\ +d(b((((ac)p)q)q) - (((((ca)b)d)p)p)q + (((((ba)c)d)p)q)q \\ +c(((((ab)d)p)q)q) - b(((((ac)d)q)q)q) \tag{3.1.84}$$

$$\langle a, \langle b, \langle c, d\rangle\rangle\rangle = c(b((((ad)p)p)p) - d(b((((ac)p)p)q) - c(((((ab)d)p)p)q) \\ +d(((((ab)c)p)q)q) - c(((((ba)d)p)p)q) + d(((((ba)c)p)q)q) \\ +(((((ca)b)d)p)q)q - (((((da)b)c)q)q)q \tag{3.1.85}$$

$$\langle a, \langle b, \langle d, c\rangle\rangle\rangle = d(b((((ac)p)p)p) - c(b((((ad)p)p)q) - d(((((ab)c)p)p)q) \\ +c(((((ab)d)p)q)q) - d(((((ba)c)p)p)q) + c(((((ba)d)p)q)q) \\ +(((((da)b)c)p)q)q - (((((ca)b)d)q)q)q \tag{3.1.86}$$

$$\langle a, \langle c, \langle b, d\rangle\rangle\rangle = c(b((((ad)p)p)p) - d(b((((ac)p)p)q) - c(((((ab)d)p)p)q) \\ +d(((((ab)c)p)q)q) - c(((((ba)d)p)p)q) + d(((((ba)c)p)q)q) \\ +(((((ca)b)d)p)q)q - (((((da)b)c)q)q)q \tag{3.1.87}$$

$$\langle a, \langle c, \langle d, b\rangle\rangle\rangle = d(c((((ab)p)p)p) - c(b((((ad)p)p)q) - d(((((ab)c)p)p)q) \\ +b(((((ac)d)p)q)q) - d(((((ca)b)p)p)q) + c(((((ba)d)p)q)q) \\ +(((((da)b)c)p)q)q - (((((ba)c)d)q)q)q \tag{3.1.88}$$

$$\langle a, \langle d, \langle b, c\rangle\rangle\rangle = d(b((((ac)p)p)p) - d(c((((ab)p)p)q) - b(((((ac)d)p)p)q) \\ +c(((((ab)d)p)q)q) - d(((((ba)c)p)p)q) + d(((((ca)b)p)q)q) \\ +(((((ba)c)d)p)q)q - (((((ca)b)d)q)q)q \tag{3.1.89}$$

$$\langle a, \langle d, \langle c, b\rangle\rangle\rangle = d(c((((ab)p)p)p) - d(b((((ac)p)p)q) - c(((((ab)d)p)p)q) \\ +b(((((ac)d)p)q)q) - d(((((ca)b)p)p)q) + d(((((ba)c)p)q)q) \\ +(((((ca)b)d)p)q)q - (((((ba)c)d)q)q)q \tag{3.1.90}$$

$$\langle b, \langle c, \langle a, d\rangle\rangle\rangle = c(b((((ad)p)p)p) - d(c((((ba)p)p)q) - b(((((ac)d)p)p)q) \\ +d(((((ba)c)p)q)q) - c(((((ab)d)p)p)q) + d(((((ca)b)p)q)q) \\ +(((((ab)c)d)p)q)q - (((((da)b)c)q)q)q \tag{3.1.91}$$

$$\langle b, \langle c, \langle d, a\rangle\rangle\rangle = d(c((((ba)p)p)p) - c(b((((ad)p)p)q) - d(((((ba)c)p)p)q) \\ +b(((((ac)d)p)q)q) - d(((((ca)b)p)p)q) + c(((((ab)d)p)q)q) \\ +(((((da)b)c)p)q)q - (((((ab)c)d)q)q)q \tag{3.1.92}$$

$$\langle b, \langle d, \langle a, c\rangle\rangle\rangle = d(b((((ac)p)p)p) - d(c((((ba)p)p)q) - b(((((ac)d)p)p)q) \\ +c(((((ba)d)p)q)q) - d(((((ab)c)p)p)q) + d(((((ca)b)p)q)q) \\ +(((((ab)c)d)p)q)q - (((((ca)b)d)q)q)q \tag{3.1.93}$$

$$\langle b, \langle d, \langle c, a\rangle\rangle\rangle = d(c((((ba)p)p)p) - d(b((((ac)p)p)q) - c(((((ba)d)p)p)q) \\ +b(((((ac)d)p)q)q) - d(((((ca)b)p)p)q) + d(((((ab)c)p)q)q) \\ +(((((ca)b)d)p)q)q - (((((ab)c)d)q)q)q \tag{3.1.94}$$

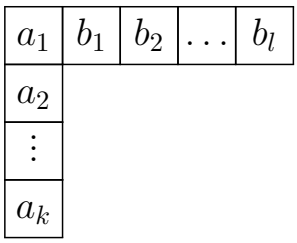
$$\langle c, \langle d, \langle a, b\rangle\rangle\rangle = d(c((((ab)p)p)p) - d(c((((ba)p)p)q) - c(((((ab)d)p)p)q) \\ +c(((((ba)d)p)q)q) - d(((((ab)c)p)p)q) + d(((((ba)c)p)q)q) \\ +(((((ab)c)d)p)q)q - (((((ba)c)d)q)q)q \tag{3.1.95}$$

$$\langle c, \langle d, \langle b, a\rangle\rangle\rangle = d(c((((ba)p)p)p) - d(c((((ab)p)p)q) - c(((((ba)d)p)p)q) \\ +c(((((ab)d)p)q)q) - d(((((ba)c)p)p)q) + d(((((ab)c)p)q)q) \\ +(((((ba)c)d)p)q)q - (((((ab)c)d)q)q)q \tag{3.1.96}$$

4. Basis elements in bicommutative algebras

4.1 Young diagrams

Let's consider an ordered set X. We investigate Young diagrams of two specific forms: (n) and $(n-k, 1^k)$, where k ranges from 1 to $n-2$, aiming to identify basis elements at degree n. These Young diagrams are populated with elements from X, satisfying conditions such as $a_1 \leq a_2 \leq \cdots \leq a_k$ and $b_1 \leq b_2 \leq \cdots \leq b_k$, with both k and l being positive integers for $a_1, \ldots, a_k, b_1, \ldots, b_l \in X$. These arrangements can then be mapped to monomials representing bicommutative base elements as follows.



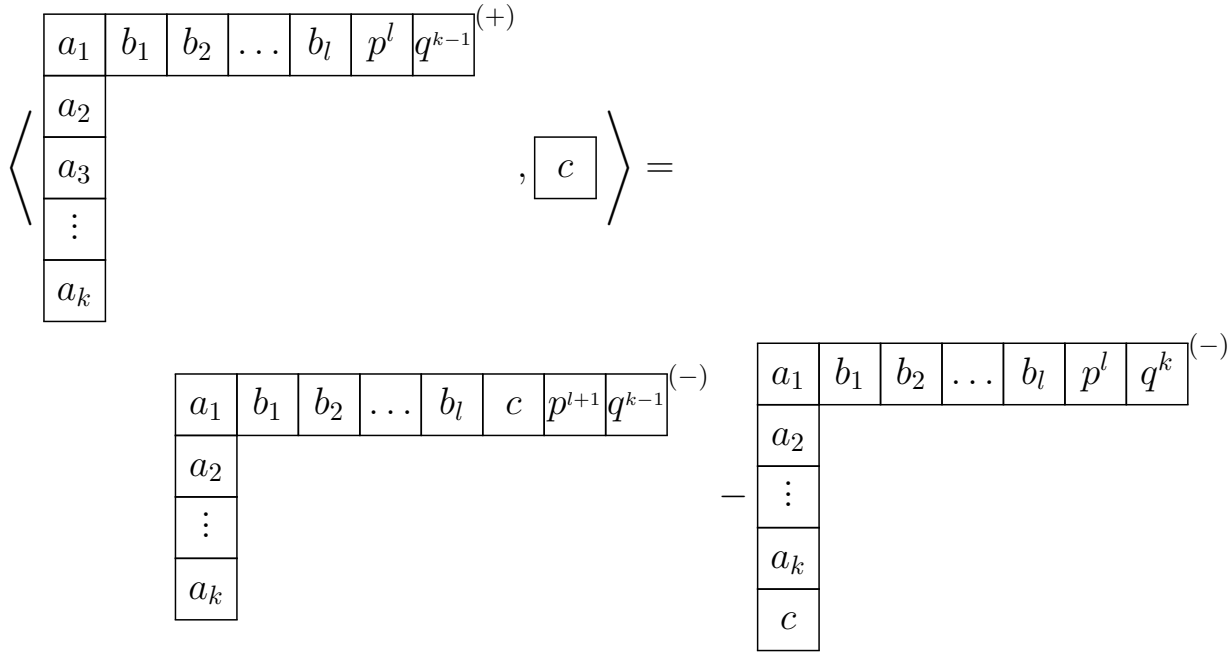
$$\longmapsto a_k(\ldots(a_2((\ldots((a_1b_1)b_2)\ldots)b_l))\ldots).$$

We present the construction of the multilinear base elements of $Bicom(\{a, b, c\})$ as an example and suppose that $a < b < c$.

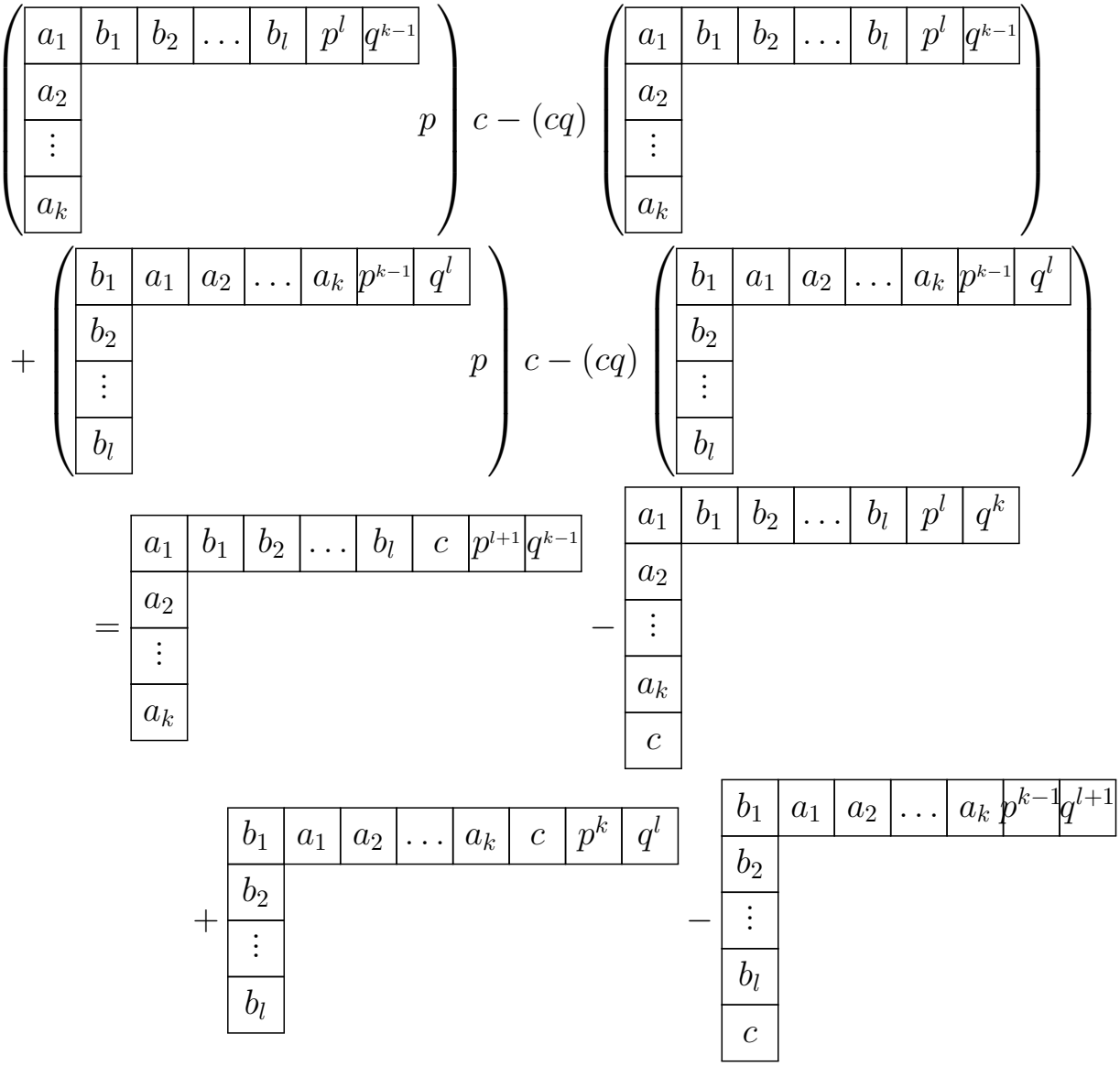
$$\boxed{a}\boxed{b}\boxed{c} \longmapsto (ab)c, \quad \boxed{b}\boxed{a}\boxed{c} \longmapsto (ba)c, \quad \boxed{c}\boxed{a}\boxed{b} \longmapsto (ca)b$$

$$\begin{array}{|c|c|}\hline a & c \\ \hline b \\ \cline{1-1}\end{array} \longmapsto b(ac), \quad \begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array} \longmapsto c(ab), \quad \begin{array}{|c|c|}\hline b & a \\ \hline c \\ \cline{1-1}\end{array} \longmapsto c(ba).$$

Proposition 1. $k + l = n$

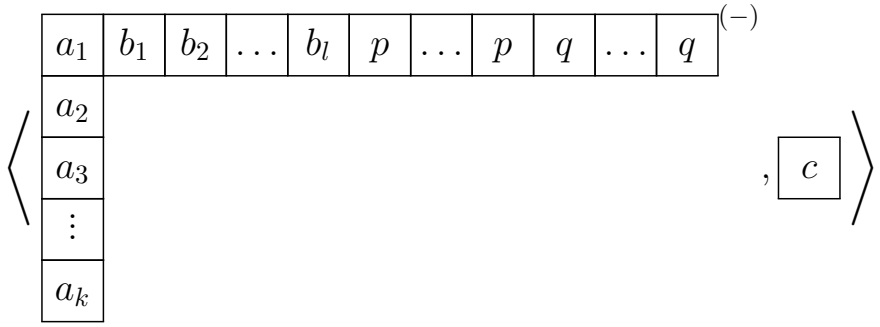


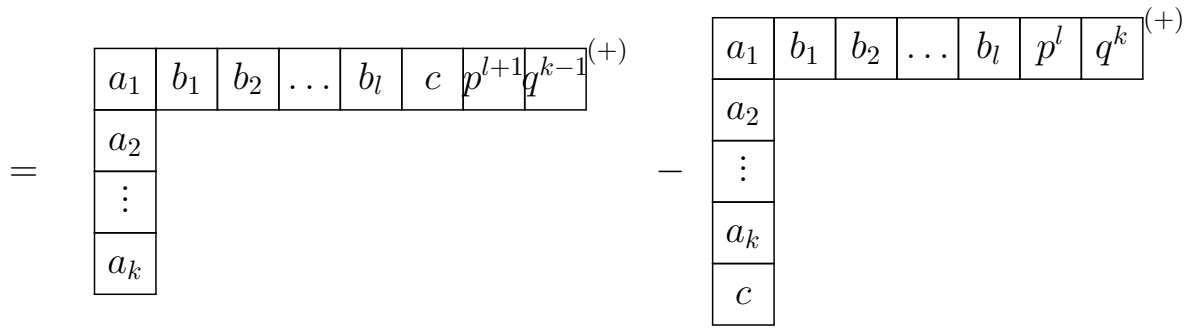
Proof. It is easy to see that



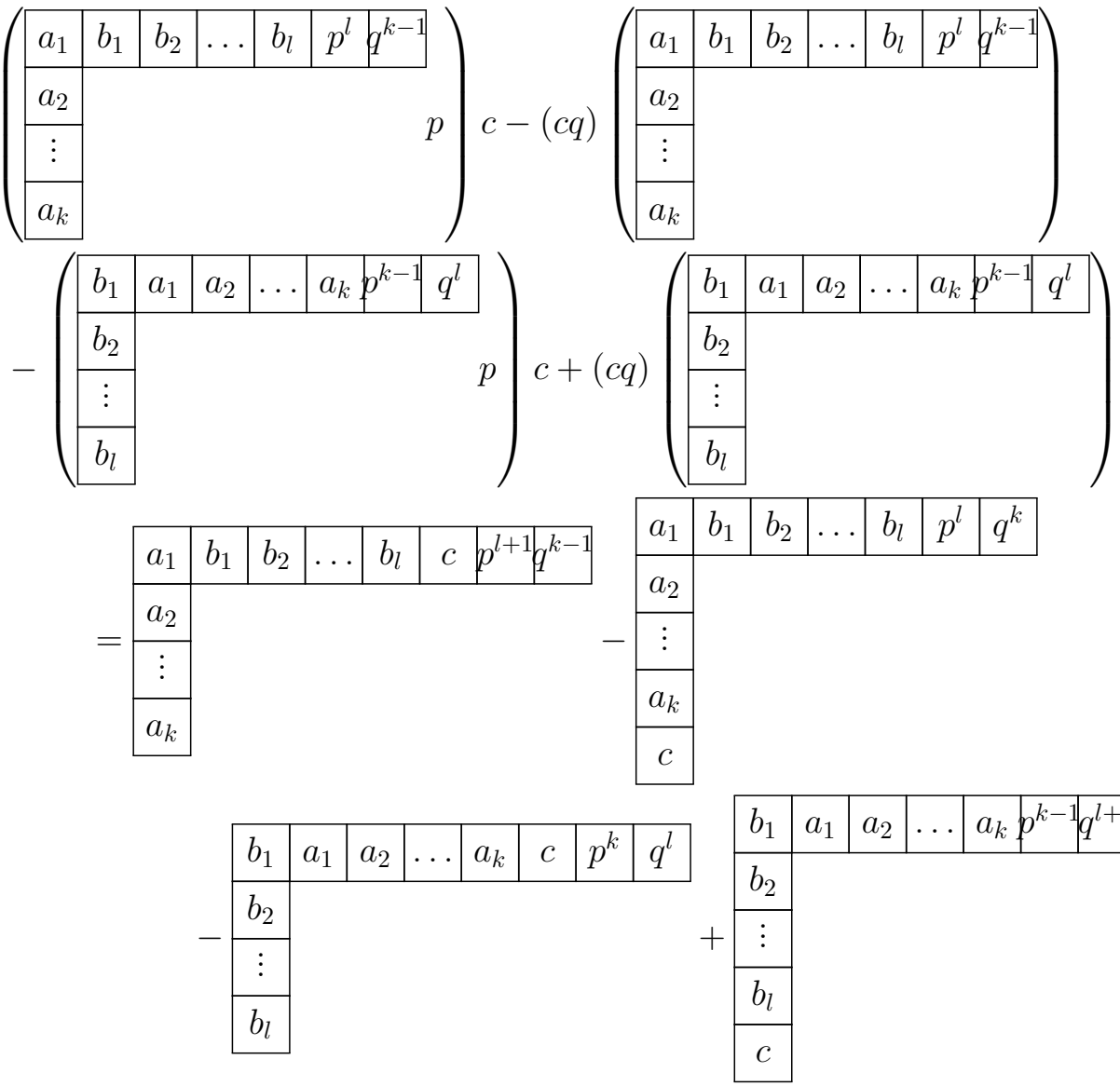
where $k+l=n$. □

Proposition 2. $k+l=n$





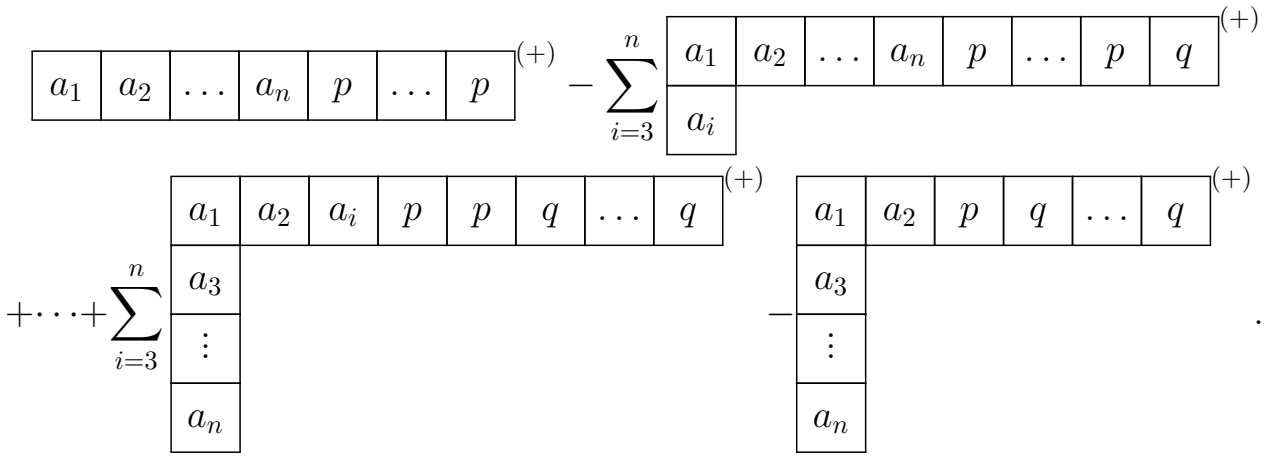
Proof. It is easy to see that



where $k + l = n$ □

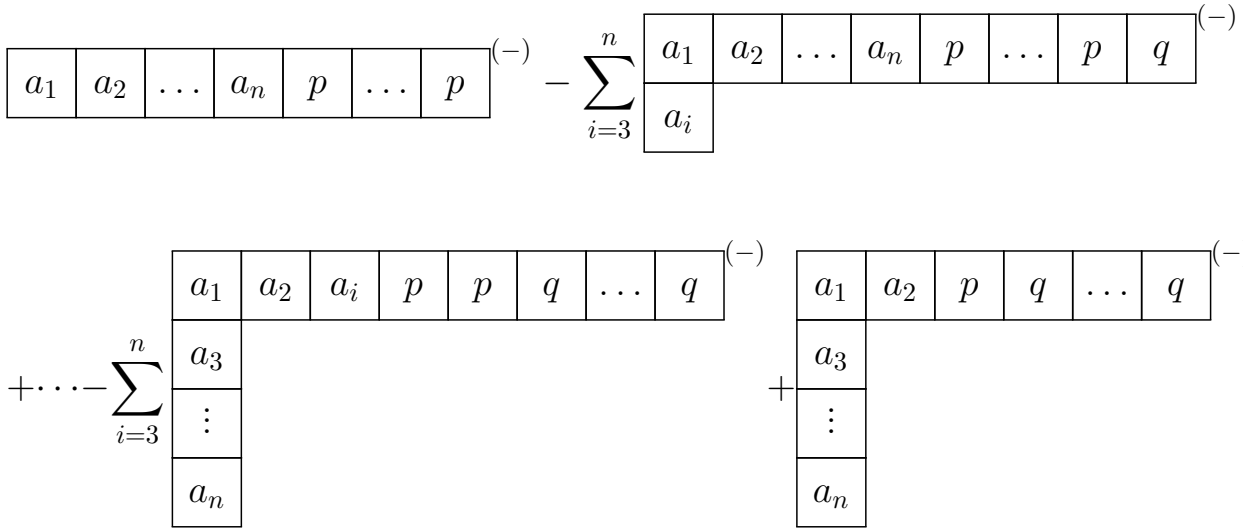
Lemma 1. If n is odd, then

$$\langle\langle\langle\langle a_1, a_2\rangle, a_3\rangle ...\rangle, a_n\rangle =$$



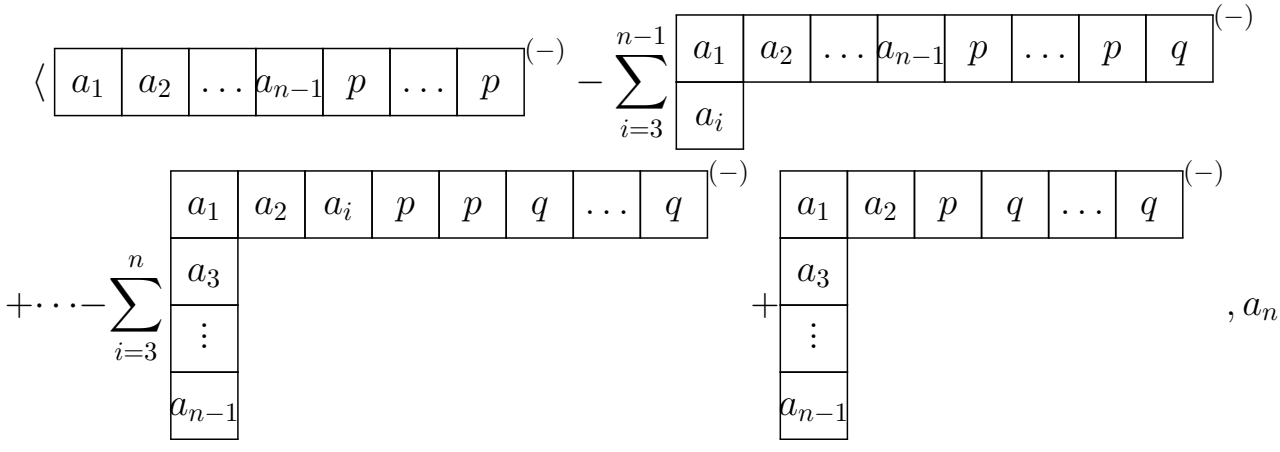
If n is even, then

$$\langle\langle \dots \langle\langle a_1, a_2\rangle, a_3\rangle \dots \rangle, a_n\rangle =$$

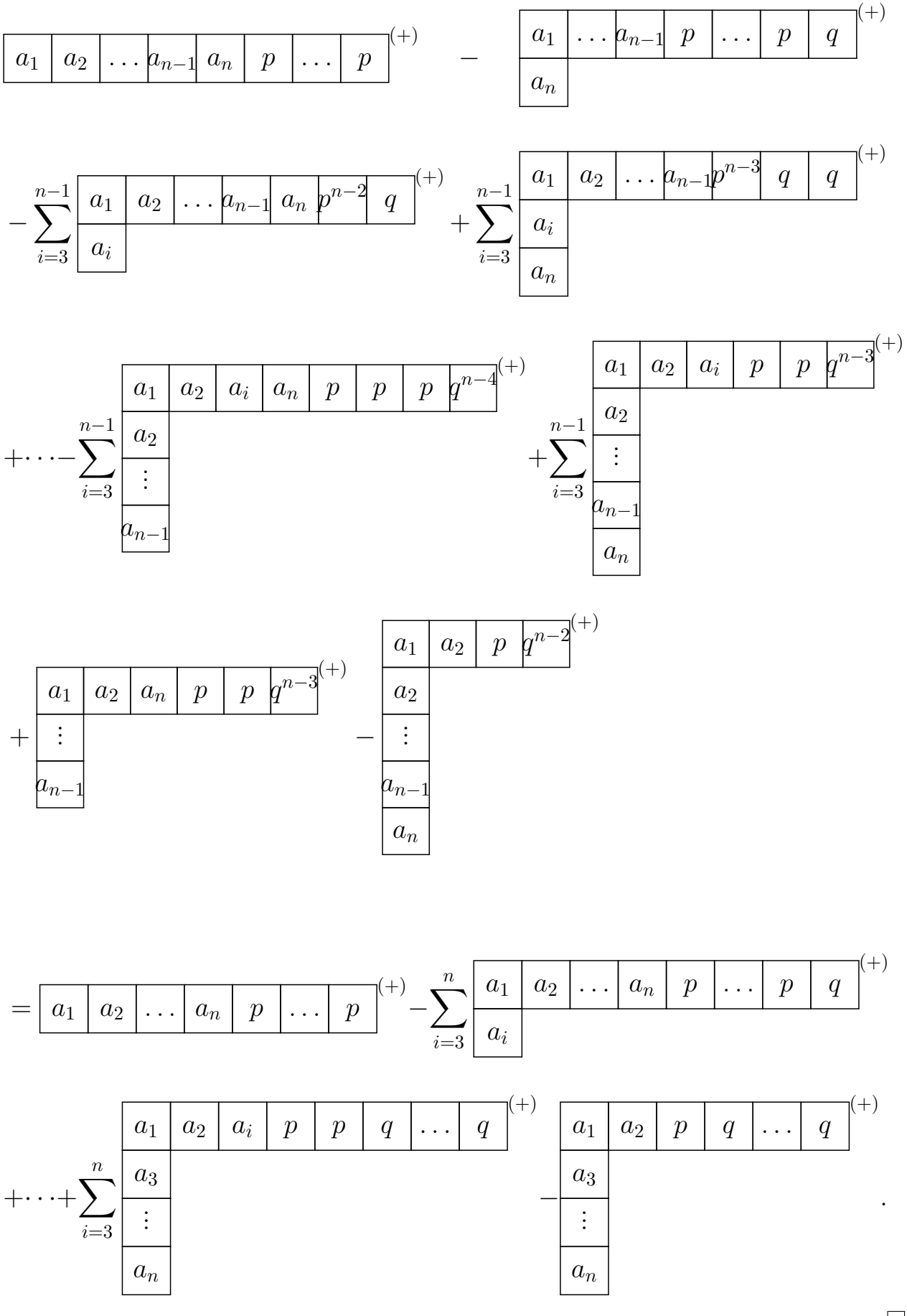


Proof. Assume that the above formulas true for $k < n$. So, $k = n - 1$. Let $n - 1$ is even, then n is odd and

$$\langle\langle \dots \langle\langle a_1, a_2\rangle, a_3\rangle, \dots, a_{n-1}\rangle, a_n\rangle =$$



By Prop 1 and Prop 2:



□

5. Conclusion

In conclusion, we consider bicommutative algebras under mutation products. We obtain that any bicommutative algebra under the mutation product satisfies Lie-admissible identity, which follows from two independent identities of degree three. Moreover, we obtain all identities of degree four and our article has been submitted to the journal "SDU Bulletin".

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АНЫҚТАМА

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